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Kuratowski's Index of Non-Compactness and Renorming in Banach Spaces

F. García*
Departamento de Matemáticas, Facultad de Matemáticas,
Universidad de Murcia, 30.100 Espinardo, Murcia, Spain
fergar@um.es

L. Oncina
Departamento de Matemáticas, Facultad de Matemáticas,
Universidad de Murcia, 30.100 Espinardo, Murcia, Spain
luis@um.es

J. Orihuela
Departamento de Matemáticas, Facultad de Matemáticas,
Universidad de Murcia, 30.100 Espinardo, Murcia, Spain
joseori@um.es

S. Troyanski
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Acad. G. Bonchev str. block 8, 1113 Sofia, Bulgaria, and
Departamento de Matemáticas, Facultad de Matemáticas,
Universidad de Murcia, 30.100 Espinardo, Murcia, Spain
stroya@um.es

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A point $x \in A \subset (X, \|\cdot\|)$ is *quasi-denting* if for every $\varepsilon > 0$ there exists a slice of A containing x with Kuratowski index less than ε. The aim of this paper is to generalize the following theorem with a geometric approach, see [19]: A Banach space such that every point of the unit sphere is quasi-denting (for the unit ball) admits an equivalent LUR norm.

Keywords: Quasi-denting points, Kuratowski's index, LUR renorming

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1. Introduction

For a bounded set B in a metric space X, the **Kuratowski index of non-compactness of** B is defined by

$$\alpha(B) := \inf\{\varepsilon > 0 : B \text{ is covered by a finite family of sets of diameter less than } \varepsilon\}$$

Obviously $\alpha(B) \leq \text{diam } (B)$ and $\alpha(B) = 0$ if, and only if, B is totally bounded in X; i.e. relatively compact when X is a complete metric space.

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If B is a closed convex and bounded subset of X, a point $x \in B$ is said to be **quasi-denting** for B if for every $\varepsilon > 0$ there exists an open half space H with $x \in H$ and such that the 'slice' $H \cap B$ has $\alpha(H \cap B) < \varepsilon$. When in the former definition we require the diameter of $H \cap B$ to be less than ε instead of $\alpha(H \cap B) < \varepsilon$, the point x is said to be **denting** for B. The notion of quasi-denting point was introduced in [4], under the name of α-denting point, in connection with the investigation of differentiability properties of convex functions in Banach spaces; the notion of denting point goes back to the early studies of sets with the Radon-Nikodým property [1] and it was used in [18] to show that a Banach space X admits an equivalent locally uniformly rotund norm if all the points in its unit sphere S_X are denting points for the unit ball B_X. For an elegant proof of this result see [17]. Let us recall that the norm $\|\cdot\|$ in X is said to be locally uniformly rotund (**LUR** for short) if

$$\lim_{n\to\infty} \|x_n - x\| = 0 \text{ whenever } \lim_{n\to\infty} \left(2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2\right) = 0.$$

For an up-to-date account of **LUR** renormings we refer to [2, 5, 7, 20]. It is not known whether X admits an equivalent **LUR** norm if every bounded set in X has a slice of arbitrarily small diameter (i.e. the Radon-Nikodým property). G. Lancien proved that X admits an equivalent **LUR** norm whenever, for every $\varepsilon > 0$, B_X is a union of complements of a decreasing transfinite but countable family C_α^ε of closed convex sets such that $C_\alpha^\varepsilon \setminus C_{\alpha+1}^\varepsilon$ is a union of slices of C_α^ε of diameter less than ε, [9, 10], see also [5].

Throughout this paper we shall denote by X a normed space and $F \subset X^*$ will be a norming subspace for it; i.e. if we define

$$|||x||| := \sup\{|f(x)| : f \in B_{X^*} \cap F\} \text{ for every } x \in X,$$

then $|||\cdot|||$ provides an equivalent norm for X. When the original norm coincides with $|||\cdot|||$ we say that F is **1-norming**. As usual we denote by $\sigma(X, F)$ the topology in X of convergence on the elements from F. We shall denote by $\mathbb{H}(F)$ the family of all $\sigma(X, F)$-open half spaces in X. So for a point x in a $\sigma(X, F)$-closed, convex and bounded subset B of X, we shall say that x in $\sigma(X, F)$-denting ($\sigma(X, F)$-quasi-denting) for B whenever the open half space in the definition of denting (quasi-denting) can be chosen from $\mathbb{H}(F)$.

The following modification of the 'Cantor derivation' is a main tool used by Lancien to obtain his result:

We fix a normed space X, a norming subspace $F \subset X^*$ and $B \subset X$ a $\sigma(X, F)$-closed, convex and bounded subset of X. Pick any $\varepsilon > 0$ and define

$$D_{\varepsilon,F}(B) := \{x \in B : \|\cdot\| -\text{diam } (H \cap B) > \varepsilon \text{ for all } H \in \mathbb{H}(F), x \in H\}$$

Again $D_{\varepsilon,F}(B)$ is a $\sigma(X, F)$-closed, convex and bounded subset of X and we can iterate the construction and define

$$D_{\varepsilon,F}^{\alpha+1}(B) := D_{\varepsilon,F}\left(D_{\varepsilon,F}^{\alpha}(B)\right), \text{ where } D_{\varepsilon,F}^{0}(B) := B$$

and

$$D_{\varepsilon,F}^{\alpha}(B) := \bigcap_{\beta<\alpha} D_{\varepsilon,F}^{\beta}(B) \text{ if } \alpha \text{ is a limit ordinal.}$$

Then we set

$$\delta_F(B,\varepsilon) := \begin{cases} \inf\{\alpha : D^{\alpha}_{\varepsilon,F}(B) = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

and $\delta_F(B) := \sup\{\delta_F(B,\varepsilon) : \varepsilon > 0\}$.

Indeed, Lancien showed that $\delta_{X^*}(B_X) < \omega_1$ (resp. $\delta_X(B_{X^*}) < \omega_1$) implies X (resp. X^*) admits an equivalent **LUR** norm (resp. dual **LUR** norm). For quasi-denting points, refining probabilistic methods, it was shown in [19] that a Banach space X also admits an equivalent **LUR** norm provided all points in the unit sphere S_X are quasi-denting points for B_X. A modification of the derivation approach has been subsequently used by M. Raja [17] who provided a transparent proof, and significative improvements, of the result of the fourth author for denting points, [18]. Nevertheless, for quasi-denting points the approach is still fully probabilistic. A first contribution in this paper will be to provide Raja's approach for quasi-denting points; i.e. to show with a geometric construction, free from probabilistic arguments, the theorem for quasi-denting points. Indeed, for a given subset S of a $\sigma(X,F)$-closed, convex and bounded subset B of a normed space X we define its dentability index (with respect to the norming subspace $F \subset X^*$) in B as follows:

$$\delta_F(S,B,\varepsilon) := \begin{cases} \inf\{\alpha : D^{\alpha}_{\varepsilon,F}(B) \cap S = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

and $\delta_F(S,B) := \sup\{\delta_F(S,B,\varepsilon) : \varepsilon > 0\}$.

In other words we want to measure how many steps of Lancien's derivation process for B are necessary to " eat out" the subset S. When all the points of the unit sphere are denting points for the unit ball of X we obviously have $\delta_{X^*}(S_X, B_X) = 1$ and Raja's approach immediately gives the following:

Theorem 1.1 ([17]). *If $\delta_F(S_X, B_X) < \omega_1$ the normed space X admits an equivalent $\sigma(X,F)$-lower semi-continuous* **LUR** *norm.*

Indeed, the following is a tool for **LUR** renormings, [15].

Theorem 1.2 ([12, 17]). *Let X be a normed space and let F be a norming subspace of its dual. Then X admits an equivalent $\sigma(X,F)$-lower semi-continuous* **LUR** *norm if, and only if, for every $\varepsilon > 0$ we can write*

$$X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$$

in such a way that for every $x \in X_{n,\varepsilon}$, there exists a $\sigma(X,F)$-open half space H containing x with diam $(H \cap X_{n,\varepsilon}) < \varepsilon$.

When in the above theorem we replace open slices with weak ($\sigma(X,F)$) open sets we arrive to the concept introduced in [8]. Namely: a normed space X is said to have a countable cover by sets of small local diameter if for every $\varepsilon > 0$,

$$X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$$

in such a way that for every $x \in X_{n,\varepsilon}$ there exists a weak $(\sigma(X,F))$ open set W containing x with diam $(W \cap X_{n,\varepsilon}) < \varepsilon$. One could replace the diameter in the definition above by Kuratowski index of non-compactness to measure the size of the set $W \cap X_{n,\varepsilon}$. In this case, since closed balls are weak $(\sigma(X,F))$-closed, one can easily show that both notions coincide.

Let us recall that a normed space has the Kadec property if the norm and weak topologies coincide on the unit sphere. Using that an extreme point of continuity is denting ([11]) we reformulate the theorem mentioned before: a rotund Banach space with the Kadec property is **LUR** renormable. It is well known that ℓ_∞ has a rotund norm but fails to have a norm with the Kadec property. R. Haydon [7] proved that $C(\Upsilon)$, Υ diadic tree, admits a norm with the Kadec property but fails to have a rotund norm if the height of the tree is bigger or equal to ω_1. However, both Kadec property and rotundity in different combinations can be replaced by something weaker. In [12] is shown that if X has a countable cover by sets of small local diameter and all points of S_X are extreme for $B_{X^{**}}$ then X admits a **LUR** norm. In [14] is shown that if X has the Kadec property and all faces of its unit sphere have Krein-Milman property the it admits a **LUR** renorming.

Our main results in this paper will provide extensions of the former results when we use the Kuratowski index of non-compactness instead of the diameter both in the derivation process of Lancien and in the statement of the former theorem. Indeed, we shall prove the following:

Theorem 1.3. *Let X be a normed space and let F be a norming subspace of its dual. Then X admits an equivalent $\sigma(X,F)$-lower semi-continuous* **LUR** *norm if, and only if, for every $\varepsilon > 0$ we can write*

$$X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$$

in such a way that for every $x \in X_{n,\varepsilon}$, there exists a $\sigma(X,F)$-open half space H containing x with $\alpha(H \cap X_{n,\varepsilon}) < \varepsilon$.

Taking advantage of homogeneity one can replace the space X in the former theorem by its unit sphere, see Theorem 4.1.

In the course of the proof we shall show that a $\sigma(X,F)$-closed, convex and bounded subset B of the normed space X has $\sigma(X,F)$-open slices of arbitrarily small diameter if, and only if, it has $\sigma(X,F)$-open slices of arbitrarily small Kuratowski index (Corollary 2.4), and therefore the index of non-compactness also gives characterizations of sets with the Radon-Nikodým property; moreover, we shall show that

$$\delta_F\left(\sigma(X,F) - \text{quasi-denting points of } B, B\right) < \omega_1$$

from where the theorem of the fourth named author [19] follows immediately (Corollary 3.7).

From the topological point of view we shall see that a normed space X admits an equivalent $\sigma(X,F)$-lower semi-continuous **LUR** norm if, and only if, the norm topology has a network $\mathcal{N} = \cup_{n=1}^{\infty}\mathcal{N}_n$ such that for every $n \in \mathbb{N}$ and for every $x \in \cup\mathcal{N}_n$ there is a $\sigma(X,F)$-open half space H with $x \in H$ such that H meets only a finite number of elements from $\mathcal{N}_n$ (Corollary 4.5), therefore turning the 'discrete' condition appearing in

[13] into a 'locally finite' one.

Throughout the paper when dealing with a normed space X and $F \subset X^*$ a norming subspace for it, in order to simplify the notation, all closures taken in X will be with respect to the $\sigma(X,F)$-topology unless otherwise stated.

2. Kuratowski's Index of non-compactness and dentability

In order to work with the index of non-compactness we need to introduce the following definition for a bounded subset B of a normed space X and a positive integer p:

$$\alpha(B,p) := \inf\{\varepsilon > 0 : B \text{ is covered by } p \text{ sets of diameter less than } \varepsilon\}$$

and we have $\alpha(B) = \inf\{\alpha(B,p) : p = 1,2,\ldots\}$.

The first result we need is the following:

Lemma 2.1. *Let X be a normed space and $F \subset X^*$ be a 1-norming subspace. Let C, C_0 and C_1 be $\sigma(X,F)$-closed, convex and bounded subsets of X. Let p be a positive integer, $\varepsilon > 0$ and $M = diam(C_0 \cup C_1)$. If we assume that*

1. *$C_0 \subset C$ and $\alpha(C_0,p) < \varepsilon' < \varepsilon$;*
2. *C is not a subset of C_1;*
3. *C is a subset of $\overline{co}(C_0,C_1)$.*

Then if r is a positive number such that $2rM + \varepsilon' < \varepsilon$ and we set

$$D_r := \{(1-\lambda)x_0 + \lambda x_1; r \leq \lambda \leq 1, x_0 \in C_0, x_1 \in C_1\},$$

then

$$C \setminus \overline{D}_r \neq \emptyset \text{ and } \alpha\left(C \setminus \overline{D}_r, p\right) < \varepsilon$$

Remark 2.2. For $p = 1$, the lemma above is just the *Bourgain-Namioka superlemma*, see [1, 3]. Following the proof of that case it is not difficult to see that it remains true for every $p \in \mathbb{N}$. Since we shall make use of some of details from the proof we shall give it in full detail.

Proof. For $0 \leq r \leq 1$ we define

$$D_r := \{(1-\lambda)x_0 + \lambda x_1 : r \leq \lambda \leq 1, x_0 \in C_0, x_1 \in C_1\}.$$

Note that D_r is convex, $\overline{D}_0 \supset C$ (by condition 3), $D_1 = C_1$ and for $0 < r < 1$ we have $\overline{D}_r \not\supset C$. Let us show the last claim.

Since $C \not\subset C_1$ we can find $x^* \in F$ such that $\sup x^*(C_1) < \sup x^*(C)$. Now if C were contained in $\overline{D}_r$ for some $r > 0$, then we would have

$$\begin{aligned} & \sup x^*(C) \leq \sup x^*(\overline{D}_r) = \sup x^*(D_r) \\ \leq \; & (1-r)\sup x^*(C_0) + r\sup x^*(C_1) \leq (1-r)\sup x^*(C) + r\sup x^*(C_1) \end{aligned}$$

thus $r\sup x^*(C) \leq r\sup x^*(C_1)$, and since $r > 0$ we would have $\sup x^*(C) \leq \sup x^*(C_1)$, a contradiction.

Since $\alpha(C_0, p) < \varepsilon' < \varepsilon$ we should have $C_0 \subset \bigcup_{i=1}^{p} B_i$ with $\operatorname{diam}(B_i) < \varepsilon'$. Finally fix $r > 0$ such that $2rM + \varepsilon' < \varepsilon$.

Notice that $C \setminus \overline{D}_r \subset \overline{D}_0 \setminus \overline{D}_r$. Also

$$D_0 \setminus \overline{D}_r \subset C_0 + B(0; rM) \subset \bigcup_{i=1}^{p} [B_i + B(0; rM)] .$$

Indeed, if $x \in D_0 \setminus \overline{D}_r$, $x = (1-\lambda)x_0 + \lambda x_1$ with $x_0 \in C_0$, $x_1 \in C_1$ and $0 \leq \lambda < r$. Then

$$\|x - x_0\| = \lambda \|x_0 - x_1\| < rM.$$

Now

$$\overline{D}_0 \setminus \overline{D}_r \subset \bigcup_{i=1}^{p} \overline{B_i + B(0; rM)}$$

and the sets $\overline{B_i + B(0; rM)}$ have diameter less than $2rM + \varepsilon' < \varepsilon$. □

An easy consequence of Lemma 2.1 for $p = 1$ is the following.

Proposition 2.3. *Let X be a normed space and $F \subset X^*$ be 1-norming subspace. If B is a $\sigma(X, F)$-closed, convex and bounded subset of X and H is a $\sigma(X, F)$-open half space with $H \cap B \neq \emptyset$ and $\alpha(H \cap B) < \varepsilon$, then there exists another $\sigma(X, F)$-open half space G with $\emptyset \neq G \cap B \subset H \cap B$ and diam$(G \cap B) < \varepsilon$.*

Proof. By induction on the integer p such that $\alpha(H \cap B, p) < \varepsilon$. For $p = 1$ there is nothing to prove. Let us assume the assertion is true for $p \leq n-1$ and write

$$H \cap B \subset B_1^H \cup B_2^H \cup \ldots \cup B_n^H$$

where every B_i^H is a $\sigma(X, F)$-closed an convex set with $\operatorname{diam}(B_i^H) < \varepsilon$. If we define

$$L_1 := \overline{\operatorname{co}}(B \setminus H, B_1^H \cap B)$$

we have two possibilities:

(i) $B = L_1$ and we can apply Lemma 2.1 for $p = 1$ to the sets $C_0 = B_1^H \cap B$ and $C_1 = B \setminus H$ to obtain a $\sigma(X, F)$-open half space G with $G \cap B_1^H \neq \emptyset$, $G \cap B \subset H \cap B$ and $\operatorname{diam}(G \cap B) < \varepsilon$.

(ii) $L_1 \subsetneq B$, then for any $y \in B \setminus L_1$ we have, by Hahn-Banach's Theorem, a $\sigma(X, F)$-open half space $\tilde{H}$ with $y \in \tilde{H}$ and $\tilde{H} \cap B \subset H \cap B$ but

$$\tilde{H} \cap B \subset B_2^H \cup B_3^H \cup \ldots \cup B_n^H$$

and if we apply the induction hypothesis to this slice the proof is done. □

As a corollary we obtain a better statement than the one given by Gilles and Moors in [4], see Theorems 4.2 and 4.3.

Corollary 2.4. *For a normed space X, a norming subspace $F \subset X^*$ and a $\sigma(X, F)$-closed, convex and bounded subset B of X, the following are equivalent:*

1. *B is $\sigma(X,F)$-dentable; i.e. B has $\sigma(X,F)$-open slices of arbitrarily small diameter;*
2. *B has $\sigma(X,F)$-open slices whose Kuratowski index of non-compactness is arbitrarily small .*

Proof. We can work with the equivalent norm $|||\cdot|||$ given by the norming subspace F and apply Proposition 2.3 for every $\varepsilon > 0$. $\square$

3. Dentability index of quasi-denting points

We are going to iterate now Bourgain-Namioka superlemma together with the former construction in Proposition 2.3 to describe when quasi-denting points are eaten out in Lancien's derivation process. For a normed space X and a norming subset $F \subset X^*$, we shall denote by $\mathbb{H}(F)$ the family of all $\sigma(X,F)$-open half spaces in X. Indeed, we shall prove the following:

Theorem 3.1. *For every $\varepsilon > 0$ there is a countable ordinal η_ε such that if X is a normed space and $F \subset X^*$ is a 1-norming subspace, then for every $\sigma(X,F)$-closed, convex and bounded subset B of X, if*

$$Q_\varepsilon := \{x \in B; \exists H \in \mathbb{H}(F), x \in H \text{ with } \alpha(H \cap B) < \varepsilon\},$$

then we have

$$\delta_F(Q_\varepsilon, B, \varepsilon) < \eta_\varepsilon < w_1$$

The proof of the theorem is based on a series of previous results. We begin with:

Lemma 3.2. *Let B be $\sigma(X,F)$-closed, convex and bounded subset of X, where F is a 1-norming subspace for X and $\varepsilon > 0$ be fixed. Let $B := L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_n$ be $\sigma(X,F)$-closed and convex. Let S be a subset of L_n, then*

$$\delta_F(S,B,\varepsilon) \leq \delta_F(L_0 \setminus L_1, B, \varepsilon) + \delta_F(L_1 \setminus L_2, L_1, \varepsilon) + \ldots$$
$$+\delta_F(L_{n-1} \setminus L_n, L_{n-1}, \varepsilon) + \delta_F(S, L_n, \varepsilon)$$

Proof. We shall prove it by induction on n. For $n = 1$ let $B = L_0 \supset L_1 \supset S$ be as in the statement and let: $\delta_F(L_0 \setminus L_1, B, \varepsilon) = \alpha$, $\delta_F(S, L_1, \varepsilon) = \beta$.

Since $D^\alpha_{\varepsilon,F}(B) \cap (B \setminus L_1) = \emptyset$ we have $D^\alpha_{\varepsilon,F}(B) \subset L_1$. Given $x \in D^{\alpha+1}_{\varepsilon,F}(B)$, we have $\operatorname{diam}(H \cap D^\alpha_{\varepsilon,F}(B)) > \varepsilon$ for every $H \in \mathbb{H}(F)$, so $\operatorname{diam}(H \cap L_1) > \varepsilon$ for every $H \in \mathbb{H}(F)$, thus $x \in D_{\varepsilon,F}(L_1)$. So

$$D^{\alpha+1}_{\varepsilon,F}(B) \cap S \subset D_{\varepsilon,F}(L_1) \cap S.$$

Since β is the first ordinal such that $D^\beta_{\varepsilon,F}(L_1) \cap S = \emptyset$ one must have $D^{\alpha+\beta}_{\varepsilon,F}(B) \cap S = \emptyset$, therefore $\delta_F(S,B,\varepsilon) \leq \alpha + \beta$.

Now suppose we have $B := L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_n \supset S$ as in the statement and suppose the formula holds for $n-1$ sets. Considering $L_0 \supset L_1 \supset S$, as we did before,

$$\delta_F(S,B,\varepsilon) \leq \delta_F(L_0 \setminus L_1, B, \varepsilon) + \delta_F(S, L_1, \varepsilon) \qquad (*)$$

If we consider now the sets $L_1 \supset L_2 \supset \ldots \supset L_n \supset S$, by the induction hypothesis

$$\delta_F(S, L_1, \varepsilon) \leq \delta_F(L_1 \setminus L_2, L_1, \varepsilon) + \ldots + \delta_F(L_{n-1} \setminus L_n, L_{n-1}, \varepsilon) + \delta_F(S, L_n, \varepsilon)$$

To finish the proof we just need to use the later inequality in $(*)$. $\square$

Lemma 3.3. *Let B be $\sigma(X,F)$-closed, convex and bounded subset of a normed space X, where F is a 1-norming subspace for X and $\varepsilon > 0$ be fixed. Let H be a $\sigma(X,F)$-open half space with*

$$\alpha(H \cap B, n) < \varepsilon \text{ for some } n > 1 \text{ fixed.}$$

Then there exists a sequence of $\sigma(X,F)$-closed, convex subsets

$$B =: B_0 \supset B_1 \supset B_2 \supset \ldots \supset B_s \supset B_{s+1} \supset \ldots$$

such that

$$H \cap B \subset (B_0 \setminus B_1) \cup (B_1 \setminus B_2) \cup \ldots \cup (B_s \setminus B_{s+1}) \cup \ldots$$

and for every $s = 0, 1, 2, \ldots$ and every $y \in B_s \setminus B_{s+1}$ there is a $\sigma(X,F)$-open half space G with $y \in G$, $G \cap B \subset H \cap B$, and

$$\alpha(G \cap B_s, p) < \varepsilon \text{ for some } p \leq n-1$$

Proof. Since $\alpha(H \cap B, n) < \varepsilon$ we can fix $\sigma(X,F)$-closed, convex non-void sets

$$\{B_1^H, B_2^H, \ldots, B_n^H\} \text{ with } \operatorname{diam}(B_i^H) < \varepsilon, \text{ for } i = 1, 2, \ldots, n$$

and $H \cap B \subset B_1^H \cup \ldots \cup B_n^H$.

Let us define

$$L_1 := \overline{\text{co}}\,(B \setminus H, B_1^H \cap B).$$

If $y \in B \setminus L_1$, Hahn-Banach's Theorem provides us with a $\sigma(X,F)$-open half space G, with $y \in G$ and $G \cap L_1 = \emptyset$, thus

$$G \cap B \subset H \cap B \text{ and } G \cap B \subset B_2^H \cup \cdots \cup B_n^H$$

and therefore $\alpha(G \cap B, p) < \varepsilon$ for some $p \leq n-1$.

Let us consider the sets $C_0^1 := B_1^H \cap B$ and $C_1 = B \setminus H$ and apply Lemma 2.1 with $p = 1$, to find $0 < r < 1$, indeed it is enough if $2r\operatorname{diam}(B) + \operatorname{diam}(B_1^H) < \varepsilon$, such that if

$$D_{r,1} := \{(1-\lambda)x_0 + \lambda x_1; r \leq \lambda \leq 1, x_0 \in C_0^1, x_1 \in C_1\}$$

we have $L_1 \setminus \overline{D}_{r,1} \neq \emptyset$ and $\operatorname{diam}(L_1 \setminus \overline{D}_{r,1}) < \varepsilon$. So for every $y \in L_1 \setminus \overline{D}_{r,1}$ we should have a $\sigma(X,F)$-open half space G with $y \in G$, $G \cap \overline{D}_{r,1} = \emptyset$, thus $G \cap B \subset H \cap B$ and $G \cap L_1 \subset L_1 \setminus \overline{D}_{r,1}$, so $\operatorname{diam}(G \cap L_1) < \varepsilon$ and $\alpha(G \cap L_1, 1) < \varepsilon$.

We set $B_1 := L_1$ and $B_2 := \overline{D}_{r,1}$. We shall iterate now the former construction to " eat out" the whole B_1^H and to reach all the points of $B \cap H$ in a countable number of steps. Let us define

$$L_2 := \overline{\text{co}}\,\left(B \setminus H, B_1^H \cap \overline{D}_r\right)$$

If $y \in \overline{D}_{r,1} \setminus L_2$, there is a $\sigma(X,F)$-open half space G with $y \in G$ and $G \cap L_2 = \emptyset$, thus $G \cap B \subset H \cap B \subset B_1^H \cup B_2^H \cup \ldots \cup B_n^H$. Moreover

$$G \cap \overline{D}_{r,1} \subset B_2^H \cup \ldots \cup B_n^H$$

since $\overline{D}_{r,1} \cap B_1^H \subset L_2$, and then $\alpha\left(G \cap \overline{D}_{r,1}, p\right) < \varepsilon$ for some $p \leq n-1$

We shall now apply again the Bourgain-Namioka superlemma with the sets

$$C_0^2 := B_1^H \cap \overline{D}_{r,1} \text{ and } C_1 := B \setminus H$$

and with the same r as above we should have diam $(L_2 \setminus \overline{D}_{r,2}) < \varepsilon$ where

$$D_{r,2} := \{(1-\lambda)x_0 + \lambda x_1; r \leq \lambda \leq 1, x_0 \in C_0^2, x_1 \in C_1\}.$$

As before, for every $y \in L_2 \setminus \overline{D}_{r,2}$ there exists a $\sigma(X,F)$-open half space G with $y \in G$, $G \cap B \subset H \cap B$ and $\alpha(G \cap L_2, 1) < \varepsilon$.

We set $B_3 := L_2$ and $B_4 := \overline{D}_{r,2}$. The process will continue by induction defining a sequence of sets

$$B = B_0 \supset L_1 \supsetneq \overline{D}_{r,1} \supset L_2 \supsetneq \overline{D}_{r,2} \supset \ldots \supset L_s \supsetneq \overline{D}_{r,s} \supset L_{s+1} \supsetneq \ldots$$

such that for every $y \in L_s \setminus \overline{D}_{r,s}$ there is a $\sigma(X,F)$-open half space G with $y \in G$, $G \cap B \subset H \cap B$ and $\alpha(G \cap L_s, 1) < \varepsilon$; and for every $y \in \overline{D}_{r,s} \setminus L_{s+1}$ there is a $\sigma(X,F)$-open half space G with $y \in G$, $G \cap B \subset H \cap B$ and $\alpha\left(G \cap \overline{D}_{r,s}, p\right) < \varepsilon$ for some $p \leq n-1$.

If $\overline{D}_{r,s_0} \cap B_1^H = \emptyset$ for some $s_0 \geq 1$, then the process stops and the sequence should be finite in that case. Note that when it happens we have

$$H \cap \overline{D}_{r,s_0} \subset B_2^H \cup \ldots \cup B_n^H$$

and $\alpha\left(H \cap \overline{D}_{r,s_0}, p\right) < \varepsilon$ for some $p \leq n-1$ too.

If the process does not stop, we shall see now that for each $y \in H \cap B$ there is an integer $s \geq 2$ such that either $y \in L_s \setminus \overline{D}_{r,s}$ or $y \in \overline{D}_{r,s-1} \setminus L_s$ whenever $y \notin (B \setminus L_1) \cup \left(L_1 \setminus \overline{D}_{r,1}\right)$. Indeed, if $H = \{x \in X : f(x) > \mu\}$, $f \in F$, then we have

$$\sup f|_{\overline{D}_{r,1}} \leq (1-r) \sup f(B_1^H \cap B) + r\mu$$

for the first step

$$\begin{aligned} &\sup f|_{\overline{D}_{r,2}} \leq (1-r) \sup f(B_1^H \cap \overline{D}_{r,1}) + r\mu \\ &\leq (1-r)[(1-r)\sup f(B_1^H \cap B) + r\mu] + r\mu = (1-r)^2 \sup f(B_1^H \cap B) + (1-r)r\mu + r\mu \end{aligned}$$

for the second step and by recurrence

$$\sup f|_{\overline{D}_{r,s}} \leq (1-r)^s \sup f(B_1^H \cap B) + r\mu[1 + (1-r) + \cdots + (1-r)^{s-1}]$$

for $s = 1, 2, \ldots$. Consequently for every y with $f(y) > \mu$, y cannot be in all the sets $\overline{D}_{r,s}$ for $s = 1, 2, \ldots$ because the former inequality should imply $f(y) \leq \mu$. Then if s is the first integer with $y \notin \overline{D}_{r,s}$ we will have either $y \in L_s \setminus \overline{D}_{r,s}$ or $y \in \overline{D}_{r,s-1} \setminus L_s$, when $s \geq 2$ and $y \in B \setminus L_1$ or $y \in L_1 \setminus \overline{D}_{r,1}$ when $s = 1$.

The lemma is finished by defining $B_{2n+1} := L_{n+1}$ and $B_{2n} := \overline{D}_{r,n}$, $n = 1, 2, \ldots$ when the process does not stop and $B_{s_0} = \overline{D}_{r,s_0}$, $B_{s_0+1} = \ldots = \emptyset$ when the process stops at the s_0-step. We have seen before that $\alpha\left(H \cap \overline{D}_{r,s_0}, p\right) < \varepsilon$ for some $p \leq n-1$ in that case too. $\square$

Proposition 3.4. *For every $\varepsilon > 0$, there exists a sequence of ordinal numbers*

$$1 =: \xi_1 < \xi_2 < \ldots < \xi_p < \ldots < \omega_1$$

such that if X is a normed space and $F \subset X^*$ is a 1-norming subspace for it, we have

$$\delta_F \left(H \cap B, B, \varepsilon\right) \leq \xi_p \tag{$*$}$$

whenever B is a $\sigma(X, F)$-closed, convex and bounded subset of X and H is a $\sigma(X, F)$-open half space with $\alpha\left(H \cap B, p\right) < \varepsilon$.

Proof. We shall define by induction on p the sequence of countable ordinals $(\xi_n)_n$. For $p = 1$ the Kuratowski index $\alpha(\cdot, 1)$ coincides with the diameter and $H \cap B$ should be *eaten* at the first step of the derivation process, i.e., $\xi_1 := 1$ verifies $(*)$.

Let us assume that we have already defined

$$\xi_1 < \xi_2 < \ldots < \xi_{n-1} < \omega_1$$

such that $(*)$ is satisfied for $p \leq n - 1$. Let us fix a $\sigma(X, F)$-closed, convex and bounded subset of X and H a $\sigma(X, F)$-open half space with

$$\alpha\left(H \cap B, n\right) < \varepsilon$$

By Lemma 3.3 we have a sequence of $\sigma(X, F)$-closed, convex subsets

$$B = B_0 \supset B_1 \supset \ldots \supset B_s \supset B_{s+1} \supset \ldots \tag{$**$}$$

such that

$$H \cap B \subset (B_0 \setminus B_1) \cup (B_1 \setminus B_2) \cup \ldots \cup (B_s \setminus B_{s+1}) \cup \ldots$$

and for every s and every $y \in B_s \setminus B_{s+1}$ there exists a $\sigma(X, F)$-open half space G, with $y \in G$, $G \cap B \subset H \cap B$, and $\alpha\left(G \cap B_s, p\right) < \varepsilon$ for some $p \leq n - 1$. By our induction assumption we should have

$$\delta_F \left(G \cap B_s, B_s, \varepsilon\right) \leq \xi_{n-1}$$

and consequently $\delta_F \left(B_s \setminus B_{s+1}, B_s, \varepsilon\right) \leq \xi_{n-1}$, $s = 0, 1, 2, \ldots$ when $(**)$ is infinite and $\delta_F \left(H \cap B_{s_0}, B_{s_0}, \varepsilon\right) \leq \xi_{n-1}$ too, when the sequence stops at the s_0-step. In any case we can apply Lemma 3.2 to obtain

$$\delta_F \left(B_s \setminus B_{s+1}, B, \varepsilon\right) \leq (s + 1)\xi_{n-1} \text{ for } s = 0, 1, 2, \ldots$$

Therefore we have

$$\delta_F \left(H \cap B, B, \varepsilon\right) \leq \sup\{(s + 1)\xi_{n-1} : s = 0, 1, 2, \ldots\} =: \xi_n$$

which finishes the induction process. □

Corollary 3.5. *For every $\sigma(X, F)$-closed, convex and bounded subset B of X, if*

$$Q_{\varepsilon,p} := \{x \in B : \exists H \in \mathbb{H}(F), x \in H \textit{ with } \alpha(H \cap B, p) < \varepsilon\}$$

then we have

$$\delta_F \left(Q_{\varepsilon,p}, B, \varepsilon\right) \leq \xi_p < \omega_1, \quad p = 1, 2, \ldots$$

Proof. $Q_{\varepsilon,p} = \cup\{H \cap B : H \in \mathbb{H}(F) \text{ and } \alpha(H \cap B, p) < \varepsilon\}$ and

$$\delta_F(Q_{\varepsilon,p}, B, \varepsilon) \leq \sup\{\delta_F(H \cap B) : H \in \mathbb{H}(F) \text{ and } \alpha(H \cap B, p) < \varepsilon\} \leq \xi_p$$

□

Now we arrive to the

Proof of Theorem 3.1. We have $Q_\varepsilon = \cup\{Q_{\varepsilon,p} : p = 1, 2, \ldots\}$, by the former corollary we see that $\delta_F(Q_{\varepsilon,p}, B, \varepsilon) \leq \xi_p$ for $p = 1, 2, \ldots$ from where it follows

$$\delta_F(Q_\varepsilon, B, \varepsilon) \leq \sup\{\xi_p : p = 1, 2, \ldots\} =: \eta_\varepsilon < \omega_1$$

because ω_1 is not the limit of a sequence of countable ordinals. □

Corollary 3.6. *There is a countable ordinal η such that if X is a normed space and $F \subset X^*$ is a norming subspace, then for every $B \subset X$ a $\sigma(X, F)$-closed convex and bounded subset of X, if Q is the sets of quasi-denting points of B, we have*

$$\delta_F(Q, B) < \eta < \omega_1$$

Proof. It is not a restriction to assume that the given norm is $|||\cdot|||$, making F a 1-norming subspace, then we have

$$\delta_F(Q, B) \leq \sup\{\eta_{\varepsilon_n}; n = 1, 2, \ldots\} =: \eta < w_1$$

where ε_n tends to 0. □

From Theorem 1.1 in the introduction we get the theorem of the fourth named author with a geometric proof in full generality:

Corollary 3.7. *If the normed space X has a norming subspace $F \subset X^*$ such that S_X is formed by quasi-denting points of B_X, in $\sigma(X, F)$, then $\delta_F(S_X, B_X) < \omega_1$ and consequently X admits an equivalent $\sigma(X, F)$-lower semi-continuous* **LUR** *norm.*

4. LUR renorming theorem

The aim of this section is to prove the following result, from where Theorem 1.3 in the introduction is a particular case. Let us recall that a subset $A \subset X$ of the normed space X is said to be a **radial set** if for every $x \in X$ there is $\rho > 0$ such that $\rho x \in A$.

Theorem 4.1. *Let X be a normed space and $F \subset X^*$ be a norming subspace for it. The following conditions are equivalent:*

1. *X admits an equivalent $\sigma(X, F)$-lower semi-continuous* **LUR** *norm;*
2. *For every $\varepsilon > 0$, $X = \cup\{X_{n,\varepsilon} : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$ and $x \in X_{n,\varepsilon}$ there exists H, $\sigma(X, F)$-open half space with $x \in H$ and $\alpha(H \cap X_{n,\varepsilon}) < \varepsilon$;*
3. *There exists a radial set $A \subset X$ such that for every $\varepsilon > 0$, $A = \cup\{A_{n,\varepsilon} : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$ and $x \in A_{n,\varepsilon}$ there exists H, $\sigma(X, F)$-open half space with $x \in H$ and $\alpha(H \cap A_{n,\varepsilon}) < \varepsilon$*

Let us observe that no convex assumption is required for the sets $\{X_{n,\varepsilon}\}$ or $\{A_{n,\varepsilon}\}$ in the decompositions above. As for the proof of Theorem 1.2 in the introduction, see [17], we need firstly a convexification argument that will reduce Theorem 4.1 to Theorem 1.2 because of the study we have done in the previous section.

We begin with a revision of Lemma 2.1 for an arbitrary set A and $x \in A$ with a half space $H \in \mathbb{H}(F)$ with $x \in H$ and $\alpha(H \cap A) < \varepsilon$ (in this case we shall say that x is an ε-$\sigma(X,F)$-quasi-denting point for A).

Lemma 4.2. *Let A be a bounded subset of the normed space X, $F \subset X^*$ be 1-norming for it. Set $M :=$ diam (A) and let $\varepsilon > 0$ be fixed. If $x \in A$ is such that there is $H = \{y \in X : g(y) > \eta\}$, with $g \in F$, $\eta \in \mathbb{R}$, $x \in H$ and $\alpha(H \cap A) < \varepsilon$, then there exists $r \in]0,1]$ which only depends upon ε and M such that we can fix a $\sigma(X,F)$-closed and convex subset $D_r(x) \subset \overline{co}(A)$ with the following properties:*

i) $\overline{co}(A) \setminus D_r(x) \neq \emptyset$;

ii) $\alpha(\overline{co}(A) \setminus D_r(x)) < 3\varepsilon$;

iii) $\sup g(D_r(x)) \leq (1-r) \sup g(\overline{co}(A)) + r\eta$

Proof. Let us choose sets $B_1, B_2, \ldots, B_n$ with diam $(B_i) < \varepsilon$ and $u_i \in B_i \cap A$, $i = 1, \ldots, n$, such that $H \cap A \subset \cup\{B_i : i = 1, \ldots, n\}$. Now let $K_\varepsilon :=$ co $(u_1, \ldots, u_n)$ and set

$$C_0 := \{y \in \overline{\text{co}}(A) : \text{dist }(y, K_\varepsilon) \leq \varepsilon\}$$

and

$$C_1 := \{y \in \overline{\text{co}}(A) : g(y) \leq \eta\} = \overline{\text{co}}(A) \setminus H.$$

As we did in Lemma 2.1, for $0 \leq r \leq 1$, let

$$D_r := \{(1-\lambda)x_0 + \lambda x_1 : r \leq \lambda \leq 1, x_0 \in C_0, x_1 \in C_1\}.$$

To obtain the conclusion one must check that the sets $C = \overline{\text{co}}(A)$, C_0 and C_1 satisfy the conditions in Lemma 2.1 to apply it.

C and C_1 are clearly bounded, $\sigma(X,F)$-closed and convex. K_ε is $\|\cdot\|$-compact, hence C_0 is $\sigma(X,F)$-closed. Since K_ε is convex it is easy to see that C_0 is also convex.

1.- $C_0 \subset \overline{\text{co}}(A)$; since K_ε can be covered by finitely many balls of arbitrary small radius, it is not difficult to check that $\alpha(C_0) \leq 2\varepsilon$.

2.- $\overline{\text{co}}(A)$ is not a subset of C_1, (since $x \notin C_1$).

3.- $\overline{\text{co}}(A) = \overline{\text{co}}(C_1 \cup C_0)$. To check it we show that co $(A) \subset$ co $(C_1 \cup C_0)$. To do so, set $B_1 =$ co $(A \cap H)$ and $B_2 =$ co $(A \setminus H)$. It is clear that co $(A) \subset$ co $(B_1 \cup B_2)$. Now since $A \cap H \subset C_0$ one must have $C_0 \supset B_1$ and clearly $B_2 \subset C_1$.

Since $\alpha(C_0) \leq 2\varepsilon$, we have $\alpha(C_0) < 3\varepsilon$ and we can take $\varepsilon' = \frac{5\varepsilon}{2}$ in Lemma 2.1 and then it will be enough to take $r < \frac{\varepsilon}{4M}$. Now the lemma applies to give the conclusion for i) and ii). Property iii) easily follows from the definition of the set $D_r =: D_r(x)$ and the fact that $g(y) \leq \eta$ for $y \in C_1$. □

We shall iterate now the former lemma to be able to ensure that ε-quasi-denting points for an arbitrary subset B should be 3ε-quasi-denting points in some convex set of a sequence $\{B_n\}$ associated to B.

Lemma 4.3 (Iteration Lemma). *Let B be a bounded subset of the normed space X, $F \subset X^*$ 1-norming such that for some $\varepsilon > 0$ fixed, every $x \in B$ is an ε-$\sigma(X,F)$-quasi-denting point for B. Then there is a sequence*

$$B_0 = \overline{co}\,(B) \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots$$

of convex, $\sigma(X,F)$-closed subsets of $\overline{co}\,(B)$ such that for every $x \in B$ there exists $p \geq 0$ satisfying $x \in B_p$ and x is a 3ε-$\sigma(X,F)$-quasi-denting point for B_p.

Proof. We shall construct by recurrence sequences of sets

$$B_0 = \overline{\mathrm{co}}\,(B) \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots$$

and $B =: \tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_n, \ldots$ such that

$$B_n := \overline{\mathrm{co}}\left(B \cap \tilde{B}_1 \cap \tilde{B}_2 \cap \ldots \cap \tilde{B}_n\right) \text{ if } B \cap \tilde{B}_1 \cap \tilde{B}_2 \cap \ldots \cap \tilde{B}_n \neq \emptyset$$

and given $x \in B$, if $x \in \left(B \cap \tilde{B}_1 \cap \tilde{B}_2 \cap \ldots \cap \tilde{B}_{n-1}\right) \setminus \tilde{B}_n$ then x is a 3ε-$\sigma(X,F)$-quasi-denting point for

$$B_{n-1} = \overline{\mathrm{co}}\left(B \cap \tilde{B}_1 \cap \tilde{B}_2 \cap \ldots \cap \tilde{B}_{n-1}\right).$$

Indeed, set $B_0 := \overline{\mathrm{co}}\,(B)$ and $\tilde{B}_0 := B$. Now fix $x \in B$, by hypothesis we fix $g_x \in F$, $\eta_x \in \mathbb{R}$ such that the half space $H_x = \{y \in X : g_x(y) > \eta_x\}$ satisfies

$$x \in H_x \cap B \text{ and } \alpha\,(H_x \cap B) < \varepsilon.$$

Let $M = \operatorname{diam}\,(B_0)$. At each point x from B together with the corresponding $H_x \in \mathbb{H}(F)$, we may apply the former lemma to obtain $D_r^1(x)$, $\sigma(X,F)$-closed and convex and $r \in [0,1]$ with the properties described in Lemma 4.2. Now define

$$\tilde{B}_1 := \bigcap_{x \in \tilde{B}_0} D_r^1(x).$$

Note that if $x \in B \setminus \tilde{B}_1$ then, there exists $y \in B$ such that $x \in \overline{\mathrm{co}}\,(B) \setminus D_r^1(y)$ and x is a 3ε-$\sigma(X,F)$-quasi-denting point for $B_0 = \overline{\mathrm{co}}\,(B)$.

Notice that if $B \cap \tilde{B}_1 = \emptyset$ we would have finished the proof since every $x \in B$ would be a 3ε-$\sigma(X,F)$-quasi-denting point for B_0. So assume $B \cap \tilde{B}_1 \neq \emptyset$, we shall define a set B_1 as

$$B_1 := \overline{\mathrm{co}}\,(B \cap \tilde{B}_1).$$

Consider the set $B \cap \tilde{B}_1$ and H_x at every point $x \in B \cap \tilde{B}_1$. Since

$$\operatorname{diam}\,(\overline{\mathrm{co}}\,(B \cap \tilde{B}_1)) \leq M \text{ and } \alpha(H_x \cap B \cap \tilde{B}_1) < \varepsilon$$

we apply Lemma 4.2 to the set $B \cap \tilde{B}_1$ and this time we will obtain sets $D_r^2(x)$ with the properties given by the lemma and r being the same r as above. Now define

$$\tilde{B}_2 := \bigcap_{x \in B \cap \tilde{B}_1} D_r^2(x)$$

As we did before, if $x \in (B \cap \tilde{B}_1) \setminus \tilde{B}_2$ there must be $y \in B \cap \tilde{B}_1$ such that

$$x \in B_1 = \overline{\mathrm{co}}\,(B \cap \tilde{B}_1) \setminus D_r^2(y)$$

and x is a 3ε-$\sigma(X, F)$-quasi-denting point for B_1. It follows now by recurrence that such sequences can be built and it will be finite if

$$B \cap \tilde{B}_1 \cap \tilde{B}_2 \cap \ldots \cap \tilde{B}_n = \emptyset$$

To finish the proof we need to show that for every $x \in B$ there exists $p \geq 0$ such that $x \in \left(B \cap \ldots \cap \tilde{B}_{p-1}\right) \setminus \tilde{B}_p$. So suppose this is not the case, i.e., there exists $x \in B$ (which will be fixed from now on), such that $x \in B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_p$ for every $p = 1, 2, \ldots$ Let us consider the sets $D_r^p(x)$ defined for the point x, g_x and η_x at each step $p = 1, 2, \ldots$ Recall from Lemma 4.2 that

$$\sup g_x\left(D_r^1(x)\right) \leq (1-r)\sup g_x(B_0) + r\eta_x.$$

So for $p = 2$, and bearing in mind that $B_1 \subset D_r^1(x)$ we have

$$\begin{aligned} & \sup g_x\left(D^2(x)\right) \leq (1-r)\sup g_x(B_1) + r\eta_x \\ \leq \quad & (1-r)\sup g_x\left(D_r^1(x)\right) + r\eta_x \leq (1-r)\left[(1-r)\sup g_x(B_0) + r\eta_x\right] + r\eta_x \\ = \quad & (1-r)^2 \sup g_x(B_0) + r\eta_x(1 + (1-r)) \end{aligned}$$

Now by induction we should have

$$\begin{aligned} & \sup g_x\left(D_r^n(x)\right) \leq (1-r)^n \sup g_x(B_0) + r\eta_x\left[1 + (1-r) + \ldots + (1-r)^{n-1}\right] \\ = \quad & (1-r)^n \sup g_x(B_0) + \eta_x(1-(1-r)^n) = \eta_x + (1-r)^n(\sup g_x(B_0) - \eta_x) \end{aligned}$$

for every integer n such that $x \in B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_{n-1}$.

Now since $(1-r)^n$ tends to 0 as n goes to infinity and $\eta_x < g_x(x)$ one can choose n large enough so that $\sup g_x(D_r^n(x)) < g_x(x)$ which is a contradiction with assuming $x \in D_r^n(x)$.

Thus, there exists $n \in \mathbb{N}$ such that $x \in B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_{n-1}$ and $x \notin D_r^n(x)$ hence $x \in (B \cap \tilde{B}_1 \cap \ldots \cap \tilde{B}_{n-1}) \setminus \tilde{B}_n$ as we wanted. □

The step connecting Kuratowski's index with dentability follows now from Theorem 3.1:

Corollary 4.4. *Let B be a bounded subset of the normed space X, $F \subset X^*$ 1-norming subspace for it such that for some $\varepsilon > 0$ fixed, every $x \in B$ is an ε-$\sigma(X, F)$-quasi-denting point for B. Then there is a countable family $\{T_n : n = 1, 2, \ldots\}$ of $\sigma(X, F)$-closed and convex subsets of $\overline{co}\,(B)$ such that for every $x \in B$ there exists $p > 0$ such that $x \in T_p$ and there is $H \in \mathbb{H}(F)$ with $x \in H$ and diam $(H \cap T_p) < 3\varepsilon$.*

Proof. If we set $B_0 \supset B_1 \supset \ldots \supset B_n \supset \ldots$ as in Lemma 4.3, we know that

$$\delta_F\left(3\varepsilon - \sigma(X, F) - \text{quasi-denting points of } B_p, B_p, 3\varepsilon\right) < \eta_{3\varepsilon} < \omega_1$$

and therefore the family of derived sets $\{D^{\beta}_{3\varepsilon,F}(B_p) : \beta < \eta_{3\varepsilon}, p = 1, 2, \ldots\}$, provides us a countable family $\{T_n : n = 1, 2, \ldots\}$ with the required properties. □

Now we arrive to the

Proof of Theorem 4.1. (1)⇒(2) Follows from Theorem 1.2 in the introduction.

(2)⇒(1) It is clear that condition (2) must be true for any equivalent norm and it is not a restriction to assume that the given norm is $|||\cdot|||$ making F a 1-norming subspace for it. Then we have the conditions of Corollary 4.5 for every set

$$X_{n,\varepsilon} \cap B(0,m) \quad n = 1,2,\ldots, m = 1,2,\ldots$$

and we will have countable families $\{T_p^{n,m,\varepsilon} : p = 1,2,\ldots\}$, $n = 1,2,\ldots$, $m = 1,2,\ldots$ such that for every $x \in X_{n,\varepsilon} \cap B(0,m)$ there is $p \geq 0$ such that

$$x \in T_p^{n,m,\varepsilon} \text{ and there is } H \in \mathbb{H}(F) \text{ with } x \in H \text{ and diam } \left(H \cap T_p^{n,m,\varepsilon}\right) < 3\varepsilon$$

If we set

$$Y_p^{n,m,\varepsilon} := \{x \in T_p^{n,m,\varepsilon} : \text{ there is } H \in \mathbb{H}(F), x \in H \text{ and diam } (H \cap T_p^{n,m,\varepsilon}) < 3\varepsilon\}$$

we have $X = \cup\{Y_p^{n,m,\varepsilon} : n,m,p = 1,2,\ldots\}$ and we have the decomposition fixed in Theorem 1.2 which is equivalent to have a $\sigma(X,F)$-lower semi-continuous **LUR** norm on X.

(2)⇒(3) Is obvious.

(3)⇒(2) Given $x \in X \setminus \{0\}$ let $r(x) > 0$ such that $r(x)x \in A$. By hypothesis, for every $k \in \mathbb{N}$, $A = \cup_n A_{n,k}$ with the property that for every $x \in A_{n,k}$ there exists $H \in \mathbb{H}(F)$, $x \in H$ such that $\alpha\left(A_{n,k} \cap H\right) < \frac{1}{k}$. For $q \in \mathbb{Q}$, $n,m,k \in \mathbb{N}$ define

$$A_{n,k}^{q,m} := \{y \in X \setminus \{0\} : r(y)y \in A_{n,k}, 0 < \frac{1}{q} - \frac{1}{m} < \frac{1}{r(y)} < \frac{1}{q}\}.$$

We shall show that $X \setminus \{0\} = \cup\{A_{n,k}^{q,m} : n,m,k \in \mathbb{N}, q \in \mathbb{Q}\}$ and for every $\varepsilon > 0$, and $x \in X \setminus \{0\}$ there exist $n,m,k \in \mathbb{N}$, $q \in \mathbb{Q}$, $H \in \mathbb{H}(F)$ with $x \in H$ such that $\alpha\left(A_{n,k}^{q,m} \cap H\right) < \varepsilon$.

So, given $\varepsilon > 0$ and $x_0 \in X \setminus \{0\}$, consider $r(x_0) > 0$ and let $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{r(x_0)\varepsilon}{2}$. For this k (fixed), let $n \in \mathbb{N}$ such that $r(x_0)x_0 \in A_{n,k}$. By the property of A, there exist $f \in F$ and $\mu \in \mathbb{R}$ such that

$$r(x_0)x_0 \in H = \{x \in X : f(x) > \mu\} \text{ and } \alpha\left(A_{n,k} \cap H\right) < \frac{1}{k}.$$

Therefore, there are sets B_i, $i = 1,\ldots,j$ with diam $(B_i) < \frac{1}{k}$ such that

$$A_{n,k} \cap H \subset \bigcup_{i=1}^{j} B_i.$$

For every $i \in \{1,\ldots,j\}$ fix $x_i \in B_i$. Take $m \in \mathbb{N}$ such that $m > \frac{2M}{\varepsilon} + r(x_0)$ and let $M = \max_i\{\|x_i\|\}$. Finally let $q \in \mathbb{Q}$ such that

$$\frac{1}{q} - \frac{1}{m} < \frac{1}{r(x_0)} < \frac{1}{q} \text{ and } f(x_0) > \frac{\mu}{q} > \frac{\mu}{r(x_0)}.$$

Take now the $\sigma(X,F)$-open half space $H' := \{x \in X : f(x) > \frac{\mu}{q}\}$. It is clear that $x_0 \in A^{q,m}_{n,k} \cap H'$. Let, for every $i \in \{1,\ldots,j\}$, $u_i = \frac{1}{r(x_0)} x_i$. Let us prove that $\alpha\left(A^{q,m}_{n,k} \cap H'\right) < 2\varepsilon$ by checking $A^{q,m}_{n,k} \cap H' \subset \bigcup_{i=1}^{j} B(u_i;\varepsilon)$. To do so take any $y \in A^{q,m}_{n,k} \cap H'$. In particular $f(y) > \frac{\mu}{q}$, hence

$$f(r(y)y) = r(y)f(y) > r(y)\frac{\mu}{q} > \mu.$$

Therefore, $r(y)y \in A_{n,k} \cap H$. So, there must be x_i, for some $i \in \{1,\ldots,j\}$ such that $\|r(y)y - x_i\| < \frac{1}{k}$ thus, $\|y - \frac{1}{r(y)}x_i\| < \frac{1}{kr(y)}$. So,

$$\begin{aligned} \|y-u_i\| &= \|y - \frac{1}{r(x_0)}x_i\| \leq \|y - \frac{1}{r(y)}x_i\| + \|\frac{1}{r(y)}x_i - \frac{1}{x_0}x_i\| \\ &< \frac{1}{k}\frac{1}{r(y)} + \|x_i\|\left|\left(\frac{1}{r(y)} - \frac{1}{r(x_0)}\right)\right| < \frac{1}{k}\left(\frac{1}{r(x_0)} + \frac{1}{m}\right) + M\frac{1}{m} < \varepsilon \end{aligned}$$

□

In order to give our last result we should introduce some terminology. Recall that in a topological space X a family of subsets of X, $\mathcal{A}$, is said to be **relatively locally finite** (resp. **isolated**) if for every $x \in \cup\{A : A \in \mathcal{A}\}$ there exists an open set $V \ni x$ such that the set $\{A : A \in \mathcal{A}, A \cap V \neq \emptyset\}$ is finite (resp. contains exactly one element). If P is any of the properties above, as usual, the family $\mathcal{A}$ is said to be σ-P if $\mathcal{A} = \cup\{\mathcal{A}_n : n \in \mathbb{N}\}$ in such a way that for each $n \in \mathbb{N}$ the family $\mathcal{A}_n$ has property P.

When dealing with a normed space X and $F \subset X^*$ norming, we shall talk of **slicely P** whenever the open set V can be chosen to be an open half space from $\mathbb{H}(F)$.

Finally a **network** for a topological space X is a collection $\mathcal{N}$ of subsets of X such that whenever $x \in U$ with U open, there exists $N \in \mathcal{N}$ with $x \in N \subset U$.

Recall that from [13] it follows that given a Banach space X and a norming subspace for it F, X admits a $\sigma(X,F)$-lower semi-continuous **LUR** equivalent norm if, and only if, the norm topology has a σ-slicely isolated network, see also [6, 15, 16].

Corollary 4.5. *Let $(X, \|\cdot\|)$ be a normed space and $F \subset X^*$ norming. The following conditions are equivalent:*

1. *The norm topology admits a σ-slicely relatively locally finite network;*
2. *X admits an equivalent $\sigma(X,F)$-lower semi-continuous* **LUR** *norm.*

Proof. By the result in [13] mentioned above we only have to show that (1)⇒(2) and this will be done through the equivalent conditions in Theorem 4.1. To do so, one may assume that the network $\mathcal{N} = \cup\{\mathcal{N}_n : n \in \mathbb{N}\}$ satisfying (1) is such that for each $n \in \mathbb{N}$ the family $\mathcal{N}_n$ consists of pairwise disjoint sets. Indeed, if this is not the case then for each $n, m \in \mathbb{N}$ we define the family

$$\mathcal{N}_n^m := \{N_1 \cap \ldots \cap N_m : N_i \in \mathcal{N}_n, i = 1,2,\ldots,m\}$$

and the sets $S_n^m := \{x \in X : x \in A \in \mathcal{N}_n^m \text{ and } \operatorname{ord}(x,\mathcal{N}_n) = m\}$. Now we set

$$\{\mathcal{N}_n^m \cap S_n^m\} := \{A \cap S_n^m : A \in \mathcal{N}_n^m\}$$

It is easy to show that for each $n, m \in \mathbb{N}$ the sets in this family are pairwise disjoint, $\cup\{\mathcal{N}_n^m \cap S_n^m : m \in \mathbb{N}\}$ is a refinement for $\mathcal{N}_n$ and $\{\mathcal{N}_n^m \cap S_n^m : n, m \in \mathbb{N}\}$ is network for the norm topology which is σ-slicely relatively locally finite.

Now fix $\varepsilon > 0$. For every positive integer n define

$$X_{n,\varepsilon} := \{x \in \cup\{N : N \in \mathcal{N}_n\} \text{ such that } x \in N \subset B(x;\varepsilon)\} \equiv$$

$$\{x \in X : \text{there exists } N \in \mathcal{N}_n \text{ with } x \in N \subset B(x;\varepsilon)\}$$

Since $\mathcal{N}$ is a network for the norm topology we have $X = \cup\{X_{n,\varepsilon} : n \in \mathbb{N}\}$. Fix $x \in X_{n,\varepsilon}$. Since the network is σ-slicely relatively locally finite, there must be $H \in \mathbb{H}(F)$ such that $x \in H$ and $H \cap \cup\{N : N \in \mathcal{N}_n\} = H \cap N_1 \cup \ldots \cup H \cap N_p$ for a finite number of sets $N_i \in \mathcal{N}_n$.

If we consider $y \in H \cap X_{n,\varepsilon}$ we have $y \in H \cap N_j$ for some $j \in \{1, 2, \ldots, p\}$, and by the very definition of $X_{n,\varepsilon}$ and the disjointness of the family $\mathcal{N}_n$, $y \in N_j \subset B(y;\varepsilon)$. So for N_j we have diam $(N_j) < 2\varepsilon$.

Therefore we have $\{p_1, p_2, \ldots, p_q\} \subset \{1, 2, \ldots, p\}$ so that $H \cap X_{n,\varepsilon} \subset N_{p_1} \cup \ldots \cup N_{p_q}$ and diam $(N_{p_i}) < 2\varepsilon$. So $\alpha(H \cap X_{n,\varepsilon}) < 2\varepsilon$ and the proof is done. □

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