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On L^1-Lower Semicontinuity in BV

Virginia De Cicco
Università di Roma, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Via Scarpa 16, 00161 Roma, Italy
decicco@dmmm.uniroma1.it

Nicola Fusco
Università di Napoli, Dipartimento di Matematica e Applicazioni, Via Cinzia, 80126 Napoli, Italy
n.fusco@unina.it

Anna Verde
Università di Napoli, Dipartimento di Matematica e Applicazioni, Via Cinzia, 80126 Napoli, Italy
anverde@unina.it

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A lower semicontinuity result is obtained for the BV extension of an integral functional of the type

$$\int_\Omega f(x, u(x), \nabla u(x))\, dx\,,$$

where the energy density f is not coercive and satisfies mild regularity assumptions.

1. Introduction

Since the celebrated paper [20] of Serrin appeared in 1961, many authors have contributed to the study of the L^1-lower semicontinuity of a functional of the type

$$\int_\Omega f(x, u(x), \nabla u(x))\, dx\,, \qquad u \in W^{1,1}(\Omega), \tag{1}$$

and of the corresponding extension (3) to $BV(\Omega)$, with the aim of understanding which are the minimal assumptions on f that guarantee lower semicontinuity.

Here, the starting point is the result, proved in [20], stating that if the integrand $f(x, u, \xi)$ is a nonnegative, continuous function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$, convex in ξ, and such that the (classical) derivatives $\nabla_x f, \nabla_\xi f, \nabla_x \nabla_\xi f$ exist and are continuous functions, then the integral functional (1) is lower semicontinuous in $W^{1,1}(\Omega)$, with respect to the L^1 convergence in the open set Ω.

In 1983 De Giorgi, Buttazzo and Dal Maso (see [10]) showed that Serrin's assumptions can be substantially weakened when dealing with autonomous (i.e. $f \equiv f(u, \xi)$) functionals of the type (1). In this case they were able to prove the L^1-lower semicontinuity without even assuming f to be continuous in u for all $\xi \in \mathbb{R}^N$. Their result is proved by approximating the integrand f with a sequence of affine functions and then using a suitable version of the chain rule in the Sobolev space $W^{1,1}(\Omega)$ in order to get the lower semicontinuity of the approximating functionals.

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On the other hand, things are more complicate if we allow f to depend explicitly on x, since convexity in ξ and continuity with respect to all variables (x, u, ξ) are not enough to ensure the lower semicontinuity of functional (1), as shown by a well known example of Aronszajin (see [19]). An even more surprising counterexample to lower semicontinuity has been recently given by Gori, Maggi and Marcellini in [18] (see also [17]). In their example, which is one dimensional, the integrand f is equal to $|a(x, u)\xi - 1|$ and the function a is Hölder continuous in x, uniformly with respect to u.

These examples clearly show that in order to get lower semicontinuity we must retain, in the spirit of Serrin's theorem recalled above, some differentiability of f with respect to x. This is done, for instance, by Gori, Maggi and Marcellini in [18]. They prove that if f is continuous in (x, u, ξ), convex in ξ and weakly differentiable in x and if for any open set $\Omega' \subset\subset \Omega$ and any bounded set $B \subset \mathbb{R} \times \mathbb{R}^N$ there exists a constant $L(\Omega', B)$ such that

$$\int_{\Omega'} \Big|\nabla_x f(x, u, \xi)\Big|\, dx \leq L(\Omega', B) \qquad \text{for every } \ (u, \xi) \in B\,, \tag{2}$$

then the functional (1) is L^1-lower semicontinuous. Also this result is proved by approximating the integrand f with a sequence of affine functions whose coefficients are explicitly given in terms of f (see Lemma 2.1 below, proved in [9]).

The lower semicontinuity result by Gori, Maggi and Marcellini has been extended in a later paper ([8]) by De Cicco and Leoni, who prove a fairly general version of the chain rule for vectors fields with divergence in L^1. Using this chain rule, they get the lower semicontinuity of the functional (1) under the assumption that f is continuous in u and convex in ξ and that the distributional gradient $\nabla_\xi f$ satisfies a suitable assumption. Namely, they require that, for $\mathcal{L}^1$-a.e. $u \in \mathbb{R}$ and $\mathcal{L}^N$-a.e. $\xi \in \mathbb{R}^N$, $\nabla_\xi f(\cdot, u, \xi)$ is locally summable in Ω, its distributional divergence $\mathrm{div}_x \nabla_\xi f(x, u, \xi)$ is locally summable in Ω and $\mathrm{div}_x \nabla_\xi f(x, u, \xi)$ belongs to $L^1_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$. These assumptions seem very close to being necessary conditions for lower semicontinuity (see, for instance, [14], [16], [3] in the BV setting, and [8] for a discussion on this subject).

In this paper, we prove a lower semicontinuity result for the functional

$$F(u, \Omega) = \int_\Omega f(x, u, \nabla u)dx + \int_\Omega \overline{f}^\infty(x, \widetilde{u}, \frac{D^c u}{|D^c u|})d|D^c u| + \int_{J_u} d\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} \overline{f}^\infty(x, t, \nu_u)dt, \tag{3}$$

where $u \in BV(\Omega)$, $\widetilde{u}$ is the approximate limit of u, $\overline{f}^\infty$ denotes the recession function of $\overline{f}$ with respect to ξ and $\overline{f}(x, u, \xi)$ is a suitable Borel function, continuous in u, convex in ξ and such that for any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ the function $\overline{f}(\cdot, u, \xi)$ coincides $\mathcal{H}^{N-1}$-a.e. in Ω with the precise representative of $f(\cdot, u, \xi)$ (see the definition given in (5)). As usual, $D^c u$ denotes the Cantor part of the measure Du, $D^c u/|D^c u|$ is the derivative of the measure $D^c u$ with respect to its total variation $|D^c u|$ and J_u is the jump set of u (the definitions of all these quantities are recalled in Section 2).

The functional F is the natural extension of the functional (1) to the space $BV(\Omega)$. In fact (see, for instance, the paper [5] by Dal Maso), under standard continuity and coercivity assumptions on f, F coincides with the *relaxed* functional of (1), that is the greatest L^1-lower semicontinuous functional on $BV(\Omega)$ coinciding with (1) on $W^{1,1}(\Omega)$.

Therefore, the study of the lower semicontinuity of the functional F is essentially parallel to the Sobolev case, though occasionally things may turn out to be more difficult to prove in the BV setting. Among the many papers devoted to the BV case we just mention here the ones which are closer to our approach, such as [6], [7], [4], [12], [13] and [15], where the lower semicontinuity result of [18] is extended to the functional F under the extra assumption that f is continuous.

Here we give a further improvement of the result of [15] by dropping the continuity of f with respect to x. Namely we prove the following result.

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ a locally bounded Carathéodory function, such that, for every $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the function $f(\cdot, u, \xi)$ is weakly differentiable in Ω. Let us assume that there exists a set $Z \subset \Omega$, with $\mathcal{L}^N(Z) = 0$, such that*

(i) $f(x, u, \cdot)$ is convex in $\mathbb{R}^N$ for every $(x, u) \in (\Omega \setminus Z) \times \mathbb{R}$;
(ii) $f(x, \cdot, \xi)$ is continuous in $\mathbb{R}$ for every $(x, \xi) \in (\Omega \setminus Z) \times \mathbb{R}^N$

and that for any open set $\Omega' \subset\subset \Omega$ and any bounded set $B \subset \mathbb{R} \times \mathbb{R}^N$ the estimate (2) *holds. Then, the functional F defined in* (3) *is lower semicontinuous in $BV(\Omega)$ with respect to the $L^1(\Omega)$ convergence.*

Beside the usual approximation from below of the integrand f by a sequence of affine functions, the proof of the above theorem uses a localization argument which allows us to prove separately the lower semicontinuity of the diffuse part and of the jump part of the functional F. However, the main technical tool used in the proof is provided by the integration by parts formula stated below. This formula does not seem to be contained in any of the similar results known in the literature and we think that could be of independent interest.

Proposition 1.2. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a bounded Borel function with compact support in $\mathbb{R}^N \times \mathbb{R}$, such that, for any $t \in \mathbb{R}$, $b(\cdot, t)$ is weakly differentiable in $\mathbb{R}^N$. Assume that*

(j) there exists a set $Z \subset \mathbb{R}^N$, with $\mathcal{L}^N(Z) = 0$, such that for every $x \in \Omega \setminus Z$ the function $b(x, \cdot)$ is continuous in $\mathbb{R}$;
(jj) there exists a constant L such that for any $t \in \mathbb{R}$

$$\int_{\mathbb{R}^N} \left| \frac{\partial b}{\partial x}(x, t) \right| dx \le L\,.$$

Then, for every $u \in BV(\mathbb{R}^N)$ and for every $\phi \in C^1_0(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla\phi(x)dx \int_0^{u(x)} b(x,t)dt &= -\int_{\mathbb{R}^N} \phi(x)dx \int_0^{u(x)} \frac{\partial b}{\partial x}(x,t)\,dt \\ &- \int_{\mathbb{R}^N} \phi(x) b(x, u\,(x)) \nabla u(x)\,dx - \int_{\mathbb{R}^N} \phi\,(x)\,\overline{b}(x, \widetilde{u}\,(x)) dD^c u \\ &- \int_{J_u} \phi(x)\nu_u(x) d\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} \overline{b}(x,t)dt\,, \end{aligned}$$

where $\overline{b}$ is defined as in (5).

2. Definitions and preliminary results

In this section we recall a few preliminary results needed in the sequel and some basic definitions of the theory of BV functions. We follow the notation used in [2] and we refer to this book for all the properties of BV functions used here.

The first lemma is an approximation result due to De Giorgi (see [9]).

Lemma 2.1. *There exists a sequence δ_j in $C_0^\infty(\mathbb{R}^N)$, with $\delta_j \geq 0$ and $\int_{\mathbb{R}^N} \delta_j(\zeta)\, d\zeta = 1$, with the property that, whenever $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ is a Carathéodory function, if we set, for any $j \in \mathbb{N}$ and any $x \in \Omega$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,*

$$\begin{aligned} a_{0,j}(x,u) &= \int_{\mathbb{R}^N} f(x,u,\zeta)\left((N+1)\delta_j(\zeta) + \langle \nabla\delta_j(\zeta), \zeta\rangle\right) d\zeta\,, \\ a_{i,j}(x,u) &= -\int_{\mathbb{R}^N} f(x,u,\zeta)\frac{\partial\delta_j}{\partial\zeta_i}(\zeta)\, d\zeta, \qquad \text{for } i = 1,\dots,N\,, \\ g_j(x,t,\xi) &= a_{0,j}(x,u) + \sum_{i=1}^{N} a_{i,j}(x,u)\xi_i\,, \end{aligned} \tag{4}$$

then, for all $(x,u) \in \Omega \times \mathbb{R}$ such that $f(x,u,\cdot)$ is convex, we have

$$f(x,u,\xi) = \sup_{j\in\mathbb{N}} \max\{g_j(x,u,\xi), 0\} \qquad \textit{for all } \xi \in \mathbb{R}^N\,.$$

Notice that if f is a Carathéodory (resp. Borel) function, then also the $a_{i,j}$ are Carathéodory (resp. Borel) functions in $\Omega \times \mathbb{R}$ and if f satisfies (2) then a similar estimate is also satisfied by the functions $a_{i,j}$ (locally uniformly with respect to u).

Let $k : \Omega \times \mathbb{R}^M \to \mathbb{R}$ be a locally bounded Carathéodory function. For any $x \in \Omega$, $z \in \mathbb{R}^M$, we set

$$\overline{k}(x,z) = \limsup_{r\to 0} \fint_{B_r(x)} k(y,z)\, dy\,. \tag{5}$$

Notice that $\overline{k}$ is a Borel function and that $\overline{k}(x,\cdot)$ is a continuous function for *any* $x \in \Omega$.

Let us recall that if $h : \mathbb{R}^N \to \mathbb{R}$ is a convex function, its *recession function* $h^\infty : \mathbb{R}^N \to \mathbb{R}$ is defined by setting

$$h^\infty(\xi) = \lim_{t\to+\infty} \frac{h(t\xi)}{t} \qquad \text{for any } \xi \in \mathbb{R}^N\,. \tag{6}$$

If $h : V \times \mathbb{R}^N \to \mathbb{R}$ is a any function, such that for any v in the set V, the function $h(v,\cdot)$ is convex, we denote by $h^\infty(v,\xi)$ the recession function of $h(v,\cdot)$. The following result is an easy consequence of the definition (6) and of Lemma 2.1 (see also [2, Lemma 2.33]).

Lemma 2.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Carathéodory function, satisfying the assumptions of Theorem 1.1. Then, for all $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$,*

$$\overline{f}^\infty(x,u,\xi) = \sup_{j\in\mathbb{N}} \max\{\langle\alpha_j(x,u),\xi\rangle, 0\}\,,$$

where for any $i = 1, \dots, N$,

$$\alpha_{i,j}(x,u) = -\int_{\mathbb{R}^N} \overline{f}(x,u,\zeta) \frac{\partial \delta_j}{\partial \zeta_i}(\zeta)\, d\zeta \,. \tag{7}$$

An immediate consequence of this lemma is that the function $(x,u,\xi) \mapsto \overline{f}^{\infty}(x,u,\xi)$ is a Borel function. Thus, the functional F in (3) is well defined.

The following result is contained in [2, Lemma 2.35].

Lemma 2.3. *Let μ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^N$ and let $\psi_j : \Omega \to [0,\infty]$, $j \in \mathbb{N}$, be Borel functions. Then*

$$\int_\Omega \sup_j \psi_j \, d\mu = \sup \left\{ \sum_{j \in J} \int_{A_j} \psi_j \, d\mu \right\} ,$$

where the supremum ranges among all finite sets $J \subset \mathbb{N}$ and all families $\{A_j\}_{j\in\mathbb{N}}$ of pairwise disjoint open sets with compact closure in Ω.

Let u be a function in $L^1_{\mathrm{loc}}(\Omega)$. We say that u is *approximately continuous* at the point $x \in \Omega$ if there exists $\widetilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |u(y) - \widetilde{u}(x)| \, dy = 0 \, ;$$

the value $\widetilde{u}(x)$ is called the *approximate limit* of u at x. In the sequel, whenever $k : \Omega \times \mathbb{R}^M \to \mathbb{R}$ is a locally bounded Carathéodory function, we shall denote by $\widetilde{k}(x,u)$ the approximate limit of $k(\cdot,u)$ at x, provided that this limit exists; in this case we have also $\widetilde{k}(x,z) = \overline{k}(x,z)$. The set C_u of all points where u is approximately continuous is a Borel set. We say that a point $x \in \Omega \setminus C_u$ is an *approximate jump point* for u if there exist $u^+(x), u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in \mathbb{S}^{N-1}$ such that $u^-(x) < u^+(x)$ and

$$\lim_{r\to 0} \fint_{B_r^+(x;\nu_u(x))} |u(y) - u^+(x)| \, dy = 0, \quad \lim_{r\to 0} \fint_{B_r^-(x;\nu_u(x))} |u(y) - u^-(x)| \, dy = 0 \, ,$$

where $B_r^+(x;\nu_u(x)) = \{y \in B_r(x) : \langle y - x, \nu_u(x)\rangle > 0\}$ and $B_r^-(x;\nu_u(x))$ is defined analogously. Also the set $J_u \subset \Omega \setminus C_u$ of all approximate jump points is a Borel set and the function $(u^+(x), u^-(x), \nu_u(x)) : J_u \to \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{N-1}$ is a Borel function.

Given a point $x \in C_u$, we say that u is *approximately differentiable* at x if there exists $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{r\to 0} \frac{1}{r^{N+1}} \int_{B_r(x)} |u(y) - \widetilde{u}(x) - \langle \nabla u(x), y - x\rangle| \, dy = 0 \, .$$

The vector $\nabla u(x)$ is called the *approximate differential* of u at x. The set of points in C_u where the approximate differential of u exists is a Borel set denoted by $\mathcal{D}_u$. It can be easily verified that $\nabla u : \mathcal{D}_u \to \mathbb{R}^N$ is a Borel map.

A function $u \in L^1(\Omega)$ is called of *bounded variation* if its distributional gradient Du is an $\mathbb{R}^N$-vector valued measure and the total variation $|Du|$ of Du is finite in Ω. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$. If $u \in BV(\Omega)$, we denote by $D^a u$ the *absolutely continuous* part of Du with respect to the Lebesgue measure $\mathcal{L}^N$. The singular part $D^s u$ can be split in two more parts, the *jump part* $D^j u$ and the *Cantor part* $D^c u$, defined by

$$D^j u = D^s u \llcorner J_u, \qquad D^c u = D^s u - D^j u\,.$$

Furthermore,

$$D^a u = \nabla u \mathcal{L}^N \llcorner \mathcal{D}_u, \qquad D^c u = Du \llcorner (C_u \setminus \mathcal{D}_u), \qquad D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u\,,$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$ (see [2, Proposition 3.92]). We shall also use the following form of the *coarea formula* for BV functions (see [11, Theorem 4.5.9])

$$\int_\Omega g\, d|Du| = \int_{-\infty}^{+\infty} dt \int_{\{u^- \le t \le u^+\}} g\, d\mathcal{H}^{N-1}\,, \tag{8}$$

where $g : \Omega \to [0, +\infty]$ is a Borel function.

We conclude this section by proving the following lemma that will be useful in the sequel.

Lemma 2.4. *Let f be a Carathéodory function satisfying the assumptions of Theorem 1.1. Let us denote by a_j the functions obtained from f as in (5) and by α_j those obtained in the same way from $\overline{f}$ as in (7). Then, for any $u \in \mathbb{R}$, there exists a Borel set $Z_u \subset \Omega$, with $\mathcal{H}^{N-1}(Z_u) = 0$, such that for any $j \in \mathbb{N}$,*

$$\overline{a}_j(x,u) = \widetilde{a}_j(x,u) = \alpha_j(x,u) \qquad \text{for all } x \in \Omega \setminus Z_u\,.$$

Proof. Let us fix $u \in \mathbb{R}$. We claim that there exists a Borel set $Z_u \subset \Omega$, with $\mathcal{H}^{N-1}(Z_u) = 0$, such that, for any $x \in \Omega \setminus Z_u$ and any $\xi \in \mathbb{R}^N$, x is a point of approximate continuity for the function $f(\cdot, u, \xi)$, hence in particular

$$\widetilde{f}(x,u,\xi) = \overline{f}(x,u,\xi) \qquad \text{for any } x \in \Omega \setminus Z_u \text{ and any } \xi \in \mathbb{R}^N\,. \tag{9}$$

To this aim, let us consider a sequence ξ_h, dense in $\mathbb{R}^N$, and recall that since the functions $f(\cdot, u, \xi_h)$ are weakly integrable in Ω, for any h there exists a Borel set $B_{u,h} \subset \Omega$ such that $\mathcal{H}^{N-1}(B_{u,h}) = 0$ and x is a point of approximate continuity for $f(\cdot, u, \xi_h)$ for any $x \in \Omega \setminus B_{u,h}$. Let us set $Z_u = \cup_h B_{u,h}$ and fix $x \in \Omega \setminus Z_u$, $\xi \in \mathbb{R}^N$. Let ξ_{k_h} be a subsequence of ξ_h converging to ξ. Since $f(\cdot, u, \cdot)$ is locally bounded and convex in ξ for $\mathcal{L}^N$-a.e. $x \in \Omega$, then for any open set $\Omega' \subset\subset \Omega$ and any compact set K in $\mathbb{R}^N$, there exists a constant L such that

$$|f(y,u,\xi) - f(y,u,\xi')| \le L|\xi - \xi'| \qquad \text{for } \mathcal{L}^N\text{-a.e. } y \in \Omega' \text{ and any } \xi, \xi' \in K\,.$$

From this inequality, we get that

$$\begin{aligned} &\limsup_{r\to 0} \fint_{B_r(x)} |f(y,u,\xi) - \widetilde{f}(x,u,\xi_{k_h})|\, dy \\ &\le \limsup_{r\to 0} \fint_{B_r(x)} |f(y,u,\xi) - f(y,u,\xi_{k_h})|\, dy + \limsup_{r\to 0} \fint_{B_r(x)} |f(y,u,\xi_{k_h}) - \widetilde{f}(x,u,\xi_{k_h})|\, dy \\ &\le c|\xi - \xi_{k_h}|\,, \end{aligned}$$

hence we easily deduce that the sequence $\widetilde{f}(x,u,\xi_{k_h})$ converges to the approximate limit of $f(\cdot,u,\xi)$ at x and thus that x is a point of approximate continuity for $f(\cdot,u,\xi)$.

To conclude the proof, we now show that x is a point of approximate continuity also for the functions $a_j(\cdot,u)$, thus obtaining that $\widetilde{a}_j(x,u)=\overline{a}_j(x,u)$, and that indeed $\widetilde{a}_j(x,u)=\alpha_j(x,u)$. In fact, using (9) and Fubini's theorem, we get

$$\begin{aligned}
\fint_{B_r(x)}|a_j(y,u)-\alpha_j(x,u)|\,dy &\le \fint_{B_r(x)}dy\Big|\int_{\mathbb{R}^N}[f(y,u,\zeta)-\overline{f}(x,u,\zeta)]\nabla\delta_j(\zeta)\,d\zeta\Big| \\
&= \fint_{B_r(x)}dy\Big|\int_{\mathbb{R}^N}[f(y,u,\zeta)-\widetilde{f}(x,u,\zeta)]\nabla\delta_j(\zeta)\,d\zeta\Big| \\
&\le \int_{\mathbb{R}^N}|\nabla\delta_j(\zeta)|\,d\zeta\,\fint_{B_r(x)}|f(y,u,\zeta)-\widetilde{f}(x,u,\zeta)|\,dy.
\end{aligned}$$

Then, the assertion follows from the Lebesgue dominated convergence theorem since, for any $\zeta\in\mathbb{R}^N$,

$$\lim_{r\to 0}\fint_{B_r(x)}|f(y,u,\zeta)-\widetilde{f}(x,u,\zeta)|\,dy=0\,.$$

$\square$

3. Proof of the Theorem 1.1

In this section we give the proof of both Proposition 1.2 and Theorem 1.1.

Proof of Proposition 1.2. Let $\varrho_\varepsilon(x)=\varepsilon^{-N}\varrho(x/\varepsilon)$, $\varepsilon>0$, a family of mollifiers, where ϱ is a nonnegative smooth function, with compact support in the unit ball B_1, such that $\int_{\mathbb{R}^N}\varrho(y)dy=1$. For any $\varepsilon>0$, $(x,t)\in\mathbb{R}^N\times\mathbb{R}$, we set

$$b_\varepsilon(x,t)=\int_{\mathbb{R}^N}\varrho_\varepsilon(x-z)b(z,t)\,dz,\qquad\qquad f_\varepsilon(x,t)=\int_0^t b_\varepsilon(x,\tau)\,d\tau\,.$$

We claim that $f_\varepsilon\in C^1_0(\mathbb{R}^N\times\mathbb{R})$. To this aim, let B_R be an open ball such that the support of b is contained in $B_R\times(-R,R)$; then the support of f_ε is contained in $B_{R+\varepsilon}\times(-R,R)$ and thus to prove the claim it is enough to show that $f_\varepsilon\in C^1(B_{R+\varepsilon}\times\mathbb{R})$. Let su fix $\sigma>0$. Since b is a Carathéodory function in $B_{R+2\varepsilon}\times\mathbb{R}$, by Scorza Dragoni's theorem there exists a compact set K such that $\mathcal{L}^N(B_{R+2\varepsilon}\setminus K)<\sigma$ and $b_{|K\times\mathbb{R}}$ is continuous. Moreover, let $\delta\in(0,\sigma)$ be such that

$$(z,t_1),(z,t_2)\in K\times\mathbb{R},\ |t_1-t_2|<\delta\quad\Longrightarrow\quad|b(z,t_1)-b(z,t_2)|<\sigma\,.$$

Let us now fix $(x_1,t_1),(x_2,t_2)\in B_{R+\varepsilon}\times\mathbb{R}$ such that $|(x_1,t_1)-(x_2,t_2)|<\delta$. Then, setting

$M = \sup |b|$, we have

$$
\begin{aligned}
&\left|\frac{\partial f_\varepsilon}{\partial t}(x_1,t_1) - \frac{\partial f_\varepsilon}{\partial t}(x_2,t_2)\right| = \left|\int_{B_\varepsilon(x_1)} \varrho_\varepsilon(x_1-z)b(z,t_1)\,dz - \int_{B_\varepsilon(x_2)} \varrho_\varepsilon(x_2-z)b(z,t_2)\,dz\right| \\
&\le \int_{B_\varepsilon(x_1)} |\varrho_\varepsilon(x_1-z)-\varrho_\varepsilon(x_2-z)||b(z,t_1)|\,dz + \int_{B_\varepsilon(x_1)\setminus B_\varepsilon(x_2)} \varrho_\varepsilon(x_2-z)|b(z,t_1)|\,dz \\
&\quad + \int_{B_\varepsilon(x_2)\setminus B_\varepsilon(x_1)} \varrho_\varepsilon(x_2-z)|b(z,t_2)|\,dz + \int_{B_\varepsilon(x_1)\cap B_\varepsilon(x_2)} \varrho_\varepsilon(x_2-z)|b(z,t_1)-b(z,t_2)|\,dz \\
&\le Mc_\varepsilon|x_1-x_2| + \int_{B_\varepsilon(x_1)\cap B_\varepsilon(x_2)\setminus K} \varrho_\varepsilon(x_2-z)|b(z,t_1)-b(z,t_2)|\,dz \\
&\quad + \int_{B_\varepsilon(x_1)\cap B_\varepsilon(x_2)\cap K} \varrho_\varepsilon(x_2-z)|b(z,t_1)-b(z,t_2)|\,dz \\
&\le Mc_\varepsilon|x_1-x_2| + 2Mc_\varepsilon\mathcal{L}^N(B_\varepsilon(x_1)\cap B_\varepsilon(x_2)\setminus K) + \sigma \le \sigma(3Mc_\varepsilon+1)
\end{aligned}
$$

where c_ε is a constant depending only on ϱ, ε and N. Since, for any $i = 1, \dots, N$,

$$\frac{\partial f_\varepsilon}{\partial x_i}(x,t) = \int_0^t d\tau \int_{\mathbb{R}^N} \frac{\partial \varrho_\varepsilon}{\partial x_i}(x-z)b(z,\tau)\,dz\,,$$

the proof of the continuity of $\partial f_\varepsilon/\partial x_i$ is similar (actually simpler). Let us now set, for any $\varepsilon > 0$,

$$v_\varepsilon(x) = \int_0^{u(x)} b_\varepsilon(x,t)\,dt.$$

By applying the chain rule to the composition of the function $f_\varepsilon \in C_0^1(\mathbb{R}^N \times \mathbb{R})$ with the BV map $x \mapsto (x, u(x))$ (see, for instance, [2, Theorem 3.96]), we get, for any $\phi \in C_0^1(\mathbb{R}^N)$,

$$
\begin{aligned}
\int_{\mathbb{R}^N} \nabla\phi(x)v_\varepsilon(x)\,dx = &-\int_{\mathbb{R}^N} \phi(x)dx\int_0^{u(x)} \frac{\partial b_\varepsilon}{\partial x}(x,t)\,dt - \int_{\mathbb{R}^N} \phi(x)b_\varepsilon(x,u(x))\nabla u(x)\,dx \\
&- \int_{\mathbb{R}^N} \phi(x)b_\varepsilon(x,\widetilde{u}(x))\,dD^c u - \int_{J_u} \phi(x)\nu_u(x)d\mathcal{H}^{N-1}\int_{u^-(x)}^{u^+(x)} b_\varepsilon(x,t)\,dt\,.
\end{aligned}
$$

Using again Scorza Dragoni's theorem one can easily prove (see, for instance the proof of formula (42) in [18]) that there exists a set $Z_1 \subset \mathbb{R}^N$, with $\mathcal{L}^N(Z_1) = 0$, such that

$$\lim_{\varepsilon\to 0^+} b_\varepsilon(x,t) = b(x,t) \qquad \text{for any } (x,t) \in (\mathbb{R}^N \setminus Z_1) \times \mathbb{R}\,.$$

From this equality we immediately get that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)b_\varepsilon(x,u(x))\nabla u\,dx = \int_{\mathbb{R}^N} \phi(x)b(x,u(x))\nabla u\,dx$$

and that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \nabla\phi(x)v_\varepsilon(x)\,dx = \int_{\mathbb{R}^N} \nabla\phi(x)\,dx\int_0^{u(x)} b(x,t)\,dt\,.$$

Notice that, by assumption (jj), for any $t \in \mathbb{R}$ the functions $\dfrac{\partial b_\varepsilon}{\partial x}(\cdot,t)$ converge in $L^1(\mathbb{R}^N)$ to $\dfrac{\partial b}{\partial x}(\cdot,t)$ and $\left\|\dfrac{\partial b_\varepsilon}{\partial x}(\cdot,t)\right\|_{L^1(\mathbb{R}^N)} \le L$ for any $\varepsilon > 0$ and $t \in \mathbb{R}$. Therefore, denoting by $I(x)$ the interval $(0,u(x))$, if $u(x) > 0$, and the interval $(u(x),0)$, otherwise, by Fubini's theorem and by the Lebesgue's theorem of dominated convergence, we get

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}&\left|\int_{\mathbb{R}^N}\phi(x)\,dx\int_0^{u(x)}\frac{\partial b_\varepsilon}{\partial x}(x,t)\,dt - \int_{\mathbb{R}^N}\phi(x)\,dx\int_0^{u(x)}\frac{\partial b}{\partial x}(x,t)\,dt\right| \\ &\le \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}dt\int_{\mathbb{R}^N}\chi_{I(x)}(t)\left|\frac{\partial b_\varepsilon}{\partial x}(x,t) - \frac{\partial b}{\partial x}(x,t)\right|dx = 0\,.\end{aligned}$$

Let us now prove that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\phi(x)b_\varepsilon(x,\widetilde{u}(x))\,dD^c u = \int_{\mathbb{R}^N}\phi(x)\overline{b}(x,\widetilde{u}(x))\,dD^c u\,. \tag{10}$$

To this aim, let us use the coarea formula (8), thus getting

$$\begin{aligned}\int_{\mathbb{R}^N}\phi(x)b_\varepsilon(x,\widetilde{u}(x)\,dD^c u &= \int_{C_u\setminus\mathcal{D}_u}\phi(x)b_\varepsilon(x,\widetilde{u}(x))\frac{D^c u}{|Du|}(x)\,d|Du| \\ &= \int_{\mathbb{R}}dt\int_{\{u^-\le t\le u^+\}}\phi(x)b_\varepsilon(x,\widetilde{u}(x))\chi_{C_u\setminus\mathcal{D}_u}(x)\frac{D^c u}{|Du|}(x)\,d\mathcal{H}^{N-1} \\ &= \int_{\mathbb{R}}dt\int_{\{\widetilde{u}=t\}\cap(C_u\setminus\mathcal{D}_u)}\phi(x)b_\varepsilon(x,t)\frac{D^c u}{|Du|}(x)\,d\mathcal{H}^{N-1}\,.\end{aligned} \tag{11}$$

Let us fix $t \in \mathbb{R}$. Since $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$ is a point of approximate continuity for $b(\cdot,t)$, $b_\varepsilon(x,t)$ converges to $\widetilde{b}(x,t)$ for $\mathcal{H}^{N-1}$-a.e. x, hence $b_\varepsilon(x,t)$ converges to $\overline{b}(x,t)$ for $\mathcal{H}^{N-1}$-a.e. x. Moreover, since $u \in BV(\mathbb{R}^N)$, by the coarea formula we have

$$\int_{-\infty}^{+\infty}\mathcal{H}^{N-1}\left(\{\widetilde{u}=t\}\cap(C_u\setminus\mathcal{D}_u)\right)dt = |Du|(C_u\setminus\mathcal{D}_u) < \infty\,. \tag{12}$$

Therefore, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0}\int_{\{\widetilde{u}=t\}\cap(C_u\setminus\mathcal{D}_u)}\phi(x)b_\varepsilon(x,t)\frac{D^c u}{|Du|}(x)\,d\mathcal{H}^{N-1} = \int_{\{\widetilde{u}=t\}\cap(C_u\setminus\mathcal{D}_u)}\phi(x)\overline{b}(x,t)\frac{D^c u}{|Du|}(x)\,d\mathcal{H}^{N-1}\,.$$

From this equation, passing to the limit in (11) and using (12) and the dominated convergence theorem, we immediately get (10).

Finally, let us consider the Radon measure $\mu = |Du| \times \mathcal{L}^1$ and define for any $i, j \in \mathbb{N}$,

$$G_{i,j} = \left\{(x,t) \in \mathbb{R}^N \times \mathbb{R} :\ \limsup_{r\to 0}\fint_{B_r(x)}|b(y,t) - q_i|dy < \frac{1}{j}\right\},$$

where q_i is a dense sequence in $\mathbb{R}$. The set

$$G = \cap_{j=1}^{\infty}\cup_{i=1}^{\infty} G_{i,j}$$

is a Borel set and $(x,t) \in G$ if and only if x is a point of approximate continuity for $b(\cdot,t)$. By assumption, the function $b(\cdot,t)$ is weakly differentiable, hence $\mathcal{H}^{N-1}(\{x \in \mathbb{R}^N : (x,t) \notin G\}) = 0$. Therefore, the complement G^c of G satisfies the equality

$$\mu(G^c) = \int_{\mathbb{R}} dt \int_{\mathbb{R}^N} \chi_{G^c}(x,t) d|Du| = 0\,. \tag{13}$$

Let us estimate

$$\begin{aligned} &\Big| \int_{J_u} \phi(x)\nu_u(x)\, d\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} b_\varepsilon(x,t)\, dt - \int_{J_u} \phi(x)\nu_u(x)\, d\mathcal{H}^{N-1} \int_{u^-(x)}^{u^+(x)} \overline{b}(x,t)\, dt \Big| \quad (14) \\ &\qquad \le \|\phi\|_\infty \int_{J_u} d|Du| \,⨍_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - \overline{b}(x,t)|\, dt\,. \end{aligned}$$

By (13) we have that

$$\lim_{\varepsilon\to 0} ⨍_{u^-(x)}^{u^+(x)} |b_\varepsilon(x,t) - \overline{b}(x,t)|\, dt = 0 \qquad \text{for } |Du|\text{-a.e. } x \in J_u$$

and from this equality we immediately get that the right hand side in (14) goes to zero, as $\varepsilon \to 0$.

This concludes the proof. □

Remark 3.1. From the above proof it is clear that the integration by parts formula stated in Proposition 1.2 still holds if we replace $\overline{b}$ by any Borel function $\beta : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that, for any $t \in \mathbb{R}$, $\overline{b}(x,t) = \beta(x,t)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$.

The proof of Theorem 1.1 is based on an argument introduced in [15].

Proof of Theorem 1.1.
Step 1. We treat separately the two terms depending on the diffuse part of Du, i.e. $D^a u + D^c u$, and the jump term $D^j u$. Let (u_n) be a sequence in $BV(\Omega)$ converging in $L^1(\Omega)$ to $u \in BV(\Omega)$. Let us fix an open set $\Omega' \subset\subset \Omega$ and a function $\eta \in C_0^1(\mathbb{R})$, with $0 \le \eta(t) \le 1$. Let K_1, K_2 be two compact sets such that

$$K_1 \subset \Omega' \cap C_u, \qquad K_2 \subset \Omega' \setminus C_u\,. \tag{15}$$

Then, we may find two open sets Ω_1, Ω_2, contained in Ω', such that

$$\Omega_1 \cap \Omega_2 = \emptyset, \qquad K_1 \subset \Omega_1, \qquad K_2 \subset \Omega_2\,. \tag{16}$$

Finally, let us denote by g_j the sequence of functions provided by Lemma 2.1. Since

$$\liminf_{n\to\infty} F(u_n,\Omega) \ge \liminf_{n\to\infty} F(u_n,\Omega_1) + \liminf_{n\to\infty} F(u_n,\Omega_2)\,, \tag{17}$$

we are going to estimate separately the two terms on the right hand side of this inequality.

Step 2. Let us fix a finite family $\{A_j\}_{j\in J}$ of disjoint open sets with the closure contained in Ω_1. Let $(\varphi_r)_{r\in\mathbb{N}}$ be a sequence in $C_0^1(\Omega_1)$, with $0 \le \varphi_r \le 1$ for all r, and, for any $j \in J$, let $(\eta_{j,s})_{s\in\mathbb{N}}$ be a sequence in $C^1_0(A_j\times\mathbb{R})$, with $0\le\eta_{j,s}\le 1$ for all j,s. Since by Lemmas 2.1 and 2.2

$$f(x,u,\xi)\ge\sum_{j\in J}g_j(x,u,\xi)\eta_{j,s}(x,u)\varphi_r(x)\qquad\text{for all }(x,u,\xi)\in(\Omega\setminus Z)\times\mathbb{R}\times\mathbb{R}^N$$

$$\overline{f}^\infty(x,u,\xi)\ge\sum_{j\in J}\langle\alpha_j(x,u),\xi\rangle\eta_{j,s}(x,u)\varphi_r(x)\qquad\text{for all }(x,u,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^N\,,$$

for any $r,s\in\mathbb{N}$, we have, by Proposition 1.2 and Remark 3.1,

$$\begin{aligned}F(u_n,\Omega_1)\ge&\sum_{j\in J}\int_{\Omega_1}a_{0,j}(x,u_n)\eta_{j,s}(x,u_n)\varphi_r dx+\sum_{j\in J}\biggl\{\int_{\Omega_1}\langle a_j(x,u_n)\eta_{j,s}(x,u_n),\nabla u_n\rangle\varphi_r dx\\&+\int_{\Omega_1}\langle\alpha_j(x,\widetilde{u}_n),\frac{D^cu_n}{|D^cu_n|}\rangle\eta_{j,s}(x,\widetilde{u}_n)\varphi_r d|D^cu_n|\\&+\int_{J_u\cap\Omega_1}\varphi_r d\mathcal{H}^{N-1}\int_{u_n^-(x)}^{u_n^+(x)}\langle\alpha_j(x,t),\nu_u(x)\rangle\eta_{j,s}(x,t)dt\biggr\}\\=&\sum_{j\in J}\int_{\Omega_1}a_{0,j}(x,u_n)\eta_{j,s}(x,u_n)\varphi_r dx-\sum_{j\in J}\biggl\{\int_{\Omega_1}dx\int_0^{u_n(x)}\langle a_j(x,t),\nabla\varphi_r(x)\rangle\eta_{j,s}(x,t)dt\\&+\int_{\Omega_1}\varphi_r dx\int_0^{u_n(x)}\mathrm{div}_x\bigl(a_j(x,t)\eta_{j,s}(x,t)\bigr)dt\biggr\}\,.\end{aligned}$$

Passing to the limit in this inequality and using Proposition 1.2 and Remark 3.1 again, we easily get

$$\begin{aligned}\liminf_{n\to\infty}F(u_n,\Omega_1)\ge&\sum_{j\in J}\int_{\Omega_1}\bigl[a_{0,j}(x,u)\eta_{j,s}(x,u)+\langle a_j(x,u)\eta_{j,s}(x,u),\nabla u\rangle\bigr]\varphi_r\,dx\\&+\sum_{j\in J}\int_{\Omega_1}\langle\alpha_j(x,\widetilde{u}(x))\eta_{j,s}(x,\widetilde{u}(x)),\frac{D^cu}{|D^cu|}\rangle\varphi_r\,d|D^cu|\\&+\sum_{j\in J}\int_{\Omega_1\cap J_u}\biggl[\int_{u^-(x)}^{u^+(x)}\langle\alpha_j(x,t)\eta_{j,s}(x,t),\nu_u(x)\rangle dt\biggr]\varphi_r\,d\mathcal{H}^{N-1}\,.\end{aligned}\tag{18}$$

From Lusin's theorem there exists a sequence $\varphi_r\in C^1_0(\Omega_1)$, with $0\le\varphi_r(x)\le1$ such that $\varphi_r(x)\to\chi_{C_u\cap\Omega_1}(x)$ for $|Du|$-a.e. $x\in\Omega_1$. Therefore passing to the limit as $r\to\infty$, inequality (18) becomes

$$\begin{aligned}\liminf_{n\to\infty}F(u_n,\Omega_1)\ge&\sum_{j\in J}\int_{\Omega_1}\bigl[a_{0,j}(x,u)\eta_{j,s}(x,u)+\langle a_j(x,u)\eta_{j,s}(x,u),\nabla u\rangle\bigr]\,dx\\&+\sum_{j\in J}\int_{\Omega_1}\langle\alpha_j(x,\widetilde{u}(x))\eta_{j,s}(x,\widetilde{u}(x)),\frac{D^cu}{|D^cu|}\rangle\,d|D^cu|\,.\end{aligned}$$

Next, taking for any $j\in J$, $\eta_{j,s}(x,t)=\gamma_{j,s}(x)\eta(t)$, with $\gamma_{j,s}(x)$ converging to $\chi_{D_j}(x)+\chi_{C_j}(x)$ for $|Du|$-a.e. $x\in A_j$, where

$$D_j=\{x\in A_j\cap\mathcal{D}_u:g_j(x,u(x),\nabla u(x))>0\}$$

$$C_j = \Big\{ x \in A_j \cap (C_u \setminus \mathcal{D}_u) : \langle \alpha_j(x, \widetilde{u}(x)), \frac{D^c u}{|D^c u|} \rangle > 0 \Big\},$$

we get immediately that

$$\liminf_{n\to\infty} F(u_n, \Omega_1) \geq \sum_{j\in J} \int_{A_j} \eta(u) \max\{g_j(x, u(x), \nabla u(x)), 0\}\, dx$$
$$+ \sum_{j\in J} \int_{A_j} \eta(\widetilde{u}(x)) \max\Big\{ \langle \alpha_j(x, \widetilde{u}(x)), \frac{D^c u}{|D^c u|}, 0 \rangle \Big\}\, d|D^c u| \;.$$

Therefore, by applying Lemma 2.3 with $\mu = |Du|$ we obtain, by Lemmas 2.1 and 2.2,

$$\liminf_{n\to\infty} F(u_n, \Omega_1) \geq \int_{\Omega_1} f(x, u, \nabla u)\eta(u)\, dx + \int_{\Omega_1} \overline{f}^{\infty}\Big(x, u, \frac{D^c u}{|D^c u|}\Big)\eta(\widetilde{u})\, d|D^c u| \;.$$

Step 3. Let us fix a finite family $\{U_j\}_{j\in J}$ of disjoint open sets with compact closure in $\Omega_2 \times \mathbb{R}$. Let $(\varphi_r)_{r\in\mathbb{N}}$ be a sequence in $C^1_0(\Omega_2)$, with $0 \leq \varphi_r \leq 1$ for all r, and let $(\eta_{j,s})_{s\in\mathbb{N}}$ be a sequence in $C^1_0(U_j)$, with $0 \leq \eta_{j,s} \leq 1$ for all j, s. Arguing as in Step 2 and letting $\varphi_r(x)$ converge to $\chi_{J_u\cap\Omega_2}(x)$ for $|Du|$-a.e. $x \in \Omega_2$, we get by Fubini's theorem

$$\liminf_{n\to\infty} F(u_n, \Omega_2) \geq \sum_{j\in J} \int_{\Omega_2\times\mathbb{R}} \langle \alpha_j(x,t)\eta_{j,s}(x,t), \nu_u(x)\rangle \chi_{[u^-(x),u^+(x)]}(t)\, d\lambda \;,$$

where λ denotes the product measure of the two σ-finite measures $\mathcal{H}^{N-1} \llcorner J_u$ and $\mathcal{L}^1$. Let A_m be an increasing sequence of Borel sets such that $\cup_m A_m = \mathbb{R}^N \times \mathbb{R}$ and $\lambda(A_m) < \infty$ for any m. Let us fix m and, for any $j \in J$, let us apply Lusin's theorem again to get a sequence $\eta_{j,s}(x,t)$ converging λ-a.e. to $\eta(t)\chi_{S_j\cap A_m}(x,t)$, where $S_j = \{(x,t) \in U_j : \langle \alpha_j(x,t), \nu_u(x)\rangle > 0\}$. Thus, from the previous inequality we obtain

$$\liminf_{n\to\infty} F(u_n, \Omega_2) \geq \sum_{j\in J} \int_{U_j\cap A_m} \eta(t)\chi_{[u^-(x),u^+(x)]}(t) \max\{\langle \alpha_j(x,t), \nu_u(x)\rangle, 0\}\, d\lambda$$

and letting $m \to \infty$

$$\liminf_{n\to\infty} F(u_n, \Omega_2) \geq \sum_{j\in J} \int_{U_j} \eta(t)\chi_{[u^-(x),u^+(x)]}(t) \max\{\langle \alpha_j(x,t), \nu_u(x)\rangle, 0\}\, d\lambda \;.$$

Therefore, by applying Lemma 2.3 with $\mu = \lambda = \mathcal{H}^{N-1} \llcorner J_u \times \mathcal{L}^1$ we obtain, by Lemma 2.2 and Fubini's theorem,

$$\liminf_{n\to\infty} F(u_n, \Omega_2) \quad\geq\quad \int_{\Omega_2} \Big[\int_{u^-(x)}^{u^+(x)} \eta(t)\overline{f}^{\infty}(x, t, \nu_u(x))dt \Big] d\mathcal{H}^{N-1} \;.$$

Letting $\eta(t) \uparrow 1$ for any $t \in \mathbb{R}$, from this inequality and from (19), we obtain, recalling (15), (16) and (17),

$$\liminf_{n\to\infty} F(u_n, \Omega) \geq F(u, K_1) + F(u, K_2) \;.$$

The result follows by letting first $K_1 \uparrow C_u$ and then $K_2 \uparrow \Omega' \setminus C_u$ and, finally, letting $\Omega' \uparrow \Omega$.

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