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On the Weak* Convergence of Subdifferentials of Convex Functions

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Let us assume that a sequence $\{f_n\}_{n=1}^{\infty}$ of proper lower semicontinuous convex functions is bounded on some open subset of a weakly compactly generated Banach space. It is shown that if $\{f_n\}_{n=1}^{\infty}$ is a Mosco converging sequence, then for every subgradient x^* of f at x there are subgradients $x_n^* \in \partial f_n(x_n)$ such that $\{x_n^*\}_{n=1}^{\infty}$ is weakly* converging to x^*.

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1. Introduction

One of the fundamental results of Convex Analysis is the Attouch Theorem. This theorem expresses the equivalence of Mosco convergence of a sequence of proper lower semicontinuous convex functions to the Painleve-Kuratowski graph convergence of its subdifferentials, see [1] for the reflexive Banach set up and [2, 3, 4] for extensions to nonreflexive cases. However, it is known fact that for any nonreflexive Banach space there is a sequence $\{f_n\}_{n=1}^{\infty}$ of proper lower semicontinuous convex functions such that $f = Mosco - \lim_{n\longrightarrow\infty} f_n$ and $(x, x^*) \in Graph(\partial f)$ but $(x, x^*) \notin PK - \lim_{n\longrightarrow\infty} Graph(\partial f_n)$, see Prop. 4.1 of [2].

In this paper we use the Mosco convergence of sequence of proper lower semicontinuous convex functions. In order to characterize this convergence we use conditions given in [2]. Namely, we say that $f = Mosco - \lim_{n\longrightarrow\infty} f_n$ if the two following conditions are satisfied:

S1 whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence weakly convergent to x, then $f(x) \leq \liminf_{n\longrightarrow\infty} f_n(x_n)$;
S2 for each $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ converging in norm to x for which $f(x) = \lim_{n\longrightarrow\infty} f_n(x_n)$.

It is not difficult to notice that if $\{f_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of lower semicontinuous convex functions and f is the pointwise limit of $\{f_n\}_{n=1}^{\infty}$ then **S1** and **S2** are satisfied.

Below we provide another example showing that even if f is the norm in l_1 space and f_n are lipschitz continuous with the constant equal to 1 yet we do not have graph convergence of $Graph(\partial f_n)$ to $Graph(\partial f)$.

Example 1.1. Let us define $f(x) := \sum_{i=1}^{\infty} |x^i|$, $f_n(x) := \sum_{i=1}^{n} |x^i| + \sum_{i=n+1}^{\infty} x^i$ for every

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$x \in l_1 := \{u = (u^1, u^2, ...) \mid u^i \in \mathbf{R}$ for every $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} |u^i| < \infty\}$ and n=1,2,..., $l_\infty := \{v = (v^1, v^2, ...) \mid v^i \in \mathbf{R}$ for every $i = 1, 2, \ldots$ and $\sup_{i=1,2,\ldots} |v^i| < \infty\}$. Let us observe that the sequence $\{f_n\}_{n=1}^{\infty}$ is nondecreasing sequence of continuous convex functions and f is the pointwise limit of it, so $f = Mosco - \lim_{n\longrightarrow\infty} f_n$. It is obvious that

$$0 \in \partial f(0) := \{h \in l_\infty \mid \sum_{i=1}^{\infty} h^i x^i \leq \sum_{i=1}^{\infty} \left|x^i\right| \text{ for every } x \in l_1\}.$$

For every

$$(u, v) \in Graph(\partial f_n) := \{(a, b) \in l_1 \times l_\infty \mid \sum_{i=1}^{\infty} b^i x^i \leq f_n(a + x) - f_n(a) \text{ for every } x \in l_1\}$$

we have $\|v\| \geq 1$. In fact, since for every $x \in l_1$

$$\sum_{i=1}^{\infty} v^i x^i \leq f_n(u + x) - f_n(u),$$

so taking

$$\bar{x}^i := \begin{cases} 0, & \text{if } i \neq n+1; \\ -1, & \text{if } i = n+1 \end{cases}$$

we get

$$-v^{n+1} \leq \sum_{i=1}^{n} \left|u^i\right| + u^{n+1} - 1 + \sum_{i=n+2}^{\infty} u^i - \sum_{i=1}^{n} \left|u^i\right| - u^{n+1} - \sum_{i=n+2}^{\infty} u^i = -1,$$

so $\sup_{i=1,2,\ldots} |v^i| \geq 1$. Hence there is no sequence $\{(x_n, x_n^*)\}_{n=1}^{\infty}$ such that $(x_n, x_n^*) \in Graph(\partial f_n)$ and $(x_n, x_n^*) \longrightarrow (0, 0)$, where the convergence is in the norm topology, so $(0, 0) \notin PK - \lim_{n\longrightarrow\infty} Graph(\partial f_n)$.

Herein, we show that if E is a weakly compactly generated Banach space and $f = Mosco- \lim_{n\longrightarrow\infty} f_n$, then for every $(x, x^*) \in Graph(\partial f)$ there is a sequence $\{(x_n, x_n^*)\}_{n=1}^{\infty}$ such that $(x_n, x_n^*) \in Graph(\partial f_n)$, $x_n \longrightarrow x, f_n(x_n) \longrightarrow f(x)$ and $x_n^* \overset{weak^*}{\longrightarrow} x^*$, where "$\overset{weak^*}{\longrightarrow}$" stands for the weak* convergence. Unfortunately, to get this result we need an additional assumption on the sequence $\{f_n\}_{n=1}^{\infty}$, namely, the sequence must be uniformly bounded on some open, nonempty subset of E.

There is also an example showing that in l_∞ space this convergence of subdifferentials can not be ensured (see Section 3).

2. Preliminaries

In the sequel E will be a real Banach space which is weakly compactly generated. We recall that a Banach space is WCG (weakly compactly generated) if there exists a weakly compact subset W of E that spans a dense linear space in E (one can always assume that W is convex), we refer to [5, 6] for detailed information on those spaces and to [8] for background on the weak topology. By E^* we denote the dual space to E, i.e. the space of the continuous linear functionals on E, see [8].

As usual for any convex function $f : E \longrightarrow \mathbf{R} \cup \{+\infty\}$ finite at x and $\epsilon \geq 0$, by $\partial_\epsilon f(x)$ we denote its ϵ-subdifferential i.e.

$$\partial_\epsilon f(x) := \{x^* \in E^* \mid \langle x^*, h\rangle \leq \epsilon + f(x+h) - f(x) \text{ for every } h \in E\}$$

and by $Graph(\partial_\epsilon f)$ its graph, $(x, x^*) \in Graph(\partial_\epsilon f) \iff x^* \in \partial_\epsilon f(x)$. When $\epsilon = 0$ ϵ-subdifferential is called subdifferential and denoted by $\partial f(x)$. Several properties of these notions can be found in [1, 6, 9]. Below we recall one of them, the Brøndsted-Rockafellar Theorem, see for example Corollary 29.2 of [9].

Theorem 2.1. *Let $f : E \longrightarrow \mathbf{R} \cup \{+\infty\}$ be a convex proper lower semicontinuous function, $\alpha, \beta > 0$, $y \in E$ with $f(y) < \infty$ and $f(y) \leq \inf_E f + \alpha\beta$. Then there exists $(x, z^*) \in Graph(\partial f)$ such that $\|x - y\| \leq \alpha$, $f(x) \leq f(y)$ and $\|z^*\| \leq \beta$.*

We end this section recalling two properties of weakly compact sets (we refer to [6, 8, 9] for the definition and applications of this notion). The first is a consequence of Eberlein-Šmulian Theorem, see Theorem 5.3.1 of [7]. It says that if $\{q_n\}_{n=1}^\infty$ is a sequence of elements of weakly compact subset of E, then it contains a subsequence weakly convergent to an element of E. The second is a consequence of the Hahn-Banach Theorem, see Theorem 3.4 b of [8], we refer also to [9] for others forms of it. Namely, if $A, B \subset E \times \mathbf{R}$ are convex, and weakly compact and closed, respectively, then there are $y^* \in E^*$ and $\mu \in \mathbf{R}$, $\delta > 0$ such that $\|y^*\| + |\mu| > 0$ and

$$\forall (a, \alpha) \in A, (b, \beta) \in B, \quad \langle y^*, a\rangle + \mu\alpha + \delta \leq \langle y^*, b\rangle + \mu\beta. \tag{1}$$

3. Results

Below we provide an outer characterization of the subdifferential of a Mosco-converging sequence. As it was mentioned in the introduction we are able to do it under an additional assumption (see assumption (ii) of the theorem below).

Theorem 3.1. *Let E be a WCG Banach space, $(x, x^*) \in E \times E^*$ be fixed and $f : E \longrightarrow \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous convex function such that $f(x) \in \mathbf{R}$, $x^* \in \partial f(x)$. Assume that $f_n : E \longrightarrow \mathbf{R} \cup \{+\infty\}$ are lower semicontinuous convex functions such that:*

(i) $f = Mosco - \lim_{n\longrightarrow\infty} f_n$;

(ii) *there is an open nonempty subset U of E and a constant $c \in \mathbf{R}$ such that for every $u \in U$ and $n \in \mathbb{N}$ we have $f_n(u) \leq c$.*

Then there are sequences $\{x_n\}_{n=1}^\infty \subset E$ and $\{x_n^*\}_{n=1}^\infty \subset E^*$ such that:

(iii) $\lim_{n\longrightarrow\infty} x_n = x$, $\lim_{n\longrightarrow\infty} f_n(x_n) = f(x)$;

(iv) $\forall n \in N, \quad x_n^* \in \partial f_n(x_n)$;

(v) $x_n^* \stackrel{weak^*}{\longrightarrow} x^*$.

Proof. Let W be a convex weakly compact subset of E such that $E = \mathrm{cl}\,\mathrm{span} W$, where "cl" stands for the topological closure in the norm topology. Put

$$Q := \mathrm{conv}\{W \cup -W\},$$

where “conv" is the convex hull (see [8], Exercise 2). It is known fact that Q is weakly compact and $E = \mathrm{cl}\bigcup_{n=1}^{\infty} nQ$. For a time being let us assume that $f(0) = 0$ and $0 \in \partial f(0)$, thus $f(x) \geq 0$ for every $x \in E$. By assumption **S2** there is a sequence $\{u_n\}_{n=1}^{\infty} \subset E$ such that $u_n \longrightarrow 0$ and $f_n(u_n) \longrightarrow 0$. We shall show that

$$\forall k \in N, \exists n(k) \in N : \forall n \geq n(k), \forall q \in Q \quad f_n(kq + u_n) + \frac{1}{k} \geq 0. \tag{2}$$

We prove it by a contradiction. Let us suppose, that we can choose a subsequence $\{q_{n_i}\}_{i=1}^{\infty}$ in Q such that for every $i \in N$

$$f_{n_i}(kq_{n_i} + u_{n_i}) + \frac{1}{k} \leq 0.$$

By the Eberlein-Šmulian Theorem there is a subsequence $\{q_{n_{i_s}}\}_{s=1}^{\infty}$ of $\{q_{n_i}\}_{i=1}^{\infty}$, which is weakly convergent to some $\bar{q} \in Q$, thus assumption **S1** implies

$$\frac{1}{k} \leq f(k\bar{q}) + \frac{1}{k} \leq \liminf_{s \longrightarrow \infty} f_{n_{i_s}}(kq_{n_{i_s}} + u_{n_{i_s}}) + \frac{1}{k} \leq 0,$$

a contradiction. Thus (2) holds true. We may assume that $n(k) < n(k+1)$, $\|u_n\| \leq \frac{1}{k}$ and $|f_n(u_n)| < \frac{1}{k}$ for every $k \in N$ and $n(k) \leq n < n(k+1)$. Hence, by (2) for every $k \in N$ and $n(k) \leq n < n(k+1)$ we get

$$\{\mathrm{epi} f_n - (u_n, f_n(u_n))\} \cap \{kQ \times \{-\frac{2}{k}\}\} = \emptyset.$$

By (1) there exists $(y_n^*, \mu_n) \in E^* \times \mathbf{R}$ such that

$$\forall (b, \beta) \in \mathrm{epi} f_n,\ (a, \alpha) \in kQ \times \{-\frac{2}{k}\}, \quad \langle y_n^*, b - u_n \rangle + \mu_n(\beta - f_n(u_n)) > \langle y_n^*, a \rangle + \mu_n \alpha. \tag{3}$$

Of course (3) implies that $\mu_n \geq 0$ for every $k \in N$ and $n(k) \leq n < n(k+1)$.

There exist $\bar{a} \in E$ and $r > 0$ such that $B(\bar{a}, 3r) := \{z \in E \mid \|z - \bar{a}\| \leq 3r\} \subseteq U$. Since $\|u_n\| \longrightarrow 0$, for some n_0 we get

$$\forall n \geq n_0, \quad B(\bar{a}, 2r) + u_n \subseteq U. \tag{4}$$

Because $E = \mathrm{cl}\bigcup_{k=1}^{\infty} kQ$, for some $k_0 > 4$ and every $k \geq k_0$ we are able to find $\tilde{q}_k \in kQ$ such that $\|\tilde{q}_k - \bar{a}\| < r$. Taking $k_1 > k_0$ such that $n(k_1) > n_0$ we have

$$\forall k \geq k_1,\ n(k) \leq n < n(k+1), \quad B(0, r) \subseteq U - u_n - kQ. \tag{5}$$

By (3) if $\mu_n = 0$, then $\langle y_n^*, b - u_n - a \rangle > 0$ for every $b \in U$, $a \in kQ$, so by (5) $y_n^* = 0$, a contradiction. Thus $\mu_n > 0$ for every $k > k_1$ and for every n such that $n(k) \leq n < n(k+1)$. By (3) with $b \in \mathrm{dom} f_n := \{y \in E \mid f_n(y) < \infty\}$, $\beta = f_n(b)$, $a = 0$ and $\alpha = -\frac{2}{k}$ we get (keep in mind that $k > k_1$ and $n(k) \leq n < n(k+1)$)

$$\forall b \in \mathrm{dom} f_n, \quad \frac{2}{k} + f_n(b) - f_n(u_n) \geq \left\langle -\mu_n^{-1} y_n^*, b - u_n \right\rangle \tag{6}$$

and similarly for $b = u_n$, $\beta = f_n(u_n)$, $a \in kQ$ and $\alpha = -\frac{2}{k}$ we get

$$\forall a \in kQ, \quad \frac{2}{k} \geq \left\langle \mu_n^{-1} y_n^*, a \right\rangle. \tag{7}$$

Assumption (ii) and (4), (6) imply

$$\forall u' \in B(\bar{a}, 2r), \quad \left\langle -\mu_n^{-1} y_n^*, u' \right\rangle \leq \frac{2}{k} + |f_n(u_n)| + f_n(u' + u_n) \leq c + 1. \tag{8}$$

It follows from (7) (keep in mind $kQ \subset (k+1)Q$ and $Q = -Q$) that

$$\forall a \in \bigcup_{k=1}^{\infty} kQ, \quad 0 = \lim_{n \longrightarrow \infty} \left\langle \mu_n^{-1} y_n^*, a \right\rangle. \tag{9}$$

Let us take any $s \geq 1$, $h \in E$. There is $\tilde{k}$ such that for every $k \geq \tilde{k}$ there are $q_k^1, q_k^2 \in Q$ such that

$$\left\| s(-h - kq_k^1) \right\| \leq r \text{ and } \left\| \bar{a} - kq_k^2 \right\| \leq r,$$

so

$$-sh = s(-h - kq_k^1) + skq_k^1 + (kq_k^2 - \bar{a}) + \bar{a} - kq_k^2 \in B(\bar{a}, 2r) + skq_k^1 - kq_k^2.$$

Hence by (8) and (9) we get

$$\limsup_{n \longrightarrow \infty} \left\langle -\mu_n^{-1} y_n^*, -sh \right\rangle \leq c + 1,$$

thus

$$\forall h \in E, \quad 0 \geq \limsup_{n \longrightarrow \infty} \left\langle \mu_n^{-1} y_n^*, h \right\rangle,$$

which gives

$$\forall h \in E, \quad 0 = \lim_{n \longrightarrow \infty} \left\langle \mu_n^{-1} y_n^*, h \right\rangle. \tag{10}$$

It follows from (6) that

$$\forall k > k_1, \; n(k) \leq n < n(k+1),$$
$$\left(\sqrt{\frac{2}{k}} \right)^2 + \inf_{b \in E} \{ f_n(b) + \left\langle \mu_n^{-1} y_n^*, b \right\rangle \} \geq f_n(u_n) + \left\langle \mu_n^{-1} y_n^*, u_n \right\rangle.$$

Since

$$\partial \left(f_n(\cdot) + \left\langle \mu_n^{-1} y_n^*, \cdot \right\rangle \right) = \partial (f_n(\cdot)) + \mu_n^{-1} y_n^*,$$

the Brøndsted-Rockafellar Theorem ensures that for every $k > k_1$ and $n(k) \leq n < n(k+1)$ we are able to find $(x_n, x_n^*) \in Graph(\partial f_n)$ such that

$$\|x_n - u_n\| \leq \sqrt{\frac{2}{k}}, \;\; f_n(x_n) + \left\langle \mu_n^{-1} y_n^*, x_n - u_n \right\rangle \leq f_n(u_n) \quad \text{and} \quad \left\| x_n^* + \mu_n^{-1} y_n^* \right\| \leq \sqrt{\frac{2}{k}}.$$

Thus $x_n \longrightarrow 0$ and by (10) $x_n^* \xrightarrow{weak^*} 0$. By the Mosco convergence $0 = f(0) \leq \liminf_{n \longrightarrow \infty} f_n(x_n)$. We have also $0 = \lim_{n \longrightarrow \infty} \langle \mu_n^{-1} y_n^*, x_n - u_n \rangle$, so

$$\limsup_{n \longrightarrow \infty} f_n(x_n) = \limsup_{n \longrightarrow \infty} \left(f_n(x_n) + \left\langle \mu_n^{-1} y_n^*, x_n - u_n \right\rangle \right) \leq \lim_{n \longrightarrow \infty} f_n(u_n) = 0,$$

which implies $0 = \lim_{n\longrightarrow\infty} f_n(x_n)$.

In order to finish the proof let us observe that if $x^* \in \partial f(x)$ then the first part of the proof can be applied to the functions $g(h) := f(x+h) - f(x) - \langle x^*, h\rangle$ and $g_n(h) := f_n(x+h) - f(x) - \langle x^*, h\rangle$. We have $\partial g(0) = \partial f(x) - x^*$ and $\partial g_n(h_n) = \partial f(x+h_n) - x^*$. Of course $h_n \longrightarrow 0$ and $g_n(h_n) \longrightarrow 0$ imply $f_n(x+h_n) \longrightarrow f(x)$. Whenever $g_n^* \in \partial g_n(h_n)$ and $\langle g_n^*, h\rangle \longrightarrow 0$ then $\langle x^* + g_n^*, h\rangle \longrightarrow \langle x^*, h\rangle$ with $x^* + g_n^* \in \partial f_n(x+h_n)$. This completes the proof.

The next example shows that the theorem is not valid in l_∞.

Example 3.2. Let us fix any $\bar{x} \in l_\infty$ such that $0 < \bar{x}^1 < \bar{x}^2 < \ldots$, $\|\bar{x}\| = 1$ and $\bar{x}^* \in \partial\|\bar{x}\|$. Let us define $f(x) := \|x\|$ and $f_n(x) := \max_{1\le i\le n} |x^i|$ for every $x \in l_\infty$ and $n \in N$. It is not difficult to see that $f_1(x) \le f_2(x) \le \ldots$ and $\lim_{n\longrightarrow\infty} f_n(x) = \|x\|$ for every $x \in l_\infty$, thus (**S1**) and (**S2**) are satisfied. Take any sequence $\{x_n\}_{n=1}^\infty \subset l_\infty$ such that $x_n \longrightarrow \bar{x}$ and $f_n(x_n) \longrightarrow f(\bar{x})$. Let us assume that for some sequence $\{x_n^*\}_{n=1}^\infty$ such that $x_n^* \in \partial f_n(x_n)$ for every $n \in N$ we have $x_n^* \overset{weak^*}{\longrightarrow} \bar{x}^*$. Let $e_i(x) := |x^i|$ for every $i = 1, 2, \ldots$ and $x = (x^1, x^2, \ldots) \in l_\infty$. Let us put $I_n(x) := \{i \in \{1, \ldots, n\} \mid f_n(x) = |x^i|\}$ for every $x \in l_\infty$. It is an easy exercise to show that

$$\partial e_i(x) = \begin{cases} \{x^* \in l_\infty^* \mid \forall u \in l_\infty, \quad \langle x^*, u\rangle = u^i\}, & \text{if } x^i > 0; \\ \{x^* \in l_\infty^* \mid \forall u \in l_\infty, \quad \langle x^*, u\rangle = -u^i\}, & \text{if } x^i < 0; \\ \{x^* \in l_\infty^* \mid \forall u \in l_\infty, \quad \langle x^*, u\rangle \le |u^i|\}, & \text{if } x^i = 0 \end{cases}$$

and

$$\forall x \in l_\infty, \quad \partial f_n(x) \subseteq \operatorname{conv}\{\bigcup_{i\in I_n(x)} \partial e_i(x)\} \tag{11}$$

(keep in mind that the sets $\partial e_i(x)$ are weak* compact and convex, see the Banach-Alaoglu Theorem of [8]). For every $n = 1, 2, \ldots$ let us define $d(n) := \min I_n(x_n)$ and $g(n) := \max I_n(x_n)$. Since $x_n \longrightarrow \bar{x}$, so $d(n) \longrightarrow \infty$ (we recall that $0 < \bar{x}^1 < \bar{x}^2 < \ldots$). We are able to find a subsequence $\{n_k\}_{k=1}^\infty$ such that $g(n_k) < d(n_{k+1})$ and by (11) multipliers $\lambda_i^{n_k} \ge 0$ such that $\sum_{i=d(n_k)}^{g(n_k)} \lambda_i^{n_k} = 1$ for every $k = 1, 2, \ldots$ and $\left\langle x_{n_k}^*, u\right\rangle = \sum_{i=d(n_k)}^{g(n_k)} \lambda_i^{n_k} u^i$ for every $u = (u^1, u^2, \ldots) \in l_\infty$. Let us define

$$h^i := \begin{cases} 1, & \text{if } d(n_k) \le i \le g(n_k), k = 1, 3, 5, \ldots; \\ -1, & \text{if } d(n_k) \le i \le g(n_k), k = 2, 4, 6, \ldots; \\ 0, & \text{otherwise} \end{cases}$$

and $h := (h^1, h^2, \ldots)$. We have

$$\limsup_{k\longrightarrow\infty} \left\langle x_{n_k}^*, h\right\rangle = 1$$

and

$$\liminf_{k\longrightarrow\infty} \left\langle x_{n_k}^*, h\right\rangle = -1.$$

This contradicts to the weak* covergence of $\{x_n^*\}_{n=1}^\infty$ to $\bar{x}^*$.

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