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Partial and Full Boundary Regularity for Minimizers of Functionals with Nonquadratic Growth

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We consider regularity at the boundary for minimizers of variational integrals whose integrands have nonquadratic growth in the gradient. Under relatively mild assumptions on the coefficients we obtain a partial regularity result. For coefficients of a more particular type, namely those satifying a particular splitting condition, we obtain full boundary regularity. The results are new for the situation under consideration. The key ingredients are a new version of the usual Gehring-type lemma, and a careful adaptation of the technique of dimension-reduction to the current setting.

1. Introduction

In this paper we are concerned with the question of boundary regularity for minimizers of variational integrals whose integrands have nonquadratic growth in the gradient. We provide a partial regularity result for a general class of integrands (see Theorem 4.2 for a precise statement), and we also show full boundary regularity for a more restricted class of minimizers (see Theorem 5.4). The results are new for the nonquadratic case (the quadratic case having been dealt with in [21]).

We consider a bounded Lipschitz domain Ω in $\mathbb{R}^n$, where we take $n \geq 2$. For a fixed exponent $p \in (1, \infty)$ we consider functionals of the form

$$F(u, \Omega) = \int_\Omega (A(x, u)Du \cdot Du)^{p/2} dx\,, \tag{1}$$

defined for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ for some $N \geq 1$. Here $Du = ((D_\alpha u^i))_{1\leq\alpha\leq n,\, 1\leq i\leq N}$ is the derivative of u. We require that the coefficients $A(\cdot\,,\,\cdot) = ((A_{ij}^{\alpha\beta}(\cdot\,,\,\cdot))_{1\leq\alpha,\beta\leq n,\, 1\leq i,j\leq N}$, are defined and uniformly continuous, uniformly elliptic and uniformly bounded on $\Omega \times \mathbb{R}^N$.

It has been known for some time that $W^{1,p}$-minimizers of F need not be everywhere regular even in the quadratic case $p = 2$, see [17]. This motivates the study of the partial

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regularity theory associated to minimizers of F. The first object is to obtain estimates on the size of the singular set $\operatorname{Sing} u$ of a minimizer u. Here $\operatorname{Reg} u$ is the set of regular points of u, i.e. the set of points for which u is continuous, and $\operatorname{Sing} u$ is the complement of $\operatorname{Reg} u$. The second object is to obtain higher regularity for u on $\operatorname{Reg} u$.

A particular class of coefficients to consider are those which satisfy a so-called splitting condition, i.e.:

(S1) $$A_{ij}^{\alpha\beta}(x,u) = G^{\alpha\beta}(x,u) g_{ij}(x,u),$$

and each of the coefficient matrices $G = (\!(G_{\alpha\beta})\!) = (\!(G^{\alpha\beta})\!)^{-1}$ and $g = (\!(g_{ij})\!)$ is uniformly continuous, uniformly elliptic and uniformly bounded on $\Omega \times \mathbb{R}^N$. In the quadratic case, (local-)minimizers of (1) with coefficients of the form (S1) were considered by Giaquinta-Giusti ([13]): the authors showed that, for such a minimizer u, there holds $\mathcal{H}^{n-2-\varepsilon}(\operatorname{Sing} u) = 0$ (where here $\mathcal{H}^s$ denotes s-dimensional Hausdorff-measure) for some $\varepsilon > 0$. For general p, Fusco-Hutchinson ($p > 2$) and Acerbi-Fusco ($1 < p < 2$) showed that there holds $\mathcal{H}^{n-p-\varepsilon}(\operatorname{Sing} u) = 0$ for some $\varepsilon > 0$: see [9, Theorem 7.1], [1, Theorem 1.2] (and the remarks following the latter theorem).

In the current paper we establish the boundary analogue of this partial regularity result. To formulate the boundary-value problem appropriately we consider a function $h \in W^{1,s}(\Omega, \mathbb{R}^N)$ for some $s > n$. We then consider u minimizing F with coefficients of the form given by (S1) subject to the boundary condition

$$u\Big|_{\partial\Omega} = h\Big|_{\partial\Omega},$$

and in Theorem 4.2 we establish that u is regular on a relatively open subset of $\overline{\Omega}$ whose complement has vanishing $\mathcal{H}^{n-p-\varepsilon}$-measure; we further show that u is Hölder continuous with Hölder exponent $1 - \frac{n}{s}$ on this regular set. This result is new for $p \neq 2$: for $p = 2$ it was shown by Jost-Meier, see [21, Lemma 2].

In order to obtain better regularity – indeed full boundary regularity – we consider a further restriction on the coefficients, namely we consider coefficients satisfying the structural conditions (S1), and additionally we assume:

(S2) G depends only on x; and G and g are symmetric, with moduli of continuity satisfying a Dini-condition.

This latter condition is defined in (64): we note here that, in particular, Hölder-continuous coefficients are included.

In the interior such a restriction makes possible an improvement of the estimate for the singular set of a bounded minimizer u of the corresponding functional F. In the quadratic case, Giaquinta-Giusti showed $\mathcal{H} - \dim(\operatorname{Sing} u) \leq n - 3$, and $\operatorname{Sing} u$ is discrete in Ω for $n = 3$ (see [14, Theorem 1, Theorem 2]). For $p > 2$ Fusco-Hutchinson showed $\mathcal{H} - \dim(\operatorname{Sing} u) \leq n - [q] - 1$ for some $q > p$, and $\operatorname{Sing} u$ is discrete in Ω for $n = [q] + 1$ (here $[q]$ is the integer part of q).

In the current paper we are able to show full boundary regularity in this situation i.e. we show (see Theorem 5.4) that u is Hölder continuous in a neigbourhood of $\partial\Omega$ with Hölder exponent $1 - \frac{n}{s}$ on this regular set. This result is new for $p \neq 2$: for $p = 2$ it was shown by Jost-Meier, see [21, Lemma 2].

We note here that minimizing problems for functionals with coefficients having the special form (S2) arise in a number of settings, for example various geometrically motivated energy functionals for maps into Riemannian manifolds, such as harmonic maps or more generally p-harmonic maps. In the particular case of energy minimizing harmonic and p-harmonic maps, interior partial regularity and full boundary regularity has been established: see [23], [24], [22], [18], [5], [6], [7]. We also note that full interior regularity (i.e. everywhere Hölder continuity) for minimizers of

$$\int_\Omega |Du|^p dx \tag{2}$$

for $p > 2$ was shown by Uhlenbeck in [28]. In fact her results are applicable to critical points of (2), and also to somewhat more general coefficients (though not as general as (S2)). See also [27] and [1] for the case $1 < p < 2$.

We next provide a brief outline of the remainder of the paper. In the next section we assemble some technical results we will need later. Most of these are elementary – albeit somewhat tricky – algebraic estimates available in the existing literature. The notable exception is the final result in that section, Theorem 2.4. This result, which is of independent interest, is a combined local and global version of the usual Gehring–type L^p–L^q estimate to be found, for example [11, Chapter V, Theorem 1.2]. The proof is provided in the appendix at the end of the paper. In Section 3 we prove higher integrability at the boundary for minimizers of variational integrals with the same structure as (1), but with constant coefficients. The key steps are deriving the global Caccioppoli-type inequality (13), and combining this with the new version of the Gehring-type estimate, Theorem 2.4. In Section 4 we prove partial boundary regularity for minimizers of F under the splitting condition (S1). The procedure is relatively standard, making use of the technique of "freezing the coefficients" to enable one to bring into play the higher-regularity estimate of the previous section. In Section 5 we establish full boundary regularity for bounded minimizers of functionals with coefficients which satisfy (S1) and (S2). The technique is that of dimension reduction, a technique originally used by Federer in the setting of geometric measure theory in [4]. The technique has been applied to bounded minimizers of functionals with coefficients satisfying (S1) and (S2) with quadratic growth to obtain the above-mentioned improvement of the estimate of singular set in the interior (see [14]) and full boundary regularity (see [21]).

We close this introductory section with a few remarks on our results and techniques. For most of the preliminary results we admit more general coefficients than those satisfying the splitting conditions (S1) and (S2), restricting the structure at each stage only as it becomes necessary. The combined local and global version of the Gehring-type estimate, Theorem 2.4, enables us to treat the boundary situation in manner which analogous to that used in the interior. In particular we are able to avoid the technical difficulties associated with the reflection-type arguments which are usually a feature of boundary-regularity results. As a consequence, we are able to treat the superquadratic case (i.e. $p > 2$) and the subquadratic case (i.e. $1 < p < 2$) simultaneously for large portions of the paper: this is not possible even in the interior using existing techniques. As noted above, the essential ingredient required to enable our new version of the Gehring-type estimate to be applied in order to produce a global higher-integrability result is a (global) Caccioppoli-type inequality of the form (13). As such, Theorem 2.4 has the potential to be applied to a range of more general partial-regularity problems. Finally, we note that a number of intermediate results, particularly in the subquadratic case, are new, and even their interior analogues have not appeared in the literature: for example, the monotonicity formula, Lemma 5.3.

2. Preliminary technical results, notation

We start with some remarks on notation. We denote n–dimensional Lebesgue measure and n–dimensional Hausdorff measure by $\mathcal{L}^n$ and $\mathcal{H}^n$, respectively. We write $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$, and further $B_\rho = B_\rho(0)$. Similarly we denote upper half balls as follows: for $x_0 \in \mathbb{R}^{n-1} \times \{0\}$ we write $B^+_\rho(x_0)$ for $\{x \in \mathbb{R}^n : x_n > 0\,, |x - x_0| < \rho\}$, and set $B^+_\rho = B^+_\rho(0)$, $B^+_1 = B^+$. For $x_0 \in \mathbb{R}^{n-1} \times \{0\}$ we further write $D_\rho(x_0)$ for $\{x \in \mathbb{R}^n : x_n = 0\,, |x - x_0| < \rho\}$, and set $D_\rho = D_\rho(0)$, $D_1 = D$. For bounded, measurable $X \subset \mathbb{R}^n$ with $\mathcal{L}^n(X) > 0$ we denote the average of a given $h \in L^1(X)$ by $⨍_X h\,dx$, i.e. $⨍_X h\,dx = \frac{1}{\mathcal{L}^n(X)} \int_X h\,dx$. In particular, we write $h_{x_0,\rho} = ⨍_{B_\rho(x_0)} h\,dx$. We let α_n denote the volume of the unit ball in $\mathbb{R}^n$, i.e. $\alpha_n = \mathcal{L}^n(B_1) = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$.

For completeness, we also note here a number of technical results which will be used in our proofs. The results are elementary – albeit technical – inequalities.

Lemma 2.1. *Let h be nonnegative and bounded on $[\rho/2, \rho]$, and satisfy*

$$h(t) \le \theta h(s) + A(s-t)^{-2} + B(s-t)^{-\kappa} + D$$

for nonnegative constants A, B, D, κ and θ with $0 < \theta < 1$, for all s and t with $\rho/2 \le t < s \le \rho$. Then there exists a constant c depending only on θ and κ such that

$$h(\rho/2) \le c\big(A\rho^{-2} + B\rho^{-\kappa} + D\big)\,.$$

Lemma 2.2. *Given nonnegative numbers R_1, A, B, α and β with $\alpha > \beta$ there exist, corresponding to every $\gamma \in [\alpha, \beta)$, a positive constant ε_0 depending only on α, γ and A and a constant c depending only on α, β, γ and A such that the following is true: whenever Φ is nonnegative and nondecreasing on $(0, R_1)$ and satisfies*

$$\Phi(\rho) \le A\left(\left(\frac{\rho}{R}\right)^\alpha + \varepsilon\right)\Phi(R) + BR^\beta \qquad \text{for all } \rho \in (0, R)$$

for some $R < R_1$ and some $\varepsilon \in (0, \varepsilon_0)$, then there holds:

$$\Phi(\rho) \le c\left[\left(\frac{\rho}{R}\right)^\beta \Phi(R) + B\rho^\beta\right] \qquad \text{for all } \rho \in (0, R).$$

See [8, Lemma 3.2] respectively [12, Chapter III, Lemma 2.1] for a proof.

Throughout the paper we shall use the functions V, $V_\mu \colon \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$V(z) = |z|^{\frac{p-2}{2}} z \qquad V_\mu(z) = \big(\mu^2 + |z|^2\big)^{\frac{p-2}{4}} z$$

(here $\mu \ge 0$ and $p > 1$). We note that $V = V_0$. The following lemma collects some algebraic properties of the functions V and V_μ we shall use in the remainder of the paper.

Lemma 2.3. *There exists a constant $c > 1$ depending only on k and p such that*

$$c^{-1}\big(\mu^2 + |z|^2 + |y|^2\big)^{\frac{p-2}{4}}|z-y| \le \big|V_\mu(z) - V_\mu(y)\big| \le c\big(\mu^2 + |z|^2 + |y|^2\big)^{\frac{p-2}{4}}|z-y| \tag{3}$$

holds for any z, $y \in \mathbb{R}^k$ and $p \in (1,2)$. Moreover there holds

$$\big(|z|^{p-2}\xi - |y|^{p-2}\eta\big)\cdot(z-y) \ge c\big|V(z) - V(y)\big|^2 \tag{4}$$

Finally in the case $p \ge 2$ we have

$$|y-z|^p \le 2^p(|y|^{p-2}y - |z|^{p-2}z)\cdot(y-z) \tag{5}$$

for any z, $y \in \mathbb{R}^k$.

The first inequality can be directly inferred from [1, Lemma 2.2], while the second one is an easy consequence of this lemma. The last inequality can be found in [19, Corollary 1].

Throughout the paper we will be considering bounded Lipschitz domains. For such a domain Ω in $\mathbb{R}^n$ this means that the boundary $\partial\Omega$ can be represented as the graph of a Lipschitz function in a neighbourhood of every boundary point (after a suitable rotation). In view of the compactness of $\partial\Omega$ these functions have a uniform Lipschitz constant Lip_Ω. The regularity of the boundary ensures that we can find a constant R_0 depending only on Ω such that $B_\rho(z) \cap \Omega$ is simply connected for all ρ with $0 < \rho \le R_0$ and any $z \in \partial\Omega$. This then allows us to deduce the existence of a positive constant γ depending only on Ω and n such that for such ρ and z there holds:

$$\frac{\mathcal{H}^{n-1}(B_\rho(z) \cap \partial\Omega)}{\mathcal{H}^{n-1}(\partial B_\rho(z) \cap \Omega)} > \gamma\,. \tag{6}$$

As a further consequence of the condition imposed on the regularity of $\partial\Omega$, we note that there exists a positive constant $\widetilde{\gamma}$ (also depending only on n and Ω) such that for $0 < \rho \le R_0$ and $z \in \partial\Omega$ there holds:

$$\frac{\mathcal{L}^n(B_\rho(z) \cap \Omega)}{\mathcal{L}^n(B_\rho(z))} \ge \widetilde{\gamma} \quad \text{and} \quad \frac{\mathcal{L}^n(B_\rho(z) \setminus \Omega)}{\mathcal{L}^n(B_\rho(z))} \ge \widetilde{\gamma}\,. \tag{7}$$

Since we also have the trivial inclusion $B_\rho(z) \cap \Omega \subset B_\rho(z)$, we deduce that the measure $\mathcal{L}^n\big|_\Omega$ satisfies a so–called *Ahlfors regularity condition*, i.e. there exists a positive constant k_Ω depending on n and Ω such that there holds:

$$k_\Omega \rho^n \le \mathcal{L}^n(B_\rho(z) \cap \Omega) \le \alpha_n \rho^n \tag{8}$$

for all $z \in \overline{\Omega}$ and $0 < \rho \le \operatorname{diam}\Omega$. We note that the constant k_Ω depends only on the similarity class of Ω, i.e. $k_{t\Omega} = k_\Omega$ for any $t > 0$.

In particular for any such ρ there holds

$$\mathcal{L}^n(B_\rho(z) \cap \Omega) \le 2^n \frac{\alpha_n}{k_\Omega}\,\mathcal{L}^n(B_{\rho/2}(z) \cap \Omega)$$

uniformly for $z \in \overline{\Omega}$, meaning that $\mathcal{L}^n\Big|_\Omega$ is a *doubling measure*. This doubling property of the measure $\mathcal{L}^n\Big|_\Omega$ implies the validity of Vitali's covering theorem (see for example [3, 2.8.7, 2.8.8]), i.e. for any covering $\mathcal{F}$ of a subset $A \subset \Omega$, consisting of "balls" $B_\rho(x) \cap \Omega$, $x \in \overline{\Omega}$ with uniformly bounded radii, there exists a countable and disjoint subfamily $\{B_{\rho_k}(x_k) \cap \Omega\}$ such that $A \subset \bigcup_k B_{5\rho_k}(x_k) \cap \Omega$.

We close this section by stating the following version of the Gehring lemma, cf. the standard version given in, for example, [11, Chapter V, Theorem 1.2].

Theorem 2.4. *Let A be a closed subset of $\overline{\Omega}$. Consider two nonnegative functions $g, f \in L^1(\Omega)$ and p with $1 < p < \infty$, and such that there holds*

$$\left(\fint_{B_{\rho/2}(x)\cap\Omega} |g|^p dx \right) \leq b^p \left[\left(\fint_{B_\rho(x)\cap\Omega} |g| dx \right)^p + \fint_{B_\rho(x)\cap\Omega} |f|^p dx \right] \tag{9}$$

for almost all $x \in \Omega \setminus A$ with $B_\rho(x) \cap A = \emptyset$, for some constant b. Then there exist constants $c = c(n, p, q, b, k_\Omega)$ and $\varepsilon = \varepsilon(n, p, b, k_\Omega)$ such that

$$\left(\fint_\Omega |\widetilde{g}|^q dx \right)^{1/q} \leq c \left[\left(\fint_\Omega |g|^p dx \right)^{1/p} + \left(\fint_\Omega |f|^q dx \right)^{1/q} \right]$$

for all $q \in [p, p+\varepsilon)$, where $\widetilde{g}(x) = \frac{\mathcal{L}^n(B_{d(x,A)}(x)\cap\Omega)}{\mathcal{L}^n(\Omega)}\, g(x)$.

The proof of this theorem is given in the appendix at the end of this paper.

In the definition of $\widetilde{g}$ we use the convention $d(x, \emptyset) = \infty$. In particular, for $A = \emptyset$, we have $\widetilde{g} = g$ and this theorem implies a global version of the usual Gehring estimate. Moreover, for $\Omega_{A,\delta} = \{x \in \Omega : d(x, A) > \delta\}$, the conclusion of the theorem can be rewritten as

$$\left(\fint_{\Omega_{A,\delta}} |g|^q dx \right)^{1/q} \leq \widetilde{c} \left[\left(\fint_\Omega |g|^p dx \right)^{1/p} + \left(\fint_\Omega |f|^q dx \right)^{1/q} \right], \tag{10}$$

where the constant $\widetilde{c}$ only depends on n, p, q, b, k_Ω and δ.

3. Higher integrability at the boundary

The first preliminary result is a higher–integrability result at the boundary for local minimizers of functionals whose integrands have p–growth in the gradient. This result was given in the quadratic case as Lemma 1 in [21]. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ we consider a functional $F(\,\cdot\,, \Omega)$ defined for $\mathbb{R}^N$–valued functions and given by

$$F(u, \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx$$

(with suitable restrictions on f and u to ensure that the integrand is locally integrable in Ω).

Here we make the (relatively mild) structural assumption:

(H1) $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0, \infty)$ is a Carathéodory-Function, i.e. $f(\,\cdot\,, \xi, z)$ is measurable in Ω for every $(\xi, z) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, $f(x, \cdot\,, \cdot\,)$ is measurable for every $x \in \Omega$, and there exist p in $(1, \infty)$ and λ, $\Lambda \in (0, \infty)$ such that:

$$\lambda |z|^p \leq f(x, \xi, z) \leq \Lambda |z|^p \qquad \text{for all } (x, \xi, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}.$$

Under this assumption, $F(v;\Omega)$ is defined for $v \in W^{1,p}(\Omega,\mathbb{R}^N)$.

In particular for the consideration of boundary regularity we restrict attention to the case that $\Omega = B_R^+$ for some $R > 0$. An $\mathbb{R}^N$–valued function v is called a *minimizer* (more precisely, a $W^{1,p}$*–minimizer in* $B_R^+ \cup D_R$) for F satisfying (H1) on B_R^+ if, for every $\varphi \in W_0^{1,p}(B_R^+,\mathbb{R}^N)$, there holds:

$$F(u;B_R^+) \le F(u+\varphi;B_R^+)\,.$$

Lemma 3.1. *Consider $u \in W^{1,p}(B_R^+,\mathbb{R}^N)$ which is a minimum of F in $B_R^+ \cup D_R$, where F satisfies* (H1) *for some $p > 1$. Further assume that there holds $u\big|_{D_R} = h\big|_{D_R}$ for some function $h \in W^{1,\overline{q}}(B_R,\mathbb{R}^N)$ for an exponent $\overline{q} > p$. Then there exists an exponent q depending only on n, N, λ, Λ and p with $p < q \le \overline{q}$ such that $u \in W^{1,q}_{\mathrm{loc}}(B_R^+,\mathbb{R}^N)$. Further for $y \in B_R^+ \cup D_R$ and $0 < \rho < R - |y|$ there holds:*

$$\left(\fint_{B_{\rho/2}(y)\cap B_R^+} |Du|^q\,dx\right)^{1/q} \le c\Big[\Big(\fint_{B_\rho(y)\cap B_R^+} |Du|^p\,dx\Big)^{1/p} + \Big(\fint_{B_\rho(y)\cap B_R^+} |Dh|^q\,dx\Big)^{1/q}\Big] \quad (11)$$

for a constant $c = c(n,N,\lambda,\Lambda,p,q)$. If $B_\rho(y) \Subset B_R^+$, this estimate can be improved to

$$\left(\fint_{B_{\rho/2}(y)} |Du|^q dx\right)^{1/q} \le c\left(\fint_{B_\rho(y)} |Du|^p dx\right)^{1/p}. \quad (12)$$

Proof. We consider $x_0 \in B_R^+ \cup D_R$, $0 < r < R - |x_0|$, and distinguish two cases.

Case 1. $x_0^n \le \frac{3r}{4}$. Here we consider t, s with $0 < t < s \le r$ and choose a cut–off function $\eta \in C_0^\infty(B_s(x_0))$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_t(x_0)$, $|\nabla \eta| \le \frac{2}{s-t}$. Since u is a local minimum of F, we have:

$$F(u;B_s(x_0)\cap B_R^+) \le F(u-\eta(u-h);B_s(x_0)\cap B_R^+)\,.$$

Using (H1), Young's inequality and the convexity of $z \mapsto |z|^p$ we see:

$$\begin{aligned}\lambda \int_{B_t(x_0)\cap B_R^+} |Du|^p dx &\le \Lambda \int_{B_s(x_0)\cap B_R^+} |D(u-\eta(u-h))|^p dx \\ \le\ & 2^{p-1}\Lambda \Bigg(\int_{(B_s(x_0)\setminus B_t(x_0))\cap B_R^+} |Du|^p dx + \int_{B_s(x_0)} \eta^p |Dh|^p dx \\ &\qquad + \frac{2^p}{(s-t)^p}\int_{B_s(x_0)\cap B_R^+} |u-h|^p dx\Bigg).\end{aligned}$$

We add $2^{p-1}\Lambda \int_{B_t(x_0)\cap B_R^+} |Du|^p dx$ to both sides of this inequality and obtain

$$\begin{aligned}\int_{B_t(x_0)\cap B_R^+} |Du|^p dx \ \le\ & \vartheta \int_{B_s(x_0)\cap B_R^+} |Du|^p dx + \int_{B_s(x_0)\cap B_R^+} |Dh|^p dx \\ &+ \frac{2^p}{(s-t)^p}\int_{B_s(x_0)\cap B_R^+} |u-h|^p dx\,,\end{aligned}$$

where we have abbreviated $\vartheta = \frac{2^{p-1}\Lambda}{\lambda+2^{p-1}\Lambda}$. Since $\vartheta \in (0,1)$ we can apply Lemma 2.1 to establish the following Caccioppoli-type inequality: there holds

$$\fint_{B_{r/2}(x_0)\cap B_R^+} |Du|^p dx \le c \left[r^{-p} \fint_{B_r(x_0)\cap B_R^+} |u-h|^p dx + \fint_{B_r(x_0)\cap B_R^+} |Dh|^p dx \right] \tag{13}$$

for all $x_0 \in B_R^+$, $0 < r < R - |x_0|$, $x_0^n \le \frac{3r}{4}$, for a constant $c = c(n,p,\lambda,\Lambda)$.

To estimate the first term on the right-hand side of (13) we extend $g = u - h$ to B_R by letting $g = 0$ on $B_R \setminus B_R^+$. Since $u = h$ on D_R we have that $g \in W^{1,p}(B_R, \mathbb{R}^N)$. Noting that $B_r(x_0) \setminus B_R^+$ contains a ball of radius $\frac{r}{8}$ we can apply the Sobolev inequality in the form given, for example, by [29, Theorem 4.4.2] to obtain

$$\begin{aligned} r^{-p} \fint_{B_r(x_0)\cap B_R^+} |u-h|^p \, dx &\le 2^n r^{-p} \fint_{B_r(x_0)} |u-h|^p \, dx \le c \left(\fint_{B_r(x_0)} |D(u-h)|^{p_*} \, dx \right)^{p/p_*} \\ &\le c \left(\fint_{B_r(x_0)\cap B_R^+} |D(u-h)|^p \, dx \right)^{p/p_*}, \end{aligned}$$

where $c = c(n,N,p)$. Inserting this into (13) and using Hölder's inequality we arrive at

$$\fint_{B_{r/2}(x_0)\cap B_R^+} |Du|^p \, dx \le c \left[\left(\fint_{B_r(x_0)\cap B_R^+} |Du|^{p_*} dx \right)^{p/p_*} + \fint_{B_r(x_0) B_R^+} |Dh|^p dx \right] \tag{14}$$

for a constant c depending only on λ, Λ, n, N and p.

Case 2. $x_0^n > \frac{3r}{4}$. Here we have $B_{3r/4}(x_0) \Subset B_R^+$. For $0 < s < t \le \frac{3}{4}r$ we consider the comparison function $u - \eta(u - u_{x_0,3r/4})$ in place of $u - \eta(u - h)$ in the above argument to obtain the inequality

$$\fint_{B_{r/2}(x_0)} |Du|^p dx \le c \left(\fint_{B_{3r/4}(x_0)} |Du|^{p_*} dx \right)^{p/p^*} \le c \left(\fint_{B_r(x_0)\cap B_R^+} |Du|^{p_*} dx \right)^{p/p_*}, \tag{15}$$

for a constant c depending only on n, N, λ, Λ and p.

Hence for any ball $B_\rho(y)$ with $y \in B_R^+ \cup D_R$ and $0 < \rho < R - |y|$ inequality (14) holds for any ball $B_r(x_0) \cap B_R^+ \setminus B_\rho(y) = \emptyset$. Therefore we can apply Theorem 2.4 with $\Omega = B_\rho(y) \cap B_R^+$ and $A = \partial B_\rho(y) \cap B_R^+$. We note that we can choose the constant k_Ω independent of ρ and R because any such Ω satisfies a uniform interior and exterior cone-condition.

The interior result follows by the same reasoning from (15). □

The next result of this chapter is a global higher-integrability result for weak solutions of certain degenerate elliptic systems with p-Laplacian type behaviour. We consider a bounded Lipschitz domain Ω in $\mathbb{R}^n$, $p > 1$ and a given function $h \in W^{1,\overline{q}}(\Omega, \mathbb{R}^N)$ where $\overline{q} > p$. We denote by $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ the solution of the Dirichlet problem

$$\left. \begin{aligned} \int_\Omega (\mathcal{A}\, Dv \cdot Dv)^{\frac{p-2}{2}} \mathcal{A}\, Du \cdot D\varphi \, dx = 0 \qquad & \text{for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^N) \\ u = h \qquad & \text{on } \partial\Omega \end{aligned} \right\} \tag{16}$$

where here the constant coefficients $\mathcal{A} = (\mathcal{A}_{ij}^{\alpha\beta})$ are assumed to satisfy

$$\mathcal{A}\, z \cdot z \geq \lambda |z|^2 \quad \text{and} \quad \mathcal{A}\, z \cdot w \leq \Lambda |z|\,|w| \quad \text{for all } z,\, w \in \mathbb{R}^{nN}. \tag{17}$$

Lemma 3.2. *Consider $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ which is a weak solution of the Dirichlet problem (16) where $p > 1$, $\mathcal{A}$ satisfies (17), and $h \in W^{1,\overline{q}}(\Omega, \mathbb{R}^N)$ for an exponent $\overline{q} > p$. Then there exists an exponent q depending only on n, N, λ, Λ, k_Ω and p with $p < q \leq \overline{q}$ such that $u \in W^{1,q}(\Omega, \mathbb{R}^N)$. Furthermore, there holds:*

$$\left(\fint_\Omega |Du|^q \, dx \right)^{1/q} \leq c \left(\fint_\Omega |Du|^p \, dx \right)^{1/p} + c \left(\fint_\Omega |Dh|^q \, dx \right)^{1/q} \tag{18}$$

for a constant $c = c(n, N, \lambda, \Lambda, k_\Omega, p, q)$.

Proof. The proof closely follows the lines of the proof of Lemma 3.1. For $x_0 \in \overline{\Omega}$ and $r > 0$ we consider seperately the two cases $\operatorname{dist}(x_0, \partial\Omega) \leq \frac{3}{4} r$ and $\operatorname{dist}(x_0, \partial\Omega) > \frac{3}{4} r$. In the first case we test our system with $\eta^p(u - h)$ where $\eta \in C_0^1(B_r(x_0))$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{r/2}(x_0)$ and $|\nabla \eta| \leq \frac{4}{r}$. The argument from Lemma 3.1 applies since in this case $\mathcal{L}^n(B_r(x_0) \setminus \Omega) \geq \gamma \alpha_n r^n$ for some $\gamma > 0$ (note that Ω is a bounded Lipschitz domain and therefore fulfills a uniform exterior cone-condition). In the second case we use $\eta^p(u - u_{x_0, 3r/4})$ as a test-function in (16), where η is a suitable cut-off function with support in $B_{3r/4}(x_0) \Subset \Omega$. With these modifications it is straightforward to show that the hypotheses of Theorem 2.4 are fulfilled with $A = \emptyset$, $g = Du$ and $f = Dh$. $\square$

Finally, for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ minimizing

$$F_{\mathcal{A}}(u) = \int_\Omega (\mathcal{A}\, Du \cdot Du)^{\frac{p}{2}} \, dx$$

with respect to the Dirichlet boundary condition $u = h$ on $\partial\Omega$ for some $h \in W^{1,\overline{q}}(\Omega, \mathbb{R}^N)$ where $\overline{q} > p > 1$ we have the following global higher-integrability result.

Lemma 3.3. *Consider $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ which is a minimum of $F_{\mathcal{A}}$ subject to the Dirichlet boundary condition $u = h$ on $\partial\Omega$ where $\mathcal{A}$ satisfies (17), and $h \in W^{1,\overline{q}}(\Omega, \mathbb{R}^N)$ for an exponent $\overline{q} > p > 1$. Then there exists some exponent q depending only on n, N, λ, Λ, k_Ω and p with $p < q \leq \overline{q}$ such that $u \in W^{1,q}(\Omega, \mathbb{R}^N)$. Furthermore, there holds:*

$$\left(\fint_\Omega |Du|^q \, dx \right)^{1/q} \leq c \left(\fint_\Omega |Du|^p \, dx \right)^{1/p} + c \left(\fint_\Omega |Dh|^q \, dx \right)^{1/q} \tag{19}$$

for a constant $c = c(n, N, \lambda, \Lambda, k_\Omega, p, q)$.

We will need the following scaled a priori estimate for p–harmonic functions.

Lemma 3.4. *Consider $v \in W^{1,p}(B_R^+, \mathbb{R}^N)$, $p > 1$, which solves*

$$\left. \begin{aligned} \operatorname{div}(|Dv|^{p-2} Dv) &= 0 \quad \textit{on } B_R^+, \\ v &= h \quad \textit{on } D_R \end{aligned} \right\}$$

for a given $h \in W^{1,s}(B_R^+, \mathbb{R}^N)$, $s > p$. Then for all $\rho \in (0, R]$ and any $\tilde{n}$ in $\big[n(1-\frac{p}{s}), n\big)$ there holds:

$$\int_{B_\rho^+} |Dv|^p dx \le c\bigg[\Big(\frac{\rho}{R}\Big)^{\tilde{n}} \int_{B_R^+} |Dv|^p dx + \rho^{n(1-\frac{p}{s})}\Big(\int_{B_R^+} |Dh|^s dx\Big)^{p/s}\bigg]$$

for a constant c depending only on n, N, p, s and $\tilde{n}$.

Proof. Denote the minimizer of $\int_{B_R^+} |Df|^p dx$ in $\{f \in W^{1,p}(B_R^+, \mathbb{R}^N) \,:\, f = v - h \text{ on } \partial B_R^+\}$ by w. Then w is a weak solution of $\operatorname{div}(|Dw|^{p-2}Dw) = 0$ on B_R^+, and further $w = 0$ on D_R. We denote by $\widetilde{w}$ the extension of w to $B_R(0)$ by odd reflection. Then $\widetilde{w}$ is p-harmonic on $B_R(0)$, and hence we have the standard estimate (which follows in the case $p > 2$ directly from [16, Theorem 3.1], and cf. also [28, Theorem 3.1], and from [1, Proposition 2.13] in the subquadratic case $1 < p < 2$):

$$\int_{B_\rho^+} |Dw|^p dx \le c\,\Big(\frac{\rho}{R}\Big)^n \int_{B_R^+} |Dw|^p dx \qquad \text{for all } 0 < \rho \le R\,, \tag{20}$$

for a constant c depending only on n, N and p.

We first consider the **super-quadratic case** $p \ge 2$. Here we calculate, using (5), the fact that v and w are weakly p-harmonic on B_R^+, Young's inequality and Hölder's inequality, for $\varepsilon > 0$:

$$\int_{B_R^+} |Dv - Dw|^p dx \;\le c\bigg[\varepsilon \int_{B_R^+} (|Dv|^p + |Dw|^p)\, dx + \varepsilon^{1-p}\Big(\int_{B_R^+} |Dh|^s dx\Big)^{p/s} R^{n(1-p/s)}\bigg] \tag{21}$$

for a constant $c = c(n, p, s)$. The minimizing property of w yields, with Hölder's inequality:

$$\int_{B_R^+} |Dw|^p dx \;\le\; c\bigg[\int_{B_R^+} |Dv|^p dx + \Big(\int_{B_R^+} |Dh|^s dx\Big)^{p/s} R^{n(1-p/s)}\bigg],$$

where c has the same dependencies as above. Combining this with (21) we have:

$$\int_{B_R^+} |Dv - Dw|^p dx \le c\bigg[\varepsilon \int_{B_R^+} |Dv|^p dx + \varepsilon^{1-p}\Big(\int_{B_R^+} |Dh|^s dx\Big)^{p/s} R^{n(1-p/s)}\bigg] \tag{22}$$

where $c = c(n, p, s)$. From (20) and (22) we infer:

$$\int_{B_\rho^+} |Dv|^p dx \le \tilde{c}\bigg[\Big(\Big(\frac{\rho}{R}\Big)^n + \varepsilon\Big) \int_{B_R^+} |Dv|^p dx + \varepsilon^{1-p}\Big(\int_{B_R} |Dh|^s dx\Big)^{p/s} R^{n(1-p/s)}\bigg],$$

for a constant $\tilde{c}$ depending on n, N, p and s.

Hence the conditions of Lemma 2.2 are fulfilled with $\Phi(\rho) = \int_{B_\rho^+} |Dv|^p dx$, $A = \tilde{c}$, $\alpha = n$ and $\beta = 1 - \frac{p}{s}$. Choosing $\varepsilon < \varepsilon_0$, with ε_0 given by Lemma 2.2, we obtain the desired conclusion.

In the **subquadratic case** $1 < p < 2$ we can apply (4) with $z = Du$, $y = Dv$ and $k = nN$, to obtain

$$\int_{B_R^+(x_0)} \Big| |Dv|^{\frac{p-2}{2}} Dv - |Dw|^{\frac{p-2}{2}} Dw \Big|^2 dx \le c \int_{B_R^+(x_0)} (|Dv|^{p-2}Dv - |Dw|^{p-2}Dw)\cdot(Dv-Dw)dx$$

with $c = c(n, N, p)$. This is the analogous estimate to (21). This means that the remainder of the proof can be carried through as in the case $p \ge 2$. $\square$

We next consider symmetric matrices g in $\mathbb{R}^{N^2}$ and G in $\mathbb{R}^{n^2}$ with

$$\begin{aligned} |\xi|^2 \le g_{ij}\xi^i\xi^j = g\,\xi\cdot\xi \le \Lambda^{1/2}|\xi|^2 \qquad &\text{for all } \xi \in \mathbb{R}^N, \\ |\eta|^2 \le G^{\alpha\beta}\,\eta_\alpha\eta_\beta = G^{-1}\eta\cdot\eta \le \Lambda^{1/2}|\eta|^2 \qquad &\text{for all } \eta \in \mathbb{R}^n. \end{aligned}$$

We set $\mathcal{A} = G^{-1} \otimes g = (G^{\alpha\beta} g_{ij}) \in \mathbb{R}^{(nN)^2}$. Note that we have

$$|\eta|^2|\xi|^2 \le \mathcal{A}\,\eta\otimes\xi\cdot\eta\otimes\xi \le \Lambda\,|\eta|^2|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N,\ \eta \in \mathbb{R}^n,$$

i.e. $\mathcal{A}$ is elliptic in the sense of Legendre–Hadamard with ellipticity constant 1 and upper bound Λ.

Lemma 3.5. *Consider a fixed exponent $p > 1$ and $v \in W^{1,p}(B_R^+, \mathbb{R}^N)$ which solves*

$$\left.\begin{aligned} \textstyle\int_{B_R^+} (\mathcal{A}\,Du\cdot Du)^{\frac{p-2}{2}}\,\mathcal{A}\,Du\cdot D\varphi\,dx = 0 \qquad &\textit{for all } \varphi \in W_0^{1,p}(B_R^+, \mathbb{R}^N) \\ u\big|_{D_R} = h \qquad & \end{aligned}\right\}$$

where h is a given function in $W^{1,s}(B_R^+, \mathbb{R}^N)$ for some $s > p$. Then for all $\rho \in (0, R]$ and any $\tilde n$ in $\big[n(1-\frac{p}{s}), n\big)$ there holds:

$$\int_{B_\rho^+} |Dv|^p dx \le c\Big[\Big(\frac{\rho}{R}\Big)^{\tilde n}\int_{B_R^+} |Dv|^p dx + \rho^{n(1-\frac{p}{s})}\Big(\int_{B_R^+} |Dh|^s dx\Big)^{p/s}\Big]$$

for a constant c depending only on n, N, p, Λ, s and $\tilde n$.

Proof. Since g and G^{-1} are positive definite and symmetric there exist invertible matrices γ and Γ such that $g\,\xi\cdot\widetilde{\xi} = \gamma\,\xi\cdot\gamma\,\widetilde{\xi}$ for any $\xi, \widetilde{\xi} \in \mathbb{R}^N$ and $G^{-1}\eta\cdot\widetilde{\eta} = \Gamma^{-1}\eta\cdot\Gamma^{-1}\widetilde{\eta}$ for any $\eta, \widetilde{\eta} \in \mathbb{R}^n$. For $y = \Gamma x$ we set $w(y) = \gamma\,u(\Gamma^{-1}y)$ and $\ell(y) = \gamma\,h(\Gamma^{-1}y)$. Then $w \in W^{1,p}(\Gamma B_R^+, \mathbb{R}^N)$ solves the Dirichlet problem

$$\left.\begin{aligned} \textstyle\int_{\Gamma B_R^+} |Dw|^{p-2}Dw\cdot D\varphi\,dy = 0 \qquad &\text{for all } \varphi \in W_0^{1,p}(\Gamma B_R^+, \mathbb{R}^N) \\ w\big|_{\Gamma D_R} = \ell \qquad & \end{aligned}\right\} \tag{23}$$

where $\ell \in W^{1,s}(\Gamma B_R^+, \mathbb{R}^N)$. Since G^{-1} has ellipticity constant 1 and upper bound $\sqrt{\Lambda} \ge 1$, we have for any $r > 0$: $TB^+_{r/\sqrt[4]{\Lambda}} \subset \Gamma B_r^+$ for some orthogonal matrix T. Therefore w solves (23) with ΓB_R^+ replaced by $TB^+_{R/\sqrt[4]{\Lambda}}$ and ΓD_R replaced by $TD_{R/\sqrt[4]{\Lambda}}$, so we can apply

Lemma 3.4 to deduce that, for all $0 < \rho \le r \le \frac{R}{\sqrt[4]{\Lambda}}$ and fixed $\tilde{n} \in [n(1-\frac{p}{s}), n)$, there holds:

$$\int_{TB_\rho^+} |Dw|^p dx \le c\left[\left(\frac{\rho}{r}\right)^{\tilde{n}} \int_{TB_r^+} |Dw|^p dx + \left(\int_{TB_r^+} |D\ell|^s dx\right)^{p/s} \rho^{n(1-p/s)}\right]$$

for a constant c depending only on n, N, Λ, p, s and $\tilde{n}$. Transforming back to the original functional yields the desired estimate. □

4. Partial boundary regularity

In this section we consider the boundary analogue of the results of [9] and [1], i.e. the case of an integrand with p-growth for some fixed exponent $p > 1$. We consider $u \in W^{1,p}(B_R^+, \mathbb{R}^N)$ which is a local minimizer for

$$\mathcal{F}(u) = \int_{B_R^+} \big(A(x,u)Du \cdot Du\big)^{p/2}\, dx$$

with $u = h$ on D_R for a given $h \in W^{1,s}(B_R^+, \mathbb{R}^N)$, where $s > p$. Here $\mathcal{F}(u)$ is given in components by $\int_{B_R^+} \big(A_{ij}^{\alpha,\beta}(x,u) D_\alpha u^i D_\beta u^j\big)^{p/2}$.

We impose the following structure conditions on A.

(C1) There exists $\Lambda > 0$ such that

$$A(x,\xi)z \cdot \widetilde{z} \le \Lambda |z|\,|\widetilde{z}| \quad \text{for all } z, \widetilde{z} \in \mathbb{R}^{nN},\ (x,\xi) \in \overline{B}_R^+ \times \mathbb{R}^N.$$

(C2) The coefficients $A(x,\xi)$ are uniformly strongly elliptic, i.e. there exists $\lambda > 0$ such that

$$A(x,\xi)z \cdot z \ge \lambda |z|^2 \qquad \text{for all } z \in \mathbb{R}^{nN},\ (x,\xi) \in \overline{B}_R^+ \times \mathbb{R}^N.$$

(C3) There holds $A \in C^0(\overline{B}_R^+ \times \mathbb{R}^N, \mathbb{R}^{nN})$, and further A is uniformly continuous, i.e. there exists a concave and nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and $\omega < 1$ such that

$$|A(x,\xi) - A(y,\eta)| \le \omega\left(|x-y|^p + |\xi - \eta|^p\right). \tag{24}$$

For the results of this section, the requirement of uniform continuity in (C3) can be relaxed; see the remarks at the end of the section. For the full regularity results of Section 5 we will require uniform continuity in (C3).

In view of the fact that we are considering minimizers, and since $A(x,\xi)z \cdot z = A(x,\xi)^t z \cdot z$, we can henceforth assume that A is symmetric, i.e. $A(x,\xi)z \cdot \tilde{z} = A(x,\xi)\tilde{z} \cdot z$ for any $x \in \overline{B}_R^+$, $\xi \in \mathbb{R}^N$, and z, $\tilde{z} \in \mathbb{R}^{nN}$.

We begin with a few remarks concerning **higher integrability.** We observe that the functional $w \mapsto \mathcal{F}(w)$ satisfies the hypothesis (H1) from Section 3 with λ replaced by $\lambda^{p/2}$ and Λ replaced by $\Lambda^{p/2}$. Therefore we can apply Lemma 3.1 to obtain that $u \in W^{1,\tilde{q}}(B_{R/2}^+, \mathbb{R}^N)$ for some $p < \tilde{q} < s$. Furthermore, we have

$$\left(⨍_{B_{R/2}^+} |Du|^{\tilde{q}}\, dx\right)^{1/\tilde{q}} \le c\left(⨍_{B_R^+} |Du|^p\, dx\right)^{1/p} + c\left(⨍_{B_R^+} |Dh|^{\tilde{q}}\, dx\right)^{1/\tilde{q}} \tag{25}$$

where $c = c(n, N, \lambda, \Lambda, p, s)$. (Applying Lemma 3.1 directly we see that the constant depends on $\tilde{q}$, but by choosing $\tilde{q}$ systematically, e.g. halfway between p and the supremum of the exponents yielded by Lemma 3.1, we see that the dependency on $\tilde{q}$ can be expressed in terms of the remaining parameters and s.) Next, we consider $v \in u + W_0^{1,p}(B^+_{R/2}, \mathbb{R}^N)$ to be a minimizer of $\int_{B^+_{R/2}} (A(0, u_{R/2})Dv \cdot Dv)^{p/2} dx$, where $u_{R/2}$ denotes $⨍_{B^+_{R/2}} u\, dx$. Then, by Lemma 3.2 we obtain that $v \in W_0^{1,q}(B^+_{R/2}, \mathbb{R}^N)$ for some $q \in (p, \tilde{q})$. Moreover, we have

$$\left(⨍_{B^+_{R/2}} |Dv|^q\, dx\right)^{1/q} \leq c\left(⨍_{B^+_{R/2}} |Dv|^p\, dx\right)^{1/p} + c\left(⨍_{B^+_{R/2}} |Du|^q\, dx\right)^{1/q}, \tag{26}$$

for c having the same dependencies as the constant in (25). Our next aim is to prove the following estimate for $Du - Dv$:

Lemma 4.1. *Under the above assumptions there holds:*

$$\begin{aligned} &⨍_{B^+_{R/2}} |Du - Dv|^p dx \\ \leq\;& c\left(⨍_{B^+_R} |Du|^p dx + \left(⨍_{B^+_R} |Dh|^s dx\right)^{p/s}\right) \omega^{\frac{q-p}{q}}\left(R^p ⨍_{B^+_R} [1 + |Du|^p] dx\right) \end{aligned} \tag{27}$$

in the case $p \geq 2$, respectively

$$\begin{aligned} &⨍_{B^+_{R/2}} |V(Du) - V(Dv)|^2 dx \\ \leq\;& c\left(⨍_{B^+_R} |Du|^p dx + \left(⨍_{B^+_R} |Dh|^s dx\right)^{p/s}\right) \omega^{\frac{q-p}{q}}\left(R^p ⨍_{B^+_R} [1 + |Du|^p] dx\right), \end{aligned} \tag{28}$$

in the case $1 < p < 2$, where the constant c depends only on n, N, p, λ, Λ and s.

After having established this lemma the remainder of the section is concerned with deducing partial boundary regularity for u under suitable restrictions on the structure of A.

Proof of Lemma 4.1. We write $\mathcal{A}$ for $A(0, u_{R/2})$, and define a function $F^0 : \mathbb{R}^N \to [0, \infty)$ via:

$$F^0(\xi) = (\mathcal{A}\xi \cdot \xi)^{p/2}. \tag{29}$$

We first look at the **superquadratic** case $p \geq 2$: The Taylor expansion for F^0 about Du yields:

$$\begin{aligned} \int_{B^+_{R/2}} F^0(Du) dx &= \int_{B^+_{R/2}} F^0(Dv) dx + \int_{B^+_{R/2}} DF^0(Dv) \cdot (Du - Dv) dx \\ &\quad + \int_{B^+_{R/2}} \left[\int_0^1 (1-s) D^2F^0(Du + s(Dv - Du)) ds (Du - Dv)\right] \cdot (Du - Dv) dx \\ &= I + II + III \end{aligned} \tag{30}$$

with the obvious labelling for I, II and III.

From the Euler equation for Dv we see:

$$II = 0\,.$$

In order to estimate III, we begin by observing:

$$(D^2F^0(\xi)\eta)\cdot\eta = p(\mathcal{A}\xi\cdot\xi)^{p/2-1}\left[\mathcal{A}\eta\cdot\eta + (p-2)\frac{(\mathcal{A}\xi\cdot\eta)^2}{\mathcal{A}\xi\cdot\xi}\right] \geq c|\xi|^{p-2}|\eta|^2\,,$$

for ξ, $\eta \in \mathbb{R}^{nN}$, $\xi \neq 0$, for a constant $c = p\lambda^{\frac{p}{2}}$. Note that the inequality also holds in the case that $\xi = 0$, because $D^2F^0(0) = 0$. Thus we can estimate III from below as follows:

$$III \geq c\int_{B^+_{R/2}}\int_0^1 (1-s)|Du + s(Dv-Du)|^{p-2}|Du-Dv|^2 ds dx\,.$$

In order to estimate the integral $\int_0^1(1-s)|Du+s(Dv-Du)|^{p-2}ds$ we consider the cases $|Dv| \geq |Du|$ and $|Dv| < |Du|$ separately. In the case $|Dv| < |Du|$ we note that for $s \in [0,1/4]$ there holds:

$$(1-s)|Du + s(Dv-Du)|^{p-2} \geq c(p)(|Du| + |Dv|)^{p-2}.$$

Thus by integrating s from 0 to $\frac{1}{4}$ in this case, and analogously from $\frac{3}{4}$ to 1 in the case $|Dv| \geq |Du|$, we can estimate

$$III \geq c(p,\Lambda)\int_{B^+_{R/2}}|Du-Dv|^p dx,$$

and hence from (30) we have:

$$\fint_{B^+_{R/2}}|Du-Dv|^p dx \leq c\fint_{B^+_{R/2}}\big(F^0(Du) - F^0(Dv)\big)dx\,, \tag{31}$$

for a constant $c = c(p,\Lambda)$. In order to estimate the integral on the left-hand side of (31), we begin by writing

$$\begin{aligned}
&\fint_{B^+_{R/2}}(F^0(Du) - F^0(Dv))dx\\
= &\fint_{B^+_{R/2}}\left[(\mathcal{A}Du\cdot Du)^{p/2} - (A(x,u)Du\cdot Du)^{p/2}\right]\,dx\\
&\quad+\fint_{B^+_{R/2}}(A(x,u)Du\cdot Du)^{p/2} - (A(x,v)Dv\cdot Dv)^{p/2}dx\\
&\quad+\fint_{B^+_{R/2}}(A(x,v)Dv\cdot Dv)^{p/2} - (\mathcal{A}Dv\cdot Dv)^{p/2}dx\\
= &\; IV + V + VI
\end{aligned} \tag{32}$$

with the obvious labelling. The minimality of u yields

$$V \le 0\,.$$

In order to estimate IV, we begin by using (24) and (C1) to observe:

$$\big|(\mathcal{A}\xi\cdot\xi)^{p/2} - (A(x,u)\xi\cdot\xi)^{p/2}\big| \le c(p,\Lambda)|\xi|^p\omega(|x|^p + |u-u_{R/2}|^p)\,. \tag{33}$$

Applying (33), (25) (with $\widetilde{q}\in(p,s]$ being the higher integrability exponent from (25)), Jensen's inequality and Poincaré's inequality for half-balls (keeping in mind that $\omega\le 1$, and that ω is nondecreasing and concave) we derive:

$$\begin{aligned}
IV &\le c\fint_{B^+_{R/2}} |Du|^p\,\omega\big((R/2)^p + |u-u_{R/2}|^p\big)dx\\
&\le c\left(\fint_{B^+_{R/2}} |Du|^{\widetilde{q}}dx\right)^{p/\widetilde{q}}\left(\fint_{B^+_{R/2}} \omega^{\frac{\widetilde{q}}{\widetilde{q}-p}}((R/2)^p + |u-u_{R/2}|^p)dx\right)^{\frac{\widetilde{q}-p}{\widetilde{q}}}\\
&\le c\left(\fint_{B^+_R}|Du|^p dx + \left(\fint_{B^+_R}|Dh|^s dx\right)^{p/s}\right)\left(\fint_{B^+_{R/2}}\omega((R/2)^p + |u-u_{R/2}|^p)dx\right)^{\frac{\widetilde{q}-p}{\widetilde{q}}}\\
&\le c\left(\fint_{B^+_R}|Du|^p dx + \left(\fint_{B^+_R}|Dh|^s dx\right)^{p/s}\right)\omega^{\frac{\widetilde{q}-p}{\widetilde{q}}}\left((R/2)^p + \fint_{B^+_{R/2}}|u-u_{R/2}|^p dx\right)\\
&\le c\left(\fint_{B^+_R}|Du|^p dx + \left(\fint_{B^+_R}|Dh|^s dx\right)^{p/s}\right)\omega^{\frac{\widetilde{q}-p}{\widetilde{q}}}\left(R^p + R^p\fint_{B^+_R}|Du|^p dx\right),
\end{aligned}$$

where here the constant c depends only on n, N, p, s, λ and Λ. In order to estimate VI we begin by calculating directly as above to see

$$VI \le c(p,\Lambda)\fint_{B^+_{R/2}} |Dv|^p\,\omega((R/2)^p + |v-v_{R/2}|^p)\,dx\,. \tag{34}$$

Similarly to above we use the higher-integrability of Dv from (26) with exponent $q\in(p,\tilde{q}]$ (note that we need here the global version valid for Lipschitz domains) to estimate the right-hand side of (34), keeping in mind $\omega\le 1$, as follows:

$$VI \le c\left(\fint_{B^+_{R/2}}|Dv|^q dx\right)^{p/q}\omega^{\frac{q-p}{q}}\left((R/2)^p + \fint_{B^+_{R/2}}|v-v_{R/2}|^p dx\right) \tag{35}$$

$$\le c\left[\fint_{B^+_{R/2}}|Dv|^p dx + \left(\fint_{B^+_{R/2}}|Du|^{\tilde{q}}dx\right)^{p/\tilde{q}}\right]\cdot\omega^{\frac{q-p}{q}}\left(R^p + R^p\fint_{B^+_R}|Dv|^p dx\right), \tag{36}$$

for a constant $c = c(n,N,\lambda,\Lambda,p,q,\tilde{q})$. Here we have also used Jensen's inequality, Poincaré's inequality (on half-balls) and Hölder's inequality.

We next note that (C1), (C2) and the minimizing property of v yield the estimate:

$$\fint_{B^+_{R/2}} |Dv|^p\,dx \le \left(\frac{\Lambda}{\lambda}\right)^{p/2}\fint_{B^+_{R/2}}|Du|^p\,dx\,. \tag{37}$$

Using this and the higher-integrability estimate for u, i.e. (25), we obtain from (35):

$$VI \leq c\left(\fint_{B_R^+} |Du|^p dx + \left(\fint_{B_R^+} |Dh|^s dx\right)^{p/s}\right) \omega^{\frac{q-p}{q}} \left(R^p + R^p \fint_{B_R^+} |Du|^p dx\right)$$

where $c = c(n, N, \lambda, \Lambda, p, s)$. Using the estimates obtained for IV, V and VI in (32), and using this in turn in (31), we finally obtain the desired estimate (27).

Now we consider the **subquadratic** case $1 < p < 2$. The fact that $1 < p < 2$ means that the second derivative $D^2F^0(z)$ (with F^0 defind in (29)) does not exist for $z = 0$, so we cannot carry over the arguments for $p \geq 2$ in the form presented above. Instead we proceed by defining a family of mollifications of F^0 via

$$F^\varepsilon(z) = (\varepsilon^2 + \mathcal{A}z \cdot z)^{p/2} \quad \text{for } z \in \mathbb{R}^{nN},$$

for $\varepsilon > 0$. Since $\mathcal{A}$ is symmetric, we have that $(z, \tilde{z}) \mapsto \mathcal{A}z \cdot \tilde{z}$ defines an inner product on $\mathbb{R}^{nN}$, meaning that we have via the Cauchy-Schwarz inequality:

$$(A(x,\xi)z \cdot \tilde{z})^2 \leq (A(x,\xi)z \cdot z)(A(x,\xi)\tilde{z} \cdot \tilde{z}) \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^N \text{ and } z,\ \tilde{z} \in \mathbb{R}^{nN}. \tag{38}$$

We then write

$$\begin{aligned}
&\int_{B_{R/2}^+} (F^0(Du) - F^0(Dv))\, dx & (39)\\
&= \int_{B_{R/2}^+} \Big[(F^0(Du) - F^0(Dv)) - (F^\varepsilon(Du) - F^\varepsilon(Dv))\Big] dx + \int_{B_{R/2}^+} \big(F^\varepsilon(Du) - F^\varepsilon(Dv)\big) dx \\
&= I_\varepsilon + II_\varepsilon & (40)
\end{aligned}$$

with the obvious labelling. The dominated convergence theorem shows immediately:

$$I_\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0. \tag{41}$$

In order to study II_ε, we begin by observing that the Taylor expansion for F^ε about Du yields:

$$\begin{aligned}
II_\varepsilon &= \int_{B_{R/2}^+} DF^\varepsilon(Dv)(Du - Dv) dx \\
&\quad + \int_{B_{R/2}^+} \int_0^1 (1-s) D^2F^\varepsilon(Dv + s(Du - Dv)) ds (Du - Dv, Du - Dv) dx \\
&= I'_\varepsilon + II'_\varepsilon & (42)
\end{aligned}$$

with the obvious labelling. We have

$$\begin{aligned}
\lim_{\varepsilon \to 0} I'_\varepsilon &= \lim_{\varepsilon \to 0} \int_{B_{R/2}^+} (\varepsilon^2 + \mathcal{A}Dv \cdot Dv)^{\frac{p-2}{2}} \mathcal{A}Dv \cdot (Du - Dv) dx \\
&= \int_{B_{R/2}^+} (\mathcal{A}Dv \cdot Dv)^{\frac{p-2}{2}} \mathcal{A}Dv \cdot (Du - Dv) dx = 0\,. & (43)
\end{aligned}$$

In order to estimate II'_ε we begin by estimating (using (38) and noting that $1 < p < 2$):

$$D^2F^\varepsilon(z)(w,w) \geq p(p-1)\Lambda^{\frac{p-2}{2}}\lambda(\mu^2+|z|^2)^{\frac{p-2}{2}}|w|^2\,,$$

where we have abbreviated $\mu = \frac{\varepsilon}{\sqrt{\Lambda}}$. Setting further $z = Dv + s(Du - Dv)$ and $w = Du - Dv$ (note that $-1/2 < \frac{p-2}{2} < 0$) and using (3) we thus have the estimate:

$$\begin{aligned}
&\int_0^1 (1-s)D^2F^\varepsilon(Dv+s(Du-Dv))(Du-Dv\,,Du-Dv)ds\\
\geq\;& \tfrac{1}{2}p(p-1)\Lambda^{\frac{p-2}{2}}\lambda(\mu^2+|Du|^2+|Dv|^2)^{\frac{p-2}{2}}|Du-Dv|^2\\
\geq\;& c\left|(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du-(\mu^2+|Dv|^2)^{\frac{p-2}{4}}Dv\right|^2\,,
\end{aligned}$$

where $c = c(n,N,p,\lambda,\Lambda)$. Applying Fatou's Lemma we can thus conclude, writing $V(z)$ for $|z|^{\frac{p-2}{2}}z$,

$$\liminf_{\varepsilon\to 0} II'_\varepsilon \geq c\int_{B^+_{R/2}} |V(Du)-V(Dv)|^2dx\,. \tag{44}$$

Allowing ε to tend to 0 in (39) and combining (44) with (41) and (43), we thus see

$$\int_{B^+_{R/2}} |V(Du)-V(Dv)|^2dx \leq c\int_{B^+_{R/2}} (F^0(Du)-F^0(Dv))dx\,, \tag{45}$$

where c depends only on n, N, p, λ and Λ. As in the case $p \geq 2$ we decompose the right-hand side of (45) as

$$\begin{aligned}
\fint_{B^+_{R/2}} (F^0(Du)-F^0(Dv))dx \;=\;& \fint_{B^+_{R/2}} \big(\mathcal{A}Du\cdot Du\big)^{p/2}-\big(A(x,u)Du\cdot Du\big)^{p/2}dx\\
&+\fint_{B^+_{R/2}} \big(A(x,u)Du\cdot Du\big)^{p/2}-\big(A(x,v)Dv\cdot Dv\big)^{p/2}dx\\
&+\fint_{B^+_{R/2}} \big(A(x,v)Dv\cdot Dv\big)^{p/2}-\big(\mathcal{A}Dv\cdot Dv\big)^{p/2}dx\\
=\;& IV+V+VI\,,
\end{aligned} \tag{46}$$

with the obvious labelling. We have $V \leq 0$ due to the minimality of u. The term IV can be estimated similarly to the case $p \geq 2$. Since $1 < p < 2$ we have via (C2), for $z \in \mathbb{R}^{nN}$, $z \neq 0$, $0 \leq s \leq 1$:

$$[(s\mathcal{A}+(1-s)A(x,u))z\cdot z]^{p/2-1} \leq (s\lambda|z|^2+(1-s)\lambda|z|^2)^{p/2-1} = (\lambda|z|^2)^{p/2-1}\,,$$

and hence, using (24):

$$\begin{aligned}
&\big|(\mathcal{A}z\cdot z)^{p/2}-(A(x,u)z\cdot z)^{p/2}\big|\\
\leq\;& \frac{p}{2}\int_0^1 [(s\mathcal{A}+(1-s)A(x,u))z\cdot z]^{p/2-1}\,|(\mathcal{A}-A(x,u))z\cdot z|ds\\
\leq\;& c|z|^p\omega\big(|x-x_0|^p+|u-u_R|^p\big)\,,
\end{aligned}$$

where c depends only on p and λ. This inequality is analogous to (33), and the remainder of the estimate for IV can be performed exactly as in the superquadratic case. Further the term VI can be estimated in exactly the same manner as in the case $p \geq 2$, meaning that we obtain from (46) and (45) the estimate (28). $\square$

We are now in a position to derive the desired $C^{0,\alpha}$ estimate.

Theorem 4.2. *Consider coefficient matrices $G^{-1} = (G^{\alpha\beta})$ and $g = (g_{ij})$ which are uniformly continuous on $\overline{B}^+_R \times \mathbb{R}^N$, and which satisfy:*

$$\begin{aligned} &|\xi|^2 \leq g(x,\tilde{\xi})\,\xi\cdot\xi \leq \Lambda^{1/2}|\xi|^2 && \text{for all } x \in B^+_R,\ \xi,\tilde{\xi} \in \mathbb{R}^N, \\ &|\eta|^2 \leq G^{-1}(x,\tilde{\xi})\eta\cdot\eta \leq \Lambda^{1/2}|\eta|^2 && \text{for all } x \in B^+_R,\ \tilde{\xi} \in \mathbb{R}^N,\ \eta \in \mathbb{R}^n. \end{aligned} \tag{47}$$

Consider further fixed $p > 1$, $s > n$, and a given function $h \in W^{1,s}(B^+_R, \mathbb{R}^N)$. Then there exist constants $\varepsilon_0 > 0$ and $R_0 > 0$ depending only on n, N, p, λ, s and $\omega(\cdot)$ such that the following is true: For $u \in W^{1,p}(B^+_R, \mathbb{R}^N)$ which minimizes

$$\int_{B^+_R} \left(G^{\alpha\beta}(x,u)g_{ij}(x,u)D_\alpha u^i D_\beta u^j\right)^{p/2} dx \quad \textit{subject to the boundary condition} \quad u\Big|_{D_R} = h$$

and which fulfills the smallness condition

$$R^{p-n}\int_{B^+_R} |Du|^p\,dx + R^{p(1-n/s)}\left(\int_{B^+_R} |Dh|^s\,dx\right)^{p/s} \leq \varepsilon_0$$

for some $0 < R \leq R_0$, there holds $u \in C^{0,1-n/s}(\overline{B}^+_{R/2}, \mathbb{R}^N)$.

Proof. The assumptions yield the existence of a bounded and continuous modulus of continuity ω which is concave, nondecreasing, and which satisfies $\omega(0) = 0$, $\omega \leq 1$, such that the coefficients $\mathcal{A} = G^{-1} \otimes g$ given in components by $\mathcal{A}^{\alpha\beta}_{ij}(x,u) = G^{\alpha\beta}(x,u)g_{ij}(x,u)$ satisfy:

$$|\mathcal{A}(x,\xi) - \mathcal{A}(\widetilde{x},\widetilde{\xi})| \leq \omega(|x-\widetilde{x}|^p + |\xi-\widetilde{\xi}|^p)$$

for all $x, \widetilde{x} \in \overline{B}^+_R$, $\xi, \widetilde{\xi} \in \mathbb{R}^N$.

Consider $x_0 \in D_R$, a half ball $B^+_r(x_0) \subset B^+_R$ and $v \in u + W^{1,p}_0\left(B^+_{r/2}(x_0), \mathbb{R}^N\right)$ which minimizes

$$\int_{B^+_{r/2}(x_0)} \left(G^{\alpha\beta}(x_0,u_{r/2})g_{ij}(x_0,u_{r/2})D_\alpha v^i D_\beta v^j\right)^{p/2} dx\,.$$

First we restrict to the **superquadratic** case $p \geq 2$: For $\tau \in (0, 1/4]$ we have

$$\begin{aligned} &(\tau r)^{p-n}\int_{B^+_{\tau r}(x_0)} |Du|^p dx \\ \leq\ & c(p)\Big[(\tau r)^{p-n}\int_{B^+_{r/2}(x_0)} |Du-Dv|^p dx + (\tau r)^{p-n}\int_{B^+_{\tau r}(x_0)} |Dv|^p dx\Big] \\ =\ & c(p)\big[I + II\big] \end{aligned}$$

with the obvious labelling. By replacing $B_{R/2}$ by $B^+_{r/2}(x_0)$ and and taking $\delta = 1 - \frac{p}{q}$ in (27) we obtain

$$I \leq c\tau^{p-n}\Big[r^{p-n}\int_{B_r^+(x_0)}|Du|^p + r^{p(1-n/s)}\Big(\int_{B_r^+(x_0)}|Dh|^s dx\Big)^{p/s}\Big]\,\omega^\delta\Big(r^p \fint_{B_r^+(x_0)}(1+|Du|^p)dx\Big)$$

for $c = c(n, N, \Lambda, p, s)$. Here we have also used the concavity of ω.

In order to estimate II we can apply Lemma 3.5 with $\rho = \tau r$, and R replaced by $r/2$ to obtain:

$$\begin{aligned} II &\leq c(\tau r)^{p-n}\Big[\tau^{\widetilde{n}}\int_{B^+_{r/2}(x_0)}|Dv|^p dx + \Big(\int_{B^+_{r/2}(x_0)}|Dh|^s dx\Big)^{p/s}(\tau r)^{n(1-p/s)}\Big] \\ &\leq c\Big[\tau^{p+\widetilde{n}-n}r^{p-n}\int_{B_r^+(x_0)}|Du|^p dx + (\tau r)^{p(1-n/s)}\Big(\int_{B_r^+(x_0)}|Dh|^s dx\Big)^{p/s}\Big]. \end{aligned}$$

Here we have also used the fact that $\int_{B_{r/2}(x_0)}|Dv|^p \leq \Lambda^{p/2}\int_{B_{r/2}(x_0)}|Du|^p dx$ (see (37)). Fixing an exponent $\tilde{n}$ in $\big[n(1-\frac{p}{s}), n\big)$ and combining these estimates we obtain:

$$\begin{aligned} &(\tau r)^{p-n}\int_{B^+_{\tau r}(x_0)}|Du|^p dx \\ \leq\;& c\tau^{p-\widetilde{n}+n}\Big[1+\tau^{-\widetilde{n}}\omega^\delta\Big(r^p\fint_{B_r^+(x_0)}(1+|Du|^p)dx\Big)\Big]\,r^{p-n}\int_{B_r^+(x_0)}|Du|^p dx \\ +\;& cr^{p(1-n/s)}\Big(\int_{B_r^+(x_0)}|Dh|^s dx\Big)^{p/s}\Big(\tau^{p-n}\omega^\delta\Big(r^p\fint_{B_r^+(x_0)}(1+|Du|^p)dx\Big)+\tau^{p(1-n/s)}\Big), \end{aligned} \tag{48}$$

where $c = c(n, N, \Lambda, p, s, \widetilde{n})$. Define now the function $\phi : (0, R - |x_0|] \to \mathbb{R}$ via

$$\phi(r) = r^{p-n}\int_{B_r^+(x_0)}|Du|^p dx\,.$$

After setting

$$\widehat{p} = p + \widetilde{n} - n,\;\; a = a(r) = \omega^\delta\Big(r^p\fint_{B_r^+(x_0)}(1+|Du|^p)dx\Big)$$

$$\text{and}\qquad b = b(r) = \Big(\int_{B_r^+(x_0)}|Dh|^s dx\Big)^{p/s}$$

equation (48) can be rewritten as

$$\phi(\tau r) \leq c\tau^{\widehat{p}}\big(1+\tau^{-\widetilde{n}}a(r)\big)\,\phi(r) + cr^{p(1-n/s)}b(r)\,\big(\tau^{p-n}a(r)+\tau^{p(1-n/s)}\big)\,.$$

We are now in a position to apply a standard iteration procedure yielding the desired result. In particular, we obtain the excess–decay estimate

$$\phi(\rho) \leq c\Big[\Big(\frac{\rho}{R}\Big)^{p(1-n/s)}\phi(r) + \rho^{p(1-n/s)}\Big(\int_{B_R^+}|Dh|^s dx\Big)^{p/s}\Big]$$

for any $0 \le \rho \le R/2$ and $x_0 \in D_{R/2}$ where the constant c depends on n, N, p, s, Λ and $\omega(\,\cdot\,)$. Combining this estimate with the analogous interior excess–decay estimate (cf. [9, Section 6]) yields, by Poincaré's inequality and Campanato's characterization of Hölder continuous functions, the desired result follows in a standard manner, cf. [21, Lemma 2].

In the **subquadratic** case $1 < p < 2$ we argue as follows: For $0 < \tau \le \frac{1}{4}$ there holds

$$\begin{aligned}(\tau r)^{p-n}\int_{B^+_{\tau r}} |Du|^p dx &= (\tau r)^{p-n}\int_{B^+_{\tau r}} |V(Du)|^2 dx \\ &\le 2(\tau r)^{p-n}\int_{B^+_{\tau r}} |V(Du)-V(Dv)|^2 dx + 2(\tau r)^{p-n}\int_{B^+_{\tau r}} |Dv|^p\, dx\end{aligned}$$

The second term on the right-hand side can be estimated in exactly the same manner as the corresponding term appearing in the case $p \ge 2$, because Lemma 3.5 is valid in the current setting. The first term can be estimated completely analogously to the corresponding term for $p \ge 2$, with inequality (28) now playing the role of (27). The remainder of the proof proceeds identically. □

Remark. As noted earlier, the results of this section can be extended to the case that A is merely assumed to be continuous in (C3), and similarly for G^{-1} and g in Theorem 4.2. In this case we have to work with moduli of continuity $\omega(M,\,\cdot\,)$ for $A(x,\xi)$ on sets of the form $\overline{B^+_R} \times \{\xi \in \mathbb{R}^N : |\xi| \le M\}$. Under the assumption of boundedness of $|u_R|$ we then obtain the analogous results; cf. [9, Sections 5,6].

5. Full boundary regularity

In this section we prove full boundary regularity for a class of minimizers. Obviously in the case $p > n$ the Sobolev embedding theorem immediately shows that an F-minimizer u is everywhere regular, and by Theorem 4.2 we immediately have that u is Hölder-continuous with Hölder exponent $1 - \frac{n}{s}$ in a neighbourhood of $\partial\Omega$.

The first result is a compactness theorem. The interior analogue is proven in the case $p = 2$ in [14, Lemma 1], and in the case $p > 2$ in [9, Lemma 8.1].

Lemma 5.1. *Consider a sequence $\{A^{(\nu)}(\,\cdot\,,\,\cdot\,)\}$ of continuous functions defined on $B^+ \times \mathbb{R}^N$ which converge uniformly on $B^+ \times \mathbb{R}^N$ to a function $A(\,\cdot\,,\,\cdot\,)$, and such that each function $A^{(\nu)}$ satisfies:*

(H1)$_{(\nu)}$ $A^{(\nu)}(x,\xi)z\cdot z \ge \lambda|z|^2$ *for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^N$, $z \in \mathbb{R}^{nN}$;*
(H2)$_{(\nu)}$ $A^{(\nu)}(x,\xi)z\cdot w \le \Lambda|z||w|$ *for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^N$, and z, $w \in \mathbb{R}^{nN}$;*
(H3)$_{(\nu)}$ $|A^{(\nu)}(x,\xi) - A^{(\nu)}(\widetilde{x},\widetilde{\xi})| \le \omega(|x-\widetilde{x}|^p + |\xi-\widetilde{\xi}|^p)$ *for all $x, \widetilde{x} \in \mathbb{R}^n$, and $\xi, \widetilde{\xi} \in \mathbb{R}^N$*

for a bounded and continuous modulus of continuity ω which is concave, nondecreasing, and which satisfies $\omega(0) = 0$, $\omega \le 1$. Consider further sequences $\{u^{(\nu)}\}$ in $W^{1,p}(B^+,\mathbb{R}^N)$ and $\{h^{(\nu)}\}$ in $W^{1,s}(B^+,\mathbb{R}^N)$ for some fixed $s > n$ such that each $u^{(\nu)}$ is a local minimum of the functional $F^{(\nu)}(\,\cdot\,,B^+)$ in $W^{1,p}(B^+,\mathbb{R}^N)$ relative to the boundary values $h^{(\nu)}$ on D, where here

$$F^{(\nu)}(v,X) = \int_X \big(A^{(\nu)}(x,v)Dv\,\cdot\,Dv\big)^{p/2} dx$$

for $X \subseteq B^+$, $v \in W^{1,p}(B^+,\mathbb{R}^N)$. We further assume $h^{(\nu)} \to h$ weakly in $W^{1,s}(B^+,\mathbb{R}^N)$ and $u^{(\nu)} \to u$ weakly in $L^p(B^+,\mathbb{R}^N)$.

Then u is a local minimum of the functional $F(\,\cdot\,, B^+)$ relative to the boundary values h on D, where for $X \subseteq B^+$,

$$F(v, X) = \int_X \big(A(x, v)Dv \cdot Dv\big)^{p/2} dx\,.$$

Further there holds for any $0 < R < 1$:

$$F(u, B_R^+) = \lim_{\nu\to\infty} F^{(\nu)}\left(u^{(\nu)}, B_R^+\right)\,. \tag{49}$$

Proof. We begin by observing that the weak L^p-convergence of the $u^{(\nu)}$'s to u yields, in the light of the Caccioppoli inequality (13) and its interior analogue, the higher integrability result Lemma 3.1 and a standard covering argument, the following bound: for each $R \in (0, 1)$ there holds

$$\int_{B_R^+} |Du^{(\nu)}|^q dx \le c(R) \tag{50}$$

for a constant $c(R)$ which can also depend on the parameters n, N, p, q, λ, Λ, s, as well as on the quantities $\sup_{\nu\ge1} \|h^{(\nu)}\|_{W^{1,s}(B^+)}$ and $\sup_{\nu\ge1} \|u^{(\nu)}\|_{L^p(B^+)}$, but which is, in particular, independent of ν. Here the exponent $q > p$ (given by Lemma 3.1) is, of course, independent of R. In view of (50) and the weak L^p-convergence of the $u^{(\nu)}$'s to u we see that $u \in W^{1,p}(B_R^+, \mathbb{R}^N)$ for all $R \in (0, 1)$, and we can pass to a subsequence, again labelled $\{u^{(\nu)}\}$, such that there holds:

$$u^{(\nu)} \to u \qquad \text{strongly in } L^p; \tag{51}$$

$$Du^{(\nu)} \to Du \qquad \text{weakly in } L^q; \tag{52}$$

$$u^{(\nu)} \to u \qquad \text{pointwise a.e.} \tag{53}$$

Since $u \in W^{1,p}(B_R^+, \mathbb{R}^N)$ the Sobolev embedding theorem yields $u \in L^q(B_R^+, \mathbb{R}^N)$, and hence by (52) there holds $u \in W^{1,q}(B_R^+, \mathbb{R}^N)$ for all $\rho \in (0, R)$, and indeed, after passing to a further subsequence, $u^{(\nu)} \to u$ strongly in L^q. (The choice of subsequences depends a priori on the radius R, but obviously a subsequence fulfilling the conditions on B_R^+ will fulfill them on B_ρ^+ for all $\rho \in (0, R)$.)

We next consider

$$\begin{aligned} F^{(\nu)}(u^{(\nu)}, B_R^+) &= \int_{B_R^+} (A(x, u)Du^{(\nu)} \cdot Du^{(\nu)})^{p/2}\, dx \\ &\quad + \int_{B_R^+} [(A^{(\nu)}(x, u^{(\nu)})Du^{(\nu)} \cdot Du^{(\nu)})]^{p/2} - [(A(x, u)Du^{(\nu)} \cdot Du^{(\nu)})]^{p/2}\, dx \\ &= I + II \end{aligned} \tag{54}$$

with the obvious labelling. To estimate II we begin by noting that the monotonicity of $t \mapsto t^{\frac{p-2}{2}}$ on $(0, \infty)$ yields the elementary estimate: $|s^{\frac{p}{2}} - t^{\frac{p}{2}}| \le \frac{p}{2}|s - t|(s^{\frac{p-2}{2}} + t^{\frac{p-2}{2}})$ for

$s, t > 0$. Using this in II, and using also $(\mathrm{H2})_{(\nu)}$, Hölder's inequality and (50), we have:

$$\begin{aligned} |II| &\leq c \int_{B_R^+} |A^{(\nu)}(x, u^{(\nu)}) - A(x,u)| \cdot |Du^{(\nu)}|^p \, dx \\ &\leq c \Big(\int_{B_R^+} |A^{(\nu)}(x, u^{(\nu)}) - A(x,u)|^{\frac{q}{q-p}} \, dx \Big)^{\frac{q-p}{q}} \Big(\int_{B_R^+} |Du^{(\nu)}|^q dx \Big)^{\frac{p}{q}} \\ &\leq c(R) \Big(\int_{B_R^+} |A^{(\nu)}(x, u^{(\nu)}) - A(x,u)|^{\frac{q}{q-p}} \, dx \Big)^{\frac{q-p}{q}}, \end{aligned} \tag{55}$$

where the constant c depends only on p, λ and Λ, and the constant $c(R)$ has the same dependencies as the constant from (50). This term tends to 0 as $\nu \to \infty$ due to (53) and the uniform convergence of the $A^{(\nu)}$'s to A (noting also that the integrand in the last line is pointwise dominated by $(2\Lambda)^{\frac{q}{q-p}}$ in view of $(\mathrm{H2})_{(\nu)}$). For I we note that $v \mapsto \int_{B_R^+} (A(x,u) Dv \cdot Dv)^{p/2} dx$ is weakly lower semicontinuous on $W^{1,p}$ in view of the convexity of the integrand in Dv. Hence we can conclude from (54):

$$F(u, B_R^+) \leq \liminf_{\nu \to \infty} F^{(\nu)}(u^{(\nu)}, B_R^+) \quad \text{for } R < 1. \tag{56}$$

We now consider a given function $w \in W^{1,p}(B^+, \mathbb{R}^N)$ which satisfies $w = u$ on $B^+ \setminus B_R^+$, $w = h$ on D. As in the interior situation, the idea is to compare the $F^{(\nu)}$-energy of w with that of $u^{(\nu)}$. This can't be done directly due to the fact that the boundary values of w and those of $u^{(\nu)}$ do not coincide on D. This motivates the following construction: we map $\overline{B_R^+}$ onto $\overline{B_R}$ with a bi-Lipschitz transformation Φ in such a manner that $\Phi \colon \overline{B_R^+} \to \overline{B_R}$ is the identity on $\partial B_R \setminus D_R$, and such that $\Phi(D_R) = \{x \in \partial B_R : x_n < 0\}$. For $\rho \in (0, R)$ we choose a cut-off function $\eta \in C_0^1(B_R)$ with $0 \leq \eta \leq 1$, and $\eta \equiv 1$ in B_ρ, and $|\nabla \eta| < \frac{2}{R-\rho}$. We then define:

$$v^{(\nu)}(x) = w(x) + (1 - \eta(\Phi(x)))\,(u^{(\nu)}(x) - u(x)) \quad \text{for } x \in \overline{B_R^+}.$$

For $x \in D_R$ we then have $v^{(\nu)}(x) = h^{(\nu)}(x)$ and for $x \in \partial B_R^+ \setminus D_R$ we have $v^{(\nu)}(x) = u^{(\nu)}(x)$, meaning that there holds, via the local $F^{(\nu)}$-minimality of $u^{(\nu)}$:

$$F^{(\nu)}(u^{(\nu)}, B_R^+) \leq F^{(\nu)}(v^{(\nu)}, B_R^+)\,. \tag{57}$$

We next wish to estimate $F^{(\nu)}(v^{(\nu)}, B_R^+)$ in terms of $F^{(\nu)}(w, B_R^+)$. To this end we estimate, with the same argument as in the beginning of estimate (55):

$$\begin{aligned} & |F^{(\nu)}(v^{(\nu)}, B_R^+) - F(v^{(\nu)}, B_R^+)| \\ \leq\ & \int_{B_R^+} \Big| (A^{(\nu)}(x, v^{(\nu)}) Dv^{(\nu)} \cdot Dv^{(\nu)})^{\frac{p}{2}} - (A(x, v^{(\nu)}) Dv^{(\nu)} \cdot Dv^{(\nu)})^{\frac{p}{2}} \Big| \, dx \\ \leq\ & c \int_{B_R^+} \big| A^{(\nu)}(x, v^{(\nu)}) - A(x, v^{(\nu)}) \big| \, |Dv^{(\nu)}|^p \, dx \end{aligned} \tag{58}$$

where $c = c(p, \lambda, \Lambda)$ In order to control this, we begin by noting:

$$|Dv^{(\nu)}| \leq |Dw| + |1 - \eta \circ \Phi| \cdot |Du^{(\nu)}| + |1 - \eta \circ \Phi| \cdot |Du| + |\nabla \eta \circ \Phi| \operatorname{Lip} \Phi |u - u^{(\nu)}|\,. \tag{59}$$

We have the following estimates:

$$\int_{B_R^+} \left|A(x, v^{(\nu)}) - A^{(\nu)}(x, v^{(\nu)})\right| \, |Dw|^p \, dx \to 0 \text{ as } \nu \to \infty$$

since the integrand tends to zero as ν tends to ∞, and is pointwise bounded by the L^1-function $2\Lambda|Dw|^p$ via $(\mathrm{H2})_{(\nu)}$; by the same reasoning there holds

$$\int_{B_R^+} \left|A(x, v^{(\nu)}) - A^{(\nu)}(x, v^{(\nu)})\right| \, |Du|^p \, dx \to 0 \text{ as } \nu \to \infty;$$

by $(\mathrm{H2})_{(\nu)}$ and (51) there holds:

$$\int_{B_R^+} \left|A(x, v^{(\nu)}) - A^{(\nu)}(x, v^{(\nu)})\right| |u^{(\nu)} - u|^p dx \le 2\Lambda \int_{B_R^+} |u^{(\nu)} - u|^p dx \to 0 \ \text{ as } \nu \to \infty;$$

and finally by using Hölder's inequality and arguing as in (55) we observe

$$\begin{aligned} & \int_{B_R^+} \left|A(x, v^{(\nu)}) - A^{(\nu)}(x, v^{(\nu)})\right| \, |Du^{(\nu)}|^p dx \\ \le \ & \Big(\int_{B_R^+} \left|A(x, v^{(\nu)}) - A^{(\nu)}(x, v^{(\nu)})\right|^{\frac{q}{q-p}} \, dx \Big)^{\frac{q-p}{q}} \Big(\int_{B_R^+} |Du^{(\nu)}|^q dx \Big)^{p/q} \to 0 \text{ as } \nu \to \infty. \end{aligned}$$

Combining these estimates with (58) and (59) we can conclude:

$$\left|F^{(\nu)}(v^{(\nu)}, B_R^+) - F(v^{(\nu)}, B_R^+)\right| \to 0 \quad \text{as } \nu \to \infty. \tag{60}$$

We wish to estimate $F(v^{(\nu)}, B_R^+)$ as $\nu \to \infty$, and to this end we define the two sets

$$\mathcal{B}_1 = \{x \in B_R^+ : \ \eta(\Phi(x)) = 0\}, \qquad \mathcal{B}_2 = \{x \in B_R^+ : \ \eta(\Phi(x)) > 0\}.$$

We then write, recalling the definition of w:

$$\begin{aligned} & \int_{B_R^+} (A(x, v^{(\nu)}) Dv^{(\nu)} \cdot Dv^{(\nu)})^{p/2} dx \\ = \ & \int_{\mathcal{B}_1} (A(x, w) Dw \cdot Dw)^{p/2} dx + \int_{\mathcal{B}_2} (A(x, v^{(\nu)}) Dv^{(\nu)} \cdot Dv^{(\nu)})^{p/2} dx \\ = \ & \int_{B_R^+} (A(x, w) Dw \cdot Dw)^{p/2} dx + I + II \,, \end{aligned} \tag{61}$$

where here

$$I = -\int_{\mathcal{B}_2} (A(x, w) Dw \cdot Dw)^{p/2} dx \,, \quad II = \int_{\mathcal{B}_2} (A(x, v^{(\nu)}) Dv^{(\nu)} \cdot Dv^{(\nu)})^{p/2} dx \,.$$

The term I can be estimated using $(\mathrm{H2})_{(\nu)}$ and the transformation rule via:

$$|I| \le \Lambda^{p/2} \int_{\mathcal{B}_2} |Dw|^p dx \le c \int_{B_R \setminus B_\rho} |D(w \circ \Phi^{-1})| \, dx$$

for a constant c depending only on p, Λ, $\operatorname{Lip}\Phi$ and $\operatorname{Lip}\Phi^{-1}$. The same argument can be applied to II, yielding:

$$|II| \le c \int_{B_R \setminus B_\rho} |D(v^{(\nu)} \circ \Phi^{-1})|^p dx$$

The integrand $|D(v^{(\nu)} \circ \Phi^{-1})|^p$ can be decomposed and controlled in a manner completely analogous to that used above to control $|Dv^{(\nu)}|^p$, yielding the estimate

$$\limsup_{\nu\to\infty} \big(|I| + |II|\big) \le c\Big[\int_{B_R \setminus B_\rho} |D(w \circ \Phi^{-1})|^p \, dx + (R^n - \rho^n)^{\frac{q-p}{q}}\Big] ,$$

for a constant c depending on the same parameters as the constant from (50) as well as on $\operatorname{Lip}\Phi$ and $\operatorname{Lip}\Phi^{-1}$. In particular the right-hand side approaches 0 as $\rho \nearrow R$. Combining this with (56), (57) and (60) we have the chain of inequalities

$$\begin{aligned} F(u, B_R^+) &\le \liminf_{\nu\to\infty} F^{(\nu)}(u^{(\nu)}, B_R^+) \le \liminf_{\nu\to\infty} F^{(\nu)}(v^{(\nu)}, B_R^+) = \liminf_{\nu\to\infty} F(v^{(\nu)}, B_R^+) \\ &\le \limsup_{\nu\to\infty} F(v^{(\nu)}, B_R^+) \le F(w, B_R^+) + o(R-\rho) , \end{aligned} \tag{62}$$

i.e. we have established $F(u, B_R^+) \le F(w, B_R^+)$, and hence that u is a local F-minimizer. If we choose $w = u$ in (62), we immediately obtain the inequality (49). □

Note that (49) yields, in the light of (54) and (55):

$$F(u, B_R^+) = \lim_{\nu\to\infty} \int_{B_R^+} (A(x,u) Du^{(\nu)} \cdot Du^{(\nu)})^{p/2} \, dx \, .$$

A similar argument shows:

$$F(u, B_R^+) = \lim_{\nu\to\infty} F(u^{(\nu)}, B_R^+) \, .$$

The compactness theorem yields a result concerning the limit of a sequence singular points of minimizers of the $F^{(\nu)}$'s. This result was shown in the special case $p = 2$ as [21, Lemma 3], and the proof for general p is very similar, but for completeness we include a sketch here.

Lemma 5.2. *Under the same conditions as Lemma* 5.1*, consider a sequence of points* $\{x^{(\nu)}\}$ *in* $B^+ \cap D$ *such that* $x^{(\nu)}$ *is a singular point of* $u^{(\nu)}$*, and* x_0 *such that* $x^{(\nu)} \to x_0$ *as* $\nu \to \infty$*. Then* x_0 *is a singular point of* u*.*

Proof. The interior situation - i.e. the case $x_0 \in B^+$ - is considered for $p = 2$ in [14, Lemma 1], and for $p > 2$ in [9, Lemma 8.1]. In view of [1, Remark 3.5], the arguments of [9, Lemma 8.1] can be carried over to the interior situation in the case $1 < p < 2$. We hence consider $x_0 \in D$, and $R' \in (0, R)$ sufficiently such large that $x^{(\nu)} \in B_{R'}^+ \cup D_{R'}$.

In view of Theorem 4.2 and the uniform boundedness of the $||h^{(\nu)}||_{W^{1,s}}$ there exist positive constants ε_1 and R_1 independent of ν such that the fact that $x^{(\nu)}$ is a singular point of $u^{(\nu)}$ implies

$$\rho^{p-n} \int_{B_\rho(x^{(\nu)}) \cap B_R^+} |Du^{(\nu)}|^p dx > \varepsilon_1^p \quad \text{for all } \rho < \min\{R_1, R - R'\}.$$

The Caccioppoli inequality (13) (applied to $u^{(\nu)}$) thus yields

$$\rho^{-n}\int_{B_\rho(x^{(\nu)})\cap B_R^+}|u^{(\nu)}-h^{(\nu)}|^p dx>\varepsilon_2^p \quad \text{for all } \rho<R_2, \tag{63}$$

where $\varepsilon_2>0$ and $R_2\in(0,\frac{1}{2}\min\{R_1,R-R'\})$ are constants independent of ν. After passing to a suitable subsequence such that $u^{(\nu)}-h^{(\nu)}$ converges strongly to $u-h$ in $L^p(B^+_{R'+R_2},\mathbb{R}^N)$, we can pass to the limit in (63), to see

$$\rho^{-n}\int_{B_\rho^+(x)}|u-h|^p dx>\varepsilon_2^p \quad \text{for all } \rho<R_2\,,$$

meaning that x_0 is a singular point of u. □

The next result is a monotonicity inequality. In general we need to somewhat restrict the class of coefficients under consideration, but note that the monotonicity of the function Φ defined below follows trivially from the monotonicity of F for $p\geq n$ for any F of the form (1).

Lemma 5.3. *Consider fixed $p\in(1,n)$ and $R\in(0,1)$. Let u be a local minimizer for the functional $F(\,\cdot\,,B_R^+)$ in $W^{1,p}(B_R^+,\mathbb{R}^N)$, where here*

$$F(v,B_t^+)=\int_{B_t^+}[A(x,v)Dv\cdot Dv]^{p/2}\,dx=\int_{B_t^+}\left[G^{\alpha\beta}(x)g_{ij}(x,v)D_\alpha v^i D_\beta v^j\right]^{p/2}\,dx\,,$$

relative to the boundary condition $u\Big|_{D_R}=\xi$ for a given constant vector $\xi\in\mathbb{R}^N$. Here the coefficients G and g are assumed to be uniformly continuous on $B_R^+\times\mathbb{R}^N$, and to satisfy the ellipticity and boundedness conditions given by (47). We further assume that the associated modulus of continuity ω (cf. the proof of Theorem 4.2) satisfies

$$\int_0^\tau\frac{\omega(s)}{s}\,ds<\infty \tag{64}$$

for some (and hence for all) $\tau>0$. For $t\in(0,R]$, set

$$\Phi(t)=\exp\Big(c_1\int_0^t\frac{\omega(s^p)}{s}\,ds\Big)t^{p-n}F(u,B_t^+)\,,$$

for a suitable constant c_1 depending only on n, p and Λ. Then the function $t\mapsto\Phi(t)$ is nondecreasing on $(0,R]$.

In the case $p\geq 2$, this can be sharpened to

$$\Phi(\sigma)-\Phi(\rho)\geq c_2\int_{B_\sigma^+\setminus B_\rho^+}|x|^{p-n}\left|\left\langle\tfrac{x}{|x|},Du\right\rangle\right|^p dx \quad \text{for all } 0<\rho<\sigma\leq R \tag{65}$$

for a suitable constant c_2 with the same dependencies as c_1.

Proof. After a suitable change of coordinates in the domain (and possibly increasing the constant Λ) we can assume that there holds $G^{\alpha\beta}(0) = \delta^{\alpha\beta}$. We define $x_t = t\frac{x}{|x|}$ and set $u_t(x) = u(x_t)$ for $x \in B_R^+, 0 < t < R$. In particular u and u_t have the same boundary values on ∂B_t^+, so the fact that u is a local F-minimizer on B_R^+ means that there holds:

$$F(u, B_t^+) \leq F(u_t, B_t^+)\,. \tag{66}$$

We write

$$\begin{aligned}
&F(u_t, B_t^+)\\
=\;& \int_{B_t^+} \left[\delta^{\alpha\beta} g_{ij}(0, u_t(x)) D_\gamma u^i(x_t) D_\kappa u^j(x_t) \frac{t^2}{|x|^2}\left(\delta_{\alpha\gamma} - \frac{x_\alpha x_\gamma}{|x|^2}\right)\left(\delta_{\beta\kappa} - \frac{x_\beta x_\kappa}{|x|^2}\right)\right]^{p/2} dx\\
+\;& \int_{B_t^+} \Bigg[\left[G^{\alpha\beta}(x) g_{ij}(x, u_t(x)) D_\gamma u^i(x_t) D_\kappa u^j(x_t) \frac{t^2}{|x|^2}\left(\delta_{\alpha\gamma} - \frac{x_\alpha x_\gamma}{|x|^2}\right)\left(\delta_{\beta\kappa} - \frac{x_\beta x_\kappa}{|x|^2}\right)\right]^{p/2}\\
&- \left[\delta^{\alpha\beta} g_{ij}(0, u_t(x)) D_\gamma u^i(x_t) D_\kappa u^j(x_t) \frac{t^2}{|x|^2}\left(\delta_{\alpha\gamma} - \frac{x_\alpha x_\gamma}{|x|^2}\right)\left(\delta_{\beta\kappa} - \frac{x_\beta x_\kappa}{|x|^2}\right)\right]^{p/2}\Bigg] dx\\
=\;& I + II\,,
\end{aligned}$$

with the obvious labelling. In order to estimate I, we begin by noting that the coarea formula yields, for functions f defined on $S_t^+ = \{x \in \partial B_t^+ \,:\, x_n > 0\}$:

$$\int_{B_t^+} |x|^{-p} f\left(t\tfrac{x}{|x|}\right) dx = \frac{t^{1-p}}{n-p} \int_{S_t^+} f(y)\, d\mathcal{H}^{n-1}(y)\,. \tag{67}$$

We further note that there holds:

$$g_{ij}(y,\xi)\left(\delta_{\gamma\kappa} - \frac{x_\gamma x_\kappa}{|x|^2}\right) w_\gamma^i \widetilde{w}_\kappa^j = g_{ij}(y,\xi)\left(\delta_{\alpha\gamma} - \frac{x_\alpha x_\gamma}{|x|^2}\right)\left(\delta_{\alpha\kappa} - \frac{x_\alpha x_\kappa}{|x|^2}\right) w_\gamma^i \widetilde{w}_\kappa^j \geq 0 \tag{68}$$

for all $y \in B^+$, $\xi \in \mathbb{R}^N$ and $\widetilde{w}$, $w \in \mathbb{R}^{nN}$. Thus we can rewrite I as

$$\begin{aligned}
I &= \int_{B_t^+} \left[g_{ij}(0, u_t) \frac{t^2}{|x|^2}\left(\delta_{\gamma\kappa} - \frac{x_\gamma x_\kappa}{|x|^2}\right) D_\gamma u_t^i D_\kappa u_t^j\right]^{p/2} dx\\
&= \frac{t}{n-p} \int_{S_t^+} \left[g_{ij}(0, u)\left(\delta_{\gamma\kappa} - \frac{x_\gamma x_\kappa}{|x|^2}\right) D_\gamma u^i D_\kappa u^j\right]^{p/2} d\mathcal{H}^{n-1}\,.
\end{aligned} \tag{69}$$

Further from (47) we have the inequality

$$g_{ij}(0, u) \frac{x_\gamma x_\kappa}{|x|^2} D_\gamma u^i D_\kappa u^j \geq \left|\left\langle \tfrac{x}{|x|}\,, Du\right\rangle\right|^2. \tag{70}$$

For $p \geq 2$ we can continue to estimate from (69) (keeping in mind (70) and the elementary inequality $(a-b)^{p/2} \leq a^{p/2} - b^{p/2}$ valid for $0 < b \leq a$ and $p \geq 2$):

$$I \leq \frac{t}{n-p}\left(\int_{S_t^+} \left[g_{ij}(0,u) D_\gamma u^i D_\gamma u^j\right]^{p/2} d\mathcal{H}^{n-1} - \int_{S_t^+} \left|\left\langle \tfrac{x}{|x|}\,, Du\right\rangle\right|^p d\mathcal{H}^{n-1}\right). \tag{71}$$

We decompose:

$$\int_{S_t^+}[g_{ij}(0,u)D_\gamma u^i D_\gamma u^j]^{p/2}d\mathcal{H}^{n-1} = \int_{S_t^+}[G^{\alpha\beta}(x)g_{ij}(x,u)D_\alpha u^i D_\beta u^j]^{p/2}d\mathcal{H}^{n-1}$$

$$+ \quad \int_{S_t^+}\Big([\delta^{\alpha\beta}g_{ij}(0,u)D_\alpha u^i D_\beta u^j]^{p/2} - [G^{\alpha\beta}(x)g_{ij}(x,u)D_\alpha u^i D_\beta u^j]^{p/2}\Big)d\mathcal{H}^{n-1} = III + IV$$

with the obvious labelling. Using in turn the elementary estimate $|s^{\frac{p}{2}} - t^{\frac{p}{2}}| \le \frac{p}{2}|s-t|(s^{\frac{p-2}{2}} + t^{\frac{p-2}{2}})$ for $s,t>0$ and (47) we estimate IV by

$$IV \le p\Lambda^{p-2}\omega(t^p)\int_{S_t^+}|Du|^p\,d\mathcal{H}^{n-1} \le p\Lambda^{p-2}\omega(t^p)\int_{S_t^+}[A(x,u)Du\cdot Du]^{p/2}\,d\mathcal{H}^{n-1}$$

and hence we obtain, for $c = c(p,\Lambda) = p\Lambda^{p-2}$,

$$I \le \frac{t}{n-p}\Big((1+c\,\omega(t^p))\int_{S_t^+}[A(x,u)Du\,\cdot\,Du]^{p/2}\,d\mathcal{H}^{n-1} - \int_{S_t^+}\Big|\Big\langle\tfrac{x}{|x|}\,,Du\Big\rangle\Big|^p\,d\mathcal{H}^{n-1}\Big)\,.$$

To estimate II we argue as in the above estimate of IV and deduce

$$II \le ct^p\omega(t^p)\int_{B_t^+}|x|^{-p}|Du_t(x)|^p\,dx\,.$$

On applying (67) and then (47) we arrive at the estimate:

$$II \le \frac{ct}{n-p}\omega(t^p)\int_{S_t^+}|Du|^p\,d\mathcal{H}^{n-1} \le \frac{ct}{n-p}\int_{S_t^+}[A(x,u)Du\cdot Du]^{p/2}\,d\mathcal{H}^{n-1}\,.$$

Combining the estimates for I and II we have

$$\begin{aligned}&\int_{B_t^+}[A(x,u)Du\cdot Du]^{p/2}\,dx\\ \le\;& \frac{t}{n-p}\Big[(1+c\omega(t^p))\int_{\partial B_t^+}[A(x,u)Du\cdot Du]^{p/2}\,d\mathcal{H}^{n-1} - \int_{S_t^+}\Big|\Big\langle\tfrac{x}{|x|}\,,Du\Big\rangle\Big|^p\,d\mathcal{H}^{n-1}\Big]\,.\end{aligned} \tag{72}$$

We now set

$$\varphi(t) = t^{p-n}\int_{B_t^+}[A(x,u)Du\cdot Du]^{p/2}\,dx\,,$$

and observe from (72) that there holds:

$$\varphi'(t) + \frac{c(n-p)\omega(t^p)\varphi(t)}{t} \ge \frac{t^{p-n}}{1+c\omega(t^p)}\int_{S_t^+}\Big|\Big\langle\tfrac{x}{|x|}\,,Du\Big\rangle\Big|^p\,d\mathcal{H}^{n-1}$$

where c is the constant from (72). Recalling the definition of Φ and the fact that $\omega \le 1$ we see that there holds:

$$\Phi'(t) \ge \frac{t^{p-n}}{1+c}\int_{S_t^+}\Big|\Big\langle\tfrac{x}{|x|}\,,Du\Big\rangle\Big|^p\,d\mathcal{H}^{n-1}\,. \tag{73}$$

Integrating (73) from ρ to σ yields the desired conclusion, i.e. (65).

In the case $1 < p \leq 2$ we see that the calculation as far as (70) remains valid. From this inequality and (68), we obtain in place of (71) the weaker estimate

$$I \leq \frac{t}{n-p} \int_{\partial B_t^+} [g_{ij}(0,u) D_\gamma u^i D_\gamma u^j]^{p/2} d\mathcal{H}^{n-1} .$$

The remainder of the proof carries through, if we replace the term $\int_{S_t^+} \left| \left\langle \frac{x}{|x|}\, , Du \right\rangle \right|^p d\mathcal{H}^{n-1}$ by 0 throughout. In particular (73) now reads $\Phi' \geq 0$, which immediately leads to the desired conclusion. □

Remark. If the coefficients are independent of x then the term II in the above calculations vanishes identically, and the desired monotonicity holds for $\Phi(t) = t^{p-n} \int_{B_t^+} \left[\delta^{\alpha\beta} g_{ij} D_\alpha u^i D_\beta u^j\right]^{p/2} dx$.

We are now in a position to prove full boundary regularity for a suitable class of minimizers. In the quadratic case this result was shown in [21, Theorem].

Theorem 5.4. *Consider fixed $p > 1$, $R \in (0,1)$, and a bounded Lipschitz-domain $\Omega \in \mathbb{R}^n$. Let u be a local minimizer for the functional $F(\,\cdot\,,\Omega)$ in $W^{1,p}(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$, where here, for $X \subset \Omega$, we define*

$$F(v,X) = \int_X \left[G^{\alpha\beta}(x) g_{ij}(x,v) D_\alpha v^i D_\beta v^j\right]^{p/2} dx ,$$

relative to the boundary condition $u\Big|_{\partial\Omega} = h$ for a given function $h \in W^{1,s}(\Omega,\mathbb{R}^N)$ for some $s > n$. Here the coefficients G and g are assumed to be uniformly continuous on $\Omega^+ \times \mathbb{R}^N$, and to satisfy the ellipticity and boundedness conditions given by (47). *We further assume that the associated modulus of continuity ω satisfied condition* (64). *Then u is Hölder continuous with Hölder-exponent $1 - \frac{n}{s}$ in some neighbourhood of $\partial\Omega$.*

Proof. We consider a fixed boundary point x_0. By a suitable bi-Lipschitz transformation $\mathbf{\Psi}$ we can map some neighbourhood $\overline{B}_{\rho_0}(x_0) \cap \overline{\Omega}$ of x_0 onto $\overline{B^+}$ such that x_0 maps to 0, $\overline{B}_{\rho_0}(x_0) \cap \partial\Omega$ to D. A straightforward calculation shows that the transformed function $\widetilde{u} = u \circ \mathbf{\Psi}^{-1} \in W^{1,p}(B^+,\mathbb{R}^N)$ is a local minimizer on B^+ of a functional which has coefficients satisfying conditions analogous to (47) and (64), relative to the boundary condition $\widetilde{u}\Big|_D = \widetilde{h} = h \circ \mathbf{\Psi}^{-1}$. Since $h \in W^{1,s}(\Omega,\mathbb{R}^N)$ we have $\widetilde{h} \in W^{1,s}(B^+,\mathbb{R}^N)$. This means that it suffices to show the desired Hölder continuity for the transformed function $\widetilde{u}$: equivalently, in view of the above remarks, we can restrict our attention to the case $\Omega = B^+$, $x_0 = 0$. As in the proof of Lemma 5.3 we can further assume that there holds $G^{\alpha\beta}(0) = \delta^{\alpha\beta}$. In view of Theorem 4.2 it suffices to show that 0 is not a singular point for u.

For each $\nu \in \mathbb{N}$ we define on B_ν^+ the rescaled functions

$$u^{(\nu)} = u(\tfrac{x}{\nu}), \qquad h^{(\nu)} = h(\tfrac{x}{\nu}),$$

and on $B_\nu^+ \cup D_\nu \times \mathbb{R}^N$ the rescaled coefficients

$$A^{(\nu)}(x,v) = A(\tfrac{x}{\nu}, v).$$

For $v \in W^{1,p}(B_\nu^+, \mathbb{R}^N)$, and $X \subset B_\nu^+$ we set

$$F^{(\nu)}(v,X) = \int_X [A^{(\nu)}(x,v)Dv \cdot Dv]^{p/2}\, dx$$

Consider now a fixed $\sigma > 0$. For all $\nu \geq \sigma$ we have that $u^{(\nu)}$ and $h^{(\nu)}$ are defined on B_σ^+. We calculate

$$\int_{B_\sigma^+} |Dh^{(\nu)}|^s\, dx = \nu^{n-s} \int_{B_{\sigma/\nu}^+} |Dh|^s\, dx \quad \to 0 \text{ as } \nu \to \infty\,. \tag{74}$$

Via Morrey's inequality we have the existence of a constant c depending on n and s such that $\sup_{B_\rho^+} |h - h(0)| \leq c_s \rho^{1-\frac{n}{s}} ||h||_{W^{1,s}}$. This means that there holds

$$\begin{aligned}
\int_{B_\sigma^+} |h^{(\nu)}|^s\, dx &= \nu^n \int_{B_{\sigma/\nu}^+} |h|^s\, dx \\
&\leq \nu^n \int_{B_{\sigma/\nu}^+} \left[|h(0)| + c\left(\tfrac{\sigma}{\nu}\right)^{1-\frac{n}{s}} \|h\|_{W^{1,s}}\right]^s dx \\
&\leq c(n,s)\sigma^n \left(|h(0)|^s + \|h\|_{W^{1,s}}^s \sigma^{s-n}\nu^{n-s}\right)\,.
\end{aligned} \tag{75}$$

In view of (74) and (75) we see that $h^{(\nu)} \to h_\infty$ strongly in $W^{1,s}(B_\sigma^+, \mathbb{R}^N)$ for some constant vector h_∞: from (75) we see that there in fact holds $h_\infty = h(0)$.

Further in view of (47) and (64) the coefficients $A^{(\nu)}$ satisfy conditions **(H1)$_{(\nu)}$** to **(H3)$_{(\nu)}$** of Lemma 5.1 (replacing λ by 1, and Λ by Λ^2), and converge uniformly on $B_\sigma^+ \times \mathbb{R}^N$ to coefficients $A^{(\infty)}$ given by

$$A_{ij}^{\alpha\beta(\infty)}(x,v) = A_{ij}^{\alpha\beta}(0,v) = \delta^{\alpha\beta} g_{ij}(0,v).$$

From the boundedness of u we see that the $u^{(\nu)}$'s are uniformly bounded in $L^p(B_\sigma^+, \mathbb{R}^N)$, and hence after passing to a weakly convergent subsequence we can apply Lemma 5.1 and a suitable diagonalization argument to deduce the existence of $\varphi \in W^{1,p}(\mathbb{R}_+^n, \mathbb{R}^N)$ with $\varphi\Big|_{\partial\mathbb{R}_+^n} = h_\infty$ which is a local minimizer of the functional

$$F^{(\infty)}(v, B_\sigma^+) = \int_{B_\sigma^+} \left[\delta^{\alpha\beta} g_{ij}(0,v) D_\alpha v^i D_\beta v^j\right]^{p/2}\, dx$$

for all $\sigma > 0$. If 0 is a singular point of u it is a singular point of each of the $u^{(\nu)}$'s and hence, by Lemma 5.2, of φ: thus the proof of the theorem will be completed if we show that 0 is a regular point for φ.

We begin by noting, from Lemma 5.1:

$$\sigma^{p-n} F^{(\infty)}(\varphi, B_\sigma^+) = \lim_{\nu\to\infty} \sigma^{p-n} F^{(\nu)}(u^{(\nu)}, B_\sigma^+) = \lim_{\nu\to\infty} (\sigma/\nu)^{p-n} F(u, B_{\sigma/\nu}^+)$$

for all $\sigma > 0$: in particular the limit on the right-hand side exists and is finite by the monotonicity formula from Lemma 5.3. This implies that the term on the left-hand side is constant, i.e. independent of σ. In particular there holds $\frac{d}{d\rho}(\rho^{p-n}F^{(\infty)}(\varphi, B_\rho^+)) = 0$, i.e. (writing $g_{ij}(\xi)$ for $g_{ij}(0,\xi)$)

$$\int_{B_\rho^+}(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j)^{p/2}\,dx = \frac{\rho}{n-p}\int_{\partial B_\rho^+}(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j)^{p/2}d\mathcal{H}^{n-1}.$$

Since φ is locally $F^{(\infty)}$-minimizing we have from Lemma 5.3, in particular from (69) (keeping in mind the remark immediately following the lemma) and using also (70):

$$\int_{B_\rho^+}(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j)^{p/2}dx \le \frac{\rho}{n-p}\int_{\partial B_\rho^+}\left(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j - \left|\frac{\partial\varphi}{\partial r}\right|^2\right)^{p/2}d\mathcal{H}^{n-1}\,.$$

By combining these last two results we see immediately that there holds $\frac{\partial\varphi}{\partial r} = 0$, i.e. φ is homogeneous of degree 0 on $\mathbb{R}^n_+$. (Such a function is called a *minimizing tangent map*, cf. [25, Chapter 3]).

The final step in the proof consists in showing that φ must be constant. To this end we consider the one parameter family of diffeomorphisms $\{\Phi_t\}_{t\ge 0}$ on $\mathbb{R}^n$, where

$$\Phi_t(x) = x + t(1-|x|)e_n.$$

For $t \ge 0$ we define $C_t = \{x \in \mathbb{R}^n : 0 \le x_n \le t(1-(|x|^2 - x_n^2)^{1/2})$ (i.e. C_t is the cone with base D and apex $(0,\dots,0,t)$). Then $\Phi_t(\overline{B^+}) = \overline{B^+}\setminus C_t$, so the 1-parameter family of functions $\{\varphi_t\}_{t\ge 0}$, with

$$\varphi_t(x) = \begin{cases} \varphi(\Phi_t^{-1}(x)) & x \in \Phi_t(\overline{B^+}) \\ \varphi(0) = h_\infty & x \in C_t \end{cases}$$

is well defined. Further there holds $\varphi_t\Big|_{\partial B_t^+} = \varphi = h(0)$. Then we calculate

$$\begin{aligned} F^{(\infty)}(\varphi_t, B^+) &= \int_{\Phi_t B^+}(g_{ij}(\varphi_t)D_\alpha\varphi_t^i D_\alpha\varphi_t^j)^{p/2}dx \\ &= \int_{B^+}\left(A(\varphi)D\varphi(D\Phi_t)^{-1}\cdot D\varphi(D\Phi_t)^{-1}\right)^{p/2}|\det D\Phi_t|dx\,, \end{aligned}$$

where we have abbreviated $A(\xi) = (\delta^{\alpha\beta}g_{ij}(\xi))$. Since φ is a local minimizer for $F^{(\infty)}$, there holds:

$$\frac{d}{dt}\Big|_{0^+}\int_{B^+}\left(A(\varphi)D\varphi(D\Phi_t)^{-1}\cdot D\varphi(D\Phi_t)^{-1}\right)^{p/2}|\det D\Phi_t|dx \ge 0\,, \tag{76}$$

assuming of course that the one-sided derivative on the left-hand side exists. In order to calculate the left-hand side, we begin by observing:

$$D\Phi_t(x) = \mathrm{Id} - t\frac{x}{|x|}\otimes e_n, \quad (D\Phi_t(x))^{-1} = \mathrm{Id} + t\frac{x}{|x|}\otimes e_n + O(t^2).$$

We then calculate

$$\frac{d}{dt}\Big|_{0^+} A(\varphi)D\varphi(D\Phi_t)^{-1}\cdot D\varphi(D\Phi_t)^{-1} = 2A(\varphi)D\varphi\cdot D\varphi\frac{x}{|x|}\otimes e_n = 2A(\varphi)D\varphi\frac{x}{|x|}\cdot D\varphi\, e_n = 0\,,$$

where here the first equality follows by the symmetry of A, the second by the matrix identity $B\cdot Cx\otimes y = Bx\cdot Cy$, and the last from the degree zero homogeneity of φ. Hence there holds:

$$\begin{aligned} &\frac{d}{dt}\Big|_{0^+}\Big[\Big(A(\varphi)D\varphi(D\Phi_t)^{-1}\cdot D\varphi(D\Phi_t)^{-1}\Big)^{p/2}|\det D\Phi_t|\Big] \\ =\quad &(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j)^{p/2}\frac{d}{dt}\Big|_0|\det D\Phi_t|dx. \end{aligned}$$

From (76) and keeping in mind the above calculation for $D\Phi_t$, we thus have

$$0\le\int_{B^+}(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j)^{p/2}\frac{d}{dt}\Big|_{0^+}|\det D\Phi_t|dx = -\int_{B^+}(g_{ij}(\varphi)D_\alpha\varphi^i D_\alpha\varphi^j)^{p/2}\frac{x^n}{|x|}dx\le 0\,.$$

This shows that φ must be constant, and completes the proof. □

Appendix: A Gehring-type lemma

The aim of this section is to prove Theorem 2.4.

For a closed subset $A\subset\overline{\Omega}$ and $h\in L^1(\Omega\setminus A)$ we define the maximal functions on $\Omega\setminus A$ by:

$$M_{d(x,A)}h(x) = \sup_{0\le\rho\le d(x,A)} ⨍_{B_\rho(x)\cap\Omega} h\,dx \quad\text{and}\quad \widetilde{M}_{\frac12 d(x,A)}h(x) = \sup_{\substack{0\le\rho\le\frac12 d(x,A)\\ y\in B_\rho(x)\cap\Omega}} ⨍_{B_\rho(y)\cap\Omega} h\,dx.$$

Here $d(\cdot,A)$ denotes the distance function to A; we adhere to the convention $d(x,\emptyset)=\infty$ for any $x\in\mathbb{R}^n$. Using (8) it is straightforward to show that there holds

$$\widetilde{M}_{\frac12 d(x,A)}h(x)\le c_3 M_{d(x,A)}h(x) \tag{77}$$

for all $x\in\Omega\setminus A$ and $h\in L^1(\Omega\setminus A)$, where we choose $c_3 = 5^n\,\frac{\alpha_n}{k_\Omega}$ (indeed, by (8), $2^n\,\frac{\alpha_n}{k_\Omega}$ suffices).

We will need the following Calderon-Zygmund covering argument.

Lemma A.1. *Consider $z\in\overline{\Omega}$, $r>0$ and a nonnegative function $h\in L^1(B_r(z)\cap\Omega)$ with $⨍_{B_r(z)\cap\Omega} h\,dx < s$. Then for any $\tau\ge 1$ there exists a countable family of pairwise disjoint balls $\{B_{\rho_k}(x_k)\}$ such that*

(i) $\rho_k\le\frac{1}{5\tau}d(x_k,\Omega\setminus B_r(z))$;

(ii) $\widetilde{M}_{\frac12 d(x,\Omega\setminus B_r(z))}h(x)\ge c_3^2 s$ *for any* $x\in\bigcup_{k=1}^\infty B_{5\tau\rho_k}(x_k)\cap\Omega$,

(iii) $h(x)\le c_3^2 s$ *for almost all* $x\in(B_r(z)\cap\Omega)\setminus\bigcup_{k=1}^\infty B_{5\rho_k}(x_k)$;

(*iv*) $\sum_{k=1}^{\infty} \mathcal{L}^n(B_{\rho_k}(x_k) \cap \Omega) \le \frac{1}{s} \int_{B_r(z)\cap\Omega} h\,dx;$

(*v*) $⨍_{B_{5\rho_k}(x_k)\cap\Omega} h\,dx \le c_4\, s$ *for any* $x_k \in B_{\frac{r}{2}}(z)$*, where the constant* c_4 *depends only on* τ*,* n *and* k_Ω*. (In the case* $B_r(z) \cap \Omega = \Omega$ *this is valid for any* x_k*.)*

The constant $c_3 = 5^n \frac{\alpha_n}{k_\Omega}$ *is from* (77).

Proof. See [2, Chapter 3, Theorem 2.2], [20, Lemma 2] for the proof in general doubling metric measure spaces, i.e. in (quasi-) metric spaces that carry a doubling measure.

Let $d(x) = d(x, \Omega \setminus B_r(z))$. The set $A_s = \{x \in B_r(z) \cap \Omega : \widetilde{M}_{d/2}h(x) > c_3^2 s\}$ is open by the lower semicontinuity of $\widetilde{M}_{d/2}h$. The definition of $\widetilde{M}_{d/2}h$ and Vitali's covering theorem yield the estimate:

$$\mathcal{L}^n(A_s) \le \frac{1}{s} \int_{B_r(z)\cap\Omega} h\,dx < \mathcal{L}^n(B_r(z) \cap \Omega), \tag{78}$$

which implies that $A_s \neq B_r(z) \cap \Omega$. If A_s is empty there is nothing to show. Otherwise we apply a Whitney–type covering argument. For this purpose we set $\rho(x) = \frac{1}{5\tau}\, d(x, \Omega \setminus A_s)$. Note $\rho(x) > 0$ for $x \in A_s$ and so by Vitali's covering theorem we can extract a countable and pairwise disjoint subfamily $B_{\rho_k}(x_k) \cap \Omega \subset A_s$ from the cover $\{B_{\rho(x)}(x) \cap \Omega : x \in A_s\}$ such that (by the choice of $\rho(x)$):

$$A_s = \bigcup_k B_{5\tau\rho_k}(x_k) \cap \Omega \qquad \text{and} \qquad \rho_k = \frac{1}{5\tau} d(x_k, \Omega \setminus A_s) \le \frac{1}{5\tau} d(x_k, \Omega \setminus B_r(z)).$$

From this construction we see immediately that (i) and (ii) hold. infer Since $\widetilde{M}_{d/2}h(x) > c_3^2 s$ for any $x \in A_s$ we see by Lebesgue's differentiation theorem (which holds since $\mathcal{L}^n\big|_\Omega$ is a doubling measure) that there holds $h(x) = \lim_{\rho \searrow 0} ⨍_{B_\rho(x)\cap\Omega} h\,dx \le \widetilde{M}_{d/2}h(x) \le c_3^2 s$ for almost all $x \in (B_r(z) \cap \Omega) \setminus A_s = (B_r(z) \cap \Omega) \setminus \bigcup_k B_{5\rho_k}(x_k)$: this shows (iii). From (78) we have

$$\sum_k \mathcal{L}^n\big((B_{\rho_k}(x_k) \cap \Omega\big) \le \mathcal{L}^n(A_s) \le \frac{1}{s} \int_{B_r(z)\cap\Omega} h\,dx\,,$$

which proves (iv).

To prove (v) we observe that $B_{20\tau\rho_k}(x_k) \cap \Omega$ has a non-empty intersection with $\Omega \setminus A_s$. There are two possible cases which can occur:

Case I. $B_{20\tau\rho_k}(x_k) \cap \Omega \subset B_r(z) \cap \Omega$. We note that this case always occurs if $B_r(z) \cap \Omega = \Omega$. Then there exists $y_k \in (B_r(z) \cap \Omega) \setminus A_s$ with $d(x_k, y_k) \le 10\tau\rho_k$. This implies

$$\begin{aligned} ⨍_{B_{5\rho_k}(x_k)\cap\Omega} h\,dx &\le \frac{\mathcal{L}^n(B_{10\tau\rho_k}(x_k) \cap \Omega)}{\mathcal{L}^n(B_{5\rho_k}(x_k) \cap \Omega)} ⨍_{B_{10\tau\rho_k}(x_k)\cap\Omega} h\,dx \\ &\le (2\tau)^n \frac{\alpha_n}{k_\Omega} \widetilde{M}_{d/2}h(y_k) \le (2\tau)^n \frac{\alpha_n}{k_\Omega} c_3^2 s. \end{aligned}$$

Case II. $B_{20\tau\rho_k}(x_k) \cap (\Omega \setminus B_r(z)) \neq \emptyset$. Then, for points x_k with $d(x_k, z) \leq \frac{r}{2}$ we have $20\tau\rho_k \geq d(x_k, \Omega \setminus B_r(z)) \geq \frac{r}{2}$ which immediately yields:

$$\fint_{B_{5\rho_k}(x_k)\cap\Omega} h\,dx \leq \frac{\mathcal{L}^n(B_r(z)) \cap \Omega)}{\mathcal{L}^n(B_{5\rho_k}(x_k) \cap \Omega)} \fint_{B_r(z)\cap\Omega} h\,dx \leq (8\tau)^n \frac{\alpha_n}{k_\Omega}\, s.$$

Combining both cases yields the result with $c_4 = c_4(n, \tau, k_\Omega)$. □

We will also require the following technical result concerning Lebesgue–Stieltjes integration.

Lemma A.2. *Consider functions h_1 and $h_2 : [1, \infty) \to [0, \infty)$ which are monotone non-increasing, with $\lim_{t\to\infty} h_1(t) = \lim_{t\to\infty} h_2(t) = 0$, and such that there holds*

$$-\int_\sigma^\infty s^{p_1}\,dh_1(s) \leq a\sigma^{p_1}[h_1(\sigma) + h_2(\sigma)]$$

for all $\sigma \in [\sigma_0, \infty)$, for constants $a > 1$, $p_1 > 0$ and $\sigma_0 \geq 1$. Then for any exponent $p_2 \in [p_1, \frac{a}{a-1}p_1)$ there holds

$$\begin{aligned} &-\int_{\sigma_0}^\infty s^{p_2}\,dh_1(s) \\ \leq\ & \frac{p_1\sigma_0^{p_2-p_1}}{ap_1 - (a-1)p_2}\left(-\int_{\sigma_0}^\infty s^{p_1}\,dh_1(s)\right) + \frac{a(p_2-p_1)}{ap_1-(a-1)p_2}\left(-\int_{\sigma_0}^\infty s^{p_2}\,dh_2(s)\right). \end{aligned}$$

Proof. The result is given in [11, Chapter V, Lemma 1.2] for the case $\sigma_0 = 1$, and the result as stated here follows after a simple rescaling argument. See also [26, Lemma], and cf. [10, Lemma 1]. □

We are now in a position to proceed to the

Proof of Theorem 2.4. The result is immediate for $g \equiv 0$, so we henceforth assume that $g \not\equiv 0$. We also assume without loss of generality that $b \geq 1$ and fix constants τ, δ, $\sigma > 1$ such that $\delta \leq \frac{\sigma}{33}$ and $\frac{\tau+\delta}{\tau-1} \leq \frac{\delta}{2}$ (for example, $\tau = 10$, $\delta = 3$ and $\sigma = 100$). We define

$$\Gamma = \left(\fint_\Omega |g|^p dx\right)^{1/p} + \left(\fint_\Omega |f|^p dx\right)^{1/p},$$

and define rescalings of g and f via:

$$G(x) = \frac{g(x)}{\Gamma}, \qquad F(x) = \frac{f(x)}{\Gamma}, \qquad \widetilde{G}(x) = \frac{\widetilde{g}(x)}{\Gamma}, \quad \text{and} \quad \widetilde{F}(x) = \frac{\widetilde{f}(x)}{\Gamma},$$

where $\widetilde{f}(x) = \frac{\mathcal{L}^n(B_{d(x,A)}(x)\cap\Omega)}{\mathcal{L}^n(\Omega)}\, f(x)$. Note that in particular this means $\fint_\Omega |G|^p dx \leq 1$, $\fint_\Omega |F|^p dx \leq 1$ and $\fint_\Omega |F + G| dx \leq 1$.

In the case $A \neq \emptyset$ we start by applying a Whitney-type decomposition argument to the open set $\Omega \setminus A$. By Vitali's covering theorem, the family $\{B_{\frac{1}{5\sigma}d(x,A)}(x) \cap \Omega : x \in \Omega \setminus A\}$ contains a countable, pairwise disjoint subfamily $\{B_{\rho_k}(x_k) \cap \Omega\}$ such that

$$\Omega \setminus A = \bigcup_k B_{5\sigma\rho_k}(x_k) \cap \Omega\,, \tag{79}$$

where $\rho_k = \frac{1}{5\sigma}\, d(x_k, A)$. In particular, for any $x \in B_{5\delta\rho_k}(x_k)$ we have

$$5(\sigma - \delta)\rho_k \leq d(x, A) \leq 5(\sigma + \delta)\rho_k. \tag{80}$$

For fixed $x \in \Omega \setminus A$ we define $M = \#\{k : x \in B_{5\delta\rho_k}(x_k) \cap \Omega\}$. In order to bound M from above we observe that for $x \in B_{5\delta\rho_k}(x_k)$ we have $|x - x_k| \leq 5\delta\rho_k \leq \frac{\delta}{\sigma} d(x_k, A) \leq \frac{\delta}{\sigma}(|x - x_k| + d(x, A))$ which implies $|x - x_k| \leq \frac{\delta}{\sigma - \delta}\, d(x, A)$. On the other hand we infer from (80) that $|x_k - x_j| \geq \frac{d(x,A)}{5(\sigma+\delta)}$ for any $x \in B_{5\delta\rho_k}(x_k) \cap B_{5\delta\rho_j}(x_j)$, $k \neq j$. Therefore M is bounded by the maximal number of points in $B_{\frac{\delta}{\sigma-\delta}\, d(x,A)}(x) \cap \Omega$ with pairwise distance $\geq \frac{d(x,A)}{5(\sigma+\delta)}$. Hence there holds

$$\begin{aligned}
\mathcal{L}^n\big(B_{\frac{2\delta}{\sigma-\delta}\, d(x,A)}(x) \cap \Omega\big) &\geq \sum_{k=1}^{M} \mathcal{L}^n\Big(B_{\frac{d(x,A)}{10(\sigma+\delta)}}(x_k) \cap \Omega\Big) \\
&\geq \frac{M\, k_\Omega\, (\sigma - \delta)^n}{\alpha_n (20\delta)^n (\sigma + \delta)^n} \mathcal{L}^n\Big(B_{\frac{2\delta}{\sigma-\delta}\, d(x,A)}(x) \cap \Omega\Big)\,,
\end{aligned}$$

which implies

$$M = \#\{k : x \in B_{5\delta\rho_k}(x_k) \cap \Omega\} \leq c_5 = \frac{\alpha_n}{k_\Omega}\, (20\delta)^n \left(\frac{\sigma + \delta}{\sigma - \delta}\right)^n. \tag{81}$$

Using the Ahlfors condition (8), (80) and since $⨍_\Omega |G|^p\, dx \leq 1$ we conclude for $t_0 = \left(\frac{\alpha_n}{k_\Omega}\right)^p \left(\frac{\sigma+\delta}{\delta}\right)^{np}$:

$$⨍_{B_{5\delta\rho_k}(x_k) \cap \Omega} |\widetilde{G}|^p\, dx \leq \frac{\alpha_n^p (5(\sigma + \delta)\rho_k)^{np}}{k_\Omega (5\delta\rho_k)^n\, \mathcal{L}^n(\Omega)^{p-1}} ⨍_\Omega |G|^p\, dx \leq t_0\,. \tag{82}$$

Now, let $x \in B_{5\delta\rho_k}(x_k)$. For ρ such that $B_{\rho/2}(x) \cap \Omega \subset B_{5\delta\rho_k}(x_k) \cap \Omega \subset \Omega \setminus A$ (with the property that $\sup_{y \in B_{\rho/2}(x)} |y - x| = \frac{\rho}{2}$) we have $\frac{\rho}{2} \leq 10\delta\rho_k$. This yields the following inclusions (recalling $\delta \leq \frac{\sigma}{33}$):

$$B_{8\rho}(x) \cap \Omega \subset B_{8\rho + 5\delta\rho_k}(x_k) \cap \Omega \subset B_{165\delta\rho_k}(x_k) \cap \Omega \subset \Omega \setminus A\,,$$

and

$$B_\rho(x) \cap \Omega \subset B_{25\delta\rho_k}(x_k) \cap \Omega\,.$$

On $B_{\rho/2}(x)\cap\Omega$ we have that $d(\cdot,A)\le 5(\sigma+\delta)\rho_k$ (see (80)), while on $B_\rho(x)\cap\Omega$ we have $d(\cdot,A)\ge 5(\sigma-5\delta)\rho_k$. Using these estimates together with (9) we obtain

$$\begin{aligned}
\fint_{B_{\rho/2}(x)\cap\Omega}|\widetilde{G}|^p\,dx &\le \frac{\alpha_n^p(5(\sigma+\delta)\rho_k)^{np}}{\mathcal{L}^n(\Omega)^p}\fint_{B_{\rho/2}(x)\cap\Omega}|G|^p\,dx\\
&\le \frac{\alpha_n^p(5(\sigma+\delta)\rho_k)^{np}}{\mathcal{L}^n(\Omega)^p}\,b^p\left[\left(\fint_{B_\rho(x)\cap\Omega}|G|dx\right)^p+\fint_{B_\rho(x)\cap\Omega}|F|^pdx\right]\\
&\le B^p\left[\left(\fint_{B_\rho(x)\cap\Omega}|\widetilde{G}|dx\right)^p+\fint_{B_\rho(x)\cap\Omega}|\widetilde{F}|^pdx\right],
\end{aligned}$$

where $B=\frac{\alpha_n}{k_\Omega}\left(\frac{\sigma+\delta}{\sigma-5\delta}\right)^n b$. Since $B_{8\rho}(x)\cap\Omega\subset\Omega\setminus A$ this implies

$$M_{d(x,\Omega\setminus B_{5\delta\rho_k}(x_k))}|\widetilde{G}|^p(x)\le B^p\Big(\big(M_{\frac18 d(x,A)}|\widetilde{G}|(x)\big)^p+M_{\frac18 d(x,A)}|\widetilde{F}|^p(x)\Big) \tag{83}$$

for almost all $x\in B_{5\delta\rho_k}(x_k)\cap\Omega$. We note that in the case $A=\emptyset$ we can consider Ω instead of the family $\{B_{5\rho_k}(x_k)\}$ and set $t_0=1$ and $B=b$.

We now consider a fixed $t\ge t_0$, and define parameters $\beta=c_3^{2/p}$ and $r=\frac{2p}{p-1}\,Bc_3^{3/p}\,t$. Note in particular that $\beta\ge 1$ and $r>t>1$. Applying Lemma A.1 with $z=x_k$, $r=5\delta\rho_k$, $h=|\widetilde{G}|^p$, $s=r^p$ (note that $\fint_{B_{5\delta\rho_k}\cap\Omega(x_k)}|\widetilde{G}|^pdx\le t_0\le t<r\le r^p$) we deduce, for each k, the existence of a family of pairwise disjoint balls $\{B_{\rho_{kj}}(x_{kj})\}$ with each $x_{kj}\in B_{5\delta\rho_k}(x_k)$ such that:

$$\rho_{kj}\le\frac{1}{5\tau}d(x_k,\Omega\setminus B_{5\delta\rho_k}(x_k))\,, \tag{84}$$

$$\widetilde{M}_{\frac12 d(x,\Omega\setminus B_{5\delta\rho_k}(x_k))}|\widetilde{G}|^p(x)\ge c_3^2\,r^p=(\beta r)^p\quad\text{for any }x\in\bigcup_j B_{5\tau\rho_{kj}}(x_{kj})\cap\Omega\,, \tag{85}$$

$$|\widetilde{G}(x)|^p\le c_3^2\,r^p=(\beta r)^p\quad\text{for almost all }x\in(B_{5\delta\rho_k}(x_k)\cap\Omega)\setminus\bigcup_{j=1}^\infty B_{5\rho_{kj}}(x_{kj}), \tag{86}$$

$$\sum_{j=1}^\infty\mathcal{L}^n\big(B_{\rho_{kj}}(x_{kj})\cap\Omega\big)\le\frac{1}{r^p}\int_{B_{5\delta\rho_k}(x_k)\cap\Omega}|\widetilde{G}|^pdx\,,\quad\text{and} \tag{87}$$

$$\fint_{B_{5\rho_{kj}}(x_{kj})\cap\Omega}|\widetilde{G}|^pdx\le\frac{c_4}{c_3^2}(\beta r)^p\quad\text{for any }x_{kj}\in B_{\frac52\delta\rho_k}(x_k), \tag{88}$$

where c_4 is the constant from Lemma A.1, (v).

Now for points x_{kj} with with the property that $B_{5\rho_{kj}}(x_{kj})\cap B_{5\rho_k}(x_k)\ne\emptyset$ we have, using (84);

$$|x_{kj}-x_k|\le 5\rho_{kj}+5\rho_k\le\frac{1}{\tau}\,d(x_{kj},\Omega\setminus B_{5\delta\rho_k}(x_k))+5\rho_k\le\frac{1}{\tau}|x_k-x_{kj}|+\frac{5\delta\rho_k}{\tau}+5\rho_k\,,$$

which immediately yields

$$|x_{kj}-x_k|\le 5\frac{\tau+\delta}{\tau-1}\,\rho_k\le\tfrac52\delta\rho_k$$

since $\frac{\tau+\delta}{\tau-1} \le \frac{\delta}{2}$. This implies in particular that (88) is valid for any pair (k,j) of indices such that $(k,j) \in \mathcal{I} := \{(k,j) : B_{5\rho_{kj}}(x_{kj}) \cap B_{5\rho_k}(x_k) \neq \emptyset\}$.

For $s \ge 0$ we set

$$A_s = \{x \in \Omega \setminus A : \, |\widetilde{G}(x)| > s\}\,.$$

Define $\mathbf{B} = \bigcup_{(k,j)\in\mathcal{I}} B_{5\rho_{kj}}(x_{kj}) \cap \Omega$. From (86) and (79) (which is also true with $\sigma = 1$) we infer the existence of a set $\mathcal{X}$ with $\mathcal{L}^n(\mathcal{X}) = 0$ such that $A_{\beta r} \setminus \mathcal{X} \subset \mathbf{B}$. By (88), the Ahlfors condition (8) (i.e. $\mathcal{L}^n(B_{5\rho_{kj}}(x_{kj}) \cap \Omega) \le c_3\, \mathcal{L}^n(B_{\rho_{kj}}(x_{kj} \cap \Omega))$ and since the balls $B_{\rho_{kj}}(x_{kj})$ are mutually disjoint for fixed k and overlap for different k at most c_5 times by (81), we deduce

$$\begin{aligned} \int_{A_{\beta r}} |\widetilde{G}|^p dx &\le \int_{\bigcup_{(k,j)\in\mathcal{I}} B_{5\rho_{kj}}(x_{kj})\cap\Omega} |\widetilde{G}|^p dx \le \frac{c_4}{c_3^2}\,(\beta r)^p \sum_{(k,j)\in\mathcal{I}} \mathcal{L}^n\big(B_{5\rho_{kj}}(x_{kj}) \cap \Omega\big) \\ &\le \frac{c_4}{c_3}\,(\beta r)^p \sum_{(k,j)\in\mathcal{I}} \mathcal{L}^n\big(B_{\rho_{kj}}(x_{kj}) \cap \Omega\big) \\ &\le \frac{c_4 c_5}{c_3}(\beta r)^p \mathcal{L}^n\Big(\bigcup_{(k,j)\in\mathcal{I}} B_{\rho_{kj}}(x_{kj}) \cap \Omega)\Big)\,. \end{aligned} \tag{89}$$

Since $r > t$ we have the inclusion $A_{\beta r} \subset A_{\beta t}$. From (89) we therefore obtain:

$$\begin{aligned} \int_{A_{\beta t}} |\widetilde{G}|^p dx &= \int_{A_{\beta t}\setminus A_{\beta r}} |\widetilde{G}|^{p-1}|\widetilde{G}|\, dx + \int_{A_{\beta r}} |\widetilde{G}|^p dx \\ &\le (\beta r)^{p-1} \int_{A_{\beta t}} |\widetilde{G}| dx + \frac{c_4 c_5}{c_3}(\beta r)^p \mathcal{L}^n\Big(\bigcup_{(k,j)\in\mathcal{I}} B_{\rho_{kj}}(x_{kj}) \cap \Omega)\Big). \end{aligned} \tag{90}$$

For any k and all $x \in \bigcup_{j=1}^{\infty} B_{5\rho_{kj}}(x_{kj}) \cap \Omega$ we apply (77) and (85) to infer:

$$M_{d(x,\Omega\setminus B_{5\delta\rho_k}(x_k))}|\widetilde{G}|^p(x) \ge \frac{1}{c_3} \widetilde{M}_{\frac12 d(x,\Omega\setminus B_{5\delta\rho_k}(x_k))}|\widetilde{G}|^p(x) \ge \frac{(\beta r)^p}{c_3} = B^p\, c_3^2\, \left(\frac{2p}{p-1}\right)^p (\beta t)^p\,. \tag{91}$$

We now define $\widehat{F}(x) = \big[M_{d(x,A)}(|\widetilde{F}|^p)(x)\big]^{1/p}$. Starting from (83), we deduce the following chain of inequalities:

$$\begin{aligned} & M_{d(x,\Omega\setminus B_{5\delta\rho_k}(x_k))}|\widetilde{G}|^p(x) \le B^p \left[\Big(M_{\frac18 d(x,A)}|\widetilde{G}|(x)\Big)^p + M_{\frac18 d(x,A)}|\widetilde{F}|^p(x)\right] \\ &\le B^p \Bigg[\sup_{0<\rho<\frac18 d(x,A)} \fint_{B_\rho(x)\cap\Omega} \Bigg(|\widetilde{G}(y)| + \sup_{0<\rho<\frac18 d(x,A)} \Big(\fint_{B_\rho(x)\cap\Omega} |\widetilde{F}(z)|^p dz\Big)^{1/p}\Bigg) dy\Bigg]^p \\ &\le B^p \Bigg[\sup_{0<\rho<\frac18 d(x,A)} \fint_{B_\rho(x)\cap\Omega} \Bigg(|\widetilde{G}(y)| + c_3^{1/p} \sup_{\substack{0<\rho<\frac18 d(x,A) \\ w\in B_\rho(y)\cap\Omega}} \Big(\fint_{B_{3\rho}(w)\cap\Omega} |\widetilde{F}(z)|^p dz\Big)^{1/p}\Bigg) dy\Bigg]^p \\ &\le c_3\, B^p \Bigg[\sup_{0<\rho<\frac18 d(x,A)} \fint_{B_\rho(x)\cap\Omega} \Bigg(|\widetilde{G}(y)| + \Big(\widetilde{M}_{\frac12 d(y,A)}|\widetilde{F}|^p(y)\Big)^{1/p}\Bigg) dy\Bigg]^p \\ &\le c_3^2 B^p \left[M_{d(x,A)}(|\widetilde{G}| + |\widehat{F}|)(x)\right]^p\,. \end{aligned} \tag{92}$$

Here the third inequality follows since $\Omega\cap B_\rho(x)\subset\Omega\cap B_{2\rho}(y)\subset\Omega\cap B_{3\rho}(w)$ for $y\in B_\rho(x)$ and $w\in B_\rho(y)$ since $\frac{\mathcal{L}^n(B_{3\rho}(w)\cap\Omega)}{\mathcal{L}^n(B_\rho(x)\cap\Omega)}\le 3^n\frac{\alpha_n}{k_\Omega}\le c_3$, and the last inequality follows from (77). By combining (91) and (92) we see that, for every $x\in\mathbf{B}$, there holds:

$$M_{d(x,A)}\big(|\widetilde{G}|+|\widehat{F}|\big)(x)>\frac{2p}{p-1}\beta t\,,$$

i.e. for every such $x\in\mathbf{B}$ we can find a ball $B_{r(x)}(x)\cap\Omega$ with radius $r(x)<d(x,A)$, such that:

$$\fint_{B_{r(x)}(x)\cap\Omega}\big(|\widetilde{G}|+|\widehat{F}|\big)\,dx\ge\frac{2p}{p-1}\beta t\,. \tag{93}$$

Vitali's covering theorem thus yields the existence of a countable, disjoint family of balls $\{B_{r_j}(y_j)\cap\Omega\}\subset\{B_{r(x)}(x)\cap\Omega\,:\,x\in\mathbf{B}\}$ with $\mathbf{B}\subset\bigcup_{j\in\mathbb{N}}B_{5r_j}(y_j)\cap\Omega$. For $s\ge 0$ we set $C_s=\{x\in\Omega:\ \widehat{F}(x)>s\}$. Using (93), we then see:

$$\begin{aligned}
&\frac{2p}{p-1}\beta t\mathcal{L}^n(B_{r_j}(y_j)\cap\Omega)\le\int_{B_{r_j}(y_j)\cap\Omega}\big(|\widetilde{G}|+|\widehat{F}|\big)\,dx\\
=&\int_{B_{r_j}(y_j)\cap A_{\beta t}}|\widetilde{G}|\,dx+\int_{(B_{r_j}(y_j)\cap\Omega)\setminus A_{\beta t}}|\widetilde{G}|\,dx\\
&\qquad+\int_{B_{r_j}(y_j)\cap C_{\beta t}}|\widehat{F}|\,dx+\int_{(B_{r_j}(y_j)\cap\Omega)\setminus C_{\beta t}}|\widehat{F}|\,dx\\
<\;&2\beta t\mathcal{L}^n(B_{r_j}(y_j)\cap\Omega)+\int_{B_{r_j}(y_j)\cap A_{\beta t}}|\widetilde{G}|\,dx+\int_{B_{r_j}(y_j)\cap C_{\beta t}}|\widehat{F}|\,dx\,.
\end{aligned}$$

Rearranging, this yields:

$$\mathcal{L}^n(B_{r_j}(y_j)\cap\Omega)\le\frac{p-1}{2\beta t}\left(\int_{B_{r_j}(y_j)\cap A_{\beta t}}|\widetilde{G}|\,dx+\int_{B_{r_j}(y_j)\cap C_{\beta t}}|\widehat{F}|\,dx\right)\,. \tag{94}$$

Keeping in mind the properties of the family $\{B_{r_j}(y_j)\}$ and using (94), we see:

$$\begin{aligned}
&\mathcal{L}^n\Big(\bigcup_{(k,j)\in\mathcal{I}}B_{5\rho_{kj}}(x_{kj})\cap\Omega\Big)=\mathcal{L}^n(\mathbf{B})\le c_3\sum_{j=1}^\infty\mathcal{L}^n\big(B_{r_j}(y_j)\cap\Omega\big)\\
\le\;&c_3\,\frac{p-1}{2\beta t}\sum_{j=1}^\infty\left(\int_{B_{r_j}(y_j)\cap A_{\beta t}}|\widetilde{G}|\,dx+\int_{B_{r_j}(y_j)\cap C_{\beta t}}|\widehat{F}|\,dx\right)\\
\le\;&c_3\,\frac{p-1}{2\beta t}\left(\int_{A_{\beta t}}|\widetilde{G}|\,dx+\int_{C_{\beta t}}|\widehat{F}|\,dx\right)\,.
\end{aligned} \tag{95}$$

Using (95) in (90) we see:

$$\begin{aligned}\int_{A_{\beta t}} |\widetilde{G}|^p\,dx &\le (\beta r)^{p-1}\int_{A_{\beta t}}|\widetilde{G}|\,dx + \frac{c_4c_5}{c_3}(\beta r)^p\mathcal{L}^n\Big(\bigcup_{(k,j)\in\mathcal{I}} B_{\rho_{kj}}(x_{kj})\cap\Omega\Big)\\
&\le (\beta r)^{p-1}\int_{A_{\beta t}}|\widetilde{G}|\,dx + \frac{p-1}{2}\,c_4c_5\,\frac{(\beta r)^p}{\beta t}\Big(\int_{A_{\beta t}}|\widetilde{G}|\,dx + \int_{C_{\beta t}}|\widehat{F}|\,dx\Big)\\
&\le \Big(1+\frac{p-1}{2}\,c_4c_5\frac{r}{t}\Big)\Big(\frac{r}{t}\Big)^{p-1}(\beta t)^{p-1}\Big(\int_{A_{\beta t}}|\widetilde{G}|\,dx + \int_{C_{\beta t}}|\widehat{F}|\,dx\Big)\\
&= \widehat{c}\,(\beta t)^{p-1}\Big(\int_{A_{\beta t}}|\widetilde{G}|\,dx + \int_{C_{\beta t}}|\widehat{F}|\,dx\Big) \qquad (96)\end{aligned}$$

for a constant $\widehat{c}$ depending only on n, p, b and k_Ω (recall $r = \frac{2p}{p-1}Bc_3^{3/p}t$).

We now define functions h, H on $[0,\infty)$ via:

$$\begin{aligned} h(s) &= \int_{A_s}|\widetilde{G}|\,dx = \int_{\{|\widetilde{G}|>s\}}|\widetilde{G}|\,dx\,;\\ H(s) &= \int_{C_s}|\widehat{F}|\,dx = \int_{\{|\widehat{F}|>s\}}|\widehat{F}|\,dx\,.\end{aligned}$$

The functions h and H are monotone nonincreasing, and there holds $\lim_{s\to\infty}h(s) = \lim_{s\to\infty}H(s) = 0$. Setting $T=\beta t$, we use (96) to see:

$$\begin{aligned} -\int_T^\infty s^{p-1}dh(s) &= \int_{\{|\widetilde{G}|>T\}}|\widetilde{G}|^p dx = \int_{\{|\widetilde{G}|>\beta t\}}|\widetilde{G}|^p dx\\
&\le \widehat{c}\,(\beta t)^{p-1}\Big(\int_{A_{\beta t}}|\widetilde{G}|dx + \int_{C_{\beta t}}|\widehat{F}|dx\Big)\\
&= \widehat{c}\,T^{p-1}\big[h(T)+H(T)\big]\,, \qquad (97)\end{aligned}$$

where the constant $\widehat{c}$ is from (96). In particular by considering $t=t_0$ (with t_0 from (82)) in (97) we are in a position to apply Lemma A.2 with $h_1=h$, $h_2=H$, $p_1=p-1$, $a=\widehat{c}$ and $\sigma_0=\beta t_0$. Then for any q with $p_2=q-1\in[p-1,\frac{\widehat{c}}{\widehat{c}-1}(p-1))$ we have from Lemma A.2 the inequality

$$-\int_{\beta t_0}^\infty s^{q-1}\,dh(s)\le c\Big(-\int_{\beta t_0}^\infty s^{p-1}\,dh(s) - \int_{\beta t_0}^\infty s^{q-1}\,dH(s)\Big)\,, \qquad (98)$$

with $c=\max\{p_1\sigma_0^{p_2-p_1}, a(p_2-p_1)\}/(ap_1-(a-1)p_2)$; in particular, c depends only on n, p, b and q. Rewriting (98), we have

$$\int_{\{|\widetilde{G}|>\beta t_0\}}|\widetilde{G}|^q dx \le c\Big(\int_{\{|\widetilde{G}|>\beta t_0\}}|\widetilde{G}|^p dx + \int_{\{|\widehat{F}|>\beta t_0\}}|\widehat{F}|^q dx\Big)\,. \qquad (99)$$

We note the obvious inequality

$$\int_{\{|\widetilde{G}|\le\beta t_0\}}|\widetilde{G}|^q dx\le(\beta t_0)^{q-p}\int_{\{|\widetilde{G}|\le\beta t_0\}}|\widetilde{G}|^p dx\,. \qquad (100)$$

Finally note that the continuity of the Hardy–Littlewood maximal–function operator M_Ω (with $M_\Omega h(x) = \sup_{\rho>0} \fint_{B_\rho(x)\cap\Omega} |h| dx$) as a map from $L^s(\Omega)$ to $L^s(\Omega)$ for $s > 1$ enables us to conclude that there holds

$$\int_\Omega |M_\Omega h|^s dx \le c \int_\Omega |h|^s dx$$

for a constant c depending only on n, s and k_Ω (for example $c = 2^s c_3 \frac{s}{s-1}$ suffices, cf. [20, p. 226]). In view of the definition of $\widehat{F}$ we apply this with $s = q/p$, $h = |F|^p$ to conclude (since $\widetilde{F} \le F$ on Ω)) that

$$\int_\Omega |\widehat{F}|^q dx = \int_\Omega \left(|M_{d(x,A)} \widetilde{F}|^p\right)^{q/p} dx \le \int_\Omega \left(|M_\Omega F|^p\right)^{q/p} dx \le c \int_\Omega |F|^q dx \tag{101}$$

for a constant c depending only on n, p, b, q and k_Ω.

Combining (99), (100) and (101) and using the fact that $\widetilde{G} \le G$ on Ω we see

$$\int_\Omega |\widetilde{G}|^q dx \le c \left(\int_\Omega |G|^p dx + \int_\Omega |F|^q dx \right)$$

for a constant c depending only on n, p, b, q and k_Ω. Multiplying through by $\Gamma^q / \mathcal{L}^n(\Omega)$ we obtain the desired inequality. □

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