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On the Relaxation of a Class of Functionals Defined on Riemannian Distances

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In this paper we study the relaxation of a class of functionals defined on distances induced by isotropic Riemannian metrics on an open subset of $\mathbb{R}^N$. We prove that isotropic Riemannian metrics are dense in Finsler ones and we show that the relaxed functionals admit a specific integral representation.

Keywords: Riemannian and Finsler metrics, relaxation, Gamma convergence

1. Introduction

In this paper we study an integral functional of the form

$$\mathcal{F}(d_a) := \int_\Omega F(x, a(x))\,\mathrm{d}x, \tag{1}$$

defined on the family $\mathcal{I}$ of distances d_a induced by isotropic, continuous Riemannian metrics through the formula

$$d_a(x,y) := \inf\left\{ \mathbb{L}_a(\gamma) \ : \ \gamma \in \mathrm{Lip}([0,1];\overline{\Omega}),\ \gamma(0)=x,\ \gamma(1)=y \right\} \tag{2}$$

for every $(x,y) \in \overline{\Omega}\times\overline{\Omega}$, where the length functional $\mathbb{L}_a$ is defined as follows

$$\mathbb{L}_a(\gamma) := \int_0^1 a(\gamma(t))|\dot\gamma(t)|\,\mathrm{d}t. \tag{3}$$

Here a ranges over the family of positive continuous functions from $\overline{\Omega}$ to the interval $[\alpha,\beta]$, where α and β are fixed positive constants. We point out that the corrispondence between such metrics and elements of $\mathcal{I}$ is injective, that is two continuous, isotropic Riemannian metrics which induce the same distance through (2) actually coincide (cf. Remark 3.2). In particular, that shows that the functional (1) is well defined.

Distances of this type have already been studied in [3, 6] and, in a more geometric framework, in [10]. The set $\mathcal{I}$ can be seen as a subspace of the space of Finslerian distances $\mathcal{D}$ (see Section 2), endowed with the metrizable topology given by the uniform convergence on compact subset of $\overline{\Omega}\times\overline{\Omega}$. It has been proved in [6] that the convergence of a sequence $(d_n)_{n\in\mathbb{N}}$ to d in this topology is equivalent to the Γ-convergence of the associated length functionals L_{d_n} to L_d with respect to the uniform convergence of curves (see Section 2 for

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definitions). The main problem arising in our study is that $\mathcal{I}$ is not closed with respect to this topology. Indeed, one can exhibit sequences of continuous metrics $(a_n)_{n\in\mathbb{N}}$ which develop an oscillatory behavior in such a way that the induced distances converge to an element d which does not belong to $\mathcal{I}$ any longer (see [1]). Therefore, it is natural to consider the *relaxed functional* of (1), namely

$$\overline{\mathcal{F}}(d) := \inf\{\liminf_n \mathcal{F}(d_n) \ : \ d_n \xrightarrow{\mathcal{D}} d,\ (d_n)_{n\in\mathbb{N}} \subset \mathcal{I}\}, \tag{4}$$

defined for every d belonging to the closure of $\mathcal{I}$, where we have denoted by $\xrightarrow{\mathcal{D}}$ the convergence with respect to the topology of $\mathcal{D}$.

In this paper we prove that the space $\mathcal{I}$ is dense in $\mathcal{D}$ and, under suitable assumptions on the integrand F in (1), that the relaxed functional (4), which is therefore defined on the whole $\mathcal{D}$, has the following integral representation:

$$\overline{\mathcal{F}}(d) = \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x, \tag{5}$$

where $\Lambda_d(x) := \sup_{|\xi|=1} \varphi_d(x,\xi)$ and φ_d is the Finslerian metric associated to d by derivation (cf. Definition 2.8). In particular, the functional $\overline{\mathcal{F}}$ will coincide with $\mathcal{F}$ on $\mathcal{I}$ (cf. Proposition 3.1).

We conclude this introduction with some considerations. Definition (4) clearly implies that $\overline{\mathcal{F}}$ is lower semicontinuous. Moreover, it can be shown that it is the greatest among all lower semicontinuous ones which are bounded from above by $\mathcal{F}$ on $\mathcal{I}$ (see [4] for various results on this topic). Therefore, in order to prove our relaxation result, we first have to show that the functional (5) is lower semicontinuous. The proof of this issue is just a technical adaptation of the arguments described in [5]. To prove its maximality, instead, we will approximate each $d \in \mathcal{D}$ by means of a sequence of suitably chosen distances $d_n \in \mathcal{I}$, namely such that

$$\limsup_n \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \le \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x.$$

Then, by a standard argument (see Section 4), the maximality of (5) follows.

Indeed, finding such an approximating sequence is a delicate matter. In fact, one should define the Riemannian metrics a_n in such a way to have Γ-convergence of the relative length functionals $\mathbb{L}_{a_n}$ to $\mathbb{L}_{\varphi_d}$ (cf. (11) and Remark 3.3) and this problem is not trivial even in the simplified situation of an isotropic Riemannian metric φ_d, i.e. such that $\varphi_d = b(x)|\xi|$ where b is a Borel function from $\overline{\Omega}$ to $[\alpha,\beta]$. It is clear, in fact, that this convergence strongly relies upon the convergence of the approximating metrics on curves, which is much finer than convergence almost everywhere in $\overline{\Omega}$. Moreover we do not have many informations on the properties of the metric φ_d; we only know it is Borel measurable and such that the associated length functional $\mathbb{L}_{\varphi_d}$ is lower semicontinuous with respect to the uniform convergence of curves (see Section 2). In the general case of a non-isotropic metric the situation is obviously more delicate.

The key idea of our proof is that it is sufficient to control the convergence of the approximating distances only on a fixed countable and dense subset of $\overline{\Omega}\times\overline{\Omega}$ (Lemma 3.4).

Therefore, when we define the Riemannian metrics, we have only to control the value of the associated distance d_n on the first n points of this set. This will be done by approximating the Finsler metric φ_d along geodesics (cf. Theorem 4.7). With regard to that, let us notice that Theorem 4.7 is not just a technical result in order to prove our main theorems, but has an interesting consequence it is worth underline: every Finsler distance $d \in \mathcal{D}$ can indeed be seen as generated by a suitable Borel measurable, isotropic Riemannian metric $a : \overline{\Omega} \to [\alpha, \beta]$ according to definition (2). In other words, by allowing the isotropic metric a to vary in a somehow "uncontrolled" way, one can recover all the possible anisotropies of φ_d.
The problem of the density of (smooth) isotropic, Riemannian metrics in Finsler ones has already been studied. The question was raised in [6], and partially answered in [3] in the case $\Omega := \mathbb{R}^N$ under the additional assumption that φ_d is lower semicontinuous. We remark that our proof does not require any extra regularity property on the Finsler metric and therefore completely answers the question (on the other hand, when $\Omega \neq \mathbb{R}^N$ some hypotheses on Ω are assumed, see condition (Ω) below). Indeed, as pointed out in [3], once the density result for continuous and isotropic Riemannian metrics is established, the analogous result for smooth ones is easily recovered via a regularization argument (cf. Remark 4.4).

The paper is organized as follows: in Section 2 we recall the main notations used in the sequel and some results on Finsler metrics, Section 3 contains some preliminary lemmas and in Section 4 we prove our main theorems.

2. Notation and preliminaries on Finsler metrics

We write here a list of symbols used throughout this paper.

Ω	an open connected subset of $\mathbb{R}^N$
$\mathbb{S}^{N-1}$	the unitary sphere of $\mathbb{R}^N$
$B_r(x)$	the open ball in $\mathbb{R}^N$ of radius r centred in x
I	the closed interval $[0,1]$
$\mathbb{R}_+$	non-negative real numbers
$\mathcal{L}^N$	the $N-$dimensional Lebesgue measure
$\mathcal{H}^N$	the $N-$dimensional Hausdorff measure
$\lvert u\rvert$	the Euclidean norm of the vector $u \in \mathbb{R}^N$
χ_E	the characteristic function of the set E
$\mathrm{argmin}(\mathcal{P})$	the set of minimizers of the problem $(\mathcal{P})$

In this paper N denotes an integer number. We will say that a set ω is *well contained* in Ω and we will write $\omega \subset\subset \Omega$ to mean that its closure $\overline{\omega}$ is contained in Ω. With the word *curve* or *path* we will always indicate a Lipschitz function from the interval $I := [0,1]$ to the closed set $\overline{\Omega}$; the family of all such curves will be denoted by $\mathrm{Lip}(I, \overline{\Omega})$. Any curve γ is always supposed to be parametrized by constant speed, i.e. in such a way that $|\dot\gamma(t)|$ is constant for $\mathcal{L}^1$-a.e. $t \in I$. We will say that a sequence of curves $(\gamma_n)_{n\in\mathbb{N}}$ (uniformly) converges to a curve γ to mean that $\sup_{t\in I} |\gamma_n(t) - \gamma(t)|$ tends to zero as n goes to infinity. We will denote by $\mathrm{Lip}_{x,y}$ the family of curves γ which join x to y, i.e. such that $\gamma(0) = x$ and $\gamma(1) = y$. We remark that, if a sequence of curves $(\gamma_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_{x,y}$ is such that $\sup_n \int_0^1 |\dot\gamma_n(t)| \mathrm{d}t < +\infty$, then, since they are all parametrized by constant speed, we have that their first derivative is bounded from above. Therefore, by applying Ascoli-Arzelà theorem, we can find a curve $\gamma \in \mathrm{Lip}_{x,y}$ such that a subsequence $(\gamma_{n_i})_{i\in\mathbb{N}}$ converges to γ. This argument will be implicitly used throughout the paper.

The function $F : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ appearing in the integrand of (1) is assumed to be continuous and to fulfill the following conditions:

$$\begin{array}{ll} \text{(i)} & \text{the function } F(x,\cdot) \text{ is convex and nondecreasing for every } x \in \Omega; \\ \text{(ii)} & \int_\Omega F(x,\beta)\,\mathrm{d}x < +\infty. \end{array} \tag{6}$$

Definition 2.1. A Borel function $\varphi : \overline{\Omega} \times \mathbb{R}^N \to [0, +\infty)$ is said to be a *Finsler metric* on $\overline{\Omega} \subset \mathbb{R}^N$ if

(i) $\varphi(x,\cdot)$ is positively 1-homogeneous for every $x \in \overline{\Omega}$;
(ii) $\varphi(x,\cdot)$ is convex on $\mathbb{R}^N$ for $\mathcal{L}^N$-a.e. $x \in \overline{\Omega}$;
(iii) for every curve $\gamma \in \mathrm{Lip}(I, \overline{\Omega})$
$$\varphi(\gamma(t), \dot\gamma(t)) = \varphi(\gamma(t), -\dot\gamma(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in I.$$

Given a Finsler metric, we can define a distance d_φ on $\overline{\Omega}$ through the formula

$$d_\varphi(x,y) := \inf\left\{\mathbb{L}_\varphi(\gamma) \mid \gamma \in \mathrm{Lip}_{x,y}\right\}, \tag{7}$$

where the Finslerian length functional $\mathbb{L}_\varphi$ is defined by

$$\mathbb{L}_\varphi(\gamma) := \int_0^1 \varphi(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

A distance deriving from a Finsler metric through (7) is said to be of *Finsler type.*

Remark 2.2. Notice that $\mathbb{L}_\varphi$ is well defined. Indeed, the map $t \mapsto \big(\gamma(t), \dot\gamma(t)\big)$ is Lebesgue measurable on I and φ is Borel measurable on $\overline{\Omega} \times \mathbb{R}^N$, hence their composition $\varphi(\gamma(t), \dot\gamma(t))$ is Lebesgue measurable. Moreover, by assumptions (i) and (iii) in Definition 2.1, $\mathbb{L}_\varphi(\gamma)$ does not depend on the chosen parametrization for γ, that is, if $\rho : I \to I$ is a C^1-diffeomorphism, then $\mathbb{L}_\varphi(\gamma\circ\rho) = \mathbb{L}_\varphi(\gamma)$

We will say that a distance function is of *geodesic type* if it satisfies the following identity:

$$d(x,y) = \inf\Big\{\mathrm{L}_d(\gamma) \mid \gamma \in \mathrm{Lip}_{x,y}\Big\} \qquad \text{for every } (x,y) \in \overline{\Omega} \times \overline{\Omega}, \tag{8}$$

where $\mathrm{L}_d(\gamma)$ denotes the classical d-length of γ, obtained as the supremum of the d-lengths of inscribed polygonal curves:

$$\mathrm{L}_d(\gamma) := \sup\Big\{\sum_i d\big(\gamma(t_i), \gamma(t_{i+1})\big) : \ 0 = t_0 < t_1 < .. < t_r = 1,\ r \in \mathbb{N}\Big\}. \tag{9}$$

Denote by $d_{\overline{\Omega}}(x,y)$ the Euclidean geodesic distance in $\overline{\Omega}$, that is $d_{\overline{\Omega}} := d_a$ according to (2), with a identically equal to 1. We remark that $d_{\overline{\Omega}}$ locally coincides with the Euclidean distance. We fix two positive constants α, β with $\beta > \alpha$ and we set

$$\mathcal{M} := \left\{\varphi \text{ Finsler metric on } \overline{\Omega} \ : \ \alpha\,|\xi| \le \varphi(x,\xi) \le \beta\,|\xi| \ \text{ for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N\right\}.$$

Then we define the family $\mathcal{D}$ of distances on $\overline{\Omega}$ generated by the metrics $\mathcal{M}$, namely $\mathcal{D} := \{d_\varphi \mid \varphi \in \mathcal{M}\}$. Obviously the set $\mathcal{I}$, made up by distances d_a defined by (2) with $a : \overline{\Omega} \to [\alpha, \beta]$ continuous, is trivially included in $\mathcal{D}$ identifying $a(x)$ with the metric $a(x)|\xi|$. It is also clear that $\alpha\, d_{\overline{\Omega}} \le d \le \beta\, d_{\overline{\Omega}}$ for every $d \in \mathcal{D}$, so such distances are locally equivalent to the Euclidean one. Moreover we have (cf. [5, Lemma 2]):

Proposition 2.3. *Let $d := d_\varphi$ for some $\varphi \in \mathcal{M}$. Then $\mathrm{L}_d(\gamma) \le \mathbb{L}_\varphi(\gamma)$ for every curve γ. In particular, d is a distance of geodesic type according to definition* (8).

Remark 2.4. The inequality in the previous proposition may be strict. For example, take $\Omega := (-1,1) \times (-1,1)$, $\Gamma := \{0\} \times [-1,1]$ and $a(x) := \chi_{\overline{\Omega}}(x) + \chi_\Gamma(x)$ for all $x \in \overline{\Omega}$. Then $d_a(y,z) = |y-z|$ for all $y, z \in \overline{\Omega}$. If now we take $\gamma(t) := (0,-1/2)(1-t) + (0,1/2)t$, it is easily seen that $\mathrm{L}_{d_a}(\gamma) = 1 < 2 = \mathbb{L}_a(\gamma)$.

By using classical results of the theory of metric spaces (see, for instance, [2, Chapter 4]), one can derive the following

Proposition 2.5. *Let $d \in \mathcal{D}$. Then the length functional L_d is lower semicontinuous on $\mathrm{Lip}(I, \overline{\Omega})$ with respect to the uniform convergence of paths, namely if $(\gamma_n)_{n\in\mathbb{N}}$ converges to γ then*

$$\mathrm{L}_d(\gamma) \le \liminf_n \mathrm{L}_d(\gamma_n).$$

In particular, for every couple of points x, y in $\overline{\Omega}$ there exists a curve $\gamma \in \mathrm{Lip}_{x,y}$ which is a path of minimal d-length, i.e. such that $\mathrm{L}_d(\gamma) = d(x,y)$.

We endow $\mathcal{D}$ with the topology given by the uniform convergence on compact subset of $\overline{\Omega} \times \overline{\Omega}$. We will write $d_n \xrightarrow{\mathcal{D}} d$ to mean that the sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ converges to $d \in \mathcal{D}$ with respect to this topology. Arguing as in the proof of Theorem 3.1 in [6], one can establish the following result:

Theorem 2.6. *Let Ω be an open subset of $\mathbb{R}^N$ such that*

$$\forall r > 0 \quad \exists C_r \ge 1 \quad \text{such that} \quad d_\Omega(x,y) \le C_r |x-y| \quad \forall\, x, y \in \overline{\Omega} \cap B_r(0). \tag{Ω}$$

Let d and d_n belong to $\mathcal{D}$ for all $n \in \mathbb{N}$. Then $d_n \xrightarrow{\mathcal{D}} d$ if and only if L_{d_n} Γ-converge to L_d on $\mathrm{Lip}(I, \overline{\Omega})$ with respect to the uniform convergence of paths. Moreover, $\mathcal{D}$ is a metrizable compact space.

Throughout this paper we will always work with sets Ω which satisfy condition (Ω). This holds, for example, whenever Ω has a locally Lipschitz boundary. In particular, then, we will always assume that $\mathcal{D}$ is compact.

Remark 2.7. If Ω satisfies assumption (Ω), then $\mathcal{D}$ can be seen as a subspace of the wider space of Finsler distances on $\mathbb{R}^N$. In fact, let us show that every distance in $\mathcal{D}$ can be extended in suitable way to a distance defined on the whole $\mathbb{R}^N$. Let $d \in \mathcal{D}$ and $\varphi \in \mathcal{M}$ such that $d = d_\varphi$ according to definition (7). Let us define

$$\overline{\varphi}(x,\xi) := \begin{cases} \varphi(x,\xi) & \text{if } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N \\ 2\beta C_n |\xi| & \text{if } x \in B_n(0) \setminus (\overline{B}_{n-1}(0) \cup \overline{\Omega}) \text{ and } \xi \in \mathbb{R}^N,\ n \in \mathbb{N} \end{cases}$$

where C_n are positive constants chosen according to condition (Ω). The Finsler metric $\overline{\varphi}$ defines a distance $\overline{d} := d_{\overline{\varphi}}$ on $\mathbb{R}^N$ through (7). We claim that $\overline{d}$ is the required extension of d. Indeed, first remark that, if γ is a curve which connects two points of $\partial\Omega$ in $\mathbb{R}^N \setminus \overline{\Omega}$ (i.e. $\gamma(0)$, $\gamma(1) \in \partial\Omega$ and $\gamma((0,1)) \subset \mathbb{R}^N \setminus \overline{\Omega}$), then, by definition of $\overline{\varphi}$ and $\mathbb{L}_{\overline{\varphi}}$, we have $\mathbb{L}_{\overline{\varphi}}(\gamma) \geq 2\beta d_{\overline{\Omega}}(\gamma(0), \gamma(1)) > d(\gamma(0), \gamma(1))$. In particular, by definition of d, there exists a curve $\widetilde{\gamma}$, with same endpoints as γ and lying in $\overline{\Omega}$, such that $\mathbb{L}_{\overline{\varphi}}(\gamma) > \mathbb{L}_{\varphi}(\widetilde{\gamma})$. Taking this into account, it is not difficult to show that, for every couple of points $x, y \in \overline{\Omega}$ and for every curve γ connecting x to y in $\mathbb{R}^N$, there exists a curve $\widetilde{\gamma}$, with same endpoints and lying in $\overline{\Omega}$, such that $\mathbb{L}_{\overline{\varphi}}(\gamma) > \mathbb{L}_{\varphi}(\widetilde{\gamma})$. Since $\overline{\varphi} = \varphi$ on $\overline{\Omega} \times \mathbb{R}^N$, this immediately gives that $\overline{d} = d$ on $\overline{\Omega} \times \overline{\Omega}$. Therefore, up to replacing the distance d with its extension $\overline{d}$, we can always assume, if needed, that d is defined on $\mathbb{R}^N \times \mathbb{R}^N$.

Definition 2.8. We define the Finsler metric φ_d associated to a distance $d \in \mathcal{D}$ by derivation as

$$\varphi_d\,(x, \xi) := \limsup_{t \to 0^+} \frac{d\,(x, x + t\xi)}{t} \qquad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N, \tag{10}$$

where we have taken Remark 2.7 into account to give a meaning to the above expression for those points x which belong to $\partial\Omega$.

The length functional L_d admits the following integral representation:

$$\mathrm{L}_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t \quad \text{for all } \gamma \in \mathrm{Lip}(I, \overline{\Omega}), \tag{11}$$

i.e. $\mathrm{L}_d = \mathbb{L}_{\varphi_d}$ on $\mathrm{Lip}(I, \overline{\Omega})$ (see [10, Theorem 2.5]). We summarize in the next proposition the main properties of φ_d. For the proof, we refer to [8, 10].

Proposition 2.9. *Let $d \in \mathcal{D}$. Then the function $\varphi_d : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$ is a Borel-measurable Finsler metric. In particular we have:*

- (i) $\varphi_d(x, \cdot)$ *is positively 1-homogeneous for every $x \in \overline{\Omega}$;*
- (ii) $|\varphi_d(x, \xi) - \varphi_d(x, \nu)| \leq \beta|\xi - \nu|$ *for every $x \in \Omega$ and every $\xi, \nu \in \mathbb{R}^N$;*
- (iii) $\varphi_d(x, \cdot)$ *is a norm on $\mathbb{R}^N$ for $\mathcal{L}^N$-a.e. $x \in \overline{\Omega}$.*

Remark 2.10. Let $d := d_\varphi$ for some $\varphi \in \mathcal{M}$. In view of Remark 2.7, we may as well assume d to be defined on $\mathbb{R}^N \times \mathbb{R}^N$, so that expression (10) makes always sense. Take a curve $\gamma \in \mathrm{Lip}(I, \overline{\Omega})$ and pick up a differentiability point $t \in (0, 1)$ for γ. Then, by arguing as in [10, Theorem 2.5], we get:

$$\varphi_d(\gamma(t), \dot{\gamma}(t)) = \limsup_{h \to 0^+} \frac{d(\gamma(t), \gamma(t + h))}{h}.$$

If t is also a Lebesgue point for $|\dot{\gamma}(s)|$, that yields:

$$\varphi_d(\gamma(t), \dot{\gamma}(t)) \leq \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \varphi(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s \leq \limsup_{h \to 0^+} \frac{\beta}{h} \int_t^{t+h} |\dot{\gamma}(s)|\,\mathrm{d}s = \beta|\dot{\gamma}(t)|.$$

In particular, we deduce that the following holds:

$$\alpha \leq \varphi_d\left(\gamma(t), \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}\right) \leq \beta \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in I.$$

Given a distance $d \in \mathcal{D}$, we define for every $x \in \Omega$

$$\Lambda_d(x) := \sup_{|\xi|=1} \varphi_d(x, \xi),$$

which represents, with analogy to the Riemannian case $\varphi_d(x, \xi) = (B(x)\xi \cdot \xi)^{\frac{1}{2}}$ with $B(x)$ a symmetric and positive definite matrix, the largest "eigenvalue" of $\varphi_d(x, \cdot)$ at the point x. We notice that $\Lambda_d(x)$ is a Borel measurable function. Indeed, if $(\xi_n)_{n\in\mathbb{N}}$ is a dense sequence in $\mathbb{S}^{N-1}$, by property (ii) of Proposition 2.9 we have that $\Lambda_d(x)$ coincides with the function $\sup_n \varphi_d(x, \xi_n)$, which is Borel measurable since it is the supremum of Borel measurable functions.

3. Preliminary results

In this section we prepare the tools which will be needed in the proof of the relaxation result.

Proposition 3.1. *Let $\varphi \in \mathcal{M}$ and $d := d_\varphi$. Then*

(i) $\varphi_d(x, \xi) \leq \varphi(x, \xi)$ *for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In particular* $\Lambda_d(x) \leq \sup_{|\xi|=1} \varphi(x, \xi)$ *for $\mathcal{L}^N$-a.e. $x \in \Omega$;*

(ii) *if $\varphi(x, \xi) := a(x)|\xi|$ with $a : \overline{\Omega} \to [\alpha, \beta]$ lower semicontinuous, then $\varphi_d(x, \xi) \geq a(x)|\xi|$ for every $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$. In particular $a(x) = \Lambda_d(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$.*

Remark 3.2. If a and b are two continuous isotropic metrics which give rise to the same distance function d through (2), then $a(x) = b(x)$ for every x in $\overline{\Omega}$. In fact, by claim (ii) of Proposition 3.1, we have that the previous equality holds almost everywhere, and therefore everywhere by the continuity of the metrics. In particular, this shows that the functional (1) is well defined.

Proposition 3.1 is essentially known (*cf.* [9, Section 6]). For the reader's convenience, we provide here a proof.

Proof. (i) In view of Remark 2.7 it is enough to prove the claim in the case $\Omega := \mathbb{R}^N$. Let us fix a vector $\xi \in \mathbb{R}^N$ and, for every $x_0 \in \Omega$, let us define the curve $\gamma_{x_0}(s) := x_0 + s\xi$. Let t be a Lebesgue point for the map $s \mapsto \varphi(\gamma_{x_0}(s), \xi)$. For $h > 0$ we have

$$\frac{1}{h}\int_t^{t+h} \varphi(\gamma_{x_0}(s), \xi)\,\mathrm{d}s = \frac{1}{h}\int_0^1 \varphi(\gamma_{x_0}(t + h\tau), h\xi)\,\mathrm{d}\tau \geq \frac{d(\gamma_{x_0}(t), \gamma_{x_0}(t) + h\xi)}{h},$$

so, by taking the limsup as $h \to 0^+$, we get $\varphi_d(\gamma_{x_0}(t), \xi) \leq \varphi(\gamma_{x_0}(t), \xi)$. Since $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ is a Lebegue point for $\varphi(\gamma_{x_0}(\cdot), \xi)$ and x_0 was arbitrarily chosen in Ω, Fubini's Theorem implies that $\varphi_d(x, \xi) \leq \varphi(x, \xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. Then we can take a dense sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ and repeat the previous argument for each ξ_n. Recalling that the functions $\varphi_d(x, \cdot)$ and $\varphi(x, \cdot)$ are continuous for almost every $x \in \Omega$, we eventually get, by the density of $(\xi_n)_{n\in\mathbb{N}}$, that $\varphi_d(x, \xi) \leq \varphi(x, \xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In particular we get

$$\Lambda_d(x) \leq \sup_{|\xi|=1} \varphi(x, \xi) \qquad \mathcal{L}^N\text{-a.e. in } \Omega. \tag{12}$$

(ii) Let now $\varphi(x,\xi) := a(x)|\xi|$ with a lower semicontinuous. Arguing as in Remark 2.7, one can assume the function a to be defined on the whole $\mathbb{R}^N$ and still lower semicontinuous. By lower semicontinuity, we have that for every fixed $x \in \overline{\Omega}$ and for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $a(y) \geq a(x) - \varepsilon$ for every $y \in B_{r_\varepsilon}(x)$. Let us fix $\xi \in \mathbb{S}^{N-1}$ and take $0 < t < t_0$. Choose a d-minimizing sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_{x,x+t\xi}$ such that $\mathbb{L}_a(\gamma_n) \leq d(x, x+t\xi) + \alpha r_\varepsilon/2$ for every n. If t_0 is small enough, the curves γ_n lie within $B_{r_\varepsilon}(x)$. Then we have for every $n \in \mathbb{N}$

$$\mathbb{L}_a(\gamma_n) = \int_0^1 a(\gamma_n)|\dot\gamma_n|\,\mathrm{d}\tau \geq (a(x)-\varepsilon)\int_0^1 |\dot\gamma_n|\,\mathrm{d}\tau \geq (a(x)-\varepsilon)t$$

and letting n go to infinity we obtain

$$\frac{d(x,x+t\xi)}{t} \geq a(x) - \varepsilon. \tag{13}$$

By taking the limsup in (13) as $t \to 0^+$ and since $\varepsilon > 0$, $x \in \overline{\Omega}$ and $\xi \in \mathbb{S}^{N-1}$ were arbitrary we obtain

$$\varphi_d(x,\xi) \geq a(x) \qquad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{S}^{N-1} \tag{14}$$

and the claim follows by the 1-homogeneity of $\varphi_d(x,\cdot)$. In particular, by taking the supremum of the left-hand side of (14) over all $\xi \in \mathbb{S}^{N-1}$ and by using (12) we get that $\Lambda_d(x) = a(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. □

Remark 3.3. If $d := d_\varphi$ and $\varphi(x,\xi) := a(x)|\xi|$ with $a : \overline{\Omega} \to [\alpha,\beta]$ lower semicontinuous, by the previous Lemma we have that for every curve γ the following holds:

$$\varphi_d(\gamma(t),\dot\gamma(t)) \geq a(\gamma(t))|\dot\gamma(t)| \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in I.$$

This inequality, coupled with (11) and Proposition 2.3, implies that $\mathbb{L}_a(\gamma) = \mathrm{L}_d(\gamma)$ for every curve γ.

The key idea used in the proof of the density result is stated in the following lemma. The proof is immediate in view of Theorem 2.6 and is omitted.

Lemma 3.4. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence contained in $\mathcal{D}$ which converges pointwise to some $d \in \mathcal{D}$ on a dense subset of $\overline{\Omega}\times\overline{\Omega}$. Then $d_n \xrightarrow{\mathcal{D}} d$.*

The next result shows that the monotone convergence of metrics implies the convergence of the induced distances.

Lemma 3.5. *Let φ and φ_n belong to $\mathcal{M}$ for all $n \in \mathbb{N}$. Then $d_{\varphi_n} \xrightarrow{\mathcal{D}} d_\varphi$ in one of the following cases:*

(i) $\varphi_n(x,\xi) := a_n(x)|\xi|$ *converge increasingly to* $\varphi(x,\xi)$ *for every* $(x,\xi) \in \overline{\Omega}\times\mathbb{R}^N$, *with* a_n *lower semicontinuous in* $\overline{\Omega}$;

(ii) $\varphi_n(x,\xi)$ *converge decreasingly to* $\varphi(x,\xi)$ *for every* $(x,\xi) \in \overline{\Omega}\times\mathbb{R}^N$.

Proof. To simplify the notations, we will write d and d_n in place of d_φ and d_{φ_n} respectively.

(i) First we observe that, by the dominated convergence theorem, the functionals $\mathbb{L}_{a_n}$ converge pointwise to $\mathbb{L}_\varphi$ (i.e. $\mathbb{L}_{a_n}(\gamma)$ converges to $\mathbb{L}_\varphi(\gamma)$ for every curve γ). Moreover, by Remark 3.3, we have that $\mathrm{L}_{d_n}(\gamma) = \mathbb{L}_{a_n}(\gamma)$ for all curves γ. Since the length functionals L_{d_n} are lower semicontinuous by Proposition 2.5, we can apply [7, Remark 5.5] to deduce that L_{d_n} Γ-converge to L_d on $\mathrm{Lip}(I, \overline{\Omega})$ with respect to the uniform convergence of paths. The claim then follows by Theorem 2.6.
(ii) By Lemma 3.4 it is sufficient to prove that $d_n(x, y)$ converges to $d(x, y)$ for every fixed $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. Then, let $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. By monotonicity we get $d(x, y) \leq \inf_n d_n(x, y)$. To show the reverse inequality, take a curve $\gamma \in \mathrm{Lip}_{x,y}$. By the monotone convergence theorem and by the definition of $d_n(x, y)$ we have

$$\mathbb{L}_\varphi(\gamma) = \inf_n \mathbb{L}_{\varphi_n}(\gamma) \geq \inf_n d_n(x, y),$$

and the claim easily follows by taking the infimum over all curves in $\mathrm{Lip}_{x,y}$. □

We end this section with the proof of two lemmas which will be useful in the sequel.

Lemma 3.6. *Let $\{(x_i, y_i) \,|\, i \in \mathbb{N}\}$ be a countable collection of points in $\overline{\Omega} \times \overline{\Omega}$. Then it is possible to find a family of curves $\{\gamma_i \,|\, \gamma_i \in \mathrm{Lip}_{x_i,y_i},\ i \in \mathbb{N}\}$ such that*

(i) $\mathrm{L}_d(\gamma_i) = d(x_i, y_i)$ *and γ_i is injective for every $i \in \mathbb{N}$;*
(ii) $\gamma_i(I) \cap \gamma_j(I)$ *is a (possibly void) disjoint finite union of closed arcs for every $i, j \in \mathbb{N}$.*

Proof. First we remark that for every $i \in \mathbb{N}$ the set

$$\mathcal{R}_i := \mathrm{argmin}\{\mathrm{L}_d(\gamma) \,|\, \gamma \in \mathrm{Lip}_{x_i,y_i}\}.$$

is non-void by Proposition 2.5. Moreover, any curve in $\mathcal{R}_i$ is injective by minimality, hence it satisfies point (i) of the claim. In order to prove the Lemma, it will be enough to show that the following holds for every $n \in \mathbb{N}$:

Claim: Let $\{\gamma_i \,|\, \gamma_i \in \mathrm{Lip}_{x_i,y_i},\ i \leq n-1\}$ be a collection of curves satisfying conditions (i)-(ii) above. Then it is possible to find $\gamma_n \in \mathrm{Lip}_{x_n,y_n}$ such that the curves $\{\gamma_i \,|\, i \leq n\}$ still satisfy conditions (i)-(ii).

For $n = 1$ the claim is satisfied by choosing a γ_1 which belongs to $\mathcal{R}_1$. Let then $n > 1$ and choose a curve σ in $\mathcal{R}_n$. For every $j \leq n-1$, let us set $t_j := \min\{t \in I \,|\, \sigma(t) \in \gamma_j(I)\}$ and $T_j := \max\{t \in I \,|\, \sigma(t) \in \gamma_j(I)\}$ (we agree that $t_j = T_j = +\infty$ if such minimum does not exist), and $J := \{j \leq n-1 \,|\, t_j < T_j < +\infty\}$. If J is void, the claim is proved by setting $\gamma_n := \sigma$. Otherwise, we can suppose, up to reordering the curves γ_j, that $t_1 = \min\{t_j \,|\, j \in J\}$. Then we define $\tau_1 \in \mathrm{Lip}_{x_n,y_n}$ to be the curve obtained by moving from $\sigma(0)$ to $\sigma(t_1)$ along σ, from $\sigma(t_1)$ to $\sigma(T_1)$ along γ_1 and from $\sigma(T_1)$ to $\sigma(1)$ along σ again. Remark that, by minimality, γ_1 is a path which connects $\sigma(t_1)$ to $\sigma(T_1)$ in the shortest way and so we have not increased the length, i.e. $\mathrm{L}_d(\tau_1) \leq \mathrm{L}_d(\sigma)$, hence $\tau_1 \in \mathcal{R}_n$. Moreover $\tau_1([0, T_1]) \cap \gamma_i(I)$ is a disjoint finite union of closed arcs for every $1 \leq i \leq n-1$. Then we set $\sigma := \tau_1 |_{[T_1,1]}$ and we repeat the above argument to obtain a curve $\tau_2 : [T_1, 1] \to \overline{\Omega}$. By iterating this procedure, we eventually find a finite number of curves $\{\tau_h \,|\, 1 \leq h \leq M\}$ for some $M < n$. Then we define

$$\gamma_n(t) := \begin{cases} \tau_1(t) & \text{if } t \in [0, T_1] \\ \tau_h(t) & \text{if } t \in [T_{h-1}, T_h] \text{ and } 1 < h < M \\ \tau_M(t) & \text{if } t \in [T_{M-1}, 1]. \end{cases}$$

By what previously observed, we have that γ_n still belongs to $\mathcal{R}_n$ and is therefore injective by minimality. Moreover, it is such that $\gamma_n(I) \cap \gamma_i(I)$ is a disjoint finite union of closed arcs for every $i \leq n-1$ by construction. The claim is thus proved. $\square$

Lemma 3.7. *Let γ be an injective Lipschitz curve, $\Gamma := \gamma((0,1)) \subset \overline{\Omega}$ and $a : \overline{\Omega} \to [\alpha, \beta]$ a Borel function. Then there exists a sequence of continuous functions $\sigma_k : \Gamma \to [\alpha, \beta]$ such that $\sigma_k(x)$ converge to $a(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma$. Moreover, for every $\varepsilon > 0$ there exists a Borel subset $B_\varepsilon \subset \Gamma$ such that $\mathcal{H}^1(\Gamma \setminus B_\varepsilon) < \varepsilon$ and σ_k converge uniformly to a on B_ε.*

Proof. The function $a \circ \gamma : (0,1) \to [\alpha, \beta]$ is Borel measurable, therefore there exists a sequence $(f_k)_{k \in \mathbb{N}}$ of continuous functions $f_k : (0,1) \to [\alpha, \beta]$ such that $f_k(t)$ converges to $a \circ \gamma(t)$ for a.e. $t \in (0,1)$. Moreover, by Severini-Egoroff's theorem [11, Section 1.2, Theorem 3], for every $\widetilde{\varepsilon} > 0$ there exist an infinitesimal sequence $(\delta_k)_{k \in \mathbb{N}}$ and a Borel set $E_{\widetilde{\varepsilon}}$ such that $\mathcal{H}^1((0,1) \setminus E_{\widetilde{\varepsilon}}) < \widetilde{\varepsilon}$ and $|f_k(t) - a \circ \gamma(t)| < \delta_k$ for every $t \in E_{\widetilde{\varepsilon}}$. The claim then follows by choosing $\widetilde{\varepsilon} := \varepsilon / \mathrm{Lip}(\gamma)$ and setting $\sigma_k(x) := f_k(\gamma^{-1}(x))$, $B_{\widetilde{\varepsilon}} := \gamma(E_{\widetilde{\varepsilon}})$. $\square$

4. Main results

Our main result is stated as follows.

Theorem 4.1. *Let $\mathcal{F}$ be the functional defined on $\mathcal{I}$ by (1), where $F : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ is a continuous function satisfying conditions (6) and Ω is an open connected subset of $\mathbb{R}^N$ enjoying condition (Ω). Then its relaxed functional (4) has the following integral representation:*

$$\overline{\mathcal{F}}(d) = \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x \tag{15}$$

for all $d \in \mathcal{D}$. In particular, $\overline{\mathcal{F}}(d) = \mathcal{F}(d)$ for all $d \in \mathcal{I}$.

The proof of the previous theorem is based on the following two results which we state separately.

Theorem 4.2. *If $d_n \xrightarrow{\mathcal{D}} d$, then* $\liminf_{n \to +\infty} \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \geq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x$.

Theorem 4.3. *Let Ω be an open connected subset of $\mathbb{R}^N$ which satisfies condition (Ω). Then the family $\mathcal{I}$ of distances induced by continuous and isotropic Riemannian metrics on $\overline{\Omega}$ is dense in $\mathcal{D}$. Moreover, for every $d \in \mathcal{D}$ we can choose a sequence $(d_n)_{n \in \mathbb{N}} \subset \mathcal{I}$ such that $d_n \xrightarrow{\mathcal{D}} d$ and*

$$\limsup_{n \to +\infty} \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \leq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x. \tag{16}$$

Remark 4.4. The class of distances induced by smooth isotropic Riemannian metrics is dense in $\mathcal{I}$. Therefore, by the theorem just stated, smooth isotropic Riemannian metrics are dense in the class of Finsler metrics. In fact, let us take a distance d in $\mathcal{I}$. Then $d = d_a$ for some continuous metric $a : \overline{\Omega} \to [\alpha, \beta]$. By Tietze's Lemma, we may extend a continuously to the whole $\mathbb{R}^N$ in such a way that $\alpha \leq a(x) \leq \beta$ for all $x \in \mathbb{R}^N$. Then, by taking a sequence of convolution kernels ρ_n, we define the sequence of smooth isotropic metrics $a_n : \overline{\Omega} \to [\alpha, \beta]$ by regularization, i.e. $a_n(x) := \rho_n * a(x)$, and we call d_n the induced distances. Since the functions a_n converge to a uniformly on compact subset of $\overline{\Omega} \times \overline{\Omega}$, it can be easily shown that the length functionals $\mathbb{L}_{a_n}$ Γ-converge to $\mathbb{L}_a$ on $\mathrm{Lip}(I, \overline{\Omega})$ with respect to the uniform convergence of curves. Then, by Remark 3.3 and Theorem 2.6, we have that $d_n \xrightarrow{\mathcal{D}} d$, as claimed.

Once Theorem 4.2 and Theorem 4.3 are proven, Theorem 4.1 will trivially follow. In fact, Theorem 4.2 gives that the functional (15) is lower semicontinuous with respect to the uniform convergence of distances, and Theorem 4.3 implies it is the greatest lower semicontinuous functional defined on $\mathcal{D}$ which is bounded from above by $\mathcal{F}$ on $\mathcal{I}$. Indeed, let $\mathcal{G}$ be another competitor and let $d \in \mathcal{D}$. Choose a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{I}$ as in the statement of Theorem 4.3. We have

$$\mathcal{G}(d) \leq \liminf_{n\to+\infty} \mathcal{G}(d_n) \leq \liminf_{n\to+\infty} \mathcal{F}(d_n) \leq \limsup_{n\to+\infty} \mathcal{F}(d_n) \leq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x,$$

hence the claim. The last statement in the claim of Theorem 4.1 is an immediate consequence of Proposition 3.1.

Let us then start by proving Theorem 4.2.

Proof of Theorem 4.2. The proof will be just sketched, since it is essentially an adaptation of the arguments described in [5, Section 3], where the case $F(x, s) := s$ was considered.

Let d_n, d be as in the statement of the theorem. We recall that the function $F : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ is continuous and fulfills conditions (6). The first result we state is the following:

- *Claim:* for every bounded Borel set $\omega \subset\subset \Omega$ and every $\xi \in \mathbb{S}^{N-1}$, we have

$$\int_\omega F\left(x, \varphi_d(x, \xi)\right)\mathrm{d}x \leq \liminf_{n\to+\infty} \int_\omega F\left(x, \varphi_{d_n}(x, \xi)\right)\mathrm{d}x\ . \tag{17}$$

The previous statement is the analogous of [5, Proposition 9] and can be proved similarly. Inequality (17) immediately gives the following:

$$\sup_{|\xi|=1} \int_\omega F\left(x, \varphi_d(x, \xi)\right)\mathrm{d}x \leq \liminf_{n\to+\infty} \int_\omega F\left(x, \Lambda_{d_n}(x)\right)\mathrm{d}x, \tag{18}$$

where we have also used the monotonicity assumption (6)-(i) made on F. In order to conclude, it will be therefore enough to prove the following:

- *Claim:* Let $\varphi \in \mathcal{M}$ and assume there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive Radon measures on Ω such that the following property holds:

$$\sup_{|\xi|=1} \int_\omega F\left(x, \varphi(x, \xi)\right)\mathrm{d}x \leq \liminf_{n\to+\infty} \mu_n\left(\omega\right) \quad \text{for every Borel set } \omega \subset\subset \Omega. \tag{19}$$

  Then

$$\int_\Omega F\big(x, \sup_{|\xi|=1} \varphi(x, \xi)\big)\mathrm{d}x \leq \liminf_{n\to+\infty} \mu_n\left(\Omega\right). \tag{20}$$

Indeed, in view of (18), the claim of the theorem would follow by applying the previous statement with $\varphi := \varphi_d$ and $\mu_n(\omega) := \int_\omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x$.

Let us prove (20). First, we reduce to consider the case of a bounded domain Ω. Indeed, if this is not the case, we take a sequence $(\Omega_l)_{l\in\mathbb{N}}$ of bounded and connected open sets well contained in Ω such that $\overline{\Omega}_l \subset \Omega_{l+1}$ and $\Omega = \bigcup_{l\in\mathbb{N}} \Omega_l$, and we notice that it is enough to prove that (20) holds for $\Omega := \Omega_l$ for each $l \in \mathbb{N}$.

Let us then assume that Ω is bounded and set $\Lambda_\varphi(x) := \sup_{|\xi|=1} \varphi(x,\xi)$ for all $x \in \Omega$. Following the proof of [5, Proposition 12], we consider three cases:

(1) let φ be continuous. Then (20) easily follows from the following lemma, which is analogous to [5, Lemma 10] and may be proved similarly.

Lemma 4.5. *Let Ω be a bounded open set and $\varphi \in \mathcal{M}$ be a continuous Finsler metric. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_{D_i^\delta} F(x, \Lambda_\varphi(x))\,\mathrm{d}x \le \sup_{|\xi|=1} \int_{D_i^\delta} \left[F(x, \varphi(x,\xi)) + \varepsilon\right]\mathrm{d}x \qquad \textit{for all } i \in \mathbb{Z}^N,$$

where we have set $D_i^\delta := \Omega \cap \delta\big(2i + [-1,1)^N\big)$.

In fact, fix $\varepsilon > 0$ and take $\delta > 0$ given by Lemma 4.5. We have:

$$\int_\Omega F(x, \Lambda_\varphi(x))\,\mathrm{d}x = \sum_{i\in\mathbb{Z}^N} \int_{D_i^\delta} F(x, \Lambda_\varphi(x))\mathrm{d}x \le \sum_{i\in\mathbb{Z}^N} \sup_{|\xi|=1} \int_{D_i^\delta} \left[F(x, \varphi(x,\xi)) + \varepsilon\right]\mathrm{d}x.$$

By assumption we get

$$\int_\Omega F(x, \Lambda_\varphi(x))\,\mathrm{d}x \le \sum_{i\in\mathbb{Z}^N} \left[\liminf_{n\to+\infty} \mu_n(D_i^\delta) + \int_{D_i^\delta} \varepsilon\,\mathrm{d}x\right] \le \liminf_{n\to+\infty} \mu_n(\Omega) + \varepsilon\mathcal{L}^N(\Omega)$$

and (20) follows since ε was arbitrary.

(2) Let φ be lower semicontinuous and $\varphi(x,\cdot)$ convex for every $x \in \overline{\Omega}$. Thanks to Lemma 2.2.3 in [4], we can find an increasing sequence of continuous Finsler metrics $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that $\varphi(x,\xi) = \sup_{k\in\mathbb{N}} \varphi_k(x,\xi)$ for all $(x,\xi) \in \overline{\Omega}\times\mathbb{R}^N$. Each metric φ_k is continuous and still satisfies condition (19), so we can use the previous step to infer

$$\sup_{k\in\mathbb{N}} \int_\Omega F(x, \Lambda_{\varphi_k}(x))\,\mathrm{d}x \le \liminf_{n\to+\infty} \mu_n(\Omega).$$

The claim then follows as, by the monotone convergence theorem, we have:

$$\sup_{k\in\mathbb{N}} \int_\Omega F(x, \Lambda_{\varphi_k}(x))\,\mathrm{d}x = \int_\Omega \sup_{k\in\mathbb{N}} F(x, \Lambda_{\varphi_k}(x))\,\mathrm{d}x = \int_\Omega F(x, \Lambda_\varphi(x))\,\mathrm{d}x.$$

(3) We now make no additional regularity hypothesis on φ: we only assume it belongs to $\mathcal{M}$. First, notice that it is not restrictive to assume that $\varphi(x,\cdot)$ is convex for every $x \in \overline{\Omega}$: it is actually sufficient to redefine the metric φ by setting $\varphi(x,\xi) := \beta|\xi|$ on a negligible Borel subset of $\overline{\Omega}$ which contains all the points where the metric is not

convex. We can then apply [5, Lemma 11]: for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \overline{\Omega}$ such that $\mathcal{L}^N(\overline{\Omega} \setminus K_\varepsilon) < \varepsilon$ and $\varphi|_{K_\varepsilon \times \mathbb{R}^N}$ is continuous. We define

$$\varphi^\varepsilon(x,\xi) := \begin{cases} \varphi(x,\xi) & \text{if } x \in K_\varepsilon, \\ \beta\,|\xi| & \text{otherwise.} \end{cases}$$

Notice that φ^ε is lower semicontinuous, so we can apply the previous step with μ_n replaced by $\widetilde{\mu}_n := \mu_n + F(x,\beta)\chi_{\Omega\setminus K_\varepsilon}(x)\,d\mathcal{L}^N$ to get

$$\begin{aligned} \int_\Omega F(x,\Lambda_\varphi(x))\,\mathrm{d}x &\le \int_\Omega F(x,\Lambda_{\varphi^\varepsilon}(x))\,\mathrm{d}x \le \liminf_{n\to+\infty} \widetilde{\mu}_n(\Omega) \\ &= \liminf_{n\to+\infty} \mu_n(\Omega) + \int_{\Omega\setminus K_\varepsilon} F(x,\beta)\,\mathrm{d}x. \end{aligned}$$

As $F(x,\beta)$ is summable over Ω (condition (6)-(ii)), the integral appearing in the most right-hand side of the above inequality goes to 0 as $\varepsilon \to 0^+$. The claim hence follows as ε was arbitrarily chosen. □

Remark 4.6. The above proof still works for slightly more general functionals. Indeed, it is sufficient that there exists a sequence of continuous functions $F_k : \Omega \times [\alpha,\beta] \to \mathbb{R}_+$ which satisfy conditions (6) and such that $F(x,\xi) = \sup_k F_k(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In fact, one can apply the above argument to each F_k to get

$$\int_\Omega F_k(x,\Lambda_d(x))\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega F(x,\Lambda_{d_n}(x))\,\mathrm{d}x,$$

and the claim immediately follows by taking the supremum over k of the left-hand side term and by the monotone convergence theorem.

We now come to the proof of Theorem 4.3: for any fixed $d \in \mathcal{D}$, we want to find a sequence $(d_n)_{n\in\mathbb{N}} \subset I$ which converges to d and enjoys (16). As pointed out in Lemma 3.4, in the approximating procedure one only needs to control convergence of distances on a dense subset of $\overline{\Omega} \times \overline{\Omega}$. To this aim, we set $S := \mathbb{Q}^N \cap \overline{\Omega}$. Obviously $S \times S$ is countable and dense in $\overline{\Omega} \times \overline{\Omega}$, so we write $S \times S := \{(x_i,y_i) \,|\, i \in \mathbb{N}\}$.

As a preliminary step, we first approximate $d \in \mathcal{D}$ with distances induced by a sequence of Borel measurable and isotropic Riemannian metrics.

Theorem 4.7. *Let $d \in \mathcal{D}$. Then there exists a decreasing sequence of Borel measurable isotropic metrics $a_n : \overline{\Omega} \to [\alpha,\beta]$ such that*

(i) $d_{a_n}(x_i,y_i) = d(x_i,y_i)$ *for each* $i \le n$;
(ii) $a_n(x) = \Lambda_d(x)$ *for* $\mathcal{L}^N$*-a.e.* $x \in \Omega$.

In particular $d_{a_n} \xrightarrow{\mathcal{D}} d$. *Moreover, if we set* $a(x) := \inf_{n\in\mathbb{N}} a_n(x)$, *we have that* $d_a = d$ *on* $\overline{\Omega} \times \overline{\Omega}$, *that is every Finsler distance is induced by a Borel measurable, isotropic Riemannian metric.*

Proof. For each $(x_i,y_i) \in S \times S$ let $\gamma_i \in \mathrm{Lip}_{x_i,y_i}$ be a path of minimal d-length, i.e. $\mathrm{L}_d(\gamma_i) = d(x_i,y_i)$. Such family of curves $\{\gamma_i \,|\, i \in \mathbb{N}\}$ can be chosen in such a way to satisfy conditions (i) and (ii) of Lemma 3.6 (this assumption is not really needed here, but will be important in the proof of Theorem 4.3). By condition (ii), each non-empty set $\gamma_i(I) \cap \gamma_j(I)$ is a disjoint finite union of closed arcs. Let us fix $n \in \mathbb{N}$ and denote by T_n the finite set given by the extreme points of such arcs for every $1 \le i \le j \le n$. Set $N_n := \cup_{i \le n} \gamma_i(I)$ and let Σ_n be a Borel $\mathcal{H}^1$-negligible subset of N_n such that

$$N_n \setminus \Sigma_n \subset \bigcup_{i=1}^{n} \left\{ \gamma_i(t) \;\middle|\; t \in I, \varphi_d\left(\gamma_i(t), \frac{\dot\gamma_i(t)}{|\dot\gamma_i(t)|}\right) \in [\alpha, \beta] \right\}.$$

Observe that, in view of Remark 2.10, such a set Σ_n always exists. Then we define the function $a_n : \overline{\Omega} \to [\alpha, \beta]$ by

$$a_n(x) := \begin{cases} \beta & \text{if } x \in \partial\Omega \setminus N_n \\ \Lambda_d(x) & \text{if } x \in \Omega \setminus N_n \\ \alpha & \text{if } x \in T_n \cup \Sigma_n \\ \varphi_d\left(\gamma_i(t), \dfrac{\dot\gamma_i(t)}{|\dot\gamma_i(t)|}\right) & \text{if } x = \gamma_i(t) \in N_n \setminus (T_n \cup \Sigma_n) \end{cases} \tag{21}$$

It can be easily checked that a_n is well defined and Borel measurable. Moreover it is clear that a_n satisfies point (ii) of the claim. Let d_{a_n} be the distance generated by the metric defined by a_n for each $n \in \mathbb{N}$. We want to prove point (i) of the claim. Let us fix an $i \le n$. Then we have

$$d_{a_n}(x_i, y_i) \le \int_0^1 a_n(\gamma_i)|\dot\gamma_i| \,\mathrm{d}t = \int_0^1 \varphi_d(\gamma_i, \dot\gamma_i) \,\mathrm{d}t = d(x_i, y_i).$$

To prove the reverse inequality, choose a curve $\sigma \in \mathrm{Lip}_{x_i, y_i}$ and, for every $1 \le j < n$, set $I_{j+1} := \{t \in I \setminus \cup_{h \le j} I_h \mid \sigma(t) \in \gamma_{j+1}(I)\}$, $I_1 := \{t \in I \mid \sigma(t) \in \gamma_1(I)\}$ and $I_0 := I \setminus \cup_{j \le n} I_j$. We remark that the vector $\dot\sigma(t)$ is tangent to $\gamma_j(I)$ at $\sigma(t)$ for $\mathcal{L}^1$-a.e. $t \in I_j$, and so $a_n(\sigma)|\dot\sigma| = \varphi_d(\sigma, \dot\sigma)$ $\mathcal{L}^1$-a.e. on I_j. Therefore we have

$$\begin{aligned} \mathbb{L}_{a_n}(\sigma) &= \int_0^1 a_n(\sigma)|\dot\sigma| \,\mathrm{d}t = \sum_{j=1}^{n} \int_{I_j} a_n(\sigma)|\dot\sigma| \,\mathrm{d}t + \int_{I_0} a_n(\sigma)|\dot\sigma| \,\mathrm{d}t \\ &\ge \sum_{j=1}^{n} \int_{I_j} \varphi_d(\sigma, \dot\sigma) \,\mathrm{d}t + \int_{I_0} \varphi_d(\sigma, \dot\sigma) \,\mathrm{d}t \ge d(x_i, y_i), \end{aligned}$$

where we have used the fact that $a_n(\sigma)|\dot\sigma| \ge \varphi_d(\sigma, \dot\sigma)$ on I_0. By passing to the infimum over all possible curves $\sigma \in \mathrm{Lip}_{x_i, y_i}$ we get the claim.

Notice that $N_n \subset N_{n+1}$, and we may as well suppose that $\Sigma_n \subset \Sigma_{n+1}$ (otherwise, replace Σ_{n+1} with $\Sigma_n \cup \Sigma_{n+1}$), therefore $(a_n)_{n \in \mathbb{N}}$ is a decreasing sequence of functions. If we set $a(x) := \inf_{n \in \mathbb{N}} a_n(x)$, by applying Lemma 3.5 we get that $d_{a_n} \xrightarrow{\mathcal{D}} d_a$. In particular we have

$$d_a(x_i, y_i) = \lim_{n \to +\infty} d_{a_n}(x_i, y_i) = d(x_i, y_i)$$

for every $i \in \mathbb{N}$. Therefore $d_a = d$ on a dense subset of $\overline{\Omega} \times \overline{\Omega}$ and hence d_a coincides with d by continuity. That concludes the proof of the claim. $\square$

The metrics $(a_n)_{n\in\mathbb{N}}$ above defined will be now used to construct the required approximating sequence of distances.

Proof of Theorem 4.3. The proof is organized in two steps.

Step 1. We first remark that the closure of $\mathcal{I}$ contains the family of distances generated by lower semicontinuous isotropic Riemannian metrics. In fact, let $b : \Omega \to [\alpha, \beta]$ be a lower semicontinuous metric. It is well known that $b(x) = \sup_{n\in\mathbb{N}} \tilde{a}_n(x)$ for suitable continuous functions $\tilde{a}_n$ (and we may as well suppose that $\alpha \le \tilde{a}_n \le \beta$ by possibly replacing the function $\tilde{a}_n$ with $\tilde{a}_n \vee \alpha$). Setting $a_n(x) := \sup_{i\le n} \tilde{a}_i(x)$, we have that $d_{a_n} \xrightarrow{\mathcal{D}} d_b$ by Lemma 3.5. Moreover, by Proposition 3.1 we have that $\Lambda_{d_b}(x) = b(x)$ and $\Lambda_{d_{a_n}}(x) = a_n(x)$ almost everywhere on Ω and therefore, by the monotonicity assumption (6)-(i) made on F, we obviously have

$$\limsup_{n\to+\infty} \int_\Omega F(x, \Lambda_{d_{a_n}}(x))\,\mathrm{d}x \le \int_\Omega F(x, \Lambda_b(x))\,\mathrm{d}x.$$

To prove the theorem, it is then sufficient to find a sequence of lower semicontinuous metrics $b_n : \Omega \to [\alpha, \beta]$ such that the generated distances d_{b_n} satisfy the claim of the theorem. Indeed, by combining the idea just described with a diagonal argument, the conclusion would follow at once.

Step 2. To get the desired approximation of the distance $d \in \mathcal{D}$ via lower semicontinuous isotropic metrics, it is enough to prove that, for every fixed $n \in \mathbb{N}$, there exists a sequence of lower semicontinuous isotropic metrics $b_k : \overline{\Omega} \to [\alpha, \beta]$ such that

(i) $\lim_{k\to+\infty} d_{b_k}(x_i, y_i) = d(x_i, y_i)$ for every $i \le n$;

(ii) $\limsup_{k\to+\infty} \int_\Omega F(x, b_k(x))\,\mathrm{d}x \le \int_\Omega F(x, a_n(x))\,\mathrm{d}x$

where a_n are the Borel isotropic metrics defined in the proof of Theorem 4.7.

In fact the desired sequence of lower semicontinuous metrics is then obtained via a diagonal argument and taking into account that $a_n(x) = \Lambda_d(x)$ almost everywhere on Ω by Theorem 4.7.

Let us then fix $n \in \mathbb{N}$ and let a_n be the Borel metric defined by (21). Keeping the notations used in the proof of Theorem 4.7, we observe that the set $N_n \setminus T_n$ is a finite, disjoint union of open arcs. Therefore, by applying Lemma 3.7 to each arc, we can find a sequence of continuous functions $\sigma_k : N_n \setminus T_n \to [\alpha, \beta]$ which converge to a_n $\mathcal{H}^1$-a.e. on $N_n \setminus T_n$. Let us set $A_k := \{x \in \overline{\Omega} \,|\, \mathrm{dist}(x, N_n) < 1/k\}$. Let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of bounded open sets well contained in Ω such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$. By Lusin's theorem we may find a sequence of closed set $K_k \subset \overline{\Omega}_k \setminus A_k$ such that $a_n\,|_{K_k}$ is continuous and $\mathcal{L}^N((\overline{\Omega}_k \setminus A_k) \setminus K_k) < 1/k$. Then we define $b_k : \overline{\Omega} \to [\alpha, \beta]$ by

$$b_k(x) := \begin{cases} \sigma_k(x) & \text{if } x \in N_n \setminus T_n \\ \alpha & \text{if } x \in T_n \\ a_n(x) & \text{if } x \in K_k \\ \beta & \text{elsewhere.} \end{cases}$$

Notice that b_k is lower semicontinuous. Moreover we have

$$\limsup_{k\to+\infty}\int_\Omega F(x,b_k(x))\,\mathrm{d}x=\limsup_{k\to+\infty}\left(\int_{K_k}F(x,a_n(x))\,\mathrm{d}x+\int_{\Omega\setminus K_k}F(x,\beta)\,\mathrm{d}x\right). \tag{22}$$

Recalling that $F(x,\beta)$ is summable over Ω (condition (6)-(ii)), we have that the second integral at the right-hand side of (22) goes to zero. In fact

$$\int_{\Omega\setminus K_k}F(x,\beta)\,\mathrm{d}x=\int_{\Omega\setminus\Omega_k}F(x,\beta)\,\mathrm{d}x+\int_{\Omega_k\setminus K_k}F(x,\beta)\,\mathrm{d}x, \tag{23}$$

and the first and second term of the right-hand side of (23) go to zero by the dominated convergence theorem and the absolute continuity of the integral respectively. Therefore

$$\limsup_{k\to+\infty}\int_\Omega F(x,b_k(x))\,\mathrm{d}x\le\int_\Omega F(x,a_n(x))\,\mathrm{d}x,$$

so point (ii) of the claim is satisfied.
Let us show now that (i) holds. For $i\le n$ we have by definition

$$d_{b_k}(x_i,y_i)\le\mathbb{L}_{b_k}(\gamma_i)=\int_0^1\sigma_k(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t,$$

therefore by the dominated convergence theorem we get

$$\begin{aligned}\limsup_{k\to+\infty}d_{b_k}(x_i,y_i)&\le\limsup_{k\to+\infty}\int_0^1\sigma_k(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t=\int_0^1 a_n(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t\\&=\int_0^1\varphi_d(\gamma_i,\dot\gamma_i)\,\mathrm{d}t=d(x_i,y_i).\end{aligned} \tag{24}$$

Now, let us take for every $k\in\mathbb{N}$ a curve $\tilde\gamma_k\in\mathrm{Lip}_{x_i,y_i}$ such that

$$\mathbb{L}_{b_k}(\tilde\gamma_k)=d_{b_k}(x_i,y_i). \tag{25}$$

Notice that such a curve exists in view of Proposition 2.5 and Remark 3.3. Once again, we remark that, by Lemma 3.6, it is not restrictive to suppose that such curves are injective. Since $\alpha\int_I|\dot{\tilde\gamma}_k|\,\mathrm{d}t\le\mathbb{L}_{b_k}(\tilde\gamma_k)$, by (25) and (24) we get that $\limsup_k\int_I|\dot{\tilde\gamma}_k|\,\mathrm{d}t<+\infty$. Let us choose an $\varepsilon>0$. By applying Lemma 3.7 to each open arc of $N_n\setminus T_n$, we can find a Borel set $B_\varepsilon\subset N_n\setminus T_n$ and an infinitesimal sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that $\mathcal{H}^1(N_n\setminus B_\varepsilon)<\varepsilon$ and $|\sigma_k(x)-a_n(x)|<\delta_k$ for every $x\in B_\varepsilon$. Let us set $I_k:=\{t\in I|\tilde\gamma_k(t)\in N_n\setminus B_\varepsilon\}$. Then $b_k(\tilde\gamma_k)\ge a_n(\tilde\gamma_k)-\delta_k$ $\mathcal{L}^1$-a.e. on $I\setminus I_k$. Let us write

$$\mathbb{L}_{b_k}(\tilde\gamma_k)=\int_{I_k}b_k(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t+\int_{I\setminus I_k}b_k(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t.$$

We remark that, as $\tilde\gamma_k(I_k)\subset N_n\setminus B_\varepsilon$ for every $k\in\mathbb{N}$, by the Area-formula we have

$$\int_{I_k}|\dot{\tilde\gamma}_k|\,\mathrm{d}t=\mathcal{H}^1(\tilde\gamma_k(I_k))\le\mathcal{H}^1(N_n\setminus B_\varepsilon)<\varepsilon.$$

Taking this remark into account we get

$$\begin{aligned}\int_{I_k} b_k(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t &= \int_{I_k} a_n(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t + \int_{I_k} (b_k(\tilde\gamma_k) - a_n(\tilde\gamma_k))|\dot{\tilde\gamma}_k|\mathrm{d}t \\ &\geq \int_{I_k} a_n(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t - (\beta-\alpha)\varepsilon.\end{aligned}$$

Then we have

$$\begin{aligned}\mathbb{L}_{b_k}(\tilde\gamma_k) &\geq \int_0^1 a_n(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\,\mathrm{d}t - \delta_k \int_{I\setminus I_k} |\dot{\tilde\gamma}_k|\,\mathrm{d}t - (\beta-\alpha)\varepsilon \\ &\geq d_{a_n}(x_i,y_i) - \delta_k \int_0^1 |\dot{\tilde\gamma}_k|\,\mathrm{d}t - (\beta-\alpha)\varepsilon\end{aligned}$$

and therefore, as $\delta_k \int_0^1 |\dot{\tilde\gamma}_k|\,\mathrm{d}t$ goes to zero when $k \to +\infty$, we obtain

$$\liminf_{k\to+\infty} d_{b_k}(x_i,y_i) \geq \liminf_{k\to+\infty} \mathbb{L}_{b_k}(\tilde\gamma_k) \geq d_{a_n}(x_i,y_i) - (\beta-\alpha)\varepsilon.$$

Since ε was arbitrary, the above inequality coupled with (24) gives the claim. □

Remark 4.8. It should be noticed that the proof of Theorem 4.3 holds under very general assumptions on the function F, namely it is sufficient to take an F which is Borel measurable and satisfies assumption (ii) of (6), and such that the function $F(x,\cdot)$ is non-decreasing for $\mathcal{L}^N$-a.e. $x \in \Omega$. This consideration, together with Remark 4.6, enables us to conclude that our relaxation result, namely Theorem 4.1, holds under the following milder conditions on $F : \Omega \times [\alpha,\beta] \to \mathbb{R}_+$:

(i) there exists a sequence of continuous functions $F_k : \Omega \times [\alpha,\beta] \to \mathbb{R}_+$ satisfying conditions (6) and such that $F(x,\xi) = \sup_{k\in\mathbb{N}} F_k(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$;

(ii) $\int_\Omega F(x,\beta)\,\mathrm{d}x < +\infty$.

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