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Γ-Convergence for the Irrigation Problem

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In this paper we study the asymptotics of the functional $F(\gamma) = \int f(x) d_\gamma(x)^p dx$, where d_γ is the distance function to γ, among all connected compact sets γ of given length, when the prescribed length tends to infinity. After properly scaling, we prove the existence of a Γ-limit in the space of probability measures, thus retrieving information on the asymptotics of minimal sequences.

1. Introduction

Assume Ω is a bounded, connected open set with Lipschitz boundary in $\mathbb{R}^d$, $d \geq 2$, and let $\Sigma(\Omega)$ denote the class of all compact, connected sets $\gamma \subset \overline{\Omega}$ of finite one-dimensional Hausdorff measure $\mathcal{H}^1(\gamma)$ (we wil often refer to this quantity as the "length of γ"). The so called "irrigation problem", i.e. the problem of minimizing $\int_\Omega d_\gamma(x)dx$, the integral of the distance function to γ, among all $\gamma \in \Sigma(\Omega)$ of prescribed length $\mathcal{H}^1(\gamma) = l$ was considered in [6] in connection with mass transportation problems (see [1]). In particular, the problem of studying the asymptotics of the minimizers as $l \to \infty$ was raised in [6]. In this paper, we study the asymptotics as $l \to \infty$ of the functionals

$$F_l(\gamma) = \begin{cases} l^{\frac{p}{d-1}} \int_\Omega f(x)\, d_\gamma(x)^p\, dx, & \text{if } \gamma \in \Sigma(\Omega) \text{ and } \mathcal{H}^1(\gamma) = l, \\ +\infty, & \text{otherwise.} \end{cases}$$

Throughout, $f \in L^1(\Omega)$ is a non negative given function, d_γ denotes the distance function to the set γ and $p > 0$ is a given number. The term $l^{\frac{p}{d-1}}$ is a normalization which prevents the functionals to degenerate (indeed, in [6] it was proved that, when $p = 1$, $\min_\gamma F_l(\gamma) = O(l^{-1/(d-1)})$ as $l \to \infty$).

A direct link to mass transportation problems is provided by the observation ([4]) that, for any set γ in $\mathbb{R}^d$ and $p \geq 1$, there holds

$$\int_\Omega f(x)\, d_\gamma(x)^p\, dx = \inf \left\{ W_p(f, \nu)^p \mid \nu \in \mathcal{M}(\mathbb{R}^d),\, \nu(\mathbb{R}^d) = \int_\Omega f,\, \operatorname{spt} \nu \subset \gamma \right\},$$

where $\mathcal{M}$ denotes the space of finite measures and W_p is the Wasserstein distance between measures of equal mass (see [1]). Our main result concerns the asymptotics as $l \to \infty$ of the functionals F_l, and can be stated in terms of Γ-convergence: we refer the reader to [5] for an introduction to this subject and for the terminology related to Γ-convergence (see also [7]). To this purpose, it is convenient to associate with $\gamma \in \Sigma(\Omega)$ the probability

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measure $\mathcal{H}^1(\gamma)^{-1}\mathcal{H}^1 \llcorner \gamma$ (i.e. normalized Hausdorff measure restricted to γ) and regard F_l as a functional defined on $\mathcal{P}(\overline{\Omega})$, the space of probability measures supported in $\overline{\Omega}$, as follows:

$$F_l(\mu) = \begin{cases} l^{\frac{p}{d-1}} \displaystyle\int_\Omega f(x)\, d_\gamma(x)^p \, dx, & \text{if } \mu = l^{-1}\mathcal{H}^1 \llcorner \gamma \text{ for some} \\ & \gamma \in \Sigma(\Omega) \text{ such that } \mathcal{H}^1(\gamma) = l, \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 1.1. *The functionals F_l defined in* (1) *Γ-converge, with respect to the weak-∗ topology on $\mathcal{P}(\overline{\Omega})$, to the functional F_∞ defined on $\mathcal{P}(\overline{\Omega})$ as*

$$F_\infty(\mu) = \theta_{d,p} \int_\Omega \frac{f(x)}{\rho(x)^{\frac{p}{d-1}}} \, dx, \tag{2}$$

where $\rho \in L^1(\Omega)$ is the density (Radon-Nikodym derivative) of μ with respect to Lebesgue measure, and $\theta_{d,p}$ is a positive constant which depends only on the dimension d and on the exponent p (the fraction in the integral is understood to be zero at those points x where $f(x)$ and $\rho(x)$ vanish simultaneously).

The constant $\theta_{d,p}$ is defined for every $d \geq 2$ and every $p > 0$ as follows:

$$\theta_{d,p} = \inf \left\{ \liminf_{n\to\infty} \mathcal{H}^1(\gamma_n)^{\frac{p}{d-1}} \int_{I^d} d_{\gamma_n}(x)^p \, dx \right\}, \tag{3}$$

where $I^d = [0,1]^d$ is the unit cube in $\mathbb{R}^d$ and the infimum is taken over all sequences of sets $\{\gamma_n\}$ such that $\gamma_n \in \Sigma(I^d)$ and $\lim_n \mathcal{H}^1(\gamma_n) = \infty$.

Formula (3) does not come as a surprise, if one expects a Γ-limit of the kind (2): indeed, the expression for $\theta_{d,p}$ can be guessed by suitable localization and scaling arguments that are customary in Γ-convergence results (we refer to [5] for more details along this line).

We point out that we are not able to compute $\theta_{d,p}$ explicitly except when $d = 2$ (Theorem 4.4), and hence in dimension $d > 2$ the Γ-limit (2) is explicit up to a multiplicative constant. This happens also in [4] for a related problem (the so called "location problem"), where connected sets are replaced by finite sets of given cardinality and a different rescaling is adopted. However, in Theorem 4.3 we provide a lower bound for $\theta_{d,p}$.

We remark that, using the techniques of this paper, the Γ-convergence result in [4] can be proved without assuming that f is semicontinuous.

Note that the Γ-limit functional F_∞ in (2) has a unique minimizer in $\mathcal{P}(\overline{\Omega})$. Indeed,

$$\min_{\mu \in \mathcal{P}(\overline{\Omega})} F_\infty(\mu) = \min_{\substack{\rho \geq 0 \\ \int_\Omega \rho \leq 1}} \theta_{d,p} \int_\Omega \frac{f(x)}{\rho(x)^{\frac{p}{d-1}}} \, dx = \theta_{d,p} \left(\int_\Omega f(x)^{\frac{d-1}{p+d-1}} \, dx \right)^{\frac{p+d-1}{d-1}},$$

obtained choosing $\rho = f^{(d-1)/(p+d-1)} / \int_\Omega f^{(d-1)/(p+d-1)}$. Therefore, since $\mathcal{P}(\overline{\Omega})$ is compact with respect to the weak-∗ topology, as a consequence of Theorem 1.1 (see §7 in [7]) we have the following

Corollary 1.2. *There holds*

$$\lim_{l\to\infty}\min_{\gamma\in\Sigma(\Omega)} F_l(\gamma)=\theta_{d,p}\left(\int_\Omega f(x)^{\frac{d-1}{p+d-1}}\,dx\right)^{\frac{p+d-1}{d-1}}.$$

Moreover, if γ_n is a minimizer of F_{l_n} and $l_n\to\infty$, then the probability measures $\mathcal{H}^1(\gamma_n)^{-1}\mathcal{H}^1 \llcorner \gamma_n$ converge in the weak-$*$ topology to the probability measure $\mu=\rho\,dx$ with $\rho=f^{(d-1)/(p+d-1)}/\int_\Omega f^{(d-1)/(p+d-1)}$.

In the case where $f\equiv 1$ this corollary formalizes the intuitive idea that, for a sequence of minimizers γ_n of larger and larger length, the "length of γ_n per unit area" should tend to a constant. In fact in two dimensions, it turns out that the comb–shaped sets C_n constructed in the proof of Theorem 4.4 are asymptotically optimal when Ω is a square, and in the case of a generic domain Ω one can build an asymptotically optimal sequence dividing Ω into small squares and reproducing the rescaled comb–shaped sets in every square (as one can see from the proof of the Γ-limsup inequality in Section 3). Finding explicit asymptotically optimal sequences remains an open problem, however, in higher dimension.

2. Preliminary results

Throughout, $I^d=[0,1]^d$ is the unit cube in $\mathbb{R}^d$ ($d\ge 2$), $|E|$ denotes the Lebesgue measure of $E\subset\mathbb{R}^d$, $\Sigma(E)$ denotes the class of all compact, connected sets $\gamma\subseteq\overline{E}$ such that $\mathcal{H}^1(\gamma)<+\infty$, and $\lfloor x\rfloor$ denotes the integer part of $x\in\mathbb{R}$.

Remark 2.1. Given a set $\Gamma\in\Sigma(I^d)$, we say that Γ is *tiling* if $\Gamma\cap\partial I^d$ coincides with the 2^d vertices of I^d. Moreover, we call *periodic $\frac{1}{k}$-extension of* Γ *inside* I^d the set

$$\Gamma^k:=\bigcup_{\substack{x\in k^{-1}\mathbb{Z}^d\\ x+k^{-1}I^d\subset I^d}}\left(x+k^{-1}\Gamma\right),$$

made of k^d copies of Γ, scaled to a factor $1/k$ and fit into I^d in the usual way. If Γ is tiling, then Γ^k remains connected and

$$\mathcal{H}^1(\Gamma^k)=k^{d-1}\mathcal{H}^1(\Gamma). \tag{4}$$

Moreover, by scaling one can check that

$$\mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}}\int_{I^d} d^p_{\Gamma^k}\le k^p\mathcal{H}^1(\Gamma)^{\frac{p}{d-1}}\sum_{\substack{x\in k^{-1}\mathbb{Z}^d\\ x+k^{-1}I^d\subset I^d}}\int_{k^{-1}I^d} d^p_{k^{-1}\Gamma}=\mathcal{H}^1(\Gamma)^{\frac{p}{d-1}}\int_{I^d} d^p_\Gamma.$$

Lemma 2.2. *Let $\Gamma\in\Sigma(I^d)$ be a tiling set, and let Γ^k denote the periodic $\frac{1}{k}$-extension of Γ. Then*

$$\mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} d^p_{\Gamma^k}\overset{*}{\rightharpoonup} g\qquad \text{in } L^\infty(I^d), \tag{5}$$

where g is a constant such that

$$g\le\mathcal{H}^1(\Gamma)^{\frac{p}{d-1}}\int_{I^d} d^p_\Gamma. \tag{6}$$

Moreover, the probability measures $\mathcal{H}^1(\Gamma^k)^{-1}\mathcal{H}^1\llcorner\Gamma^k$ converge to the Lebesgue measure, in the weak-$*$ topology of $\mathcal{P}(I^d)$.

Proof. Let $\widehat{\Gamma} = \bigcup_{x\in\mathbb{Z}^d}(x+\Gamma)$ denote the periodic extension of Γ, and let $g_k(x) = d_{\widehat{\Gamma}}(kx)$. It is well known that $g_k^p \overset{*}{\rightharpoonup} \widehat{g}$ in $L^\infty(I^d)$ where

$$\widehat{g} = \int_{I^d} d_{\widehat{\Gamma}}^p \leq \int_{I^d} d_\Gamma^p. \tag{7}$$

One can easily check that $d_{\Gamma^k}(x) = k^{-1}g_k(x)$ if $x \in I^d$ and $1/k < d_{\partial I^d}(x)$, and hence $k^p d_{\Gamma^k}^p \overset{*}{\rightharpoonup} \widehat{g}$ in $L^\infty_{\mathrm{loc}}(I^d)$. Combining this with the uniform bound

$$k^p d_{\Gamma^k}(x)^p \leq g_k(x) = d_{\widehat{\Gamma}}(kx) \leq \|d_{\widehat{\Gamma}}\|_{L^\infty(\mathbb{R}^d)} \quad \forall x \in I^d,$$

one obtains that in fact $k^p d_{\Gamma^k}^p \overset{*}{\rightharpoonup} \widehat{g}$ in $L^\infty(I^d)$. Therefore, one obtains (5) and (6) using (4), (7) and letting $g = \mathcal{H}^1(\Gamma)^{\frac{p}{d-1}}\widehat{g}$.

Finally, the last part of the claim is immediate. □

Proposition 2.3. *Given $\varepsilon > 0$, for every l large enough (depending on ε) there exists a set $C \in \Sigma(I^d)$ such that C is tiling, $\mathcal{H}^1(C) = l$ and*

$$\mathcal{H}^1(C)^{\frac{p}{d-1}} \int_{I^d} d_C^p \leq (1+\varepsilon)\theta_{d,p},$$

where $\theta_{d,p}$ is defined by (3).

Proof. Given $\varepsilon > 0$, by the definition of $\theta_{d,p}$ (3) there exists a set $\gamma \in \Sigma(I^d)$ such that

$$\mathcal{H}^1(\gamma)^{\frac{p}{d-1}} \int_{I^d} d_\gamma^p \;<\; (1+\varepsilon/4)\theta_{d,p}, \tag{8}$$

$$\left(\frac{\mathcal{H}^1(\gamma) + 2^d\sqrt{d}}{\mathcal{H}^1(\gamma)}\right)^{\frac{p}{d-1}} \;<\; \frac{1+\varepsilon/2}{1+\varepsilon/4}. \tag{9}$$

Replacing γ with the set $(1-2\delta)\gamma + (\delta,\ldots,\delta)$, where $\delta \in (0,1)$ is so small that (8), (9) still hold, we may suppose that $\gamma \cap \partial I^d = \emptyset$. Letting $\Gamma = \gamma \cup \bigcup_{i=1}^{2^d} s_i$, where $s_i \subset I^d$ is the shortest segment joining γ to the i-th vertex of I^d, we have that $\Gamma \in I^d$ is tiling and (8), (9) yield

$$\mathcal{H}^1(\Gamma)^{\frac{p}{d-1}} \int_{I^d} d_\Gamma^p \leq \left(\mathcal{H}^1(\gamma) + 2^d\sqrt{d}\right)^{\frac{p}{d-1}} \int_{I^d} d_\gamma^p < (1+\varepsilon/2)\theta_{d,p}. \tag{10}$$

If $l > 0$ is large enough, letting $k = \left\lfloor (l/\mathcal{H}^1(\Gamma))^{1/(d-1)} \right\rfloor$ we have

$$l \leq \mathcal{H}^1(\Gamma)(k+1)^{d-1}, \qquad \left(1+\frac{1}{k}\right)^p \leq \frac{1+\varepsilon}{1+\varepsilon/2}. \tag{11}$$

Let Γ^k be the periodic $\frac{1}{k}$-extension of Γ inside I^d: since Γ is tiling by construction, we have from Remark 2.1 and (10)

$$\mathcal{H}^1(\Gamma^k) = k^{d-1}\mathcal{H}^1(\Gamma), \quad \mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} \int_{I^d} d_{\Gamma^k}^p \leq (1+\varepsilon/2)\theta_{d,p}. \tag{12}$$

To complete the proof, set $C := \Gamma^k \cup \Delta$, where $\Delta \subset I^d$ is any compact set such that $\mathcal{H}^1(\Gamma^k \cup \Delta) = l$ and $\Gamma^k \cup \Delta$ is connected. Then (12), (11) yield

$$\mathcal{H}^1(C)^{\frac{p}{d-1}} \int_{I^d} d_C^p = l^{\frac{p}{d-1}} \int_{I^d} d^p_{\Gamma^k \cup \Delta} \leq \left(\frac{l}{\mathcal{H}^1(\Gamma^k)} \right)^{\frac{p}{d-1}} \mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} \int_{I^d} d^p_{\Gamma^k}$$
$$\leq \left(\frac{l}{k^{d-1}\mathcal{H}^1(\Gamma)} \right)^{\frac{p}{d-1}} (1+\varepsilon/2)\theta_{d,p} \leq \left(1 + \frac{1}{k}\right)^p (1+\varepsilon/2)\theta_{d,p} \leq (1+\varepsilon)\theta_{d,p}.$$

□

Definition 2.4. For every integer $k \geq 1$, we call *grid of order* k the set $G_k \subset I^d$ of those $(x_1, \ldots, x_d) \in I^d$ such that kx_i is an integer number for every coordinate $i \in \{1, \ldots, d\}$ except at most one. One can check that G_k is made of $d(k+1)^d$ unitary segments, each orthogonal to some face of I^d. Moreover, G_k is connected,

$$\mathcal{H}^1(G_k) = d(k+1)^d, \qquad \text{and} \qquad d_{G_k}(x) \leq \frac{C}{k} \quad \forall x \in I^d, \tag{13}$$

where C depends only on the dimension.

Lemma 2.5. *Given h points $y_1, \ldots, y_h$ in the unit cube I^d, there exists a connected compact set $E \subset I^d$ such that $y_i \in E$, $1 \leq i \leq h$ and moreover*

$$\mathcal{H}^1(E) \leq Ch^{(d-1)/d},$$

where C depends only on the dimension d.

Proof. For $k \geq 1$, let G_k denote the grid of order k (see Definition 2.4). Letting $F_k = G_k \cup \bigcup_{i=1}^h s_i$, where s_i is the shortest segment with one endpoint in G_k and the other equal to y_i, (13) yields $\mathcal{H}^1(s_i) \leq C/k$ and hence

$$\mathcal{H}^1(F_k) = \mathcal{H}^1(G_k) + \sum_{i=1}^h \mathcal{H}^1(s_i) \leq C\frac{h}{k} \leq d(k+1)^{d-1} + C\frac{h}{k}.$$

Hence, it suffices to let $E = F_k$ with the optimal choice $k = \lfloor h^{1/d} \rfloor$. □

The following result is the key step in the proof of the Γ-liminf inequality.

Proposition 2.6. *Let Q be any closed cube in $\mathbb{R}^d$. For every sequence $\{\gamma_n\} \subset \Sigma(\mathbb{R}^d)$ such that $\lim_n \mathcal{H}^1(\gamma_n \cap Q) = +\infty$, there holds*

$$\liminf_n \left(\mathcal{H}^1(\gamma_n \cap Q)\right)^{\frac{p}{d-1}} \int_Q d^p_{\gamma_n} \geq |Q|^{1+\frac{p}{d-1}}\theta_{d,p}. \tag{14}$$

Proof. Take $\{\gamma_n\}$ as in the statement to be proved and let $l_n := \mathcal{H}^1(\gamma_n \cap Q)$. By scaling and translating, we may suppose that $Q = I^d$ is the unit cube and furthermore that $\gamma_n \setminus Q \neq \emptyset$, because the subsequence of those $\gamma_n \subseteq Q$ fulfills (14) by the definition of $\theta_{d,p}$

(see (3)). Moreover, passing to a subsequence, we may suppose that the liminf in (14) is a finite limit, and hence that

$$M := \sup_n l_n^{\frac{p}{d-1}} \int_Q d_{\gamma_n}^p < +\infty. \tag{15}$$

Pick $x_n \in Q$ such that $r_n := d_{\gamma_n}(x_n) = \max_Q d_{\gamma_n}$. Then clearly $B(x_n, r_n) \cap \gamma_n = \emptyset$, and hence

$$\int_Q d_{\gamma_n}^p \geq \int_{Q\cap B(x_n,r_n)} d_{\gamma_n}^p \geq \int_{Q\cap B(x_n,r_n)} d_{\partial B(x_n,r_n)}^p \geq C r_n^{p+d},$$

where C depends only on d and p. Comparing with (15), we find that

$$\max_Q d_{\gamma_n} = r_n \leq \frac{T}{l_n^{p/(p+d)(d-1)}} \quad \forall n, \tag{16}$$

where T depends on d, p and M.

Now take a generic $\gamma \in \Sigma(\mathbb{R}^d)$ such that $\gamma \setminus Q \neq \emptyset$, let $l := \mathcal{H}^1(\gamma \cap Q) > 0$ and consider the following construction, which will later be repeated for each γ_n. For small $\varepsilon > 0$, let γ^ε denote the union of all connected components of $\gamma \cap Q$ whose length is at least ε, and let Q_ε denote the cube of side $1-\varepsilon$ concentric with Q. We claim that

$$r := \sup_{x\in Q} d_\gamma(x) \leq \varepsilon \quad \Rightarrow \quad d_\gamma \equiv d_{\gamma^\varepsilon} \text{ in } Q_{4\varepsilon}. \tag{17}$$

Indeed, $d_\gamma \leq d_{\gamma^\varepsilon}$ is obvious since $\gamma^\varepsilon \subseteq \gamma$. To prove the opposite inequality, take any $x \in Q_{4\varepsilon}$, and let $y \in \gamma$ be such that $|x-y| = d_\gamma(x)$. By $r \leq \varepsilon$, we have $y \in Q$ and, if A is the connected component of $\gamma \cap Q$ which contains y, we have $A \cap \partial Q \neq \emptyset$ (recall that γ is connected and $\gamma \setminus Q \neq \emptyset$), and hence

$$\mathcal{H}^1(A) \geq d_{\partial Q}(y) \geq d_{\partial Q}(x) - |x-y| \geq 2\varepsilon - d_\gamma(x) \geq 2\varepsilon - r \geq \varepsilon.$$

Therefore, $A \subseteq \gamma^\varepsilon$ and, since $y \in A$, we also have $d_\gamma \geq d_{\gamma^\varepsilon}$ in $Q_{4\varepsilon}$ and (17) follows. Moreover, γ^ε has at most l/ε connected components, hence by Lemma 2.5 we can find $E \subset Q$ such that

$$E \cup \gamma^\varepsilon \text{ is connected and } \mathcal{H}^1(E) \leq C \left(\frac{l}{\varepsilon}\right)^{\frac{d-1}{d}}, \tag{18}$$

where C depends only on the dimension. For every $k \in \mathbb{N}$, let $G_k^\varepsilon := G_k \setminus Q_{4\varepsilon}$ where G_k is the grid of step k (Definition 2.4). One can check that that G_k^ε is connected and moreover

$$k\varepsilon \geq 1 \quad \Rightarrow \quad \mathcal{H}^1(G_k^\varepsilon) \leq C\varepsilon k^{d-1} \quad \text{and} \quad \sup_{Q\setminus Q_{4\varepsilon}} d_{G_k^\varepsilon} \leq C/k, \tag{19}$$

where C depends only on the dimension. Set $\Gamma = \Gamma(\varepsilon, k) := \gamma^\varepsilon \cup E \cup G_k^\varepsilon \cup s$, where $s \subset Q$ is a segment such that Γ is connected (recall (18)). If we suppose that the right hand sides of the implications (17), (19) are satisfied, we have

$$\begin{aligned}\int_Q d_\gamma^p &\geq \int_{Q_{4\varepsilon}} d_\gamma^p = \int_{Q_{4\varepsilon}} d_{\gamma^\varepsilon}^p \geq \int_{Q_{4\varepsilon}} d_\Gamma^p = \int_Q d_\Gamma^p - \int_{Q\setminus Q_{4\varepsilon}} d_\Gamma^p \\ &\geq \int_Q d_\Gamma^p - C\varepsilon \sup_{Q\setminus Q_{4\varepsilon}} d_\Gamma^p \geq \int_Q d_\Gamma^p - C\varepsilon \sup_{Q\setminus Q_{4\varepsilon}} d_{G_k^\varepsilon}^p \geq \int_Q d_\Gamma^p - C\frac{\varepsilon}{k^p},\end{aligned} \tag{20}$$

where C depends only on the dimension. Moreover, we find using (18) and the right hand side of (19)

$$\begin{aligned}\mathcal{H}^1(\Gamma) &\le \mathcal{H}^1(\gamma^\varepsilon) + \mathcal{H}^1(E) + \mathcal{H}^1(G_k^\varepsilon) + \mathcal{H}^1(s)\\ &\le \mathcal{H}^1(\gamma) + C\left(\frac{l}{\varepsilon}\right)^{\frac{d-1}{d}} + C\varepsilon k^{d-1} + \operatorname{diam}(Q).\end{aligned} \tag{21}$$

Now, if we perform the above construction with $\gamma = \gamma_n$, $\Gamma = \Gamma_n(\varepsilon_n, k_n)$, with the choice $\varepsilon_n = T/l_n^{\frac{p}{(p+d)(d-1)}}$ (where T is the constant appearing in (16)) and $k_n = l_n^{\frac{1}{d-1}}$, then (16) implies that $r_n \le \varepsilon_n$, hence $d_{\gamma_n} \equiv d_{\gamma_n^\varepsilon}$ by (17), and (at least for n large enough since $l_n \to \infty$) $k_n\varepsilon_n \ge 1$, hence also the inequalities in (19) are available. Therefore, the estimates in (20) carry over to γ_n and Γ_n, and we obtain

$$\begin{aligned}\lim_n l_n^{\frac{p}{d-1}} \int_Q d_{\gamma_n}^p &\ge \left(\liminf_n \frac{l_n}{\mathcal{H}^1(\Gamma_n)}\right)^{\frac{p}{d-1}} \left(\liminf_n \mathcal{H}^1(\Gamma_n)^{\frac{p}{d-1}} \int_Q d_{\Gamma_n}^p\right)\\ &\quad - CT \limsup_n l_n^{-\frac{p}{(p+d)(d-1)}} \ge \theta_{d,p} \left(\liminf_n \frac{l_n}{\mathcal{H}^1(\Gamma_n)}\right)^{\frac{p}{d-1}},\end{aligned} \tag{22}$$

since $\Gamma_n \subseteq Q$ and Γ_n is connected (recall (3)). To complete the proof, it suffices to observe that (21) yields

$$\begin{aligned}\mathcal{H}^1(\Gamma_n) &\le \mathcal{H}^1(\gamma_n) + C\left(\frac{l_n}{\varepsilon_n}\right)^{\frac{d-1}{d}} + C\varepsilon_n k_n^{d-1} + \operatorname{diam}(Q)\\ &= l_n + \frac{C}{T^{\frac{d-1}{d}}} l_n^{1-\frac{1}{p+d}} + CT l_n^{1-\frac{p}{(p+d)(d-1)}} + \operatorname{diam}(Q),\end{aligned}$$

and hence the last liminf in (22) is bounded below by 1. □

3. Proof of the Γ-convergence result

This section is devoted to the proof of Theorem 1.1. As usual, we define for every probability measure $\mu \in \mathcal{P}(\overline{\Omega})$

$$\Gamma^-(\mu) := \inf\left\{\liminf_n F_{l_n}(\mu_n)\right\}, \quad \Gamma^+(\mu) := \inf\left\{\limsup_n F_{l_n}(\mu_n)\right\}$$

(the so called Γ-liminf e Γ-limsup of the sequence of functionals F_l), where both infima are taken over all sequences of positive numbers $l_n \to \infty$ and all sequences of measures $\{\mu_n\}$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{P}(\overline{\Omega})$. We refer the reader to §1.6 in [5] for the definitions of Γ-liminf and Γ-limsup: note that, since the weak-$*$ topology on $\mathcal{P}(\overline{\Omega})$ is metrizable, we can restrict to the sequential definitions of Γ^- and Γ^+ (see p. 26 in [5], and §8 in [7]).

The proof of Theorem 1.1 is divided into two steps.

Step 1: $\Gamma^-(\mu) \ge F_\infty(\mu) \quad \forall \mu \in \mathcal{P}(\overline{\Omega})$.

To prove this, take $\mu \in P(\Omega)$, a sequence $\mu_n \overset{*}{\rightharpoonup} \mu$ and a sequence of positive numbers $l_n \to \infty$. We have to prove that $\liminf_n F_{l_n}(\mu_n) \ge F_\infty(\mu)$, hence we may assume (considering a subsequence) that $F_{l_n}(\mu_n) < +\infty$ for all n. Due to (1), this reduces to assuming that $\mu_n = l_n^{-1}\mathcal{H}^1 \llcorner \gamma_n$ for suitable sets $\gamma_n \in \Sigma(\Omega)$ such that $\mathcal{H}^1(\gamma_n) = l_n$.

We first prove that

$$\liminf_n \left(\mathcal{H}^1(\gamma_n)\right)^{\frac{p}{d-1}} \int_Q f d^p_{\gamma_n} \geq \theta_{d,p} \int_Q \frac{f}{\rho^{\frac{p}{d-1}}} \tag{23}$$

for every cube $Q \subset \Omega$, where ρ is the density of μ with respect to Lebesgue measure. Arguing by contradiction, suppose that for some cube Q (possibly passing to a subsequence)

$$\exists \quad \lim_n \left(\mathcal{H}^1(\gamma_n)\right)^{\frac{p}{d-1}} \int_Q f d^p_{\gamma_n} < \theta_{d,p} \int_Q \frac{f}{\rho^{\frac{p}{d-1}}}. \tag{24}$$

By scaling and translating, we may assume that $Q = I^d$ is the unit cube. Take arbitrary $\varepsilon > 0$, let $k_n = \left\lfloor \varepsilon l_n^{1/(d-1)} \right\rfloor$ for n sufficiently large and consider G_{k_n}, the grid of order k_n (see Definition 2.4). Letting $w_n = l_n^{\frac{p}{d-1}} d^p_{\gamma_n \cup G_{k_n}}$ we have from the second equation in (13)

$$\sup_Q w_n \leq l_n^{\frac{p}{d-1}} \sup_Q d^p_{G_{k_n}} \leq C \left(\frac{l_n^{\frac{1}{d-1}}}{k_n} \right)^p$$

where C depends only on the dimension. Therefore, since $k_n \sim \varepsilon l_n^{1/(d-1)}$ as $n \to \infty$, $\|w_n\|_{L^\infty(Q)} \leq C_\varepsilon$ uniformly in n. Thus (passing to a subsequence) we may suppose that $w_n \overset{*}{\rightharpoonup} w$ in $L^\infty(Q)$ for some $w \in L^\infty(Q)$, and hence

$$\lim_n l_n^{\frac{p}{d-1}} \int_Q f\, d^p_{\gamma_n} \geq \lim_n \int_Q f\, w_n = \int_Q f\, w. \tag{25}$$

Seeking a contradiction, we estimate w from below, as follows. Let $Q_\delta \subset Q$ be an arbitrary closed cube of side δ, and set $\Gamma_n := \gamma_n \cup G_{k_n} \cup s_n$, where s_n is any segment with one endpoint in γ_n and the other in G_n, such that Γ_n is connected in $\mathbb{R}^d$. We have

$$\begin{aligned}
\int_{Q_\delta} w &= \lim_n \int_{Q_\delta} w_n = \lim_n l_n^{\frac{p}{d-1}} \int_{Q_\delta} d^p_{\gamma_n \cup G_{k_n}} \geq \liminf_n l_n^{\frac{p}{d-1}} \int_{Q_\delta} d_{\Gamma_n} \\
&\geq \left(\liminf_n \frac{l_n}{\mathcal{H}^1(\Gamma_n \cap Q_\delta)} \right)^{\frac{p}{d-1}} \left(\liminf_n \mathcal{H}^1(\Gamma_n \cap Q_\delta)^{\frac{p}{d-1}} \int_{Q_\delta} d^p_{\Gamma_n} \right) \\
&\geq \left(\liminf_n \frac{l_n}{\mathcal{H}^1(\Gamma_n \cap Q_\delta)} \right)^{\frac{p}{d-1}} \theta_{d,p} |Q_\delta|^{1+\frac{p}{d-1}},
\end{aligned} \tag{26}$$

having used Proposition 2.6 in the last passage. To estimate from below the last liminf, we observe that

$$\mathcal{H}^1(s_n) \leq \operatorname{diam}(\Omega), \qquad \lim_n \frac{\mathcal{H}^1(Q_\delta \cap G_{k_n})}{l_n} = d|Q_\delta|\varepsilon^{d-1} \tag{27}$$

(the second equation follows easily from the definition of the grid G_{k_n}, see Definition 2.4, and from $k_n \sim \varepsilon l_n^{1/(d-1)}$). Therefore, using (27)

$$\limsup_n \frac{\mathcal{H}^1(\Gamma_n \cap Q_\delta)}{l_n} \le \limsup_n \frac{\mathcal{H}^1(s_n) + \mathcal{H}^1(Q_\delta \cap G_{k_n}) + \mathcal{H}^1(Q_\delta \cap \gamma_n)}{l_n}$$
$$= d|Q_\delta|\varepsilon^{d-1} + \limsup_n \frac{\mathcal{H}^1(Q_\delta \cap \gamma_n)}{l_n} \le d|Q_\delta|\varepsilon^{d-1} + \mu(Q_\delta),$$

since Q_δ is closed and $l_n^{-1}\mathcal{H}^1 \llcorner \gamma_n \overset{*}{\rightharpoonup} \mu$ by assumption. Therefore, combining the last estimates with (26) we find

$$\frac{1}{|Q_\delta|}\int_{Q_\delta} w \ge \theta_{d,p}\left(\frac{1}{d\varepsilon^{d-1} + \frac{\mu(Q_\delta)}{|Q_\delta|}}\right)^{\frac{p}{d-1}} \qquad \forall Q_\delta \subset Q.$$

Finally, taking Q_δ centered at $x \in Q$ and letting Q_δ shrink around x, we obtain that

$$w(x) \ge \theta_{d,p}\left(\frac{1}{d\varepsilon^{d-1} + \rho(x)}\right)^{\frac{p}{d-1}} \quad \text{for a.e. } x \in Q.$$

Plugging this estimate into (25) yields

$$\lim_n l_n^{\frac{p}{d-1}} \int_Q f\, d_{\gamma_n}^p \ge \theta_{d,p}\int_Q \frac{f}{(d\varepsilon^{d-1} + \rho)^{\frac{p}{d-1}}}$$

and, letting $\varepsilon \to 0$, we find a contradiction comparing with (24). Thus, (23) is satisfied for every cube $Q \subset \Omega$. Now consider any finite family of disjoint cubes $\{Q_j\}$, $Q_j \subseteq \Omega$. We have using (23)

$$\begin{aligned}\liminf_n l_n^{\frac{p}{d-1}} \int_\Omega f d_{\gamma_n}^p &\ge \liminf_n \sum_j l_n^{\frac{p}{d-1}} \int_{Q_j} f d_{\gamma_n}^p \ge \sum_j \liminf_n l_n^{\frac{p}{d-1}} \int_{Q_j} f d_{\gamma_n}^p \\ &\ge \theta_{d,p} \sum_j \int_{Q_j} \frac{f}{\rho^{\frac{p}{d-1}}} = \theta_{d,p}\int_{\bigcup_j Q_j} \frac{f}{\rho^{\frac{p}{d-1}}}\end{aligned}$$

and our claim follows since the family of cubes is arbitrary.

Step 2: $\Gamma^+(\mu) \le F_\infty(\mu) \quad \forall \mu \in \mathcal{P}(\overline{\Omega})$.

Recalling (1) and (2), we have to prove that, given a probability measure $\mu \in \mathcal{P}(\overline{\Omega})$ and positive numbers $l_n \to \infty$, for every $\varepsilon > 0$ one can find a sequence $\{\gamma_n\} \subset \Sigma(\Omega)$ such that

$$\mathcal{H}^1(\gamma_n) = l_n, \qquad \limsup_n l_n^{\frac{p}{d-1}} \int_\Omega d_{\gamma_n}^p \le (1+\varepsilon)\theta_{d,p}\int_\Omega \frac{f}{\rho^{\frac{p}{d-1}}},$$

where ρ is the absolutely continuous part of μ.

We first prove this claim under the extra assumption that μ is absolutely continuous, positive and piecewise constant, namely, we assume that

$$d\mu = \rho\, dx, \qquad \rho = \sum_{j=0}^m \rho_j \chi_{E_j}, \quad E_0 = \Omega \setminus \bigcup_{j=1}^m E_j, \tag{28}$$

where the ρ_j's are positive numbers and the E_j's ($j > 0$) are disjoint open cubes of side $\delta > 0$, having vertices on the lattice $\delta\mathbb{Z}^d$, such that $\overline{E_j} \subset \Omega$. By scaling, we may further assume that $\delta = 1$.

Take $\varepsilon > 0$. If $\lambda > 0$ is large enough, then Proposition 2.3 (invoked with $l = \lambda\rho_j$, $j = 0, \dots, m$) yields $m + 1$ connected compact sets $C_0, \dots, C_m$, such that each C_j is contained in the unit cube I^d, C_j is tiling and

$$\mathcal{H}^1(C_j) = \lambda\rho_j, \quad \mathcal{H}^1(C_j)^{\frac{p}{d-1}} \int_{I^d} d(x, C_j)^p \, dx \le (1+\varepsilon)\theta_{d,p} \tag{29}$$

for all $j = 0, \dots, m$.

For every integer $k > 0$, set

$$\Gamma^k := \bigcup_{j=0}^{m} \bigcup_{\substack{x \in k^{-1}\mathbb{Z}^d \\ x + k^{-1}I^d \subset \Omega \cap \overline{E_j}}} \left(x + k^{-1}C_j\right). \tag{30}$$

Since Ω is connected and bounded, the union of all closed cubes having side k^{-1} and vertices on $k^{-1}\mathbb{Z}^d$ is connected for k large enough; therefore, as C_j is tiling for every j we obtain that Γ^k is connected for large k.

Denoting by U_k the union of all closed cubes of side k^{-1}, with vertices on $k^{-1}\mathbb{Z}^d$ and contained in $\Omega_k \cap \overline{E_0}$, we have from (30), (29), (28)

$$\mathcal{H}^1(\Gamma^k) = |U_k| k^d \frac{\lambda\rho_0}{k} + \sum_{j=1}^{m} k^d \frac{\lambda\rho_j}{k} = k^{d-1}\lambda\mu\left(U_k \cup \bigcup_{i=1}^{m} E_i\right) \le k^{d-1}\lambda. \tag{31}$$

Since $\partial\Omega$ is Lipschitz hence Lebesgue-negligible, one can easily check that

$$\frac{\mathcal{H}^1 \llcorner \Gamma^k}{\mathcal{H}^1(\Gamma^k)} \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{P}(\overline{\Omega}). \tag{32}$$

Now let $\Gamma^k_j = \Gamma^k \cap \overline{E_j}$, $0 \le j \le m$. Observing that Γ^k_j is, when $1 \le j \le m$, the periodic $\frac{1}{k}$-extension of C_j inside the cube E_j, from Lemma 2.2 and the inequality in (29) we find

$$\mathcal{H}^1(\Gamma^k_j)^{\frac{p}{d-1}} d^p_{\Gamma^k_j} \overset{*}{\rightharpoonup} g_j \quad \text{in } L^\infty(E_j) \quad \text{and} \quad g_j \le (1+\varepsilon)\theta_{d,p}, \quad 1 \le j \le m. \tag{33}$$

Similarly, reasoning as in the proof of Lemma 2.2 and scaling, one can check that if $h \in \mathbb{N}$ is fixed, since U_h is a finite union of cubes of side $1/h$ there holds

$$\mathcal{H}^1(\Gamma^k_0 \cap U_h)^{\frac{p}{d-1}} d^p_{\Gamma^k_0} \overset{*}{\rightharpoonup} g_0 \quad \text{in } L^\infty(U_h) \quad \text{and} \quad g_0 \le (1+\varepsilon)\theta_{d,p}. \tag{34}$$

Recalling (30) and the fact that $\partial\Omega$ is Lipschitzian, it is easy to check that

$$\sup_\Omega d_{\Gamma^k} \le \frac{M}{k} \tag{35}$$

for some constant M independent of k. Hence, for natural numbers $k \geq h > 0$, by splitting and using (35) we find

$$\mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} \int_\Omega d^p_{\Gamma^k} f \leq \mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} \frac{M^p}{k^p} \int_{E_0 \setminus U_h} f + \left(\frac{\mathcal{H}^1(\Gamma^k)}{\mathcal{H}^1(\Gamma^k \cap U_h)} \right)^{\frac{p}{d-1}}$$
$$\times \mathcal{H}^1(\Gamma^k \cap U_h)^{\frac{p}{d-1}} \int_{U_h} d^p_{\Gamma^k_0} f + \sum_{j=1}^m \left(\frac{\mathcal{H}^1(\Gamma^k)}{\mathcal{H}^1(\Gamma^k_j)} \right)^{\frac{p}{d-1}} \mathcal{H}^1(\Gamma^k_j)^{\frac{p}{d-1}} \int_{E_j} d^p_{\Gamma^k_j} f$$

By (31) we have $\mathcal{H}^1(\Gamma^k) \leq \lambda k^{d-1}$ and hence we find using (32), (33), (34),

$$\limsup_k \mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} \int_\Omega d^p_{\Gamma^k} f \leq \lambda^{\frac{p}{d-1}} M^p \int_{E_0 \setminus U_h} f$$
$$+ \left(\frac{1}{\mu(U_h)} \right)^{\frac{p}{d-1}} \int_{U_h} g_0 f + \sum_{j=1}^m \left(\frac{1}{\mu(E_j)} \right)^{\frac{p}{d-1}} \int_{E_j} g_j f.$$

Since h is arbitrary and $U_h \uparrow E_0$ as $h \to \infty$, we obtain since $g_j \leq (1+\varepsilon)\theta_{d,p}$

$$\limsup_k \mathcal{H}^1(\Gamma^k)^{\frac{p}{d-1}} \int_\Omega d^p_{\Gamma^k} f \leq (1+\varepsilon)\theta_{d,p} \int_\Omega \frac{f}{\rho^{\frac{p}{d-1}}}. \tag{36}$$

Finally, we construct $\{\gamma_n\}$ of length l_n starting from Γ^k, as follows. Denoting by k_n the integer part of $(l_n/\lambda)^{1/(d-1)}$ for n large enough, since $\mu(U_k) \uparrow \mu(E_0)$ as $k \to \infty$, from (31) one obtains that $\mathcal{H}^1(\Gamma^{k_n}) \leq l_n$ and $\mathcal{H}^1(\Gamma^{k_n}) \sim l_n$ as $n \to \infty$. Therefore, we can set $\gamma_n := \Gamma^{k_n} \cup S_n$, where S_n is any connected compact set such that $\mathcal{H}^1(S_n) = l_n - \Gamma^{k_n}$ and $S_n \cap \Gamma^{(k_n)}$ is non-empty but $\mathcal{H}^1$-negligible, so that γ_n is connected and $\mathcal{H}^1(\gamma_n) = l_n$. Since then $\mathcal{H}^1(S_n) = o(l_n)$ and $\mathcal{H}^1(\Gamma^{k_n}) \sim l_n$ as $n \to \infty$, using (32) one can check that

$$\frac{\mathcal{H}^1 \llcorner \gamma_n}{\mathcal{H}^1(\gamma_n)} \stackrel{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{P}(\overline{\Omega}).$$

Moreover, we have using $\mathcal{H}^1(\Gamma^{k_n}) \sim \mathcal{H}^1(\gamma_n)$, $\gamma_n \supseteq \Gamma^{k_n}$ and (36)

$$\limsup_n \mathcal{H}^1(\gamma_n)^{\frac{p}{d-1}} \int_\Omega d^p_{\gamma_n} f = \limsup_n \mathcal{H}^1(\Gamma^{k_n})^{\frac{p}{d-1}} \int_\Omega d^p_{\gamma_n} f$$
$$\leq \limsup_n \mathcal{H}^1(\Gamma^{k_n})^{\frac{p}{d-1}} \int_\Omega d^p_{\Gamma^{k_n}} f \leq (1+\varepsilon)\theta_{d,p} \int_\Omega \frac{f}{\rho^{\frac{p}{d-1}}}.$$

Since ε is arbitrary, this shows that $\Gamma^+(\mu) \leq F_\infty(\mu)$ when μ is of the kind (28). To prove the statement for general $\nu \in \mathcal{P}(\overline{\Omega})$, one can argue by density, as usual (see Remark 1.29 in [5]). Indeed, given $\nu \in \mathcal{P}(\overline{\Omega})$, there exist $\nu_k \in \mathcal{P}(\overline{\Omega})$, each of the kind (28), such that

$$\nu_k \stackrel{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{P}(\overline{\Omega}) \text{ and} \qquad F_\infty(\nu_k) \to F_\infty(\nu).$$

Since Γ^+ is a fortiori lower semicontinuous (see [7], Prop. 6.8), we find

$$\Gamma^+(\nu) \leq \liminf \Gamma^+(\nu_k) \leq \liminf F_\infty(\nu_k) = F_\infty(\nu)$$

and also the general case follows.

4. Some estimates on $\theta_{d,p}$

The following lemma, which we haven't found in the literature, is a consequence of the proof of Theorem 4.4.8 in [3]: it suffices to approximate the first k Lipschitz curves therein constructed by piecewise-affine functions, with k large enough. See also [8].

Lemma 4.1. *Given a connected compact set $\gamma \subset \mathbb{R}^d$ with $\mathcal{H}^1(\gamma) < +\infty$, there exists a sequence of connected sets γ_j such that each γ_j is the union of a finite number of segments, $\mathcal{H}^1(\gamma_j) \le \mathcal{H}^1(\gamma)$ and $\gamma_j \to \gamma$ in the Hausdorff distance.*

Lemma 4.2. *Let γ be a compact connected subset of $\mathbb{R}^d$ with $\mathcal{H}^1(\gamma) < \infty$. Then*

$$\left|\{x \in \mathbb{R}^d \,:\, d_\gamma(x) \le t\}\right| \le \mathcal{H}^1(C)\omega_{d-1}t^{d-1} + \omega_d t^d, \tag{37}$$

where ω_k denotes the volume of the unit ball in $\mathbb{R}^k$.

Proof. For every $E \subset \mathbb{R}^d$, set $A_t(E) = \{x \in \mathbb{R}^d \,:\, d_E(x) < t\}$. We first suppose that $\gamma = \bigcup_{i=1}^m s_i$ where each s_i is a segment. Let $\gamma^j = \bigcup_{i=1}^j s_i$. Since γ is connected, we may suppose that $s_{j+1} \cap \gamma^j \neq \emptyset$ for $j < m$. For a single segment s,

$$|A_t(s)| = \mathcal{H}^1(s)\omega_{d-1}t^{d-1} + \omega_d t^d, \tag{38}$$

and hence the claim of the lemma is true if $m = 1$. Now suppose that

$$|A_t(\gamma^j)| \le \mathcal{H}^1(\gamma^j)\omega_{d-1}t^{d-1} + \omega_d t^d \tag{39}$$

for some $j < m$, and let us prove the same estimate with $j + 1$ in place of j. We have

$$\begin{aligned} |A_t(\gamma^{j+1})| &= |A_t(\gamma^j \cup s_{j+1})| = |A_t(\gamma^j) \cup A_t(s_{j+1})| = \\ &= |A_t(\gamma^j)| + |A_t(s_{j+1})| - |A_t(\gamma^j) \cap A_t(s_{j+1})| \le \\ &\le \left(\mathcal{H}^1(\gamma^j) + \mathcal{H}^1(s_{j+1})\right)\omega_{d-1}t^{d-1} + 2\omega_d t^d - |A_t(\gamma^j) \cap A_t(s_{j+1})| \end{aligned}$$

having used (39) and (38). Now it suffices to observe that, since $\gamma^j \cap s_{j+1} \neq \emptyset$, $A_t(\gamma^j) \cap A_t(s_{j+1})$ contains a ball of radius t. Therefore the claim follows by induction on m.

The general case follows from Lemma 4.1, approximating γ by union of segments in the Hausdorff distance (which implies the uniform convergence of the corresponding distance functions), and observing that the functional $|A_t(\gamma)|$ is lower semicontinuous in this topology (see [2], Prop. 2.1). □

Theorem 4.3. *For every $p > 0$ it holds*

$$\theta_{d,p} \ge \frac{(d-1)}{(p+d-1)\omega_{d-1}^{\frac{p}{d-1}}}.$$

Proof. Consider $C \in \Sigma(I^d)$, let $l = \mathcal{H}^1(C)$ and let A_t denote the set of those points $x \in \mathbb{R}^d$ such that $d_C(x) < t$. By Lemma 4.2,

$$|A_t \cap I^d| \le l\omega_{d-1}t^{d-1}\left(1 + \frac{t\omega_d}{l\omega_{d-1}}\right) \le l\omega_{d-1}t^{d-1}\left(1 + \frac{\sqrt{d}\omega_d}{l\omega_{d-1}}\right), \quad t \in (0, \sqrt{d})$$

and hence, raising to the power $p/(d-1)$,

$$|A_t \cap I^d|^{\frac{p}{d-1}} \leq (l\omega_{d-1})^{\frac{p}{d-1}} t^p \left(1 + \frac{K}{l}\right)^{\frac{p}{d-1}}, \quad t \in (0, \sqrt{d}) \tag{40}$$

where K depends only on p, d. Now using $|\nabla d_C| = 1$ and the coarea formula, we have

$$|A_t \cap I^d| = \int_0^t P_s \, ds, \qquad \int_{A_t \cap I^d} d_C^p = \int_0^t s^p P_s \, ds, \quad t > 0$$

where P_s is the perimeter of A_t in I^d, and hence

$$\frac{d}{dt}|A_t \cap I^d| = P_t, \qquad \frac{d}{dt}\int_{A_t \cap I^d} d_C^p = t^p P_t, \quad t > 0.$$

Therefore, multiplying (40) by P_t we obtain that

$$\frac{d}{dt}|A_t \cap I^d|^{\frac{p+d-1}{d-1}} \leq \frac{p+d-1}{d-1}(l\omega_{d-1})^{\frac{p}{d-1}} \left(1 + \frac{K}{l}\right)^{\frac{p}{d-1}} \frac{d}{dt}\int_{A_t \cap I^d} d_C^p,$$

for every $t \in (0, \sqrt{d})$. Now, since clearly $\sup_{I^d} d_C \leq \operatorname{diam} I^d = \sqrt{d}$, integrating the last inequality over $(0, \sqrt{d})$ we obtain

$$1 = |I^d| \leq \frac{p+d-1}{d-1}(l\omega_{d-1})^{\frac{p}{d-1}} \left(1 + \frac{K}{l}\right)^{\frac{p}{d-1}} \int_{I^d} d_C^p.$$

Since $C \in \Sigma(I^d)$ is arbitrary and K does not depend on C, for every sequence $C_n \in \mathcal{K}(Q)$ such that $l_n = \mathcal{H}^1(C_n) \to \infty$ there holds

$$\frac{d-1}{(p+d-1)\omega_{d-1}^{\frac{p}{d-1}}} \leq \liminf_n l_n^{\frac{p}{d-1}} \int_Q d_{C_n}^p,$$

and the claim follows recalling the definition of $\theta_{d,p}$. □

Theorem 4.4. *In two dimensions,*

$$\theta_{2,p} = \frac{1}{2^p(p+1)}.$$

Proof. Let S_n be the subset of the closed unit square in $\mathbb{R}^2$ made of $n+1$ equi-spaced vertical segments of unit length, and let $C_n = S_n \cup B$, where B is the base of the square. Clearly, C_n is connected and $\mathcal{H}^1(C_n) = n + 2$. Moreover,

$$\int_{I^d} d_{C_n}^p \leq \int_{I^d} d_{S_n}^p = 2n \int_0^{\frac{1}{2n}} t^p \, dt = \frac{1}{(p+1)(2n)^p}.$$

Therefore,

$$\liminf \mathcal{H}^1(C_n)^p \int_{I^d} d_{C_n}^p \leq \liminf \frac{(n+2)^p}{(p+1)(2n)^p} = \frac{1}{2^p(p+1)}.$$

This proves that $\theta_{2,p} \leq 1/2^p(p+1)$, whereas the opposite inequality follows from Theorem 4.3 with $d = 2$. □

References

[1] L. Ambrosio: Lecture notes on optimal transport problems, in: Mathematical Aspects of Evolving Interfaces (Funchal, 2000), P. Colli et al. (eds.), Lect. Notes Math. 1812, Springer, Berlin (2003) 1–52.

[2] L. Ambrosio, I. Fonseca, P. Marcellini, L. Tartar: On a volume constrained variational problem, Arch. Ration. Mech. Anal. 149(1) (1999) 23–47.

[3] L. Ambrosio, P. Tilli: Topics on Analysis in Metric Spaces, Oxford Univ. Press, Oxford (2004).

[4] G. Bouchitté, C. Jimenez, M. Rajesh: Asymptotique d'un problème de positionnement optimal, C. R. Math. Acad. Sci. Paris, Ser. I 335(10) (2002) 853–858.

[5] A. Braides: Γ-Convergence for Beginners, Oxford Univ. Press, Oxford (2002).

[6] G. Buttazzo, E. Oudet, E. Stepanov: Optimal transportation problems with free Dirichlet regions, Progr. Nonlinear Diff. Equations Appl. 51 (2002) 41–65.

[7] G. Dal Maso: An Introduction to Γ–Convergence, Birkhäuser, Boston (1993).

[8] D. H. Fremlin: Embedding spaces of finite length in R^3, J. London Math. Soc., II Ser. 49(1) (1994) 150–162.