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The Hamilton-Jacobi Equation of Minimal Time Control

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We study the solutions of the Hamilton-Jacobi equation that arise in connection with minimal time control, in a new global framework. These solutions, for which we establish existence using the minimal time function as a function of two variables, turn out to be closely related to time-geodesic trajectories.

Keywords: Minimal time function, viscosity solutions, geodesic trajectories, proximal analysis, monotonicity of trajectories, nonsmooth analysis

1. Introduction

In this paper, we consider a control system governed by a differential inclusion. We are given a multifunction F mapping $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$, and a time interval $[a, b]$. Associated with F is the differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [a, b]. \tag{1}$$

A *solution* $x(\cdot)$ of (1) is taken to mean an absolutely continuous function $x : [a, b] \longrightarrow \mathbb{R}^n$ which, together with $\dot{x}$, its derivative with respect to t, satisfies (1). For brevity, we will refer to any absolutely continuous function x from $[a, b]$ to $\mathbb{R}^n$ as an *arc* on $[a, b]$. We also refer to an arc x satisfying (1) as a *trajectory* of F.
The *bilateral minimal time function* $T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, +\infty]$ is defined as follows, for $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$, $T(\alpha, \beta)$ is the minimum time taken by a trajectory to go from α to β (when no such trajectory exists, $T(\alpha, \beta)$ is taken to be $+\infty$). We set

$$\mathcal{R}^0_+ := \{\alpha \in \mathbb{R}^n \ : \ T(0, \alpha) < +\infty\},$$

the set of points attainable by trajectories beginning at 0. The *lower Hamiltonian* $h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (resp. *upper Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$) corresponding to F is defined as follows

$$h(x, p) := \min_{v \in F(x)} \langle p, v \rangle \quad (\text{resp. } H(x, p) := \max_{v \in F(x)} \langle p, v \rangle).$$

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Given a lower semicontinuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ and a point x in the effective domain of f, that is, the set

$$\operatorname{dom} f := \{x' : f(x') < +\infty\},$$

we say that a vector $\zeta \in \mathbb{R}^n$ is a *proximal subgradient* of f at x if there exists $\sigma \geq 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x \rangle,$$

for all y in a neighborhood of x. The set of such ζ, which could be empty, is denoted by $\partial_P f(x)$ and referred to as the *proximal subdifferential*. Then the *proximal superdifferential* of an upper semicontinuous function f at x is defined as $\partial^P f(x) = -\partial_P(-f)(x)$. The Proximal Density Theorem asserts that $\partial_P f(x) \neq \emptyset$ for all x in a dense subset of $\operatorname{dom} f$. We also define the limiting proximal subgradient of f at $x \in \operatorname{dom} f$ by

$$\partial_L f(x) := \{\lim \zeta_i : \zeta_i \in \partial_P f(x_i), x_i \longrightarrow x \quad \text{and} \quad f(x_i) \longrightarrow f(x)\}.$$

For more information about the preceding definitions see [10].

This article focuses on the following problem for the Hamilton-Jacobi equation:

$$1 + h(x, \partial_P \varphi(x)) = 0 \;\; \forall x \in \mathcal{R}_+^0, \;\; \varphi(0) = 0 \qquad (*)$$

A solution of $(*)$ means a lower semicontinuous function $\varphi : \mathcal{R}_+^0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that $\varphi(0) = 0$ and for every $x \in \mathcal{R}_+^0$, for every $\zeta \in \partial_P \varphi(x)$ (if any), we have $h(x, \zeta) + 1 = 0$. This is equivalent to the statement that φ is a viscosity solution (see [11]) of the following Hamilton-Jacobi equation:

$$H(x, -\varphi'(x)) - 1 = 0 \;\; \forall x \in \mathcal{R}_+^0, \;\; \varphi(0) = 0,$$

see [8] and [10] for the proof of the equivalence. When F admits a standard control representation $F(x) = f(x, U)$, then this assumes the form

$$\max\{\langle f(x, u), -\varphi'(x) \rangle \;:\; u \in U\} - 1 = 0,$$

a familiar object of study in connection with the dynamic programming approach to optimal control, see for example [2].

When $(*)$ is modified by deleting the origin from the domain of the problem, there results a well-studied problem. Essentially one finds that the solution is the familiar (unilateral) minimum time function $T(\cdot, 0)$. However, $T(\cdot, 0)$ is *never* a solution on a set containing the origin, since necessarily we have $0 \in \partial_P T(\cdot, 0)(0)$ and $h(0, 0) = 0$. We refer the reader to [2], [5] and [21] for discussions of this case.

In this article we explore (apparently for the first time) the consequence of including the origin in the domain. Our goal is to identify a framework in which such *global* solutions can be usefully studied. In contrast to the classical case, it turns out to be the *bilateral* minimal time function that plays a central role in determining the solutions of $(*)$ which, as we shall see, are closely linked to global geodesic trajectories.

The layout of this article is as follows. In the next section we present our notations and hypotheses. We give some results about the monotonicity of trajectories and the relation with solutions of $(*)$ in Section 3. Section 4 is devoted to the continuity and Lipschitz continuity of $T(\cdot,\cdot)$. In Section 5, we show the existence of (minimal) solutions of $(*)$ and then we study the regularity of solutions and the linear case in Section 6. The relation between semigeodesic trajectories and solutions is examined in Section 7. In Section 8, we define a dual equation for $(*)$ and we give necessary and sufficient conditions for the existence of a geodesic passing through the origin.

2. Notations and hypotheses

The Euclidean norm is denoted $\|\cdot\|$, and $\langle,\rangle$ is the usual inner product. For $\rho > 0$, $B(0;\rho) := \{x \in \mathbb{R}^n : \|x\| < \rho\}$ and $\bar{B}(0;\rho) := \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$. The open (resp. closed) unit ball in $\mathbb{R}^n$ is denoted B (resp. $\bar{B}$). For a set $S \subset \mathbb{R}^n$, $\operatorname{int} S$ is the interior of S. We define

$$\mathcal{R}_+^\beta(t) := \{\alpha \in \mathbb{R}^n \ : \ T(\beta,\alpha) < t\}, \quad t > 0,$$

the set of points reachable from β in time less than t.
Similarly, we introduce

- $\mathcal{R}_+^\beta := \bigcup_{t>0} \mathcal{R}_+^\beta(t) = \{\alpha \in \mathbb{R}^n \ : \ T(\beta,\alpha) < +\infty\}$,
- $\mathcal{R}_-^\beta(t) := \{\alpha \in \mathbb{R}^n \ : \ T(\alpha,\beta) < t\}, \quad t > 0$,
- $\mathcal{R}_-^\beta := \bigcup_{t>0} \mathcal{R}_-^\beta(t) = \{\alpha \in \mathbb{R}^n \ : \ T(\alpha,\beta) < +\infty\}$,
- $\mathcal{R}(t) := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ T(\alpha,\beta) < t\}, \quad t > 0$,
- $\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t) = \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ T(\alpha,\beta) < +\infty\}$.

The basic hypotheses in force throughout the article are the following. We assume that for $x \in \mathbb{R}^n$, $F(x)$ is a nonempty, compact convex set, and that F satisfies the *linear growth condition*: for some positive constants γ and c, and for all $x \in \mathbb{R}^n$,

$$v \in F(x) \Longrightarrow \|v\| \leq \gamma\|x\| + c.$$

The multifunction F is also taken to be *locally Lipschitz*: every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that

$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

Under these hypotheses, any trajectory can be extended indefinitely both forward and backward, so all trajectories can be considered as being defined on $]-\infty,+\infty[$.

We say that F is β-STLC (β-*small-time locally controllable*), if $\beta \in \operatorname{int} \mathcal{R}_-^\beta(t) \ \forall \ t > 0$ (ie. $\forall \ t > 0 \ \exists \ \delta > 0$ such that $T(\cdot,\beta) < t$ on $B(\beta,\delta)$). There is a considerable literature devoted to this property, see for example [2] and [18]. We can find in these references that $[0 \in \operatorname{int} F(\beta) \Longrightarrow F$ is β-STLC$\Longrightarrow 0 \in F(\beta)]$ and that the following statements are equivalent:

(i) F is β-STLC.
(ii) $T(\cdot,\beta)$ is continuous at β.
(iii) $\mathcal{R}_-^\beta$ is open and $T(\cdot,\beta)$ is continuous in $\mathcal{R}_-^\beta$ and for any $\alpha_0 \in \partial\mathcal{R}_-^\beta$ we have

$$\lim_{\alpha \longrightarrow \alpha_0} T(\alpha,\beta) = +\infty.$$

We posit throughout the article that $-F$ is 0-STLC. Then in view of the above, and by time reversal, we have:

- $\mathcal{R}^0_+$ is open.
- $T(0,\cdot)$ is continuous in $\mathcal{R}^0_+$.
- For any $\alpha_0 \in \partial\mathcal{R}^0_+$ we have $\lim_{\alpha \longrightarrow \alpha_0} T(0,\alpha) = +\infty$.[1]

3. Monotonicity of trajectories

Let Ω be an open subset of $\mathbb{R}^n$. The class of lower semicontinuous functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ which are not identically $+\infty$ is denoted $\mathcal{F}(\Omega)$. We say that (φ, F) is *strongly increasing* on Ω if for any trajectory x on an interval $[a,b]$ for which $x([a,b]) \subset \Omega$, we have

$$\varphi(x(s)) \leq \varphi(x(t)) \ \ \forall s,t \in [a,b], s \leq t.$$

The system (φ, F) is said to be *weakly decreasing* on Ω if for every $\alpha \in \Omega$ there is a trajectory x on a nontrivial interval $[a,b]$ satisfying

$$x(a) = \alpha, \varphi(x(t)) \leq \varphi(\alpha) \quad \forall t \in [a,b];$$

by reducing b if necessary we may also arrange to have $x([a,b]) \subset \Omega$. In each case, one obtains an equivalent definition by requiring the inequality to hold on $[a,\tau[$, where $\tau \in]a,+\infty]$ is the *exit time* of the trajectory x from Ω: the supremum of all $T > 0$ having the property that $x([a,T]) \subset \Omega$. The following proposition is proven in [10, Chapter 4, §6].

Proposition 3.1. *Let $\varphi \in \mathcal{F}(\Omega)$. The system (φ, F) is strongly increasing on Ω iff*

$$h(x, \partial_P \varphi(x)) \geq 0 \quad \forall x \in \Omega,$$

and weakly decreasing on Ω iff

$$h(x, \partial_P \varphi(x)) \leq 0 \quad \forall x \in \Omega.$$

If φ satisfies both these properties, then for any $\alpha \in \Omega \cap \operatorname{dom}\varphi$ there is a trajectory x satisfying $x(0) = \alpha$ and

$$\varphi(x(t)) = \varphi(\alpha) \quad \forall t \in [0,\tau[,$$

where $\tau \in]0,+\infty]$ is the exit time of the trajectory x from Ω.

Now let φ be a solution of $(*)$. Applying the preceding proposition to the function $\varphi + t$ and the multifunction $F(x) \times \{1\}$ (and $\Omega := \mathcal{R}^0_+ \times \mathbb{R}$), we deduce that the system $(\varphi + t, F \times \{1\})$ is both weakly and strongly increasing. Because of the nature of the t-dependence here, the latter property (for example) amounts to saying that for any trajectory x of F on an interval $[0,T]$ such that $x([0,T]) \subset \mathcal{R}^0_+$ (this being equivalent to $x(0) \in \mathcal{R}^0_+$), we have

$$\varphi(x(s)) + s \leq \varphi(x(t)) + t \ \ \forall s,t \in [0,T], s \leq t.$$

4. Regularity of the bilateral minimal time function

In this section[2] we study the regularity of $T(\cdot,\cdot)$. We can easily verify that $T(\cdot,\cdot)$ satisfies the following:

[1] It is these properties that make $(*)$ a suitable framework for the global study that we carry out.
[2] It is not necessary to assume in this section that $-F$ is 0-STLC.

- $T(\cdot,\cdot)$ is lower semicontinuous.
- If $T(\alpha,\beta) < +\infty$ then the minimum defining $T(\alpha,\beta)$ is attained.
- For all (α,β,γ) we have the following triangle inequality:

$$T(\alpha,\beta) \le T(\alpha,\gamma) + T(\gamma,\beta).$$

Now we begin to study the continuity of $T(\cdot,\cdot)$. In the following proposition, we give a necessary and sufficient conditions for $T(\cdot,\cdot)$ to be continuous at a point (α,β).

Proposition 4.1. *Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have:*

(i) *$T(\cdot,\cdot)$ is continuous at $(\alpha,\alpha) \Longleftrightarrow F$ and $-F$ are α-STLC.*
(ii) *Assume that one of the following conditions holds:*
 1. $(\alpha,\beta) \in \mathcal{R}$, F is α-STLC and $-F$ is β-STLC.
 2. F and $-F$ are α-STLC and $\beta \in \mathcal{R}_+^\alpha$.
 3. F and $-F$ are β-STLC and $\alpha \in \mathcal{R}_-^\beta$.
 Then $T(\cdot,\cdot)$ is continuous at (α,β).

Proof. (i) Let $\alpha \in \mathbb{R}^n$. Then we have

$$T(\alpha,\cdot) \text{ and } T(\cdot,\alpha) \text{ are continuous in } \alpha \Longleftrightarrow F \text{ and } -F \text{ are } \alpha\text{-STLC}.$$

But using the triangle inequality we also have

$$T(\alpha,\cdot) \text{ and } T(\cdot,\alpha) \text{ are continuous at } \alpha \Longleftrightarrow T(\cdot,\cdot) \text{ is continuous at } (\alpha,\alpha).$$

The result follows.
(ii) 1) Let $(\alpha,\beta) \in \mathcal{R}$ and suppose that F and $-F$ are respectively α-STLC and β-STLC. Then we know that $\mathcal{R}_-^\alpha$ and $\mathcal{R}_+^\beta$ are open. Using the fact that $(\alpha,\beta) \in \mathcal{R}$ we get that $(\alpha,\beta) \in \mathcal{R}_-^\alpha \times \mathcal{R}_+^\beta \subset \mathcal{R}$. Hence $(\alpha,\beta) \in \operatorname{int}\mathcal{R}$. Now let (α_n,β_n) be a sequence such that $(\alpha_n,\beta_n) \longrightarrow (\alpha,\beta)$. By the triangle inequality we have

$$T(\alpha_n,\beta_n) \le T(\alpha_n,\alpha) + T(\alpha,\beta) + T(\beta,\beta_n) \tag{2}$$

Then by the continuity of $T(\cdot,\alpha)$ and $T(\beta,\cdot)$ we get that $T(\cdot,\cdot)$ is upper semicontinuous and hence continuous.
2) Clearly we have $(\alpha,\beta) \in \operatorname{int}\mathcal{R}$ since $(\alpha,\beta) \in \mathcal{R}_-^\alpha \times \mathcal{R}_+^\alpha \subset \mathcal{R}$. Now let (α_n,β_n) be a sequence such that $(\alpha_n,\beta_n) \longrightarrow (\alpha,\beta)$. By the triangle inequality we have

$$T(\alpha_n,\beta_n) \le T(\alpha_n,\alpha) + T(\alpha,\beta_n).$$

Since $T(\cdot,\alpha)$ and $T(\alpha,\cdot)$ are continuous in $\mathcal{R}_-^\alpha$ and $\mathcal{R}_+^\alpha$ respectively, the result follows as above.
3) We proceed as in 2) and we find the result. □

For the global continuity we have the following proposition which asserts that the continuity of $T(\cdot,\cdot)$ at every point of the diagonal $\mathcal{D} := \{(\alpha,\alpha) : \alpha \in \mathbb{R}^n\}$ is equivalent to the continuity in $\mathcal{R}$.

Proposition 4.2. *The following statements are equivalent:*

(i) $\mathcal{R}$ *is open,* $T(\cdot,\cdot)$ *is continuous in* $\mathcal{R}$ *and for any* $(\alpha_0,\beta_0)\in\partial\mathcal{R}$ *we have*

$$\lim_{(\alpha,\beta)\longrightarrow(\alpha_0,\beta_0)} T(\alpha,\beta)=+\infty.$$

(ii) $T(\cdot,\cdot)$ *is continuous at* (α,α) *for all* $\alpha\in\mathbb{R}^n$*.*
(iii) F *and* $-F$ *are* β*-STLC for all* $\beta\in\mathbb{R}^n$*.*

Proof. Clearly we have (i)$\Longrightarrow$(ii).
(ii)$\Longrightarrow$(iii): Follows from (i) of Proposition 4.1.
(iii)$\Longrightarrow$(i): The first part ($\mathcal{R}$ is open and $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$) follows from (ii) of Proposition 4.1. Now we show the second part. Let $(\alpha_0,\beta_0)\in\partial\mathcal{R}$. Suppose that there exist a sequence $(\alpha_n,\beta_n)\in\mathcal{R}$ and a constant K such that $(\alpha_n,\beta_n)\longrightarrow(\alpha_0,\beta_0)$ and $T_n:=T(\alpha_n,\beta_n)\le K$. Since $0\le T_n\le K$ we can assume that T_n converges to $\bar{T}\in[0,K]$. Now let x_n be a trajectory of F on $[0,+\infty[$ which satisfies $x_n(0)=\alpha_n$ and $x_n(T_n)=\beta_n$. By the compactness property of trajectories, there exists a subsequence of x_n (we do not relabel) which converges uniformly on $[0,\bar{T}]$ to a trajectory $\bar{x}$ of F. Hence $\bar{x}(0)=\alpha_0$ and $\bar{x}(\bar{T})=\beta_0$. Therefore $(\alpha_0,\beta_0)\in\mathcal{R}$ and this contradicts the fact that $\mathcal{R}$ is open. □

We proceed to study the Lipschitz continuity of $T(\cdot,\cdot)$. First we recall that for $\beta\in\mathbb{R}^n$, the (unilateral) minimal time function $T(\cdot,\beta)$ is locally Lipschitz in $\mathcal{R}_+^\beta$ iff $0\in\operatorname{int}F(\beta)$ (see [2], [4], [19] and [21]). The following proposition studies the (local) Lipschitz continuity of $T(\cdot,\cdot)$.

Proposition 4.3. *Let* $(\alpha,\beta)\in\mathbb{R}^n\times\mathbb{R}^n$*. Then we have:*

1. *If* $(\alpha,\beta)\in\mathcal{R}$ *then*
   $[\ 0\in$ *int* $F(\alpha)$ *and* $0\in$ *int* $F(\beta)\ ]\Longrightarrow T(\cdot,\cdot)$ *is Lipschitz near* (α,β)*,*
2. $0\in$ *int* $F(\alpha)\Longleftrightarrow T(\cdot,\cdot)$ *is Lipschitz near* (α,α)*,*

Proof. 1) Let $(\alpha,\beta)\in\mathcal{R}$ and assume that $0\in\operatorname{int}F(\alpha)$ and $0\in\operatorname{int}F(\beta)$. By a simple continuity argument there exist $\rho>0$ and $\delta>0$ such that

$$\left.\begin{array}{ll} h(\alpha',\gamma)<-\delta & \text{for all } \alpha'\in B(\alpha;\rho) \text{ and for every unit vector } \gamma \\ h(\beta',\gamma)<-\delta & \text{for all } \beta'\in B(\beta;\rho) \text{ and for every unit vector } \gamma \end{array}\right\} \qquad (+)$$

Since $(\alpha,\beta)\in\operatorname{int}\mathcal{R}$ (see Proposition 4.1) we arrange to have

$$(\alpha,\beta)\in B(\alpha;\rho)\times B(\beta;\rho)\subset\mathcal{R}.$$

Now let $(\alpha',\beta')\in B(\alpha;\rho)\times B(\beta;\rho)$ and let $(\theta,\xi)\in\partial_P T(\cdot,\cdot)(\alpha',\beta')$. Then

$$(\theta,\xi)\in\partial_P T(\cdot,\beta')(\alpha')\times\partial_P T(\alpha',\cdot)(\beta').$$

Hence

$$h(\alpha',\theta)\ge-1 \ \text{ and } \ h(\beta',-\xi)\ge-1.^{3}$$

[3]This follows since $T(\cdot,\alpha)$ is a solution on $\mathbb{R}^n$ of the Hamilton-Jacobi inequality

$$1+h(x,\partial_P\varphi(x))\ge 0.$$

Using (+) we get that

$$-1 \leq h(\alpha', \theta) \leq -\delta\|\theta\| \text{ and } -1 \leq h(\beta', -\xi) \leq -\delta\|\xi\|.$$

Hence

$$\|\theta\| \leq \frac{1}{\delta} \text{ and } \|\xi\| \leq \frac{1}{\delta}.$$

Therefore $\partial_P T(\cdot, \cdot)$ is bounded on $B(\alpha; \rho) \times B(\beta; \rho)$. Then by [10, Theorem 1.7.3] $T(\cdot, \cdot)$ is Lipschitz on $B(\alpha; \rho) \times B(\beta; \rho)$.

2) The necessary condition follows from 1) and the sufficient condition follows from the fact that if $T(\cdot, \cdot)$ is Lipschitz near (α, α) then $T(\cdot, \alpha)$ is Lipschitz near α and this is equivalent to $0 \in \operatorname{int} F(\alpha)$. □

The following proposition gives a necessary and sufficient condition for $T(\cdot, \cdot)$ to be locally Lipschitz in $\mathcal{R}$. As in the continuity case, if $T(\cdot, \cdot)$ is Lipschitz near every point of $\mathcal{D}$ then $T(\cdot, \cdot)$ is locally Lipschitz in $\mathcal{R}$.

Proposition 4.4. *The following statements are equivalent:*

(i) $\mathcal{R}$ *is open and* $T(\cdot, \cdot)$ *is locally Lipschitz in* $\mathcal{R}$.
(ii) $T(\cdot, \cdot)$ *is Lipschitz near* (α, α) *for all* $\alpha \in \mathbb{R}^n$.
(iii) $0 \in \operatorname{int} F(\beta)$ *for all* $\beta \in \mathbb{R}^n$.

Proof. Clearly we have (i)⟹(ii).
(ii)⟹(iii): Follows from Proposition 4.3.
(iii)⟹(i): By Proposition 4.2 we have $\mathcal{R}$ is open, and by Proposition 4.3, $T(\cdot, \cdot)$ is locally Lipschitz in $\mathcal{R}$. □

Remark 4.5. We can find more properties for the bilateral minimal time function in [14] (semiconvexity, differentiability, characterization by a system of partial Hamilton-Jacobi equations....).

5. Existence of solutions

We begin this section by the following proposition which gives some properties of a solution of $(*)$.

Proposition 5.1. *Let* φ *a solution of* $(*)$. *Then we have:*

1. $T(\alpha, \beta) + \varphi(\beta) \geq \varphi(\alpha)$, *for all* α, $\beta \in \mathcal{R}^0_+$.
2. $T(\alpha, 0) \geq \varphi(\alpha) \geq -T(0, \alpha)$, *for all* $\alpha \in \mathcal{R}^0_+$.
3. $\mathcal{R}^0_- \cap \mathcal{R}^0_+ \subset \operatorname{dom} \varphi$.
4. *For every* $\alpha \in \operatorname{dom} \varphi$ *there exists a trajectory* x *of* F *such that* $x(0) = \alpha$ *and*

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

Proof. The statements 2) and 3) follow immediately from 1). For the proof of 1), let φ a solution of $(*)$ and let α, $\beta \in \mathcal{R}^0_+$. We can assume that $(\alpha, \beta) \in \mathcal{R}$, then let $\bar{x} : [0, +\infty[\longrightarrow \mathbb{R}^n$ be a trajectory which realizes the (finite) minimal time from α to β, that is, $\bar{x}(0) = \alpha$ and $\bar{x}(T(\alpha, \beta)) = \beta$. Because the system $(\varphi + t, F \times \{1\})$ is strongly increasing in the sense explained above, we get that

$$\varphi(\alpha) = \varphi(\bar{x}(0)) \leq T(\alpha, \beta) + \varphi(\bar{x}(T(\alpha, \beta))) = T(\alpha, \beta) + \varphi(\beta).$$

The statement 1) follows.
Now we give the proof of 4). We consider $\alpha \in \operatorname{dom}\varphi$. Since the system $(t+\varphi, \{1\} \times F)$ is strongly increasing and weakly decreasing and using the fact that every trajectory which begins at α remains in $\mathcal{R}_+^0$, there exists a trajectory $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F in $\mathcal{R}_+^0$ such that $x(0) = \alpha$ and

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

This completes the proof. □

Now we define the following set (of subsets of $\mathcal{R}_+^0$):
$\mathcal{G} := \{\Gamma \subset \mathcal{R}_+^0 :$ there exists a sequence $\beta_i \in \Gamma$ such that $T(0, \beta_i) \longrightarrow +\infty\}$, which is nonempty since for any $\alpha_0 \in \partial\mathcal{R}_+^0$ we have

$$\lim_{\alpha \longrightarrow \alpha_0} T(0, \alpha) = +\infty.$$[4]

The following theorem implies that the set of solutions of $(*)$ is nonempty. We use the set $\mathcal{G}$ and the function $T(\cdot, \cdot)$ for the construction of solutions.

Theorem 5.2. *Let $\Gamma \in \mathcal{G}$ and let $\varphi_\Gamma : \mathcal{R}_+^0 \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be the function defined as follows:*

$$\varphi_\Gamma(\alpha) := \liminf_{\substack{\alpha' \longrightarrow \alpha,\ \beta \in \Gamma \\ T(0,\beta) \longrightarrow +\infty}} [T(\alpha', \beta) - T(0, \beta)].$$

Then φ_Γ is a solution of $(*)$.

Proof. By the triangle inequality we have that for all α', $\beta \in \mathcal{R}_+^0$

$$T(\alpha', \beta) - T(0, \beta) \geq -T(0, \alpha'), \tag{3}$$

then using the continuity of $T(0, \cdot)$ we get that $\varphi_\Gamma(0) \geq 0$ and $\varphi_\Gamma(\alpha) > -\infty$ for all $\alpha \in \mathcal{R}_+^0$. But clearly we have $\varphi_\Gamma(0) \leq 0$, then $\varphi_\Gamma(0) = 0$. The function φ_Γ is lower semicontinuous by the definition. Let us show that this function satisfies the Hamilton-Jacobi equation. It is sufficient to show that the function $t + \varphi_\Gamma$ is strongly increasing and weakly decreasing. We begin by weakly decreasing.
Let $\alpha \in \operatorname{dom}\varphi_\Gamma$. Then there exist two sequences α_i and β_i in $\mathcal{R}_+^0$ such that $\alpha_i \longrightarrow \alpha$, $T(0, \beta_i) \longrightarrow +\infty$ and

$$\varphi_\Gamma(\alpha) = \lim_{i \longrightarrow +\infty} [T(\alpha_i, \beta_i) - T(0, \beta_i)].$$

Then $T(\alpha_i, \beta_i) \longrightarrow +\infty$ and for i sufficiently large there exists a trajectory x_i of F on $[0, +\infty[$ such that $x_i(0) = \alpha_i$ and $x_i(T(\alpha_i, \beta_i)) = \beta_i$ (x_i realizes the minimum time between α_i and β_i). Then for i sufficiently large we have

$$T(x_i(t), \beta_i) = T(\alpha_i, \beta_i) - t, \ \ \forall t \in [0, 1]$$

and then

$$T(x_i(t), \beta_i) - T(0, \beta_i) = T(\alpha_i, \beta_i) - T(0, \beta_i) - t, \ \ \forall t \in [0, 1]. \tag{4}$$

By the compactness of trajectories we can assume that there exists a trajectory $\bar{x}$ of F on $[0, +\infty[$ such that x_i converges uniformly to $\bar{x}$ on $[0, 1]$. Then

$$\bar{x}(0) = \lim_{i \longrightarrow +\infty} x_i(0) = \lim_{i \longrightarrow +\infty} \alpha_i = \alpha.$$

[4]If $\mathcal{R}_+^0$ is unbounded then we have $\lim_{\|\alpha\| \longrightarrow +\infty} T(0, \alpha) = +\infty$ (by the linear growth condition).

Moreover, taking $i \longrightarrow +\infty$ in (4) we get that

$$\varphi_\Gamma(\bar{x}(t)) + t \leq \varphi_\Gamma(\alpha), \quad \forall t \in [0,1].$$

The weak decrease follows.

To prove strong increase, we consider an interval $[a,b] \subset]-\infty,+\infty[$ and a trajectory x of F such that $x([a,b]) \subset \mathcal{R}^0_+$. It is sufficient to show that

$$\varphi_\Gamma(x(b)) + b \geq \varphi_\Gamma(x(t)) + t, \;\; \forall t \in [a,b].$$

We set $\alpha = x(b)$ and we consider the sequence α_i and β_i as above. Letting $t \in [a,b]$, we invoke continuous dependence on the initial (or terminal) condition to deduce the existence of a sequence x_i of trajectories of F on $[t,b]$ such that $x_i(b) = \alpha_i$ and

$$\lim_{i \longrightarrow +\infty} x_i(t) = x(t).$$

By the triangle inequality we have

$$T(x_i(t),\beta_i) \leq b - t + T(\alpha_i,\beta_i)$$

then

$$T(x_i(t),\beta_i) - T(0,\beta_i) + t \leq T(\alpha_i,\beta_i) - T(0,\beta_i) + b.$$

Taking $i \longrightarrow +\infty$ we get

$$\varphi_\Gamma(x(t)) + t \leq \varphi_\Gamma(x(b)) + b.$$

The strong increase follows. □

We denote by φ_0 the function φ_Γ corresponding to the choice $\mathcal{R}^0_+$ of Γ.

Theorem 5.3. *The function φ_0 defined above is the minimal solution of* $(*)$.

Proof. By Theorem 5.2, φ_0 is a solution of $(*)$. For the minimality, let φ be a solution of $(*)$ and let $\alpha \in \operatorname{dom}\varphi$ (we can take $\alpha \in \operatorname{dom}\varphi$ since we need to show that $\varphi_0(\alpha) \leq \varphi(\alpha)$). By Proposition 5.1, there exists a trajectory $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F in $\mathcal{R}^0_+$ such that $x(0) = \alpha$ and

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

Hence

$$T(0,x(t)) \geq -\varphi(x(t)) = t - \varphi(\alpha), \quad \forall t \geq 0,$$

and then

$$\lim_{t \longrightarrow +\infty} T(0,x(t)) = +\infty.$$

Therefore

$$\varphi_0(\alpha) \leq \lim_{t \longrightarrow +\infty} [T(\alpha,x(t)) - T(0,x(t))] \leq \lim_{t \longrightarrow +\infty} [t - t + \varphi(\alpha)] = \varphi(\alpha),$$

which completes the proof. □

Now we give some examples. We show in the first that the Hamilton-Jacobi equation $(*)$ does not necessarily admit a maximal solution.

Example 5.4. For $n \in \mathbb{N}^*$, let $F(x) := C$, for all $x \in \mathbb{R}^n$, where $C \subset \mathbb{R}^n$ is a nonempty, convex and compact set with $0 \in \operatorname{int} C$. In this case $h(x,\zeta) = h_C(\zeta)$, $\forall (x,\zeta) \in \mathbb{R}^n \times \mathbb{R}^n$, where h_C is the lower support function of C:

$$h_C(\zeta) := \min\{\langle \zeta, c\rangle \ : \ c \in C\}.$$

The bilateral minimal time function is defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$T(\alpha,\beta) = g_C(\beta - \alpha),$$

for all $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$, where g_C is the Minkowski gauge of C:

$$g_C(x) := \min\{\lambda \geq 0 \ : \ \frac{x}{\lambda} \in C\}.$$

Then $\mathcal{R}_+^0 = \mathbb{R}^n$ and the Hamilton-Jacobi equation $(*)$ becomes the "eikonal equation":

$$1 + h_C(\partial_P \varphi(x)) = 0, \ \forall x \in \mathbb{R}^n, \ \ \varphi(0) = 0.$$

For every $\theta \in \mathbb{R}^n \setminus \{0\}$, the function

$$\varphi_\theta(x) = \langle \frac{-\theta}{h_C(\theta)}, x\rangle$$

is evidently a solution of $(*)$. The upper envelope of all such solutions is seen to be the function $x \mapsto g_C(-x) = T(x,0)$. In view of Proposition 5.1, this coincides with the upper envelope of all solutions of $(*)$. It follows that no maximal solution to $(*)$ exists.

The lower envelope of all solutions exhibited above is given by $\varphi_0 = -g_C$. This is a concave function nondifferentiable at 0, so that $\partial_P \varphi_0(x) \neq \emptyset$ implies $x \neq 0$ and $\partial_P \varphi_0(x) = \{\varphi_0'(x)\}$. Then since $1 + h_C(\partial_P T(\cdot,0)(x)) = 0$ for all $x \neq 0$ we get that

$$1 + h_C(\partial_P \varphi_0(x)) = 0 \quad \forall x \in \mathbb{R}^n,$$

so that φ_0 is a solution of $(*)$. Since φ_0 is a lower bound on all solutions of $(*)$, it is revealed as the minimal solution.

Example 5.5. For $n = 1$ we consider $F(x) := [-|x-1|, +|x-1|]$. In this case $\mathcal{R}_+^0 = \{\alpha \in \mathbb{R} : \alpha < 1\}$ and $h(x,\zeta) = -|\zeta||x-1|$, $\forall (x,\zeta) \in \mathbb{R} \times \mathbb{R}$. For $\alpha \in \mathcal{R}_+^0$ we calculate

$$T(\alpha,\beta) = \begin{cases} \ln(1-\alpha) - \ln(1-\beta) & \text{if } \beta > \alpha \\ \ln(1-\beta) - \ln(1-\alpha) & \text{if } \beta < \alpha \end{cases}$$

Then for $\Gamma_1 = [0,1[$ (resp. $\Gamma_2 =]-\infty, 0]$) the corresponding solution of $(*)$, obtained as in Theorem 5.2, is:

$$\varphi_1(x) = \ln(1-x) \ \ (\text{resp. } \ \varphi_2(x) = -\ln(1-x)),$$

and the minimal solution φ_0 is calculated to be $-|\ln(1-x)|$.

6. Regularity of solutions and the linear case

In this section we study the regularity of a solution of $(*)$. We begin by the following proposition which gives a sufficient condition for the continuity and the Lipschitz continuity of a solution φ of $(*)$ at a point $\alpha \in \operatorname{dom}\varphi$.

Proposition 6.1. *Let φ be a solution of $(*)$ and let $\alpha \in \operatorname{dom}\varphi$. Then we have:*

1. *F is α-STLC $\Longrightarrow \varphi$ is continuous at α.*
2. *$0 \in int F(\alpha) \Longrightarrow \varphi$ is Lipschitz near α.*

Proof. 1) Since $\mathcal{R}^0_+$ is open and F is α-STLC there exists $\rho > 0$ such that

$$B(\alpha;\rho) \subset \mathcal{R}^0_+ \quad \text{and} \quad B(\alpha;\rho) \subset \mathcal{R}^\alpha_-.$$

Then for $\beta \in B(\alpha;\rho)$ and by Proposition 5.1 we have

$$\varphi(\beta) \leq \varphi(\alpha) + T(\beta,\alpha) < +\infty, \tag{5}$$

hence φ is finite on $B(\alpha;\rho)$. Now let α_i be a sequence such that $\alpha_i \longrightarrow \alpha$ and let $\varepsilon > 0$. By the lower semicontinuity of φ it is sufficient to prove that for i sufficiently large we have

$$\varphi(\alpha_i) \leq \varphi(\alpha) + \varepsilon,$$

but this follows immediately from (5) since $T(\cdot,\alpha)$ is continuous in $\mathcal{R}^\alpha_-$.
2) Since $0 \in \operatorname{int} F(\alpha)$ and by Proposition 4.3 we have that $T(\cdot,\cdot)$ is Lipschitz near (α,α). Hence there exist $\rho > 0$ and $K > 0$ such that $T(\cdot,\cdot)$ is K-Lipschitz on $B(\alpha;\rho) \times B(\alpha;\rho)$. By 1) and since $[0 \in \operatorname{int} F(\alpha) \Longrightarrow F$ is α-STLC$]$, we can assume that $B(\alpha;\rho) \subset \mathcal{R}^0_+$ and that φ is finite on $B(\alpha;\rho)$. We claim that φ is K-Lipschitz on $B(\alpha;\rho)$. Indeed, let α_1, $\alpha_2 \in B(\alpha;\rho)$. By Proposition 5.1 we have

$$-T(\alpha_1,\alpha_2) \leq \varphi(\alpha_1) - \varphi(\alpha_2) \leq T(\alpha_2,\alpha_1),$$

but since $T(\cdot,\cdot)$ is K-Lipschitz on $B(\alpha;\rho) \times B(\alpha;\rho)$ we have

$$|T(\alpha_2,\alpha_1)| = |T(\alpha_2,\alpha_1) - T(\alpha_1,\alpha_1)| \leq K\|\alpha_1 - \alpha_2\|,$$

and

$$|T(\alpha_1,\alpha_2)| = |T(\alpha_1,\alpha_2) - T(\alpha_1,\alpha_1)| \leq K\|\alpha_1 - \alpha_2\|.$$

Then

$$-K\|\alpha_1 - \alpha_2\| \leq \varphi(\alpha_1) - \varphi(\alpha_2) \leq K\|\alpha_1 - \alpha_2\|,$$

and the Lipschitz continuity follows. □

Proposition 6.2. *We have the following statements:*

1. *Assume that F is β-STLC for all $\beta \in \mathcal{R}^0_+ \cap \mathcal{R}^0_-$. Then all solutions of $(*)$ are continuous in the open set $\mathcal{R}^0_+ \cap \mathcal{R}^0_-$.*
2. *Assume that $0 \in int F(\beta)$ for all $\beta \in \mathcal{R}^0_+ \cap \mathcal{R}^0_-$. Then all solutions of $(*)$ are locally Lipschitz in the open set $\mathcal{R}^0_+ \cap \mathcal{R}^0_-$.*

Proof. Since $0 \in \mathcal{R}^0_+ \cap \mathcal{R}^0_-$, we have that F is 0-STLC in 1) and 2). Then $\mathcal{R}^0_-$ is open and hence $\mathcal{R}^0_+ \cap \mathcal{R}^0_-$ is open. By Proposition 5.1 we have $\mathcal{R}^0_+ \cap \mathcal{R}^0_- \subset \operatorname{dom}\varphi$ for all φ a solution of $(*)$. Then by Proposition 6.1 we find the two results. □

We recall that for $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ an extended real-valued function and for $x \in \operatorname{dom} f$, the *directional derivative* of f at x in the direction $v \in \mathbb{R}^n$ is defined by

$$f'(x; v) := \lim_{t \longrightarrow 0} \frac{f(x+tv) - f(x)}{t},$$

when the limits exists. For f Lipschitz near x and given $v \in \mathbb{R}^n$, one defines the *generalized directional derivative* of f at x in the direction v as

$$f^\circ(x; v) := \limsup_{\substack{y \longrightarrow x \\ t \downarrow 0}} \frac{f(y+tv) - f(y)}{t}.$$

The *generalized gradient* of f at x (f still assumed Lipschitz near x), is the following (nonempty) set

$$\partial f(x) := \{\xi \in \mathbb{R}^n : f^\circ(x; v) \geq \langle \xi, v \rangle \ \ \forall v \in \mathbb{R}^n\}.$$

For more information about the preceding definitions see [7] or [10]. A function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *regular* at $x \in \mathbb{R}^n$, if it is Lipschitz near x and admits directional derivatives $\varphi'(x; v)$ at x for all v, with $\varphi'(x; v) = \varphi^\circ(x; v)$. If $U \subset \mathbb{R}^n$ is open, then we say that φ is regular in U if it is regular at x for all $x \in U$. A necessary condition for the regularity of φ at x is $\partial_L \varphi(x) = \partial \varphi(x)$. We say that φ is *semiconvex* on an open set $U \subset \mathbb{R}^n$ if it is continuous on U and for all $x_0 \in U$ there exist δ, $C > 0$ such that

$$2\varphi(\frac{x+y}{2}) - \varphi(x) - \varphi(y) \leq C\|x - y\|^2 \ \ \forall x, y \in B(x_0; \delta).$$

We can show (see [5]) that a semiconvex function is regular. For more information about these properties, see [5], [7] and [10].
We proceed to introduce a new property that will play an important role. A continuous function φ is said to be *mildly regular* at a point x if it satisfies

$$\partial^P \varphi(x) \subset \partial_L \varphi(x).$$

The following proposition gives sufficient conditions for a continuous function to be mildly regular at a point x.

Proposition 6.3. *Let φ be continuous in a neighborhood of a point x. Suppose that one of the following conditions holds:*

1. *φ is regular at x,*
2. *φ is differentiable at x,*
3. *$\partial_P \varphi(x)$ is nonempty.*

Then φ is mildly regular at x.

Proof. 1) If φ is regular at x then we have

$$\partial^P \varphi(x) = -\partial_P(-\varphi)(x) \subset -\partial(-\varphi)(x) = \partial \varphi(x) = \partial_L \varphi(x),$$

so φ is mildly regular at x.
2) If φ is differentiable at x, then we have $\partial^P \varphi(x) \subset \{\varphi'(x)\}$. Moreover by Subbotin's Theorem (see [10, Theorem 3.4.2]) we have $\varphi'(x) \in \partial_L \varphi(x)$. Then $\partial^P \varphi(x) \subset \partial_L \varphi(x)$ and this gives that φ is mildly regular at x.

3) Suppose that $\partial_P\varphi(x)$ is nonempty. Then there are two cases. First, if $\partial^P\varphi(x)$ is empty, then φ is mildly regular at x. Second, if $\partial^P\varphi(x)$ is nonempty, then $\partial^P\varphi(x)$ and $\partial_P\varphi(x)$ are simultaneously nonempty, which implies that φ is differentiable at x (see [10]). By 2) φ is mildly regular at x. □

We remark that the property of mild regularity, like the stronger ones of regularity or semiconvexity, can be thought of intuitively as one which rules out "concave corners". Its first use here is in the following theorem which gives an important characterization for the function φ_0.

Theorem 6.4. *Suppose that the function $T(0,\cdot)$ is mildly regular on $\mathcal{R}_+^0 \setminus \{0\}$. Then φ_0 coincides with $-T(0,\cdot)$.*

Proof. By Proposition 5.1 we have that $\varphi_0 \geq -T(0,\cdot)$ then by the minimality of φ_0 it is sufficient to prove that $-T(0,\cdot)$ is a solution of $(*)$. First we have that $-T(0,0) = 0$. Let us show that $-T(0,\cdot)$ satisfies the Hamilton-Jacobi equation of $(*)$. Let $\alpha \in \mathcal{R}_+^0$, then there are two cases:

Case 1. $\alpha \neq 0$.
Let $\zeta \in \partial_P(-T(0,\cdot))(\alpha)$. Then $-\zeta \in \partial^P T(0,\cdot)(\alpha) \subset \partial_L T(0,\cdot)(\alpha)$, since $T(0,\cdot)$ is mildly regular at α. But we have

$$1 + h_{-F}(\alpha, \partial_P T(0,\cdot)(\alpha)) = 0,^{5}$$

Hence since h_{-F} is continuous and ∂_L is constructed from ∂_P by a limiting process we get that

$$1 + h_{-F}(\alpha, -\zeta) = 0,$$

and then

$$1 + h(\alpha, \zeta) = 0.$$

Case 2. $\alpha = 0$.
We claim that $\partial_P(-T(0,\cdot))(0) = \emptyset$. Indeed, if not then $\partial^P T(0,\cdot)(0) \neq \emptyset$. Hence since $0 \in \partial_P T(0,\cdot)(0)$ we get that $T(0,\cdot)$ is differentiable at 0 and we have

$$\partial_P T(0,\cdot)(0) = \{0\},$$

and this gives a contradiction since $0 \in \text{int}\, \partial_P T(0,\cdot)(0)$[6]. □

Corollary 6.5. *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set such that $0 \in intU$. Then φ_0 is semiconcave on $\mathcal{R}_+^0$ and coincides with $-T(0,\cdot)$. Moreover, if we assume that ∂U is of class C^1 then $\varphi_0 \in C^1(\mathcal{R}_+^0 \setminus \{0\})$.

[5]This follows since $T(0,\cdot)$ is a solution on $\mathbb{R}^n \setminus \{0\}$ of the Hamilton-Jacobi equation

$$1 + h_{-F}(x, \partial_P\varphi(x)) = 0,$$

where h_{-F} is the lower Hamiltonian corresponding to $-F$.
[6]$\partial_P T(0,\cdot)(0) = \{\zeta \in \mathbb{R}^n : h(0,\zeta) \geq -1\}$, see [21, Theorem 5.1].

Proof. Clearly F satisfies our hypotheses ($-F$ is 0-STLC since $0 \in \operatorname{int} F(0)$) and when F has the stated form, it is known that the function $T(0,\cdot)$ is semiconvex on $\mathcal{R}_+^0$, see [4, Theorem 4.1]. Then $T(0,\cdot)$ is mildly regular on $\mathcal{R}_+^0$ and by Theorem 6.4 we find the result. When ∂U is of class C^1 it is known that $T(0,\cdot) \in C^1(\mathcal{R}_+^0 \setminus \{0\})$ (see [4, Corollary 5.10])[7] then $\varphi_0(\cdot) = -T(0,\cdot) \in C^1(\mathcal{R}_+^0 \setminus \{0\})$. □

It is known that if ∂U is not of class C^1 in the preceding corollary then $T(0,\cdot)$ (and thus φ_0) fails in general to be in $C^1(\mathcal{R}_+^0 \setminus \{0\})$, see [4, Example 3.5]

Remark 6.6. Let us return to Example 5.4 and Example 5.5. We remark that in these two examples we have $\varphi_0(\cdot) = -T(0,\cdot)$. This follows from that fact that in these examples the function $T(0,\cdot)$ is mildly regular. In the next section we give an example in which $\varphi_0(\cdot)$ does not coincide with $-T(0,\cdot)$, see Example 7.8.

7. Semigeodesics

Let $\alpha \in \mathbb{R}^n$. A trajectory $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F is a semigeodesic from α iff $x(0) = \alpha$ and $T(x(s),x(t)) = t-s$ for all $s \leq t \in [0,+\infty[$. In this section we present the relationship between the solutions of $(*)$ and semigeodesic trajectories. We begin with the following proposition which affirms that for a given solution φ of $(*)$ (there exist such solutions by Theorem 5.2), there exists a semigeodesic from every point $\alpha \in \operatorname{dom}\varphi$.

Proposition 7.1. *Let φ be a solution of $(*)$. Then for every $\alpha \in \operatorname{dom}\varphi$ there exists a semigeodesic x from α such that*

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

Proof. Let φ be a solution of $(*)$ and let $\alpha \in \operatorname{dom}\varphi$. By Proposition 5.1, there exists a trajectory $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F in $\mathcal{R}_+^0$ such that $x(0) = \alpha$ and

$$\varphi(x(t)) + t = 0, \quad \forall t \geq 0. \tag{6}$$

We claim that x is a semigeodesic from α. Indeed, let $s \leq t \in [0,+\infty[$, then by (6) and Proposition 5.1 we have

$$T(x(s),x(t)) \geq \varphi(x(s)) - \varphi(x(t)) = t - s,$$

but

$$T(x(s),x(t)) \leq t - s,$$

therefore $T(x(s),x(t)) = t - s$. □

Remark 7.2. The preceding proposition implies that under our hypotheses, there exists at least one semigeodesic from the origin. This can be deduced directly as follows. We consider a sequence α_n in $\mathcal{R}_+^0$ such that $T(0,\alpha_n) \longrightarrow +\infty$ (this sequence exists since $-F$ is 0-STLC). Let x_n be the trajectory of F which realizes the minimum time between 0 and α_n. By the compactness property of trajectories, there exists a trajectory x of F on $[0,+\infty[$ such that $x(0) = 0$ and x_n converges uniformly to x on compact interval. We claim that x is a semigeodesic from 0. Indeed, let $t \in [0,+\infty[$, then since $T(0,\alpha_n) \longrightarrow +\infty$ there exists n_t such that for $n \geq n_t$ we have

$$T(0,x_n(t)) = t.$$

[7] This result was first conjectured in [12] and then proved in [3].

Using the continuity of $T(0,\cdot)$ we get that

$$T(0,x(t)) = \lim_{n\longrightarrow+\infty} T(0,x_n(t)) = \lim_{n\longrightarrow+\infty} t = t,$$

and this shows that x is a semigeodesic from 0.

Remark 7.3. Using Proposition 7.1, we can show that under our hypotheses the Hamilton-Jacobi equation $(*)$ does not necessarily admit a solution on $\mathbb{R}^n$ (if we consider lower semicontinuous functions and exclude the value $-\infty$). Let us give an example. For $n=1$ we consider $F(x) := -x + [-1,1]$ for all $x \in \mathbb{R}$. In this case, we have $\mathcal{R}^0_+ =]-1,1[$ and

$$T(0,x) = \begin{cases} -\ln(1-x) & \text{if } 0 \le x < 1, \\ -\ln(1+x) & \text{if } -1 < x \le 0. \end{cases}$$

There exist only two semigeodesics from the origin, namely

- $x(t) = 1 - e^{-t}$, $t \ge 0$.
- $y(t) = e^{-t} - 1$, $t \ge 0$.

Now assume that there exists a lower semicontinuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ which is a solution of $(*)$ on $\mathbb{R}^n$, we shall derive a contradiction. By Proposition 7.1, there exists a semigeodesic z from the origin such that

$$\varphi(z(t)) + t = 0 \quad \forall t \ge 0.$$

We assume that $z = x$ (the case $z = y$ follows using the same argument). Then we have

$$\varphi(1-e^{-t}) = -t \quad \forall t \ge 0.$$

Hence if $t \longrightarrow +\infty$ then

$$\varphi(1-e^{-t}) \longrightarrow -\infty,$$

and this gives a contradiction since $(1-e^{-t}) \longrightarrow 1$ and φ is lower semicontinuous on $\mathbb{R}$.

The following theorem proves that semigeodesics from 0 are closely related to the minimal solution φ_0 of $(*)$.

Theorem 7.4. *Let $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ be a trajectory of F from 0. Then the following statements are equivalent:*

1. *The trajectory x is a semigeodesic from 0.*
2. *For all $t \ge 0$, we have $\varphi_0(x(t)) + t = 0$.*

Proof. 2)$\Longrightarrow$1): Follows immediately by Proposition 5.1 (as in the proof of the preceding proposition).
1)$\Longrightarrow$2): Since $T(0,x(t)) = t$ and by the definition of φ_0 we have that for all $s \ge 0$

$$\varphi_0(x(s)) \le \liminf_{t\longrightarrow+\infty} [T(x(s),x(t)) - T(0,x(t))] = \liminf_{t\longrightarrow+\infty} [t - s - t] = -s,$$

then

$$\varphi_0(x(s)) + s \le 0.$$

The reverse inequality follows by the strong increase property. □

In the following proposition, we present the relationship between an arbitrary solution of $(*)$ and a solution of the type φ_Γ.

Proposition 7.5. *Let φ be any solution of $(*)$. Then there exists a solution φ_Γ of $(*)$ of the type provided by Theorem 5.2 such that $\varphi_\Gamma \geq \varphi \geq \varphi_0$, and a semigeodesic from 0 along which φ, φ_Γ and φ_0 all coincide.*

Proof. Let φ be any solution of $(*)$, then by Proposition 7.1 there exists a semigeodesic x from 0 such that

$$\varphi(x(t)) + t = 0, \quad \forall t \geq 0.$$

Let $\Gamma := \{x(t) \; : \; t \geq 0\}$, then since $T(0, x(t)) = t$ we have $\Gamma \in \mathcal{G}$. We consider the solution φ_Γ of $(*)$ corresponding to Γ. Then we have:

$$\varphi_\Gamma(\alpha) = \liminf_{\substack{\alpha' \longrightarrow \alpha \\ t \longrightarrow +\infty}} [T(\alpha', x(t)) - T(0, x(t))].$$

But using Proposition 5.1 and since $T(0, x(t)) = t$ and $\varphi(x(t)) + t = 0$ we get that

$$T(\alpha', x(t)) - T(0, x(t)) \geq \varphi(\alpha') - \varphi(x(t)) - t = \varphi(\alpha').$$

Then $\varphi_\Gamma(\alpha) \geq \varphi(\alpha)$ since φ is lower semicontinuous.
By Theorem 7.4, φ and φ_0 agree along $x(\cdot)$. But for any $\tau > 0$, we have

$$\begin{aligned} \varphi_\Gamma(x(\tau)) &\leq \liminf_{t \longrightarrow +\infty} [T(x(\tau), x(t)) - T(0, x(t))] \\ &= \liminf_{t \longrightarrow +\infty} [t - \tau - t] \\ &= -\tau \\ &= \varphi(x(\tau)) \leq \varphi_\Gamma(x(\tau)), \end{aligned}$$

which establishes that φ and φ_Γ agree along $x(\cdot)$. □

In Corollary 6.5, we have proved that in the linear case the function φ_0 coincides with $-T(0, \cdot)$. For $\alpha \in \mathcal{R}_+^0$, the following theorem gives a necessary and sufficient conditions for $\varphi_0(\alpha)$ to be equal to $-T(0, \alpha)$ at a given point a.

Theorem 7.6. *Let $\alpha \in \mathcal{R}_+^0$. Then the following statements are equivalent:*

1. *The point α lies on a semigeodesic from 0.*
2. $\varphi_0(\alpha) = -T(0, \alpha)$.

Proof. 1)$\Longrightarrow$2): Let $\alpha \in \mathcal{R}_+^0$ and assume that there exists a semigeodesic x from 0 and $t \geq 0$ such that $x(t) = \alpha$. By Theorem 7.4, we have $\varphi_0(x(t)) + t = 0$. But $T(0, \alpha) = t$, then $\varphi_0(\alpha) = -T(0, \alpha)$.
2)$\Longrightarrow$1): Let $\alpha \in \mathcal{R}_+^0$ and assume that $\varphi_0(\alpha) = -T(0, \alpha)$. We can assume that $\alpha \neq 0$. Since $\varphi_0(\alpha)$ is finite and by Proposition 7.1 there exists a semigeodesic y from α such that

$$\varphi_0(y(t)) + t = \varphi_0(\alpha) = -T(0, \alpha), \quad \forall t \geq 0. \tag{7}$$

Let $x : [0, T(0, \alpha)] \longrightarrow \mathbb{R}^n$ be the minimal trajectory between 0 and α and let z be the trajectory of F on $[0, +\infty[$ obtained by concatenating x and y. We claim that z is the required semigeodesic from 0. Indeed, by Theorem 7.4 it is sufficient to prove that

$$\varphi_0(z(t)) + t = 0, \quad \forall t \geq 0.$$

We note that z is defined as follows, $z(t) = x(t)$ for $t \in [0, T(0,\alpha)]$ and $z(t) = y(t - T(0,\alpha))$ for $t \geq T(0,\alpha)$. Then by (7) we have

$$\varphi_0(z(t)) + t = 0, \quad \forall t \geq T(0,\alpha).$$

For $t \in [0, T(0,\alpha)[$ and by Proposition 5.1, we have

$$\varphi_0(z(t)) \geq -T(0, z(t)) = -T(0, x(t)) = -t,$$

hence

$$\varphi_0(z(t)) + t \geq 0 = \varphi_0(\alpha) + T(0,\alpha).$$

The reverse inequality follows by the strong increase property.
This completes the proof. □

Corollary 7.7. *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set such that $0 \in \operatorname{int} U$. Then every point in $\mathcal{R}_+^0$ lies on a semigeodesic from 0.

Proof. Follows from Corollary 6.5 and Theorem 7.6. □

In the following example, we show that $\varphi_0(\cdot)$ does not always coincide with $-T(0,\cdot)$. We also prove that $\varphi_0(\cdot)$ can take positive values.

Example 7.8. We take $n = 2$ and we define the following two multifunctions:

1.
$$F_1(x,y) = \begin{cases} \{(\dfrac{x^2 - y^2}{x^2 + y^2}, \dfrac{2xy}{x^2 + y^2})\} & \text{if } y \neq 0 \\ \{(1,0)\} & \text{if } y = 0 \end{cases}$$

2. $F_2(x,y) = \bar{B}$ for all $(x,y) \in \mathbb{R}^2$.

Now we consider the multifunction F defined by the following:

- $F(x,y) = F_2(x,y)$ for all (x,y) such that $\|(x,y)\| \leq 1$.
- $F(x,y) = F_1(x,y)$ for all (x,y) such that $\|(x,y)\| \geq 2$.
- $F(x,y) = \{(2-r)v_2 + (r-1)v_1 : v_1 \in F_1(x,y) \text{ and } v_2 \in F_2(x,y)\}$ for all (x,y) such that $1 < \|(x,y)\| := r < 2$.

Clearly F satisfies our hypotheses ($0 \in \operatorname{int} F(0)$). We note that for all $(x,y) \in \mathbb{R}^2 \backslash \{(\alpha, 0) : \alpha \in \mathbb{R}\}$, $F_1(x,y)$ is the unit tangent vector at (x,y) (pointing clockwise if $y < 0$ and counterclockwise if $y > 0$) to the unique circle centered on the y-axis and passing through both (x,y) and the origin. This implies that in the region $\|(x,y)\| > 2$, the trajectories of F move along such circles.

Claim 7.9. $\mathcal{R}_+^0 = \mathbb{R}^2 \backslash]-\infty, -2] \times \{0\}$.

Proof. It can be seen without much difficulty that we have

$$\mathbb{R}^2 \backslash]-\infty, -1[\times \{0\} \subset \mathcal{R}_+^0 \subset \mathbb{R}^2 \backslash]-\infty, -2] \times \{0\}.$$

Let us prove that the points of the form $(-a, 0)$ where $1 < a < 2$ are in $\mathcal{R}_+^0$. We fix $\varepsilon \in]0,1[$ and we consider the multifunction F_ε defined exactly like F but replaced $\bar{B}$ by $(1-\varepsilon)\bar{B}$. We can easily verify that we have:

- $F_\varepsilon(\cdot) \subset F(\cdot)$.
- There exist $\omega > 0$ and $\nu > 0$ such that

$$F_\varepsilon(x, y) + \nu\bar{B} \subset F(x, y) \ \ \forall (x, y) \in B((-a, 0); \omega),$$

and $B((-a, 0); \omega) \subset \{(\alpha, \beta) \ : \ 1 < \alpha^2 + \beta^2 < 4\}$.

Now let $K > 1$ be the Lipschitz constant of F_ε on $\bar{B}((-a, 0); \omega)$ and let $\mu > 0$ such that $\mu \leq \{\frac{\nu}{2K}, \frac{\omega}{2}\}$.

We consider a point (x, y) and an arc $z(\cdot)$ which verify:

- $\|(x, y) - (-a, 0)\| = \frac{\mu}{2}$, $x < -a$ and $y > 0$.
- $z : [0, +\infty[\longrightarrow \mathbb{R}^2$ and satisfies the following differential inclusion:

$$\dot{z}(t) \in F_1(z(t)) \ \ a.e. \ t \in [0, +\infty[, \ \ z(0) = (x, y).$$

We take $T > 0$ and $(x', y') \in \mathbb{R}^2$ such that $(x', y') := z(T) \in \{(-a, \beta) : \beta > 0\}$. Clearly we have $T \geq \|(x, y) - (x', y')\|$ and $z(t) \in \bar{B}((-a, 0); \frac{\mu}{2})$ for all $t \in [0, T]$. Then by the definition of F_ε, the arc $z(\cdot)$ is a trajectory of F_ε and taking (x, y) sufficiently near $(-a - \frac{\mu}{2}, 0)$, we can assume that

$$\frac{\|(x', y') - (-a, 0)\|}{T} \leq \frac{\|(x', y') - (-a, 0)\|}{\|(x, y) - (x', y')\|} \leq \frac{\nu}{2}. \tag{8}$$

Now we define the arc $w(\cdot)$ by

$$w(t) := z(t) + \frac{(-a, 0) - (x', y')}{T} t \ \ \forall t \in [0, T].$$

Then for all $t \in [0, T]$ we have

$$\|w(t) - (-a, 0)\| \leq \|z(t) - (-a, 0)\| + \|(-a, 0) - (x', y')\| < \mu + \mu \leq \omega,$$

hence

$$w(t) \in B((-a, 0); \omega) \ \ \forall t \in [t_0, T].$$

Using the fact that F_ε is K-Lipschitz on $\bar{B}((-a, 0); \omega)$ we get that for all $t \in [0, T]$

$$\begin{aligned} F_\varepsilon(z(t)) &\subset F_\varepsilon(w(t)) + K\|z(t) - w(t)\|\bar{B} \\ &\subset F_\varepsilon(w(t)) + K\mu\bar{B} \\ &\subset F_\varepsilon(w(t)) + \frac{\nu}{2}\bar{B}. \end{aligned}$$

This gives that

$$F_\varepsilon(z(t)) + \frac{\nu}{2}\bar{B} \subset F_\varepsilon(w(t)) + \nu\bar{B} \subset F(w(t)) \ \ \forall t \in [0, T]. \tag{9}$$

On the other hand, we have

$$\dot{w}(t) = \dot{z}(t) + \frac{(-a, 0) - (x', y')}{T} \ \ a.e. \ t \in [0, T].$$

Then by (8) and (9), we get that a.e. $t \in [0,T]$ we have

$$\dot{w}(t) = \dot{z}(t) + \frac{(-a,0)-(x',y')}{T} \in F_\varepsilon(z(t)) + \frac{\nu}{2}\bar{B} \subset F(w(t)),$$

and this shows that $w(\cdot)$ is a trajectory of F on $[0,T]$. By the definition of $w(\cdot)$ we have $w(0)=(x,y)$ and $w(T)=(-a,0)$, whence $(-a,0)\in\mathcal{R}_+^{(x,y)}$. Since $(x,y)\in\mathcal{R}_+^0$ we get that $(-a,0)\in\mathcal{R}_+^0$, and this completes the proof of the claim. □

Claim 7.10. *The trajectory $z(t)=(t,0)$, $t\in[0+\infty[$ is the unique semigeodesic from the origin.*

Proof. Clearly the trajectory $z(t)=(t,0)$, $t\in[0+\infty[$ is a semigeodesic from the origin since all velocities are bounded by 1 in norm, and no arc between two points is shorter than a straight line. To prove the uniqueness, first we remark that there exists $-2<b<-1$ such that the set $S:=\{(\alpha,\beta):\alpha^2+\beta^2<4 \text{ and } -2<\alpha<b\}$ satisfies the following: for all $(x,y)\in S$ and for all $v=(v_1,v_2)\in F(x,y)$ we have $v_1>0$. Using this fact, the continuity of $T(0,\cdot)$ and the fact that if z is a semigeodesic from 0 then $T(0,z(t))\longrightarrow+\infty$, we can easily establish our claim (details are omitted). □

Claim 7.11. *$\varphi_0(\cdot)$ does not coincide with $-T((0,0),\cdot)$*

Proof. Since $z(t)=(t,0)$, $t\in[0+\infty[$ is the unique semigeodesic from the origin and by Theorem 7.6 we get that φ_0 agrees with $-T((0,0),\cdot)$ at points of the form $(x,0)$ $(x\geq 0)$, and is strictly greater otherwise. □

Claim 7.12. *φ_0 takes positive values.*

Proof. We consider the point $(-\frac{1}{2},0)$, and we remark that as for the origin, there exists only one semigeodesic from this point, namely the trajectory $w(t)=(t-\frac{1}{2},0)$, $t\in[0+\infty[$. Then since $(-\frac{1}{2},0)\in\mathcal{R}_+^0\cap\mathcal{R}_-^0\subset\operatorname{dom}\varphi_0$ and by Proposition 7.1 we have

$$\varphi_0(-\frac{1}{2},0)=t+\varphi_0(w(t)),$$

hence for $t=\frac{1}{2}$ we get that $\varphi_0(-\frac{1}{2},0)=\frac{1}{2}+\varphi_0(0,0)=\frac{1}{2}>0$. □

8. Geodesics and the dual problem

A trajectory $x:]-\infty,+\infty[\longrightarrow\mathbb{R}^n$ of F is a geodesic iff

$$T(x(s),x(t))=t-s,$$

for all $s\leq t\in]-\infty,+\infty[$. We have proved in the preceding section that the solutions of $(*)$ are closely linked to semigeodesic trajectories. A natural question concerns the relationship between solutions of $(*)$ and geodesic trajectories.

We remark that in each of the three examples above (5.4, 5.5 and 7.8), there is in fact a geodesic through the origin[8]. We now give an example to show that such a geodesic need not exist in general.

[8]See Remark 8.7 for the proof of the existence of a geodesic through the origin for these examples.

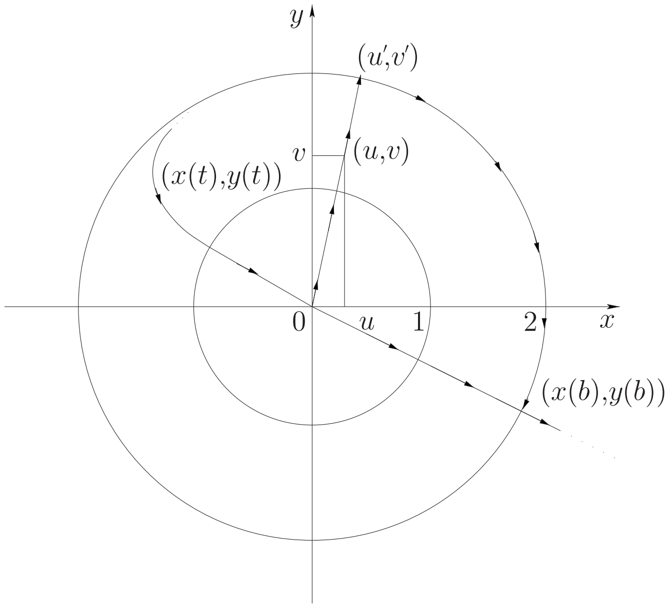

Figure 8.1: Example 8.1

Example 8.1. We take $n = 2$ and we define the following two multifunctions:

1. $F_1(x, y)$ is the closed segment between the points $\dfrac{(x, y)}{\sqrt{x^2 + y^2}}$ and $\dfrac{(y, -x)}{\sqrt{x^2 + y^2}}$ if $(x, y) \neq (0, 0)$ and the set $\{(0, 0)\}$ if $(x, y) = (0, 0)$.
2. $F_2(x, y) = \bar{B}$ for all $(x, y) \in \mathbb{R}^2$.

Now we consider the multifunction F defined by the following:

- $F(x, y) = F_2(x, y)$ for all (x, y) such that $\|(x, y)\| \leq 1$.
- $F(x, y) = F_1(x, y)$ for all (x, y) such that $\|(x, y)\| \geq 2$.
- $F(x, y) = \{(2 - r)v_2 + (r - 1)v_1 : v_1 \in F_1(x, y) \text{ and } v_2 \in F_2(x, y)\}$ for all (x, y) such that $1 < \|(x, y)\| := r < 2$.

Clearly F satisfies our hypotheses and it is easily seen that we have

$$\mathcal{R}^0_+ = \mathbb{R}^2, \;\; \mathcal{R}^0_- = 2B.$$

Now assume that there exists $(x(t), y(t))$ a geodesic passing through the origin at $t = 0$; we shall derive a contradiction. Since points in the complement of $2B$ cannot be steered to $(0, 0)$ we have that

$$\|(x(t), y(t))\| < 2, \; \forall t < 0.$$

Since $(0, 0) \in \text{int}\, F(0, 0)$ we have $T((0, 0), \cdot)$ is continuous on $\mathbb{R}^2$ and then bounded above on $2B$. Then there exists a first $b > 0$ such that $\|(x(b), y(b))\| = 2$. But for all $(u, v) \in 2B$ we have

$$\begin{aligned} T((u, v), (x(b), y(b)) &\leq T((u, v), (u', v')) + T((u', v'), (x(b), y(b)) \\ &\leq 2 + 4\pi, \end{aligned}$$

where (u', v') is as in Figure 8.1.
It follows that $T((x(t), y(t)), (x(b), y(b))) = b - t$ is bounded for $t < 0$ and this gives the desired contradiction.

We refer to the following as the *dual equation* to $(*)$:

$$1 + h(x, \partial^P \psi(x)) = 0, \ \forall x \in \mathcal{R}^0_-, \ \varphi(0) = 0. \qquad (*-)$$

A solution of $(*-)$ refers to an upper semicontinuous function. It is easy to see that ψ satisfies $(*-)$ iff $\varphi = -\psi$ is a solution of the version of $(*)$ obtained by replacing F by $-F$. Since the trajectories of $-F$ correspond to trajectories of F in reversed time, our previous results, applied to $(*-)$, give rise to consequences for F with respect to past (rather than future) time. For this purpose, the following hypothesis is relevant:

$$F \text{ is 0-STLC.}$$

To illustrate the use of the dual problem, suppose that ψ is a solution of $(*-)$, and that F is 0-STLC. Then we deduce the existence of a trajectory x of F on $]-\infty, 0]$ such that $x(0) = 0$ and

$$\psi(x(t)) + t = 0, \ \forall t \leq 0.$$

In the following theorem we use the dual problem to show the existence of a geodesic passing through the origin.

Theorem 8.2. *Assume that the following hypotheses hold:*

1. *F is 0-STLC.*
2. *$\mathcal{R}^0_- \subset \mathcal{R}^0_+$.*
3. *There exists solutions φ and ψ of $(*)$ and $(*-)$ respectively such that $\varphi \geq \psi$ on $\mathcal{R}^0_-$.*

Then there exists a geodesic passing through 0.

Proof. There exists a trajectory x with $x(0) = 0$ such that

$$\varphi(x(t)) + t = 0 \ \ \forall t \geq 0,$$

and a trajectory y with $y(0) = 0$ such that

$$\psi(y(t)) + t = 0 \ \ \forall t \leq 0.$$

We have then

$$\varphi(y(t)) + t \geq 0 \ \ \forall t \leq 0,$$

but the opposite inequality holds by strong increase. Then the trajectory z defined on $]-\infty, +\infty[$ by concatenating y and x satisfies

$$\varphi(z(t)) + t = 0 \ \ \forall t \in \mathbb{R}.$$

This gives using Proposition 5.1 that z is a geodesic. □

Remark 8.3. We can replace the second and third hypotheses of the preceding theorem by the following hypotheses and we find the same result:

1. $\mathcal{R}^0_+ \subset \mathcal{R}^0_-$.
2. There exist solution φ and ψ of $(*)$ and $(*-)$ respectively such that $\varphi \leq \psi$ on $\mathcal{R}^0_+$.

Remark 8.4. In Example 8.1 we have that F and $-F$ are 0-STLC ($0 \in \operatorname{int} F(0)$) and $\mathcal{R}^0_- \subset \mathcal{R}^0_+$, but there is no geodesic passing through 0. This shows the necessity of the third hypothesis in Theorem 8.2.

Corollary 8.5. *Assume that the following hypotheses hold:*

1. *F is is 0-STLC.*
2. *$\mathcal{R}^0_- \subset \mathcal{R}^0_+$.*
3. *There exists a continuous solution φ of $(*)$ which is mildly regular on $\mathcal{R}^0_+$.*

Then there exists a geodesic passing through 0.

Proof. The function φ satisfies

$$1 + h(x, \partial_L \varphi(x)) = 0,\ \forall x \in \mathcal{R}^0_+,\ \ \varphi(0) = 0,$$

since h is continuous, and since ∂_L is constructed from ∂_P by a limiting process. Then since φ is mildly regular on $\mathcal{R}^0_+$, we get that φ is a solution of $(*-)$. The result follows from Theorem 8.2. □

The pointwise upper envelope $E(\cdot)$ of all solutions of $(*)$ defines a lower semicontinuous function on $\mathcal{R}^0_+$ that is bounded above by $T(\cdot, 0)$. This function will play an important role to give a necessary and sufficient condition for the existence of a geodesic passing through the origin.

Theorem 8.6. *Assume that the following hypotheses hold:*

1. *$\mathcal{R}^0_- \subset \mathcal{R}^0_+$.*
2. *F and $-F$ are β-STLC for all $\beta \in \mathcal{R}^0_-$.*[9]

Then the following statements are equivalent:

(*i*) *There exists a geodesic passing through* 0.

(*ii*) $\limsup_{\substack{\alpha \in \mathcal{R}^0_- \\ T(\alpha,0) \longrightarrow +\infty}} [E(\alpha) - T(\alpha, 0)] = 0.$

Proof. (i)$\Longrightarrow$(ii): Let x be a geodesic passing through the origin. Set $\Gamma := \{x(t) : t \geq 0\}$, and consider the solution φ_Γ of $(*)$. Set $\alpha_i = x(-i)$. Then for any $\alpha \in \mathcal{R}^0_+$ and for $t > 0$ we have

$$\begin{aligned} T(\alpha, x(t)) - T(0, x(t)) &= T(\alpha_i, x(t)) - T(0, x(t)) + T(\alpha, x(t)) - T(\alpha_i, x(t)) \\ &\geq T(\alpha_i, 0) - T(\alpha_i, \alpha). \end{aligned}$$

Using the definition of φ_Γ and since $T(\cdot, \cdot)$ is continuous at $(\alpha_i, \alpha_i) \in \mathcal{R}^0_- \times \mathcal{R}^0_+$ we get that

$$\varphi_\Gamma(\alpha_i) \geq T(\alpha_i, 0).$$

Then $E(\alpha_i) = T(\alpha_i, 0)$. Since $T(\alpha_i, 0) \longrightarrow +\infty$ and $\alpha_i \in \mathcal{R}^0_-$, the result follows.
(ii)$\Longrightarrow$(i): Let α_i be a sequence in $\mathcal{R}^0_-$ such that $T(\alpha_i, 0) \longrightarrow +\infty$ and $E(\alpha_i) - T(\alpha_i, 0) \longrightarrow 0$. Then for each i there exists a solution φ_i of $(*)$ such that

$$\varphi_i(\alpha_i) \geq T(\alpha_i, 0) - \varepsilon_i,$$

[9]Since $\mathcal{R}^0_- \subset \mathcal{R}^0_+$ and using Proposition 4.1 we can show that this condition is equivalent to the continuity of $T(\cdot, \cdot)$ on $\mathcal{R}^0_- \times \mathcal{R}^0_+$.

where ε_i is a positive sequence converging to 0. Set $\tau_i = T(\alpha_i, 0)$, and let x_i be an optimal trajectory on the interval $[-\tau_i, 0]$ joining α_i to 0. By Proposition 7.1 we can extend x_i to $[0, +\infty[$ by a trajectory satisfying

$$\varphi_i(x_i(t)) + t = 0,\ \forall t \geq 0.$$

Since $x_i(t) \in \mathcal{R}^0_- \subset \mathcal{R}^0_+$ $\forall t \in [-\tau_i, 0]$ and by the strong increasing property, we have that for any $t \in [-\tau_i, 0]$,

$$\begin{aligned} 0 = \varphi(x_i(0)) + 0 &\geq \varphi_i(x_i(t)) + t \\ &\geq \varphi_i(x_i(-\tau_i)) - \tau_i \\ &= \varphi_i(\alpha_i) - T(\alpha_i, 0) \\ &\geq -\varepsilon_i. \end{aligned}$$

We deduce that

$$-\varepsilon_i \leq \varphi_i(x_i(t)) + t \leq 0,\ \forall t \in [-\tau_i, +\infty[.$$

By Proposition 5.1, we get that for any two points $s \leq t \in [-\tau_i, +\infty[$ we have

$$t - s \geq T(x_i(s), x_i(t)) \geq t - s - \varepsilon_i. \tag{10}$$

By the compactness property of trajectories, we can assume that the sequence x_i converges uniformly on bounded intervals to a trajectory x. We claim that x is a geodesic. Indeed, let $s \in]-\infty, 0]$ and let $t \in [0, +\infty[$. We have $x(s) \in \mathcal{R}^0_-$ and $x(t) \in \mathcal{R}^0_+$. Then $T(\cdot, \cdot)$ is continuous at $(x(s), x(t))$ and by (10) we get that

$$T(x(s), x(t)) = t - s,$$

which completes the proof. □

Remark 8.7. In this remark, we show how to prove the existence of a geodesic passing through the origin for the examples 5.4, 5.5 and 7.8.

1. **Example 5.4**. In this example we have that:
   - $\mathcal{R}^0_+ = \mathcal{R}^0_- = \mathbb{R}^n$.
   - $T(\cdot, \cdot)$ is continuous in $\mathcal{R} = \mathbb{R}^n \times \mathbb{R}^n$.
   - $E(\cdot) = T(\cdot, 0)$.

   Then by Theorem 8.6 there exists a geodesic passing through the origin.
2. **Example 5.5**. In this example we have that:
   - F is 0-STLC ($0 \in \operatorname{int} F(0)$).
   - $\mathcal{R}^0_+ = \mathcal{R}^0_- = \{\alpha \in \mathbb{R} : \alpha < 1\}$.
   - The function $\varphi(\cdot) = \ln(1 - \cdot)$ is a solution of $(*)$ which is mildly regular on $\mathcal{R}^0_+$.

   Then by Corollary 8.5 there exists a geodesic passing through the origin.
3. **Example 7.8**. In this example, clearly the trajectory $z(t) = (t, 0)$ is a geodesic passing through the origin.

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