Let us assume that a sequence \(\{f_n\}_{n=1}^{\infty}\) of proper lower semicontinuous convex functions is bounded on some open subset of a weakly compactly generated Banach space. It is shown that if \(\{f_n\}_{n=1}^{\infty}\) is a Mosco converging sequence, then for every subgradient \(x^*\) of \(f\) at \(x\) there are subgradients \(x_n^* \in \partial f_n(x_n)\) such that \(\{x_n^*\}_{n=1}^{\infty}\) is weakly* converging to \(x^*\).

Keywords: Subdifferentials, convex functions, Attouch’s theorem

1991 Mathematics Subject Classification: 49J52

1. Introduction

One of the fundamental results of Convex Analysis is the Attouch Theorem. This theorem expresses the equivalence of Mosco convergence of a sequence of proper lower semicontinuous convex functions to the Painleve-Kuratowski graph convergence of its subdifferentials, see [1] for the reflexive Banach set up and [2, 3, 4] for extensions to nonreflexive cases. However, it is known fact that for any nonreflexive Banach space there is a sequence \(\{f_n\}_{n=1}^{\infty}\) of proper lower semicontinuous convex functions such that \(f = \text{Mosco} - \lim_{n \to \infty} f_n\) and \((x, x^*) \in \text{Graph}(\partial f)\) but \((x, x^*) \notin PK - \lim_{n \to \infty} \text{Graph}(\partial f_n)\), see Prop. 4.1 of [2].

In this paper we use the Mosco convergence of sequence of proper lower semicontinuous convex functions. In order to characterize this convergence we use conditions given in [2]. Namely, we say that \(f = \text{Mosco} - \lim_{n \to \infty} f_n\) if the two following conditions are satisfied:

- **S1** whenever \(\{x_n\}_{n=1}^{\infty}\) is a sequence weakly convergent to \(x\), then \(f(x) \leq \lim \inf_{n \to \infty} f_n(x_n)\);
- **S2** for each \(x \in E\) there exists a sequence \(\{x_n\}_{n=1}^{\infty}\) converging in norm to \(x\) for which \(f(x) = \lim \inf_{n \to \infty} f_n(x_n)\).

It is not difficult to notice that if \(\{f_n\}_{n=1}^{\infty}\) is a nondecreasing sequence of lower semicontinuous convex functions and \(f\) is the pointwise limit of \(\{f_n\}_{n=1}^{\infty}\) then **S1** and **S2** are satisfied.

Below we provide another example showing that even if \(f\) is the norm in \(l_1\) space and \(f_n\) are lipschitz continuous with the constant equal to 1 yet we do not have graph convergence of \(\text{Graph}(\partial f_n)\) to \(\text{Graph}(\partial f)\).

**Example 1.1.** Let us define \(f(x) := \sum_{i=1}^{\infty} |x_i|\), \(f_n(x) := \sum_{i=1}^{n} |x_i| + \sum_{i=n+1}^{\infty} x_i\) for every

*The author would like to thank the unknown referee for several remarks improving the presentation.*
Let us observe that the sequence \( \{f_n\}_{n=1}^{\infty} \) is nondecreasing sequence of continuous convex functions and \( f \) is the pointwise limit of it, so \( f = \text{Mosco-} \lim_{n \to \infty} f_n \). It is obvious that

\[
0 \in \partial f(0) := \{ h \in l_\infty : \sum_{i=1}^{\infty} h^* x^i \leq \sum_{i=1}^{\infty} |x^i| \text{ for every } x \in l_1 \}.
\]

For every

\[
(u, v) \in \text{Graph}(\partial f_n) := \{(a, b) \in l_1 \times l_\infty : \sum_{i=1}^{\infty} b^i x^i \leq f_n(a + x) - f_n(a) \text{ for every } x \in l_1 \}
\]

we have \( \|v\| \geq 1 \). In fact, since for every \( x \in l_1 \)

\[
\sum_{i=1}^{\infty} v^i x^i \leq f_n(u + x) - f_n(u),
\]

so taking

\[
\bar{x}^i := \begin{cases} 0, & \text{if } i \neq n + 1; \\ -1, & \text{if } i = n + 1 
\end{cases}
\]

we get

\[
-v^{n+1} \leq \sum_{i=1}^{n} |u^i| + u^{n+1} - 1 + \sum_{i=n+2}^{\infty} u^i - \sum_{i=1}^{n} |u^i| - u^{n+1} - \sum_{i=n+2}^{\infty} u^i = -1,
\]

so \( \sup_{i=1,2,...} |v^i| \geq 1 \). Hence there is no sequence \( \{(x_n, x^*_n)\}_{n=1}^{\infty} \) such that \((x_n, x^*_n) \in \text{Graph}(\partial f_n)\) and \((x_n, x^*_n) \rightharpoonup (0,0)\), where the convergence is in the norm topology, so \((0,0) \notin PK - \lim_{n \to \infty} \text{Graph}(\partial f_n)\).

Herein, we show that if \( E \) is a weakly compactly generated Banach space and \( f = \text{Mosco-} \lim_{n \to \infty} f_n \), then for every \((x, x^*) \in \text{Graph}(\partial f)\) there is a sequence \( \{(x_n, x^*_n)\}_{n=1}^{\infty} \) such that \((x_n, x^*_n) \in \text{Graph}(\partial f_n)\), \( x_n \rightharpoonup x, f_n(x_n) \rightharpoonup f(x) \) and \( x^*_n \rightharpoonup x^* \), where \( \rightharpoonup \) stands for the weak* convergence. Unfortunately, to get this result we need an additional assumption on the sequence \( \{f_n\}_{n=1}^{\infty} \), namely, the sequence must be uniformly bounded on some open, nonempty subset of \( E \).

There is also an example showing that in \( l_\infty \) space this convergence of subdifferentials can not be ensured (see Section 3).

2. Preliminaries

In the sequel \( E \) will be a real Banach space which is weakly compactly generated. We recall that a Banach space is WCG (weakly compactly generated) if there exists a weakly compact subset \( W \) of \( E \) that spans a dense linear space in \( E \) (one can always assume that \( W \) is convex), we refer to [5, 6] for detailed information on those spaces and to [8] for
background on the weak topology. By $E^*$ we denote the dual space to $E$, i.e. the space of the continuous linear functionals on $E$; see [8].

As usual for any convex function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ finite at $x$ and $\epsilon \geq 0$, by $\partial^\epsilon f(x)$ we denote its $\epsilon$-subdifferential i.e.

$$\partial^\epsilon f(x) := \{ x^* \in E^* \mid \langle x^*, h \rangle \leq \epsilon + f(x + h) - f(x) \text{ for every } h \in E \}$$

and by $\text{Graph}(\partial^\epsilon f)$ its graph, $(x, x^*) \in \text{Graph}(\partial^\epsilon f) \iff x^* \in \partial^\epsilon f(x)$. When $\epsilon = 0$ $\epsilon$-subdifferential is called subdifferential and denoted by $\partial f(x)$. Several properties of these notions can be found in [1, 6, 9]. Below we recall one of them, the Brøndsted-Rockafellar Subdifferential is called subdifferential and denoted by $\partial f(x)$. Below we recall one of them, the Brøndsted-Rockafellar

**Theorem 2.1.** Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex proper lower semicontinuous function, $\alpha, \beta > 0$, $y \in E$ with $f(y) < \infty$ and $f(y) \leq \inf_E f + \alpha \beta$. Then there exists $(x, z^*) \in \text{Graph}(\partial f)$ such that $\|x - y\| \leq \alpha$, $f(x) \leq f(y)$ and $\|z^*\| \leq \beta$.

We end this section recalling two properties of weakly compact sets (we refer to [6, 8, 9] for the definition and applications of this notion). The first is a consequence of Eberlein-$\tilde{\text{S}}$mulian Theorem, see Theorem 5.3.1 of [7]. It says that if $\{q_n\}_{n=1}^\infty$ is a sequence of elements of weakly compact subset of $E$, then it contains a subsequence weakly convergent to an element of $E$. The second is a consequence of the Hahn-Banach Theorem, see Theorem 3.4 b of [8], we refer also to [9] for others forms of it. Namely, if $A, B \subset E \times \mathbb{R}$ are convex, and weakly compact and closed, respectively, then there are $y^* \in E^*$ and $\mu \in \mathbb{R}$, $\delta > 0$ such that $\|y^*\| + |\mu| > 0$ and

$$\forall (a, \alpha) \in A, (b, \beta) \in B, \quad \langle y^*, a \rangle + \mu \alpha + \delta \leq \langle y^*, b \rangle + \mu \beta. \quad (1)$$

### 3. Results

Below we provide an outer characterization of the subdifferential of a Mosco-converging sequence. As it was mentioned in the introduction we are able to do it under an additional assumption (see assumption (ii) of the theorem below).

**Theorem 3.1.** Let $E$ be a WCG Banach space, $(x, x^*) \in E \times E^*$ be fixed and $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function such that $f(x) \in \mathbb{R}$, $x^* \in \partial f(x)$. Assume that $f_n : E \rightarrow \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous convex functions such that:

(i) $f = \text{Mosco - } \lim_{n \rightarrow \infty} f_n$;

(ii) there is an open nonempty subset $U$ of $E$ and a constant $c \in \mathbb{R}$ such that for every $u \in U$ and $n \in \mathbb{N}$ we have $f_n(u) \leq c$.

Then there are sequences $(x_n)_{n=1}^\infty \subset E$ and $(x_n^*)_{n=1}^\infty \subset E^*$ such that:

(iii) $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$;

(iv) $\forall n \in \mathbb{N}$, $x_n^* \in \partial f_n(x_n)$;

(v) $x_n^* \rightarrow x^*$.

**Proof.** Let $W$ be a convex weakly compact subset of $E$ such that $E = \text{cl span}W$, where “cl” stands for the topological closure in the norm topology. Put

$$Q := \text{conv}\{W \cup -W\},$$


where “conv” is the convex hull (see [8], Exercise 2). It is known fact that $Q$ is weakly compact and $E = \text{cl} \bigcup_{n=1}^{\infty} nQ$. For a time being let us assume that $f(0) = 0$ and $0 \in \partial f(0)$, thus $f(x) \geq 0$ for every $x \in E$. By assumption S2 there is a sequence $\{u_n\}_{n=1}^{\infty} \subset E$ such that $u_n \rightarrow 0$ and $f_n(u_n) \rightarrow 0$. We shall show that

$$\forall k \in N, \exists n(k) \in N : \forall n \geq n(k), \forall q \in Q \quad f_n(kq + u_n) + \frac{1}{k} \geq 0. \quad (2)$$

We prove it by a contradiction. Let us suppose, that we can choose a subsequence $\{q_{n_i}\}_{i=1}^{\infty}$ in $Q$ such that for every $i \in N$

$$f_{n_i}(kq_{n_i} + u_{n_i}) + \frac{1}{k} \leq 0.$$ 

By the Eberlein-Šmulian Theorem there is a subsequence $\{q_{n_i}\}_{i=1}^{\infty}$ of $\{q_{n_i}\}_{i=1}^{\infty}$, which is weakly convergent to some $\tilde{q} \in Q$, thus assumption S1 implies

$$\frac{1}{k} \leq f(k\tilde{q}) + \frac{1}{k} \leq \liminf_{s \rightarrow \infty} f_{n_i}(kq_{n_i} + u_{n_i}) + \frac{1}{k} \leq 0,$$

a contradiction. Thus (2) holds true. We may assume that $n(k) < n(k+1)$, $\|u_n\| \leq \frac{1}{k}$ and $|f_n(u_n)| \leq \frac{1}{2}$ for every $k \in N$ and $n(k) \leq n < n(k+1)$. Hence, by (2) for every $k \in N$ and $n(k) \leq n < n(k+1)$ we get

$$\{\text{epi} f_n - (u_n, f_n(u_n))\} \cap \{kQ \times \{-\frac{2}{k}\}\} = \emptyset.$$

By (1) there exists $(y_n^*, \mu_n) \in E^* \times \mathbb{R}$ such that

$$\forall (b, \beta) \in \text{epi} f_n, \quad (a, \alpha) \in kQ \times \{-\frac{2}{k}\}, \quad \langle y_n^*, b - u_n \rangle + \mu_n(\beta - f_n(u_n)) > \langle y_n^*, a \rangle + \mu_n \alpha. \quad (3)$$

Of course (3) implies that $\mu_n \geq 0$ for every $k \in N$ and $n(k) \leq n < n(k+1)$.

There exist $\tilde{a} \in E$ and $r > 0$ such that $B(\tilde{a}, 3r) := \{z \in E \mid \|z - \tilde{a}\| \leq 3r\} \subseteq U$. Since $\|u_n\| \rightarrow 0$, for some $n_0$ we get

$$\forall n \geq n_0, \quad B(\tilde{a}, 2r) + u_n \subseteq U. \quad (4)$$

Because $E = \text{cl} \bigcup_{k=1}^{\infty} kQ$, for some $k_0 > 4$ and every $k \geq k_0$ we are able to find $\tilde{q}_k \in kQ$ such that $\|\tilde{q}_k - \tilde{a}\| < r$. Taking $k_1 > k_0$ such that $n(k_1) > n_0$ we have

$$\forall k \geq k_1, \quad n(k) \leq n < n(k+1), \quad B(0, r) \subseteq U - u_n - kQ. \quad (5)$$

By (3) if $\mu_n = 0$, then $\langle y_n^*, b - u_n - a \rangle > 0$ for every $b \in U, a \in kQ$, so by (5) $y_n^* = 0$, a contradiction. Thus $\mu_n > 0$ for every $k > k_1$ and for every $n$ such that $n(k) \leq n < n(k+1)$. By (3) with $b \in \text{dom} f_n := \{y \in E \mid f_n(y) < \infty\}$, $\beta = f_n(b)$, $a = 0$ and $\alpha = -\frac{2}{k}$ we get (keep in mind that $k > k_1$ and $n(k) \leq n < n(k+1)$)

$$\forall b \in \text{dom} f_n, \quad \frac{2}{k} + f_n(b) - f_n(u_n) \geq \langle -\mu_n^{-1}y_n^*, b - u_n \rangle. \quad (6)$$
and similarly for \( b = u_n, \beta = f_n(u_n), a \in kQ \) and \( \alpha = -\frac{2}{k} \) we get

\[
\forall a \in kQ, \quad \frac{2}{k} \geq \langle \mu_n^{-1} y_n^*, a \rangle.
\] (7)

Assumption (ii) and (4), (6) imply

\[
\forall u' \in B(\tilde{a}, 2r), \quad \langle -\mu_n^{-1} y_n^*, u' \rangle \leq \frac{2}{k} + |f_n(u_n)| + f_n(u' + u_n) \leq c + 1.
\] (8)

It follows from (7) (keep in mind \( kQ \subset (k+1)Q \) and \( Q = -Q \)) that

\[
\forall a \in \bigcup_{k=1}^{\infty} kQ, \quad 0 = \lim_{n \to \infty} \langle \mu_n^{-1} y_n^*, a \rangle.
\] (9)

Let us take any \( s \geq 1, h \in E \). There is \( \tilde{k} \) such that for every \( k \geq \tilde{k} \) there are \( q_k^1, q_k^2 \in Q \) such that

\[
\|s(-h - kq_k^1)\| \leq r \quad \text{and} \quad \|\tilde{a} - kq_k^2\| \leq r,
\]

so

\[
-sh = s(-h - kq_k^1) + skq_k^1 + (kq_k^2 - \tilde{a}) + \tilde{a} - kq_k^2 \in B(\tilde{a}, 2r) + skq_k^1 - kq_k^2.
\]

Hence by (8) and (9) we get

\[
\limsup_{n \to \infty} \langle -\mu_n^{-1} y_n^*, -sh \rangle \leq c + 1,
\]

thus

\[
\forall h \in E, \quad 0 \geq \limsup_{n \to \infty} \langle \mu_n^{-1} y_n^*, h \rangle,
\]

which gives

\[
\forall h \in E, \quad 0 = \lim_{n \to \infty} \langle \mu_n^{-1} y_n^*, h \rangle.
\] (10)

It follows from (6) that

\[
\forall k > k_1, n(k) \leq n < n(k + 1), \quad \left(\frac{\sqrt{2}}{k}\right)^2 + \inf_{b \in E} \{f_n(b) + \langle \mu_n^{-1} y_n^*, b \rangle\} \geq f_n(u_n) + \langle \mu_n^{-1} y_n^*, u_n \rangle.
\]

Since

\[
\partial \left(f_n(\cdot) + \langle \mu_n^{-1} y_n^*, \cdot \rangle\right) = \partial(f_n(\cdot)) + \mu_n^{-1} y_n^*,
\]

the Bröndsted-Rockafellar Theorem ensures that for every \( k > k_1 \) and \( n(k) \leq n < n(k+1) \) we are able to find \( (x_n, x_n^*) \in \text{Graph}(\partial f_n) \) such that

\[
\|x_n - u_n\| \leq \sqrt{\frac{2}{k}} \cdot f_n(x_n) + \langle \mu_n^{-1} y_n^*, x_n - u_n \rangle \leq f_n(u_n) \quad \text{and} \quad \|x_n^* + \mu_n^{-1} y_n^*\| \leq \sqrt{\frac{2}{k}}.
\]

Thus \( x_n \to 0 \) and by (10) \( x_n^* \rightharpoonup 0 \). By the Mosco convergence \( 0 = f(0) \leq \liminf_{n \to \infty} f_n(x_n) \). We have also \( 0 = \lim_{n \to \infty} \langle \mu_n^{-1} y_n^*, x_n - u_n \rangle \), so

\[
\limsup_{n \to \infty} f_n(x_n) = \limsup_{n \to \infty} \left( f_n(x_n) + \langle \mu_n^{-1} y_n^*, x_n - u_n \rangle \right) \leq \lim_{n \to \infty} f_n(u_n) = 0,
\]
which implies 0 = \lim_{n \to \infty} f_n(x_n).

In order to finish the proof let us observe that if \( x^* \in \partial f(x) \) then the first part of the proof can be applied to the functions \( g(h) := f(x + h) - f(x) - \langle x^*, h \rangle \) and \( g_n(h) := f_n(x + h) - f_n(x) - \langle x^*, h \rangle \). We have \( \partial g(0) = \partial f(x) - x^* \) and \( \partial g_n(h_n) = \partial f(x + h_n) - x^* \).

Of course if \( h_n \to 0 \) and \( g_n(h_n) \to 0 \) imply \( f_n(x + h_n) \to f(x) \). Whenever \( g_n^* \in \partial g_n(h_n) \) and \( \langle g_n^*, h \rangle \to 0 \) then \( \langle x^* + g_n^*, h \rangle \to \langle x^*, h \rangle \) with \( x^* + g_n^* \in \partial f_n(x + h_n) \). This completes the proof.

The next example shows that the theorem is not valid in \( l_\infty \).

**Example 3.2.** Let us fix any \( \bar{x} \in l_\infty \) such that \( 0 < \bar{x}^1 < \bar{x}^2 < \ldots, \| \bar{x} \| = 1 \) and \( \bar{x}^* \in \partial \| \bar{x} \| \). Let us define \( f(x) := \| x \| \) and \( f_n(x) := \max_{1 \leq i \leq n} | x^i | \) for every \( x \in l_\infty \) and \( n \in N \). It is not difficult to see that \( f_1(x) \leq f_2(x) \leq \ldots \) and \( \lim_{n \to \infty} f_n(x) = \| x \| \) for every \( x \in l_\infty \), thus (S1) and (S2) are satisfied. Take any sequence \( \{ x_n \}_{n=1}^\infty \subset l_\infty \) such that \( x_n \to \bar{x} \) and \( f_n(x_n) \to f(\bar{x}) \). Let us assume that for some sequence \( \{ x_n^* \}_{n=1}^\infty \) such that \( x_n^* \in \partial f_n(x_n) \) for every \( n \in N \) we have \( x_n^* \to \bar{x}^* \). Let \( e_i(x) := | x^i | \) for every \( i = 1, 2, \ldots \) and \( x = (x^1, x^2, \ldots) \in l_\infty \). Let us put \( I_n(x) := \{ i \in \{ 1, \ldots, n \} \mid f_n(x) = | x^i | \} \) for every \( x \in l_\infty \) It is an easy exercise to show that

\[
\partial e_i(x) = \begin{cases} 
\{ x^* \in l_\infty \mid \forall u \in l_\infty, \langle x^*, u \rangle = u^i \}, & \text{if } x^i > 0; \\
\{ x^* \in l_\infty \mid \forall u \in l_\infty, \langle x^*, u \rangle = -u^i \}, & \text{if } x^i < 0; \\
\{ x^* \in l_\infty \mid \forall u \in l_\infty, \langle x^*, u \rangle \leq | u^i | \}, & \text{if } x^i = 0
\end{cases}
\]

and

\[
\forall x \in l_\infty, \partial f_n(x) \subseteq \text{conv} \bigcup_{i \in I_n(x)} \partial e_i(x) \tag{11}
\]

(keep in mind that the sets \( \partial e_i(x) \) are weak* compact and convex, see the Banach-Alaoglu Theorem of [8]). For every \( n = 1, 2, \ldots \) let us define \( d(n) := \min I_n(x_n) \) and \( g(n) := \max I_n(x_n) \). Since \( x_n \to \bar{x} \), so \( d(n) \to \infty \) (we recall that \( 0 < \bar{x}^1 < \bar{x}^2 < \ldots \)). We are able to find a subsequence \( \{ n_k \}_{k=1}^\infty \) such that \( g(n_k) < d(n_{k+1}) \) and by (11) multipliers \( \lambda^k_i \geq 0 \) such that \( \sum_{i=d(n_k)}^{g(n_k)} \lambda^k_i = 1 \) for every \( k = 1, 2, \ldots \) and \( \langle x^*_{n_k}, u \rangle = \sum_{i=d(n_k)}^{g(n_k)} \lambda^k_i u^i \) for every \( u = (u^1, u^2, \ldots) \in l_\infty \). Let us define

\[
h^i := \begin{cases} 
1, & \text{if } d(n_k) \leq i \leq g(n_k), k = 1, 3, 5, \ldots; \\
-1, & \text{if } d(n_k) \leq i \leq g(n_k), k = 2, 4, 6, \ldots; \\
0, & \text{otherwise}
\end{cases}
\]

and \( h := (h^1, h^2, \ldots) \). We have

\[
\limsup_{k \to \infty} \langle x^*_{n_k}, h \rangle = 1
\]

and

\[
\liminf_{k \to \infty} \langle x^*_{n_k}, h \rangle = -1.
\]

This contradicts to the weak* convergence of \( \{ x_n^* \}_{n=1}^\infty \) to \( \bar{x}^* \).
References


