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Closing the Duality Gap in Linear Vector Optimization

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Using a set-valued dual cost function we give a new approach to duality theory for linear vector optimization problems. We develop the theory very close to the scalar case. Especially, in contrast to known results, we avoid the appearance of a duality gap in case of $b = 0$. Examples are given.

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1. Introduction

For many reasons, duality assertions are very important in optimization theory from the theoretical as well as from the numerical point of view. The duality theory of linear programming may serve as a model of what one can expect in the best case: A dual program can be stated explicitly and if it is solvable, so is the original problem, and vice versa. In this case, the optimal values coincide, i.e. no duality gap occurs. Establishing results of this type was an important task of linear programming theory right from the beginning; compare e.g. G. L. Dantzig's book [5].

Duality for multicriteria optimization problems is more complicated than for single objective optimization problems. Usually, duality theorems are proved by scalarizing the cost function. Proceeding in this way, it is possible to apply duality assertions for scalar optimization problems and finally, the conclusions have to be translated back into the context of the vector-valued case. The last step often requires some additional assumptions.

The first attempts to duality in linear vectorial programming seems to be Kornbluth [13], Isermann [10], [11], Rödder [19], and with improvements and generalizations to convex problems, Brumelle [1]. More recent expositions are Jahn [15], Göpfert/Nehse [8] and Ehrgott [6]. But until now, there is no satisfactory strong duality theorem. Compare the counterexamples in [15], p. 204f and [1].

In this note, we extend the duality theory in linear vectorial programming by using a Lagrangean approach without any scalarization. Our dual problem has set-valued nature because the "inner" problem is the problem to determine a set of efficient elements de-

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pending on a dual variable, and solving the "outer" problem means that one has to find a set of efficient points which is maximal in some sense. Regarding this set-valued nature, the aim of our paper is to use methods of set-valued optimization for deriving duality assertions in linear vector optimization.

2. Basic notation and results

2.1. Basic notation

Let $p > 0$ be an integer. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^p$ the usual linear space of real p-tupels $x = (x_1, \dots, x_p)^T$, $y = (y_1, \dots, y_p)^T$. We frequently use the notation and results of Rockafellar's monograph [18] and, as there (p. 3), "everything takes place in $\mathbb{R}^p$".

For the convenience of the reader we shall rewrite some basic definitions.

A nonempty set $K \subset \mathbb{R}^p$ is said to be a cone iff it is closed under nonnegative scalar multiplication, i.e. $\alpha K \subset K$ whenever $\alpha \geq 0$. A cone $K \subset \mathbb{R}^p$ is convex if and only if $K + K \subset K$. Here and throughout the paper, the sum of two subsets of a vector space is understood to be the usual Minkowski sum. For a nonempty set $M \subset \mathbb{R}^p$, the set $\operatorname{cone} M = \mathbb{R}_+ M$ denotes the cone generated by M consisting of all nonnegative multiples of elements of M.

A cone is said to be *pointed* iff $K \cap -K = \{0\}$. A convex pointed cone K generates a partial ordering (i.e. a reflexive, antisymmetric, transitive relation) in $\mathbb{R}^p$ by $y_1 \leq_K y_2$ iff $y_2 - y_1 \in K$.

Let $M \subset \mathbb{R}^p$ be a nonempty set. An element $\bar{y} \in M$ is called *efficient* with respect to K iff

$$\forall y \in M : \; \bar{y} - y \notin K \setminus \{0\} . \tag{1}$$

This is equivalent to $(\bar{y} - K \setminus \{0\}) \cap M = \emptyset$.

The set of efficient elements of M with respect to K is denoted by $\operatorname{Eff}[M; K]$. By convention, we associate "minimization" dealing with $\operatorname{Eff}[M; K]$ and "maximization" dealing with $\operatorname{Eff}[M; -K]$.

Let M be a nonempty convex subset of $\mathbb{R}^p$. The set 0^+M of all $y \in \mathbb{R}^p$ such that $M + y \subset M$ is a convex cone. It is called the recession cone of M. Moreover, a nonempty closed convex set $M \subset \mathbb{R}^p$ is bounded if and only if $0^+M = \{0\}$ (see [18], Theorem 8.1, 8.4). If M is a closed set and K a closed convex cone, the sum $M + K$ is closed if $0^+M \cap -K = \{0\}$ (see [18], Corollary 9.1.2).

A set $M \subset \mathbb{R}^p$ is said to be *lower externally stable* with respect to K iff $M \subset \operatorname{Eff}[M; K] + K$. This property is sometimes called *domination property* and has been studied by several authors. Compare Luc [14] and the references therein.

In our finite dimensional context, the following equivalences are valid. Note that in the following lemma the cone K does not have to be pointed.

Lemma 2.1. *Let M be a nonempty closed convex set and K be a closed convex cone. Then the following statements are equivalent.*

(*i*) $\operatorname{Eff}[M; K] \neq \emptyset$,

(*ii*) $0^+M \cap -K = \{0\}$,

(iii) *M is lower externally stable with respect to K.*

Proof. (i)⇒(ii): Assume the contrary of (ii), namely $0^+M \cap -K \neq \{0\}$. So there is $\tilde{y} \in 0^+M \cap -K$, $\tilde{y} \neq 0$. Hence we have for each $y \in M$

$$(y + \tilde{y}) - y = \tilde{y} \in -K \setminus \{0\}$$

as well as $y + \tilde{y} \in M$, hence no element of M can be efficient in contradiction to (i).

(ii)⇒(iii): Let $0^+M \cap -K = \{0\}$. For every $\bar{y} \in M$ we have $(\bar{y} - K) \cap M \neq \emptyset$. Hence

$$0^+[(\bar{y} - K) \cap M] = 0^+(\bar{y} - K) \cap 0^+M = -K \cap 0^+M = \{0\}$$

(see [18] Corollary 8.3.3) and consequently $(\bar{y} - K) \cap M$ is bounded (see [18] Theorem 8.4). So $(\bar{y} - K) \cap M$ is a compact section of M since K and M are closed. Hence

$$\emptyset \neq \mathrm{Eff}[(\bar{y} - K) \cap M; K] \subset \mathrm{Eff}[M; K]$$

by [15], Lemma 6.2 a) and Theorem 6.3 c) and consequently $\bar{y} \in \mathrm{Eff}[M; K] + K$.

(iii)⇒(i): Obviously. □

Note that the above lemma collects a number of scattered results working under different assumptions. The reader may compare e.g. [14], Corollary 3.9, 4.5 and Proposition 5.12.

2.2. Linear vector optimization

Let $n \geq 1$ be an integer and $\mathcal{X} \subset \mathbb{R}^n$. Let us consider a function $F : \mathcal{X} \to \mathbb{R}^p$. Denoting $F(\mathcal{X}) := \cup_{x \in \mathcal{X}} \{F(x)\}$, the basic problem of vector optimization is to determine and/or characterize the sets

$$\mathrm{Eff}[F(\mathcal{X}); K] \quad \text{and} \quad \{x \in \mathcal{X} : F(x) \in \mathrm{Eff}[F(\mathcal{X}); K]\}. \tag{VOP}$$

Even if F is a set-valued map from $\mathcal{X}$ to the power set of $\mathbb{R}^p$, (VOP) is well-defined by setting $F(\mathcal{X}) = \cup_{x \in \mathcal{X}} F(x)$. In this way, set-valued optimization, as considered in e.g. [3], [14] and [16] may be subsumed under vector optimization.

Roughly speaking, we are in the framework of linear vector optimization if the set M is the image under a linear mapping of a set that can be described by finitely many affine functions.

To be precise, we give the following assumptions which are considered to be in force throughout the paper.

Standing assumptions:

- m, n, p are positive integers;
- $C \in \mathbb{R}^{p \times n}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$;
- $K \subset \mathbb{R}^p$ be a closed convex pointed cone containing more than one point.

Note that the ordering cone K is not necessarily polyhedral convex. Recall that we understand a set to be polyhedral convex iff it is the intersection of finitely many closed half spaces. For details see [18], Section 19.

We shall write $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ for the non-negative orthant of $\mathbb{R}^n$. We denote by $\mathcal{X} := \{x \in \mathbb{R}^n : Ax = b\} \cap \mathbb{R}^n_+$ the set of admissible elements and by $C(\mathcal{X}) := \{y = Cx : x \in \mathcal{X}\} \subset \mathbb{R}^p$ the image set of our problem. The linear vector optimization problem is to determine (and investigate) the set

$$\operatorname{Eff}[C(\mathcal{X}); K]. \tag{P}$$

An element of $\operatorname{Eff}[C(\mathcal{X}); K]$ is called *efficient* for the linear vector programming problem (P). An element $\bar{x} \in \mathcal{X}$ such that $C\bar{x} \in \operatorname{Eff}[C(\mathcal{X}); K]$ is called a solution of (P).

In scalar linear optimization, i.e. $p = 1$, $K = \mathbb{R}^1_+$, there exists a complete duality theory featuring a dual problem such that exactly one of the following four cases occurs: 1) Admissible elements exist for the primal as well as for the dual problem, both problems have a solution, the optimal values are equal (strong duality). 2), 3) The set of admissible elements is empty either for the primal or for the dual problem; the other one is unbounded. 4) There are no admissible elements neither for the primal nor for the dual problem. Moreover, the dual of the dual problem is the original problem, in this sense the scalar theory is "symmetric". See e.g. [5], [20], [14].

A duality theory for the linear vector optimization problem (P) can be found for example in Isermann [10], [11], Brumelle [1], Göpfert/Nehse [8] and, in an infinite dimensional framework, in Jahn [15] and [2]. In all these references the assumption $b \neq 0$ was supposed to ensure strong duality results. Neither of these references gives a complete enumeration of possible cases for a linear vector optimization problem parallel to the scalar case.

We shall improve the theory in three directions: First, introducing a set-valued dual cost function we can avoid the assumption $b \neq 0$ and nevertheless, we have no duality gap. Secondly, we provide weak duality assertions and give a full strong duality theorem like it is well-known in scalar linear programming. We complete the theory by presenting a case study parallel to the scalar linear optimization. Moreover, we base our proofs onto a separation property which seems to be new in the context of vector optimization.

Note that we do not interpret (P) in the sense of a parametrized optimization problem using a scalarization right from the beginning. In contrast, we avoid a scalarization up to the point where a separation argument is needed. Compare the proof of Theorem 3.7 below.

2.3. Separation lemma

The following strict separation lemma is a basic tool for the proof of the strong duality results. It might be of independent interest since it differs slightly from well-known formulations including a strict separation property at both sides of the separating hyperplane.

Let $\Gamma \subset \mathbb{R}^p$ be a convex cone. The dual cone of the cone Γ is understood to be the set

$$\Gamma^* = \left\{\gamma \in \mathbb{R}^p : \forall y \in \Gamma : \gamma^T y \geq 0\right\}.$$

Note that a closed convex cone $\Gamma \subset \mathbb{R}^p$ is pointed iff $\operatorname{int} \Gamma^* \neq \emptyset$.

Lemma 2.2. *Let $\Gamma \subset \mathbb{R}^p$ be a closed convex pointed cone.*

(i) *If $M \subset \mathbb{R}^p$ is a polyhedral convex set with $M \cap \Gamma = \{0\}$ then there is a $\gamma \in \mathbb{R}^p \setminus \{0\}$ such that*

$$\gamma^T y \geq 0 > \gamma^T k \quad \forall y \in M, \forall k \in \Gamma \setminus \{0\}. \tag{2}$$

(ii) *If $M \subset \mathrm{I\!R}^p$ is a nonempty closed convex set with $M \cap \Gamma = \emptyset$ and $0^+M \cap \Gamma = \{0\}$ then there is a $\gamma \in \mathrm{I\!R}^p \setminus \{0\}$ such that*

$$\gamma^T y > 0 > \gamma^T k \quad \forall y \in M, k \in \Gamma \setminus \{0\}. \tag{3}$$

Proof. (i) We have $\operatorname{cone} M \cap \Gamma = \{0\}$ since otherwise there would exist $\alpha > 0$ and $\hat{y} \in M$ with $\tilde{y} := \alpha\hat{y} \in \Gamma \setminus \{0\}$, hence $\hat{y} = \frac{1}{\alpha}\tilde{y} \in M \cap \Gamma \setminus \{0\}$ in contradiction to $M \cap \Gamma = \{0\}$. Since M is a polyhedral convex set containing the origin, $\operatorname{cone} M$ is polyhedral too, hence a closed convex cone (see [18], Corollary 19.7.1 and Theorem 19.1). Thus we can apply a separation theorem (see [15], Theorem 3.22) to the cones Γ and $\operatorname{cone} M$ which states the existence of a $\gamma \in \mathrm{I\!R}^p \setminus \{0\}$ with (2).

(ii) M does not contain the origin since $M \cap \Gamma = \emptyset$ and $0 \in \Gamma$. Thus

$$\operatorname{cl}(\operatorname{cone} M) = \operatorname{cone} M \cup 0^+M$$

(see Theorem 9.6 in [18]). We have

$$\operatorname{cone} M \cap \Gamma = \{0\}$$

since otherwise there would exist $\alpha > 0, \hat{y} \in M$ such that $\tilde{y} := \alpha\hat{y} \in \Gamma \setminus \{0\}$, hence $\hat{y} = \frac{1}{\alpha}\tilde{y} \in M \cap \Gamma$ in contradiction to $M \cap \Gamma = \emptyset$. Together with $0^+M \cap \Gamma = \{0\}$ this implies

$$\operatorname{cl}(\operatorname{cone} M) \cap \Gamma = (\operatorname{cone} M \cup 0^+M) \cap \Gamma = (\operatorname{cone} M \cap \Gamma) \cup (0^+M \cap \Gamma) = \{0\}. \tag{4}$$

Since $\operatorname{cl}(\operatorname{cone} M)$ and Γ are closed convex cones, $\operatorname{int} \Gamma^* \neq \emptyset$ and (4) holds true we can apply a separation theorem (see [15], Theorem 3.22) to Γ and $\operatorname{cl}(\operatorname{cone} M)$ which states the existence of a $\gamma_1 \in \mathrm{I\!R}^p \setminus \{0\}$ with

$$\gamma_1^T y \geq 0 > \gamma_1^T k \quad \forall y \in M, k \in \Gamma \setminus \{0\}.$$

Due to $0^+M \cap \Gamma = \{0\}$ the nonempty disjoint closed convex sets M and Γ have no common direction of recession, hence they can be separated strongly (see [18], Corollary 11.4.1), i.e. there exists $\gamma_2 \in \mathrm{I\!R}^p \setminus \{0\}$ such that

$$\gamma_2^T y > \gamma_2^T k \quad \forall y \in M, k \in \Gamma$$

which implies

$$\gamma_2^T y > 0 \geq \gamma_2^T k \quad \forall y \in M, k \in \Gamma$$

since Γ is a cone containing 0. With $\gamma := \gamma_1 + \gamma_2$ we obtain

$$\gamma^T y = \gamma_1^T y + \gamma_2^T y > 0 > \gamma_1^T k + \gamma_2^T k = \gamma^T k$$

for all $y \in M$ and $k \in \Gamma \setminus \{0\}$ and $\gamma = 0$ is not possible. $\square$

3. Lagrange duality in linear vector optimization

3.1. Definitions and basic results

Constructing a dual problem to (P) using a suitable Lagrange function is one classical approach. The dual cost function is the Lagrangean minimized with respect to the original variables. The justification of this approach stems from weak and strong duality theorems. We are going to generalize this procedure to linear vector optimization problems of type (P), i.e. to

$$\left.\begin{array}{c} \operatorname{Eff}\left[C\left(\mathcal{X}\right);K\right] \\ \mathcal{X}=\left\{x\in \mathbb{R}^n:\ Ax=b\right\}\cap \mathbb{R}^n_+ \\ K\subset \mathbb{R}^p \text{ a closed convex pointed cone.}\end{array}\right\} \tag{P}$$

The Lagrange function for $x\in \mathbb{R}^n_+$, $U\in \mathbb{R}^{m\times p}$ with values in $\mathbb{R}^p$ is defined as usual:

$$L\left(x,U\right):=Cx+U^T\left(b-Ax\right)=\left(C-U^TA\right)x+U^Tb.$$

We define the dual cost function (with respect to x "minimized" Lagrangean) by

$$G\left(U\right):=\operatorname{Eff}\left[L\left(\mathbb{R}^n_+,U\right);K\right]=U^Tb+\operatorname{Eff}\left[\left(C-U^TA\right)\mathbb{R}^n_+;K\right].$$

Note that G is a set-valued mapping from $\mathbb{R}^{m\times p}$ in $2^{\mathbb{R}^p}$. It can easily be seen that for $x\in \mathbb{R}^n_+$

$$\operatorname{Eff}\left[L\left(x,\mathbb{R}^{m\times p}\right);-K\right]=\left\{\begin{array}{ccc}\{Cx\} & : & x\in\mathcal{X}\\ \emptyset & : & x\notin\mathcal{X}\end{array}\right.$$

i.e. we obtain the original problem from a "maximization" of the Lagrangean with respect to the dual variables. First, we give elementary properties of $G\left(U\right)$.

Proposition 3.1. *We have*

$$G\left(U\right)=\left\{\begin{array}{ccc}\emptyset & : & \left(C-U^TA\right)\mathbb{R}^n_+\cap -K\backslash\{0\}\neq\emptyset\\ U^Tb+D\left(U\right) & : & \left(C-U^TA\right)\mathbb{R}^n_+\cap -K\backslash\{0\}=\emptyset\end{array}\right.$$

where $D\left(U\right):=\operatorname{Eff}\left[\left(C-U^TA\right)\mathbb{R}^n_+;K\right]$ *is a cone. Moreover,* $D\left(U\right)=\{0\}$ *if and only if* $\left(C-U^TA\right)\mathbb{R}^n_+\subset K$.

Proof. To show that $G\left(U\right)=\emptyset$ if there exists $\bar{x}\in\mathbb{R}^n_+$ such that $\left(C-U^TA\right)\bar{x}\in -K\backslash\{0\}$ we note that $\left(C-U^TA\right)\bar{x}\in 0^+\left(C-U^TA\right)\mathbb{R}^n_+$. Hence we can apply Lemma 2.1 (i) and (ii) which yields the desired assertion.

Considering the other case, i.e. $\left(C-U^TA\right)\mathbb{R}^n_+\cap -K\backslash\{0\}=\emptyset$, we first note that $0\in D\left(U\right)=\operatorname{Eff}\left[\left(C-U^TA\right)\mathbb{R}^n_+;K\right]$ by definition.

Assuming that $\bar{y}\in D\left(U\right)$, i.e. $\bar{y}-\left(C-U^TA\right)x\notin K\backslash\{0\}$ for all $x\in\mathbb{R}^n_+$ we may conclude that $\bar{y}-\left(C-U^TA\right)\frac{x}{\alpha}\notin K\backslash\{0\}$ for all $\alpha>0$, $x\in\mathbb{R}^n_+$. This implies $\alpha\bar{y}-\left(C-U^TA\right)x\notin K\backslash\{0\}$ for all $\alpha>0$, $x\in\mathbb{R}^n_+$. Hence $\alpha\bar{y}\in D\left(U\right)$ and $D\left(U\right)$ is a cone.

The "if"-part of the last assertion is obvious. The "only if" part follows from the external stability of $\left(C-U^TA\right)\mathbb{R}^n_+$ guaranteed by Lemma 2.1 (i), (iii). □

The cone $D\left(U\right)$ is not necessarily convex. An example is given below.

Definition 3.2. If $G(U)$ is not the empty set the corresponding U is called *dual admissible*. The set

$$\mathcal{U} := \left\{ U \in \mathrm{I\!R}^{m\times p} : \left(C - U^T A\right) \mathrm{I\!R}^n_+ \cap -K \setminus \{0\} = \emptyset \right\}.$$

is called the *domain* of the dual cost function G.

We denote $G(\mathcal{U}) = \bigcup_{U\in\mathcal{U}} G(U)$. Our dual problem belonging to the linear vector optimization problem (P) reads as follows:

$$\left.\begin{array}{c} \mathrm{Eff}\left[G(\mathcal{U})\,; -K\right] \\ \mathcal{U} = \left\{ U \in \mathrm{I\!R}^{m\times p} : \left(C - U^T A\right) \mathrm{I\!R}^n_+ \cap -K \setminus \{0\} = \emptyset \right\}. \end{array}\right\} \tag{D}$$

Remark 3.3. 1. If $K = \mathrm{I\!R}^p_+$ we have $D(U) = \{0\}$ if and only if $U^T A \leq C$ in $\mathrm{I\!R}^{p\times n}_+$. It is well known that the set $\bar{\mathcal{U}} := \left\{ U \in \mathrm{I\!R}^{m\times p} : U^T A \leq C \right\}$ is to small to generate a satisfactory duality theory. Hence the dual of (P) should not be just $\mathrm{Eff}\left[\bar{\mathcal{U}}^T b; -K\right]$, a linear vector maximization problem.

2. The attempt to enlarge the set of dual admissible variables to $\mathcal{U}$ (instead of $\bar{\mathcal{U}}$) obtaining $\mathrm{Eff}\left[\mathcal{U}^T b; -K\right]$ as dual problem was not completely successful as well. The counterexamples of Brumelle and Jahn, quoted in Section 4, show that especially in case of $b = 0$ something goes wrong with duality. However, in case $b \neq 0$ there are strong duality results in the sense that the sets $\mathrm{Eff}\left[C(\mathcal{X})\,; K\right]$ and $\mathrm{Eff}\left[\mathcal{U}^T b; -K\right]$ coincide.

3. Several authors, e.g. Corley [3], Luc [14] and Jahn [15], share the understanding of the dual of a vector optimization problem to be set-valued in nature as explained at the beginning of Section 2.2. But there seems to be no complete duality theory even in the linear case using this approach up to now.

4. Problem (D) includes the more general cost function $G(U)$. In this case, $b = 0$ does not imply $G(U) = 0$ for all $U \in \mathcal{U}$. Using this construction we are able to give strong duality results as well as a complete case study like it is well-known from scalar linear programming. Compare Theorem 3.7 and Theorem 3.14 below.

3.2. Weak duality

The following weak duality theorem extends several well-known concepts in vector optimization, e.g. Iwanow/Nehse [12], Göpfert/Nehse [8]. The new feature lies in the fact that a set-valued dual cost function $G(U)$ is involved, hence the weak duality relation is valid for "more" elements of the space $\mathrm{I\!R}^p$. Luc [14] also used set-valued dual cost functions. He obtained similar weak duality results for a different dual problem: the set of dual admissible elements in [14], chap. 5.1 is always a cone.

A very general approach using set-valued objectives for a dual of a vector optimization problem can be found in Göpfert et al. [9].

Theorem 3.4. Weak duality theorem.
If $x \in \mathcal{X}$ and $U \in \mathcal{U}$ then

$$G(U) \cap (Cx + K \setminus \{0\}) = \emptyset. \tag{5}$$

Proof. Assume the contrary of (5), i.e.

$$\exists \bar{x} \in \mathcal{X}, \bar{U} \in \mathcal{U} : \ \bar{y} \in G\left(\bar{U}\right) \cap (C\bar{x} + K \setminus \{0\})\,.$$

Then there exists $z \in D(\bar{U}) = \mathrm{Eff}\left[\left(C - \bar{U}^T A\right)\mathbb{R}^n_+; K\right]$ such that $\bar{y} = \bar{U}^T b + z$. We have $z - y \notin K \backslash \{0\}$ for all $y \in \left(C - \bar{U}^T A\right)\mathbb{R}^n_+$ by efficiency and therefore, choosing $y = \left(C - \bar{U}^T A\right)\bar{x}$, we obtain $z - \left(C - \bar{U}^T A\right)\bar{x} \notin K \backslash \{0\}$. This implies

$$\bar{y} - \bar{U}^T b - \left(C - \bar{U}^T A\right)\bar{x} = \bar{y} - C\bar{x} \notin K \backslash \{0\}$$

contradicting the assumption. This proves (5). □

Remark 3.5. Of course, (5) is equivalent to

$$G(\mathcal{U}) \cap (C(\mathcal{X}) + K \backslash \{0\}) = \emptyset. \tag{6}$$

Moreover, the following equivalences hold true:

$$\begin{aligned} G(\mathcal{U}) \cap (C(\mathcal{X}) + K \setminus \{0\}) = \emptyset \;&\Leftrightarrow\; (G(\mathcal{U}) - K \setminus \{0\}) \cap C(\mathcal{X}) = \emptyset \\ &\Leftrightarrow\; (G(\mathcal{U}) - K \setminus \{0\}) \cap (C(\mathcal{X}) + K) = \emptyset \\ &\Leftrightarrow\; (G(\mathcal{U}) - K) \cap (C(\mathcal{X}) + K \setminus \{0\}) = \emptyset \end{aligned}$$

hence (5) can be replaced by (6) or any of the equivalent relationships.

Corollary 3.6. Sufficient optimality condition
The following inclusions hold true:

$$G(\mathcal{U}) \cap C(\mathcal{X}) \subset \mathrm{Eff}[C(\mathcal{X}); K] \quad \textit{and} \quad G(\mathcal{U}) \cap C(\mathcal{X}) \subset \mathrm{Eff}[G(\mathcal{U}); -K].$$

Proof. If $\bar{y} \in G(\mathcal{U}) \cap C(\mathcal{X})$ then we have $G(\mathcal{U}) \cap (\bar{y} + K \setminus \{0\}) = \emptyset$ by (6) hence $\bar{y} \in \mathrm{Eff}[G(\mathcal{U}); -K]$ and $\bar{y} \notin C(\mathcal{X}) + K \setminus \{0\}$, i.e. $C(\mathcal{X}) \cap (\bar{y} - K \setminus \{0\}) = \emptyset$, hence $\bar{y} \in \mathrm{Eff}[C(\mathcal{X}); K]$. □

3.3. Strong duality

Next, we propose strong duality theorems. The idea is to remain close to scalar optimization: The optimal values of the primal and the dual problem turn out to be the same, and each point being optimal for (P) is also optimal for (D) and vice versa.

If $A \neq 0$ even more can be said: The image space $\mathbb{R}^p$ can be parted into three disjoint sets: Points which are "strictly greater" than optimal values of the primal problem, i.e. elements of $\mathrm{Eff}[C(\mathcal{X}); K]$, points which are "not greater or equal" than optimal values of the dual problem, i.e. elements of $\mathrm{Eff}[G(\mathcal{U}); -K]$, and the set consisting of efficient points for both of the problems.

Theorem 3.7. *Under the standing assumptions and if* $\mathrm{Eff}[C(\mathcal{X}); K] \neq \emptyset$ *and* $A \neq 0$, *then*

$$(C(\mathcal{X}) + K \setminus \{0\}) \cup (G(\mathcal{U}) - K \setminus \{0\}) \cup (G(\mathcal{U}) \cap C(\mathcal{X})) = \mathbb{R}^p. \tag{7}$$

This theorem does not exclude the case $b = 0$. We need the following two lemmas for the proof.

Lemma 3.8. *Let* $b, v \in \mathbb{R}^m$, $\gamma, y \in \mathbb{R}^p$ *be given vectors with* $b, \gamma \neq 0$ *and* $\gamma^T y = v^T b$. *Then there is a solution* $U \in \mathbb{R}^{m \times p}$ *of the system*

$$U\gamma = v, \qquad U^T b = y. \tag{8}$$

Proof. This lemma is a very special case of Theorem 2.3.3 of [17] which is standard in the theory of generalized inverses. □

Lemma 3.9. *Let $A \in \mathbb{R}^{m\times n}$ with $A \neq 0$, $b \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ be given and assume that the scalar linear optimization problem $\min\{\gamma^T x : Ax = b, x \in \mathbb{R}^n_+\}$ has a solution with value α. Then there is $\bar{x} \in \mathbb{R}^n_+$ with $A\bar{x} \neq b$ such that*

$$\min\left\{\gamma^T x + 0\cdot\lambda : \ Ax = b,\ x - \bar{x}\cdot\lambda \geq 0,\ (x,\lambda) \in \mathbb{R}^{n+1}\right\} = \alpha.$$

Proof. If $b \neq 0$ then $\bar{x} = 0$ meets all requirements.

Now, let $b = 0$. If $\min\{\gamma^T x : Ax = 0, x \in \mathbb{R}^n_+\} =: \alpha$ exists then $\alpha = 0$. We will show that one of the unit vectors e^j, $j = 1, \ldots, n$, satisfies the requirements. Assume the contrary, namely

$$\min\{\gamma^T x : Ax = 0, x - e^j\lambda \geq 0\} < 0 = \alpha$$

for all $j \in J := \{j \in \{1, ..., n\} : Ae^j \neq 0\}$. This means for all $j \in J$ there is $\hat{x}^j \in \mathbb{R}^n$ with $A\hat{x}^j = 0$, $\hat{x}^j_i \geq 0$ for all $i \neq j$ and $\gamma^T\hat{x}^j < 0$. The n vectors $\hat{x}^j$ for $j \in J$ and e^j for $j \in I := \{j \in \{1, ..., n\} : Ae^j = 0\}$ all belong to the proper linear subspace $\{x \in \mathbb{R}^n : Ax = 0\}$ of $\mathbb{R}^n$ hence they are linearly dependent. Consequently, there are n numbers $\lambda_1, \ldots, \lambda_n$, not simultaneously equal to 0, such that

$$\sum_{j\in J} \lambda_j \hat{x}^j + \sum_{j\in I} \lambda_j e^j = 0.$$

Since the vectors e^j, $j \in I$, are linearly independent at least one of the λ_j's, $j \in J$, has to be non-zero. We will now consider $\tilde{x} := \sum_{j\in J} |\lambda_j|\, \hat{x}^j$. For $i \in I$ we have $\hat{x}^j_i \geq 0$ for all $j \in J$ hence $\tilde{x}_i \geq 0$. We define $J_+ := \{j \in J : \lambda_j \geq 0\}$ and $J_- := \{j \in J : \lambda_j < 0\}$. If $i \in J$ we have $e^j_i = 0$ for $j \in I$, hence

$$\sum_{j\in J_-} \lambda_j \hat{x}^j_i + \sum_{j\in J_+} \lambda_j \hat{x}^j_i = 0.$$

For $i \in J_-$ we obtain

$$\sum_{j\in J_+} \lambda_j \hat{x}^j_i \geq 0$$

since $\lambda_j \geq 0$ and $\hat{x}^j_i \geq 0$ for all $j \in J_+$. Hence

$$\tilde{x}_i = \sum_{j\in J_-} (-\lambda_j)\hat{x}^j_i + \sum_{j\in J_+} \lambda_j \hat{x}^j_i = 2\sum_{j\in J_+} \lambda_j \hat{x}^j_i \geq 0.$$

For $i \in J_+$ we obtain

$$\sum_{j\in J_-} \lambda_j \hat{x}^j_i \leq 0$$

since $\lambda_j < 0$ and $\hat{x}^j_i \geq 0$ for all $j \in J_-$. Hence

$$\tilde{x}_i = \sum_{j\in J_-} (-\lambda_j)\hat{x}^j_i + \sum_{j\in J_+} \lambda_j \hat{x}^j_i = -2\sum_{j\in J_-} \lambda_j \hat{x}^j_i \geq 0.$$

Thus we have shown $\tilde{x} \in \mathbb{R}^n_+$. Moreover, $A\tilde{x} = 0$ and $\gamma^T\tilde{x} < 0$ in contradiction to $\min\{\gamma^T x : Ax = 0,\, x \in \mathbb{R}^n_+\} = 0$. □

Proof of Theorem 3.7. Let $\bar{y} \notin C(\mathcal{X}) + K \setminus \{0\}$. We have to show that $\bar{y} \in (G(\mathcal{U}) - K \setminus \{0\}) \cup (G(\mathcal{U}) \cap C(\mathcal{X}))$. We will consider the two cases $\bar{y} \in C(\mathcal{X})$ and $\bar{y} \notin C(\mathcal{X})$. In the first case we will show that $\bar{y} \in G(\mathcal{U}) \cap C(\mathcal{X})$ and in the latter case we will show that $\bar{y} \in G(\mathcal{U}) - K \setminus \{0\}$.

(i) Let $\bar{y} \in C(\mathcal{X})$. Then there exists $\bar{x} \in \mathcal{X}$ with

$$\forall U \in \mathbb{R}^{m \times p} : \ \bar{y} = C\bar{x} = U^T b + (C - U^T A)\bar{x},$$

hence $\bar{y} \in U^T b + (C - U^T A)\mathbb{R}^n_+$ for all $U \in \mathbb{R}^{m \times p}$.

The conditions $\bar{y} \in C(\mathcal{X})$, $\bar{y} \notin C(\mathcal{X}) + K \setminus \{0\}$ imply $(C(\mathcal{X}) - \bar{y}) \cap -K = \{0\}$. Hence we can apply Lemma 2.2, (i) with $M = C(\mathcal{X}) - \bar{y}$ and $\Gamma = -K$ since K is closed convex pointed. Hence there exists a $\gamma \in \mathbb{R}^p \setminus \{0\}$ such that

$$\forall y \in C(\mathcal{X}) - \bar{y},\ k \in -K \setminus \{0\} : \ \gamma^T y \geq 0 > \gamma^T k. \tag{9}$$

This means $\gamma^T \bar{y} = \min\{\gamma^T Cx : \ Ax = b, x \in \mathbb{R}^n_+\}$ since $\bar{y} \in C(\mathcal{X})$. Hence by duality theory for scalar linear programming there is $v_0 \in \mathbb{R}^m$ with $A^T v_0 \leq C^T \gamma$ and $b^T v_0 = \gamma^T \bar{y}$. Thus

$$v_0^T b + (\gamma^T C - v_0^T A)x \geq \gamma^T \bar{y}$$

for all $x \in \mathbb{R}^n_+$. Since $\gamma \neq 0$ there is a solution $U_0 \in \mathbb{R}^{m \times p}$ of $\gamma^T U^T = v_0^T$. Hence

$$\gamma^T \left(U_0^T b + (C - U_0^T A)x \right) \geq \gamma^T \bar{y}$$

for all $x \in \mathbb{R}^n_+$ and consequently

$$\forall y \in U_0^T b + (C - U_0^T A)\mathbb{R}^n_+ : \ \gamma^T y \geq \gamma^T \bar{y}.$$

By (9) we have $\gamma^T y < \gamma^T \bar{y}$ for all $y \in \bar{y} - K \setminus \{0\}$. This implies $U_0^T b + (C - U_0^T A)\mathbb{R}^n_+ \cap (\bar{y} - K \setminus \{0\}) = \emptyset$, hence $\bar{y} \in G(U_0) = \operatorname{Eff}[U_0^T b + (C - U_0^T A)\mathbb{R}^n_+; K]$. So we have shown $\bar{y} \in G(\mathcal{U}) \cap C(\mathcal{X})$.

(ii) Let $\bar{y} \notin C(\mathcal{X})$. Together with $\bar{y} \notin C(\mathcal{X}) + K \setminus \{0\}$ this yields $\bar{y} \notin C(\mathcal{X}) + K$. Hence $(C(\mathcal{X}) - \bar{y}) \cap -K = \emptyset$. Moreover, we have

$$0^+(C(\mathcal{X}) - \bar{y}) \cap -K = 0^+ C(\mathcal{X}) \cap -K = \{0\}$$

by Lemma 2.1 since $\operatorname{Eff}[C(\mathcal{X}); K] \neq \emptyset$. So we can apply Lemma 2.2, (ii) to the sets $M = C(\mathcal{X}) - \bar{y}$ and $\Gamma = -K$ since K is pointed. Hence there exists $\gamma \in \mathbb{R}^p \setminus \{0\}$ such that

$$\forall y \in C(\mathcal{X}), k \in -K \setminus \{0\} : \ \gamma^T (y - \bar{y}) > 0 > \gamma^T k.$$

So the functional $\gamma^T y$ is bounded below on $C(\mathcal{X})$. Therefore the scalar linear optimization problem

$$\min\left\{\gamma^T Cx : \ Ax = b, \ x \in \mathbb{R}^n_+\right\}$$

has a solution, say $x_0 \in \mathcal{X}$ with value $\alpha := \gamma^T C x_0 > \gamma^T \bar{y}$. Take arbitrary $k_0 \in K \setminus \{0\}$ (hence $\gamma^T k_0 > 0$) and define

$$y_0 := \bar{y} + \frac{\alpha - \gamma^T \bar{y}}{\gamma^T k_0} k_0 \in \bar{y} \in K \setminus \{0\}.$$

So we have $\bar{y} \in y_0 - K \setminus \{0\}$ and $\gamma^T y_0 = \alpha$. By Lemma 3.9 there exists $\bar{x} \in \mathbb{R}^n_+$ with $A\bar{x} \neq b$ such that the scalar linear optimization problem

$$\min\{\gamma^T Cx + 0 \cdot \lambda : Ax = b, x - \bar{x} \cdot \lambda \geq 0\}$$

has a solution with value α. The corresponding dual problem is

$$\max\left\{b^T v : \; A^T v \leq C^T \gamma, \bar{x}^T A^T v = \bar{x}^T C^T \gamma\right\}$$

having a solution as well, say v_0 and we have $v_0^T b = \alpha = \gamma^T y_0$ hence

$$\gamma^T (y_0 - C\bar{x}) = v_0^T (b - A\bar{x}).$$

Since $c, b - A\bar{x} \neq 0$ we can apply Lemma 3.8 which guarantees the existence of a solution $U_0 \in \mathbb{R}^{m \times p}$ of the system

$$\begin{aligned} U\gamma &= v_0 \\ U^T(b - A\bar{x}) &= y_0 - C\bar{x}. \end{aligned}$$

For this solution holds

$$y_0 = U_0^T b + (C - U_0^T A)\bar{x} \in U_0^T b + (C - U_0^T A)\mathbb{R}^n_+$$

and

$$\gamma^T (U_0^T b + (C - U_0^T A)x) = v_0^T b + (\gamma^T C - v_0^T A)x \geq v_0^T b = \gamma^T y_0$$

for all $x \in \mathbb{R}^n_+$ hence

$$(y_0 - K \setminus \{0\}) \cap \left(U_0^T b + (C - U_0^T A)\mathbb{R}^n_+\right) = \emptyset$$

since $\gamma^T k < 0$ for all $k \in -K \setminus \{0\}$. So we have shown

$$y_0 \in G(U_0) = \mathrm{Eff}[U_0^T b + (C - U_0^T A)\mathbb{R}^n_+; K]$$

hence

$$\bar{y} \in G(U_0) - K \setminus \{0\} \subset G(\mathcal{U}) - K \setminus \{0\}.$$

□

Corollary 3.10. *Under the assumptions of Theorem* 3.7 *the following properties hold:*

(i) $\mathrm{Eff}[C(\mathcal{X}); K] = \mathrm{Eff}[G(\mathcal{U}); -K] = C(\mathcal{X}) \cap G(\mathcal{U})$

(ii) $(G(\mathcal{U}) - K \setminus \{0\}) \cup (C(\mathcal{X}) + K) = (G(\mathcal{U}) - K) \cup (C(\mathcal{X}) + K \setminus \{0\}) = \mathbb{R}^p$.

Proof. (i) By Corollary 3.6 it remains to show that

$$\mathrm{Eff}[C(\mathcal{X}); K] \subset C(\mathcal{X}) \cap G(\mathcal{U}) \quad \text{and} \quad \mathrm{Eff}[G(\mathcal{U}); -K] \subset C(\mathcal{X}) \cap G(\mathcal{U}).$$

Of course, $y \in \mathrm{Eff}[C(\mathcal{X}); K]$ implies $y \in C(\mathcal{X})$ and $y \notin C(\mathcal{X}) + K \setminus \{0\}$. By Theorem 3.4, (ii) and Remark 3.5 $y \in C(\mathcal{X})$ implies $y \notin G(\mathcal{U}) - K \setminus \{0\}$. Hence $y \in C(\mathcal{X}) \cap G(\mathcal{U})$ by Theorem 3.7.

On the other hand, $y \in \mathrm{Eff}[G(\mathcal{U}); -K]$ implies $y \in G(\mathcal{U})$ and $y \notin G(\mathcal{U}) - K \setminus \{0\}$. By Theorem 3.4, (ii) $y \in G(\mathcal{U})$ implies $y \notin C(\mathcal{X}) + K \setminus \{0\}$. Hence $y \in C(\mathcal{X}) \cap G(\mathcal{U})$ by Theorem 3.7.

(ii) The statement follows from Theorem 3.7 since

$$(C(\mathcal{X}) + K \setminus \{0\}) \cup (C(\mathcal{X}) \cap G(\mathcal{U})) \subset (C(\mathcal{X}) + K \setminus \{0\}) \cup C(\mathcal{X}) = C(\mathcal{X}) + K$$

and

$$(G(\mathcal{U}) - K \setminus \{0\}) \cup (C(\mathcal{X}) \cap G(\mathcal{U})) \subset (G(\mathcal{U}) - K \setminus \{0\}) \cup G(\mathcal{U}) = G(\mathcal{U}) - K.$$

□

We have $C(\mathcal{X}) + K = \mathrm{Eff}[C(\mathcal{X}); K] + K$ since $C(\mathcal{X})$ is lower externally stable. Corollary 3.10 tells us that we have

$$G(\mathcal{U}) - K \setminus \{0\} = \mathbb{R}^p \setminus (\mathrm{Eff}[C(\mathcal{X}); K] + K).$$

This set consists of points being not greater than points of $\mathrm{Eff}[G(\mathcal{U}); -K]$.

Proposition 3.11. *Under the standing assumptions and if $A = 0$, $\mathcal{X} \neq \emptyset$ then*

$$\mathrm{Eff}[C(\mathcal{X}); K] = \mathrm{Eff}[G(\mathcal{U}); -K] = C(\mathcal{X}) \cap G(\mathcal{U}).$$

Proof. In this case, we have $\mathcal{X} = \mathbb{R}^n_+$ and therefore

$$G(U) = \mathrm{Eff}[C\left(\mathbb{R}^n_+\right); K] = \mathrm{Eff}[C(\mathcal{X}); K]$$

for all $U \in \mathbb{R}^{m \times p}$ hence

$$\mathrm{Eff}[G(\mathcal{U}); -K] \subset G(\mathcal{U}) = \mathrm{Eff}[C(\mathcal{X}); K] \subset C(\mathcal{X})$$

and consequently

$$\mathrm{Eff}[G(\mathcal{U}); -K] \subset C(\mathcal{X}) \cap G(\mathcal{U}) = \mathrm{Eff}[C(\mathcal{X}); K].$$

Moreover $G(\mathcal{U}) = \mathrm{Eff}[C(\mathcal{X}); K]$ implies $\hat{y} - \bar{y} \notin K \setminus \{0\}$ for all $\hat{y} \in G(\mathcal{U})$ and $\bar{y} \in C(\mathcal{X})$. Hence

$$C(\mathcal{X}) \cap G(\mathcal{U}) \subset \mathrm{Eff}[G(\mathcal{U}); -K].$$

□

Proposition 3.12. *Under the standing assumptions and if $\mathcal{X} = \emptyset$ then* $\mathrm{Eff}[G(\mathcal{U}); -K] = \emptyset$.

Proof. If $\mathcal{X} = \emptyset$ then $b \neq 0$ since otherwise $0 \in \mathcal{X}$. Moreover, by Farkas' lemma, $\mathcal{X} = \emptyset$ is equivalent with the existence of $\hat{v} \in \mathbb{R}^m$ such that $A^T\hat{v} \leq 0$ and $b^T\hat{v} > 0$.

We have to show that for each $y \in G(\mathcal{U})$ there exists $\hat{y} \in G(\mathcal{U})$ such that $\hat{y} - y \in K \setminus \{0\}$. Let $\bar{y} \in G(\mathcal{U})$. Then there is $\bar{U} \in \mathcal{U}$ such that

$$\bar{y} \in G(\bar{U}) = \bar{U}b + \mathrm{Eff}[(C - \bar{U}^T A)\mathbb{R}^n_+; K]$$

i.e.

$$(\bar{U}^T b + (C - \bar{U}^T A)\mathbb{R}^n_+ - \bar{y}) \cap -K = \{0\}\,.$$

We can apply Lemma 2.2, (i) with $M = \bar{U}^T b + (C - \bar{U}^T A)\mathbb{R}^n_+ - \bar{y}$ and $\Gamma = -K$ and obtain the existence of $\gamma \in \mathbb{R}^p \setminus \{0\}$ such that

$$\gamma^T(\bar{U}^T b + (C - \bar{U}^T A)x - \bar{y}) \geq 0 \quad \forall x \in \mathbb{R}^n_+ \tag{10}$$

and

$$\gamma^T k < 0 \quad \forall k \in -K \setminus \{0\}. \tag{11}$$

Let $\bar{v} := \bar{U}\gamma$. Then (10) implies in particular $\bar{v}^T b \geq \gamma^T \bar{y}$ and $\gamma^T C - \bar{v}^T A \geq 0$.

Take arbitrary $k_0 \in K \setminus \{0\}$ (hence $\gamma^T k_0 > 0$) and define

$$\hat{y} := \bar{y} + \frac{(\bar{v} + \hat{v})^T b - \gamma^T \bar{y}}{\gamma^T k_0} k_0.$$

We obtain $\hat{y} - \bar{y} \in K \setminus \{0\}$ and $\gamma^T \hat{y} = (\bar{v} + \hat{v})^T b$. Hence by Lemma 3.8 the system

$$U\gamma = \bar{v} + \hat{v}, \qquad U^T b = \hat{y}$$

has a solution, say $\hat{U}$. For this solution we have

$$\hat{y} = \hat{U}^T b \in \hat{U}^T b + (C - \hat{U}^T A)\mathbb{R}^n_+$$

and

$$\gamma^T(\hat{U}^T b + (C - \hat{U}^T A)x - \hat{y}) = \left(\gamma^T C - (\bar{v} + \hat{v})^T A\right) x \geq 0$$

for all $x \in \mathbb{R}^n_+$ hence

$$(\hat{U}^T b + (C - \hat{U}^T A)\mathbb{R}^n_+ - \hat{y}) \cap -K \setminus \{0\} = \emptyset$$

by (11). Consequently

$$\hat{y} \in \hat{U}^T b + \mathrm{Eff}[(C - \hat{U}^T A)\mathbb{R}^n_+; K] = G(\hat{U}) \subset G(\mathcal{U}).$$

□

To complete the theory, the case $\mathcal{X} \neq \emptyset$, $\mathrm{Eff}\,[C\,(\mathcal{X})\,; K] = \emptyset$ has to be investigated.

Proposition 3.13. *Under the standing assumption and if $\mathcal{X} \neq \emptyset$ and $\mathrm{Eff}\,[C\,(\mathcal{X})\,; K] = \emptyset$ then $\mathcal{U} = \emptyset$.*

Proof. Since $\mathcal{X}$ is polyhedral convex we have

$$0^+\,(C\,(\mathcal{X})) = C\left(0^+\,(\mathcal{X})\right) = \left\{Cx : x \in \mathbb{R}^n_+, Ax = 0\right\} \subset (C - U^T A)\mathbb{R}^n_+$$

for all $U \in \mathbb{R}^{m \times p}$. Thus

$$(C - U^T A)\mathbb{R}^n_+ \cap (-K \setminus \{0\}) \supset 0^+(C\,(\mathcal{X})) \cap (-K \setminus \{0\}) \neq \emptyset$$

by Lemma 2.1, hence $\mathcal{U} = \emptyset$. □

We summarize the situation in the following theorem.

Theorem 3.14. Duality in Linear Vector Optimization
Under the standing assumptions the statement

$$\mathrm{Eff}[C(\mathcal{X});K] = \mathrm{Eff}[G(\mathcal{U});-K] = C(\mathcal{X}) \cap G(\mathcal{U})$$

is always true. Moreover, the following relationships are valid:

(*i*) *Let* $\mathcal{X} \neq \emptyset$. *Then* $\mathrm{Eff}[C(\mathcal{X});K] = \emptyset$ *if and only if* $\mathcal{U} = \emptyset$.
(*ii*) *Let* $\mathcal{U} \neq \emptyset$. *Then* $\mathrm{Eff}[G(\mathcal{U});-K] = \emptyset$ *if and only if* $\mathcal{X} = \emptyset$.

Proof. It is a consequence of Proposition 3.12, Proposition 3.13, Corollary 3.10 and Proposition 3.11. □

Thus we arrived at a complete description of the possible cases like it is well-known in scalar linear optimization ($p = 1$, $K = \mathrm{IR}^1_+$). The main features of the theory, above all the case $b = 0$, are illustrated by simple examples in the next section.

4. Examples

4.1. A standard example

Consider the problem

$$\mathrm{Eff}\left[\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \ x_1 + x_2 = 1,\ x \in \mathrm{IR}^2_+\right\}; \mathrm{IR}^2_+\right]. \tag{12}$$

Here we have $p = n = 2$ and $m = 1$. The meaning of C, A, b is clear, we have $U = (u_1, u_2)$.

It is easy to see that each admissible element $x \in \mathcal{X}$ is a solution of (12) and $C(\mathcal{X}) = \mathrm{Eff}[C(\mathcal{X});K]$. The set $\mathcal{U}^T \subset \mathrm{IR}^2$ of dual admissible elements is the union of the triangle with corners $(0,0)^T$, $(1,0)^T$ and $(0,1)^T$ and the complement of IR^2_+.

For a given dual admissible U, the set $U^Tb + \left(C - U^TA\right)\mathrm{IR}^2_+$ is the convex hull of the two rays originating in $U^Tb = U^T$ and running through $(1,0)$ and $(0,1)$, respectively. Considering $U = (1/2, 1/4)$ we see that $D(U) = \mathrm{Eff}\left[\left(C - U^TA\right)\mathrm{IR}^2_+; K\right]$ and $G(U) = U^Tb + D(U)$ are not necessarily convex. We have

$$G(\mathcal{U}) \cap C(\mathcal{X}) = \left\{x \in \mathrm{IR}^2_+ : \ x_1 + x_2 = 1\right\} = C(\mathcal{X}) = \mathrm{Eff}[C(\mathcal{X});K] = \mathrm{Eff}[G(\mathcal{U});-K].$$

This example matches the situation of Theorem 3.7. Since $b \neq 0$, the strong duality results e.g. of [8], Theorem 2.39 and [1], Theorem 4.2 also apply.

4.2. Brumelle's example

Consider the problem

$$\mathrm{Eff}\left[\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot x : \ x \in \mathrm{IR}^1_+\right\}; \mathrm{IR}^2_+\right]. \tag{13}$$

Here we have $p = 2$, $n = 1$ and $m = 0$ (no equality constraints). The meaning of C is clear, and $A = 0$, $b = 0$.

We have $\mathcal{X} = \mathbb{R}^1_+$, $K = \mathbb{R}^2_+$ and $C(\mathcal{X}) = \mathrm{Eff}\,[C(\mathcal{X}); K]$. Moreover, $G(U) = D(U) = \mathrm{Eff}\,[C(\mathcal{X}); K] = C(\mathcal{X})$ for each U since $b = 0$.

We are in the framework of Proposition 3.11 with $G(\mathcal{U}) = C(\mathcal{X})$. Note that the strong duality in the sense of Proposition 3.11 is true although former results do not apply since $b = 0$. Compare e.g. [1], [15].

4.3. Jahn's example

Consider the problem

$$\mathrm{Eff}\left[\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} :\ x_1 + x_2 = 0\right\}; \mathbb{R}^2_+\right]. \tag{14}$$

Here we have $p = n = 2$, $m = 1$ and no non-negativity constraints. The meaning of C and A is clear. Note that again $b = 0$.

Since there are no non-negativity restrictions to x, there is something to do to reach the framework of problem (P). This can be done by different methods. The well-known procedure from scalar linear programming is to replace the variable x by $x' - x''$ and $x', x'' \in \mathbb{R}^2_+$. Another possibility consists of redefining the dual cost function by

$$G(U) := \mathrm{Eff}\,[L(\mathbb{R}^n, U); K] = U^T b + \mathrm{Eff}\left[\left(C - U^T A\right)\mathbb{R}^n; K\right]$$

and going into the developments of the Sections 2 and 3.

We come out with the fact that it is necessary for $U = (u_1, u_2)$ being dual admissible that the matrix $C - U^T A$ is singular. By simple calculations one can find that $\mathcal{U} = \{U :\ u_1 + u_2 = 1\}$ and $C(\mathcal{X}) = D(U)$ for each $U \in \mathcal{U}$.

The validity of our weak and strong duality formulas are easy to check. Again, the "classical" dual problem to minimize the function $U^T b \equiv 0$ subject to certain constraints does not give any useful information.

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