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Strong Martingale Type and Uniform Smoothness

Jörg Wenzel
Department of Mathematics and Applied Mathematics,
University of Pretoria, Pretoria 0002, South Africa
wenzel@minet.uni-jena.de

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We introduce stronger versions of the usual notions of martingale type $p \leq 2$ and cotype $q \geq 2$ of a Banach space X and show that these concepts are equivalent to uniform p-smoothness and q-convexity, respectively. All these are metric concepts, so they depend on the particular norm in X.

These concepts allow us to get some more insight into the fine line between X being isomorphic to a uniformly p-smooth space or being uniformly p-smooth itself.

Instead of looking at Banach spaces, we consider linear operators between Banach spaces right away. The situation of a Banach space X can be rediscovered from this by considering the identity map of X.

Keywords: (Strong) martingale type, (strong) martingale cotype, uniform convexity, uniform smoothness

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1. Introduction

In several recent papers Kato *et al.* [3, 6] introduced the concept of strong Rademacher type and cotype and related those concepts to the uniform smoothness and convexity properties of the underlying Banach space.

In particular for $1 \leq p \leq 2$, a Banach space X has *strong Rademacher type* p if there is a constant c such that

$$\Big\| \sum_{k=0}^{n} x_k r_k \Big| L_p \Big\| \leq \Big(\|x_0\|^p + c^p \sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}$$

for all $n \in \mathbb{N}$ and elements $x_0, \dots, x_n \in X$. Here $r_0, \dots, r_n$ denotes the sequence of Rademacher functions.

It turns out [6, Theorem 3] that X is uniformly p-smooth if and only if it has strong Rademacher type p, moreover the constants involved coincide (see Section 2 for the definition of p-smoothness and q-convexity).

A dual result holds for strong Rademacher cotype q and uniform q-convexity, where $q \geq 2$.

Surely, strong Rademacher type p implies usual Rademacher type p and the same goes for the respective cotype properties. Also, both convexity and smoothness as well as the strong Rademacher type and cotype properties depend on the particular norm in X.

It is well known [4, 5], that X is uniformly p-smooth for some equivalent norm, if and only if X has martingale type p. So the more natural notion connecting to uniform p-smoothness would be something like *strong martingale type* p.

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In this paper, we define strong martingale type p and show, that it is equivalent (with the same constants) to X being uniformly p-smooth. Again, strong martingale type p implies usual martingale type p but now we also have a partial converse. It follows, that one can renorm a Banach space with martingale type p so as to have strong martingale type p in the new norm. This is not so if one only considers Rademacher type, since there is an example, due to James [2] of a nonreflexive Banach space of Rademacher type 2. Since the space is non reflexive, it cannot be uniformly p-smooth for any $p > 1$ and consequently not have strong Rademacher type p for any $p > 1$ in any equivalent norm.

All results have dual variants for strong martingale cotype q and uniform q-convexity.

Hardly any of the results in this paper are really new, they are just old ones and generalizations of old ones, put into the right perspective. Moreover, our direct proof for the renorming result in the martingale type p case (Theorem 3.7) seems to be new.

2. Notation and concepts

For $k = 0, 1, \dots,$ the *dyadic intervals*

$$\Delta_k^{(i)} := \Big[\frac{i}{2^k}, \frac{i+1}{2^k}\Big) \quad \text{where } i = 0, \dots, 2^k - 1,$$

generate the dyadic σ-algebra denoted by $\mathcal{F}_k$.

For a Banach space X, we consider *dyadic martingales* $(f_0, \dots, f_n)$, defined on $[0, 1)$, taking values in X, and adapted to the *dyadic filtration* $\mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_n$. We let $f_{-1} \equiv 0$ and denote by $d_k := f_k - f_{k-1}$ the *differences* or increments of this martingale.

By

$$\|f|L_p\| := \Big(\int_0^1 \|f(t)\|^p \, dt\Big)^{1/p}$$

we denote the L_p-norm of a function $f : [0, 1) \to X$. Given another function $f' : [0, 1) \to X'$, we write

$$\langle f, f'\rangle := \int_0^1 \langle f(t), f'(t)\rangle \, dt,$$

where $\langle x, x'\rangle$ is used to denote the duality between X and X'.

For $q \geq 2$, an operator $T : X \to Y$ is *uniformly q-convex* if there is a constant c such that

$$\Big\|\frac{Tx_+ - Tx_-}{2}\Big\| \leq c\Big(\frac{\|x_+\|^q + \|x_-\|^q}{2} - \Big\|\frac{x_+ + x_-}{2}\Big\|^q\Big)^{1/q} \tag{1}$$

for all $x_+, x_- \in X$. Equivalently, letting $x = (x_+ - x_-)/2$ and $x_0 = (x_+ + x_-)/2$, we can rephrase the definition as

$$\|Tx\| \leq c\Big(\frac{\|x + x_0\|^q + \|x - x_0\|^q}{2} - \|x_0\|^p\Big)^{1/q} \tag{1'}$$

for all $x, x_0 \in X$.

The operator has *strong martingale cotype* q if there is a constant c such that

$$\Big(\|x\|^q + \frac{1}{c^q}\sum_{k=1}^{n}\|Td_k|L_q\|^q\Big)^{1/q} \le \Big\|\sum_{k=0}^{n} d_k\Big|L_q\Big\| \tag{2}$$

for all sequences of X-valued differences $d_0, d_1, \ldots, d_n$ of dyadic martingales such that $d_0 \equiv x$.

Note that following Pisier [5] an operator has *martingale cotype* q if there is a constant c such that

$$\Big(\frac{1}{c^q}\sum_{k=0}^{n}\|Td_k|L_q\|^q\Big)^{1/q} \le \Big\|\sum_{k=0}^{n} d_k\Big|L_q\Big\| \tag{3}$$

for all sequences of X-valued differences $d_0, \ldots, d_n$ of dyadic martingales. So by $\|T\| \le c$ strong martingale cotype q implies martingale cotype q. We will see, that the reverse is only true if one passes to an equivalent norm in X.

We now repeat the definitions above in the dual case. For $1 \le p \le 2$, an operator $T : X \to Y$ is *uniformly p-smooth* if there is a constant c such that

$$\Big(\frac{\|y + Tx\|^p + \|y - Tx\|^p}{2} - \|y\|^p\Big)^{1/p} \le c\|x\| \tag{4}$$

for all $x \in X$ and $y \in Y$.

The operator has *strong martingale type* p if there is a constant c such that

$$\Big\|y + \sum_{k=1}^{n} Td_k\Big|L_p\Big\| \le \Big(\|y\|^p + c^p\sum_{k=1}^{n}\|d_k|L_p\|^p\Big)^{1/p} \tag{5}$$

for all sequences of X-valued differences $d_1, \ldots, d_n$ of dyadic martingales and $y \in Y$.

Note that following Pisier [5] an operator has *martingale type* p if there is a constant c such that

$$\Big\|\sum_{k=0}^{n} Td_k\Big|L_p\Big\| \le \Big(c^p\sum_{k=0}^{n}\|d_k|L_p\|^p\Big)^{1/p} \tag{6}$$

for all sequences of X-valued differences $d_0, \ldots, d_n$ of dyadic martingales. So by letting $y = Tx$ and since $\|T\| \le c$, strong martingale type p implies martingale type p. We will see, that the reverse is only true if one passes to an equivalent norm in Y.

For $1 \le p < \infty$, we denote by p' the *dual index* of p, given by $1/p + 1/p' = 1$.

3. Results

Theorem 3.1. *An operator T is uniformly q-convex with constant c if and only if T has strong martingale cotype q with the same constant c.*

Proof. Assume first that T has strong martingale cotype q. Define a martingale $d_0 + d_1$ by letting $d_0 \equiv x_0$ and $d_1 := x_1 r_1$. Applying (2) to this martingale yields

$$\Big(\|x_0\|^q + \frac{1}{c^q}\|Tx_1\|^q\Big)^{1/q} \le \Big(\frac{\|x_0 + x_1\|^q + \|x_0 - x_1\|^q}{2}\Big)^{1/q}.$$

The substitution $x_+ := x_0 + x_1$ and $x_- := x_0 - x_1$ then yields (1).

Conversely, if $d_0, \dots, d_n$ is any sequence of differences of a dyadic X-valued martingale such that $d_0 \equiv x$, write $f_k = \sum_{h=0}^{k} d_h$. Then f_{k-1} is constant on the intervals $\Delta_{k-1}^{(j)}$, while the (constant) value that d_k takes on the left half $\Delta_k^{(2j)}$ of $\Delta_{k-1}^{(j)}$ is the negative of the (constant) value that d_k takes on the right half $\Delta_k^{(2j+1)}$ of $\Delta_{k-1}^{(j)}$. Hence

$$\int\limits_{\Delta_{k-1}^{(j)}} \|f_{k-1}(t) + d_k(t)\|^q \, dt = \int\limits_{\Delta_{k-1}^{(j)}} \|f_{k-1}(t) - d_k(t)\|^q \, dt$$

and we have

$$\big\|f_k\big|L_q\big\| = \big\|f_{k-1} + d_k\big|L_q\big\| = \big\|f_{k-1} - d_k\big|L_q\big\|. \tag{7}$$

From (1') we get

$$\|Td_k(t)\|^q \le c^q \Big(\frac{\|f_{k-1}(t) + d_k(t)\|^q + \|f_{k-1}(t) - d_k(t)\|^q}{2} - \big\|f_{k-1}(t)\big\|^q \Big)$$

which, when integrated over t and using (7) gives

$$\|Td_k|L_q\|^q \le c^q \big(\|f_k|L_q\|^q - \|f_{k-1}|L_q\|^q \big).$$

Take the sum over $k = 1, \dots, n$ of these inequalities to get (2). □

Theorem 3.2. *An operator T is uniformly p-smooth with constant c if and only if T has strong martingale type p with the same constant c.*

Proof. Assume first that T has strong martingale type p. Define a martingale difference by letting $d_1 \equiv x r_1$. Applying (5) to this martingale yields

$$\Big(\frac{\|y + Tx\|^p + \|y - Tx\|^p}{2} \Big)^{1/p} \le \big(\|y\|^p + c^p \|x\|^p \big)^{1/p},$$

which immediately yields (4).

Conversely, if $d_1, \dots, d_n$ is any sequence of differences of a dyadic X-valued martingale, write $f_k = \sum_{h=1}^{k} d_h$. Then $y + Tf_{k-1}$ is constant on the intervals $\Delta_{k-1}^{(j)}$, while the (constant) value that Td_k takes on the left half $\Delta_k^{(2j)}$ of $\Delta_{k-1}^{(j)}$ is the negative of the (constant) value that Td_k takes on the right half $\Delta_k^{(2j+1)}$ of $\Delta_{k-1}^{(j)}$. Hence

$$\int\limits_{\Delta_{k-1}^{(j)}} \|y + Tf_{k-1}(t) + d_k(t)\|^p \, dt = \int\limits_{\Delta_{k-1}^{(j)}} \|y + Tf_{k-1}(t) - d_k(t)\|^p \, dt$$

and we have

$$\|y + Tf_k|L_p\| = \|y + Tf_{k-1} + Td_k|L_p\| = \|y + Tf_{k-1} - Td_k|L_p\|. \tag{8}$$

From (4) we get

$$\frac{\|y + Tf_{k-1}(t) + Td_k(t)\|^p + \|y + Tf_{k-1}(t) - Td_k(t)\|^p}{2} \le \|y + Tf_{k-1}(t)\|^p + c^p \|d_k(t)\|^p$$

which when integrated over t and using (8) gives

$$\|y + Tf_k|L_p\|^p \leq \|y + Tf_{k-1}|L_p\|^p + c^p\|d_k|L_p\|^p.$$

Take the sum over $k = 1, \ldots, n$ of these inequalities to get (5). □

Proposition 3.3. *An operator T has strong martingale type p if and only if its dual T' has strong martingale cotype p'. And an operator T has strong martingale cotype q if and only if its dual T' has strong martingale type q'.*

Proof. Assume that T' is of strong martingale cotype p'.

Given a sequence $d_1, \ldots, d_n$ of dyadic martinagle differences and $y \in Y$, we find an $\mathcal{F}_n$-measurable function g_n with values in Y' such that

$$\Big\|y + \sum_{k=1}^n Td_k\Big|L_p\Big\| = \Big\langle y + \sum_{k=1}^n Td_k, g_n\Big\rangle \quad \text{and} \quad \|g_n|L_{p'}\| \leq 1 + \epsilon.$$

Writing

$$e_k := \mathbb{E}(g_n|\mathcal{F}_k) - \mathbb{E}(g_n|\mathcal{F}_{k-1}),$$

we obtain another sequence of dyadic martingale differences and we have

$$\Big\langle y + \sum_{k=1}^n Td_k, g_n\Big\rangle = \langle y, e_0\rangle + \sum_{k=1}^n \langle Td_k, e_k\rangle.$$

Consequently

$$\Big\|y + \sum_{k=1}^n Td_k\Big|L_p\Big\| \leq \Big(\|y\|^p + c^p \sum_{k=1}^n \|d_k|L_p\|^p\Big)^{1/p} \Big(\|e_0\|^{p'} + \frac{1}{c^{p'}} \sum_{k=1}^n \|T'e_k|L_{p'}\|^{p'}\Big)^{1/p'}.$$

Now applying the strong martingale cotype p' property of T' we get

$$\|e_0\|^{p'} + \frac{1}{c^{p'}} \sum_{k=1}^n \|T'e_k|L_{p'}\|^{p'} \leq \Big\|\sum_{k=0}^n e_k\Big|L_{p'}\Big\|^{p'} \leq (1+\epsilon)^{p'},$$

which proves that

$$\Big\|y + \sum_{k=1}^n Td_k\Big|L_p\Big\| \leq (1+\epsilon)\Big(\|y\|^p + c^p \sum_{k=1}^n \|d_k|L_p\|^p\Big)^{1/p}.$$

Letting $\epsilon \to 0$ proves that T has strong martingale type p with the same constant.

To see the other implication, given a sequence of Y'-valued dyadic martingale differences $d_0, \ldots, d_n$ such that $d_0 \equiv y'$, let

$$\lambda := \frac{c\|y'\|^{p'-1}}{\Big(\Big\|\sum_{k=0}^n d_k\Big|L_{p'}\Big\|^{p'} - \|y'\|^{p'}\Big)^{1/p}}.$$

We find X'-valued dyadic martingale differences $e_1, \dots, e_n$ such that

$$\Big(\sum_{k=1}^n \|T'd_k|L_{p'}\|^{p'}\Big)^{1/p'} = \sum_{k=1}^n \langle T'd_k, e_k\rangle \quad \text{and} \quad \Big(\sum_{k=1}^n \|e_k|L_p\|^p\Big)^{1/p} \le 1+\epsilon.$$

Moreover we find $y \in Y$ such that $\|y'\| = \langle y', y\rangle$ and $\|y\| \le 1+\epsilon$.

We can then write

$$\begin{aligned}\Big(\sum_{k=1}^n \|T'd_k|L_{p'}\|^{p'}\Big)^{1/p'} &= \Big\langle y' + \sum_{k=1}^n d_k, \lambda y + \sum_{k=1}^n Te_k\Big\rangle - \lambda\langle y', y\rangle \\ &\le \Big\|\sum_{k=0}^n d_k\Big|L_{p'}\Big\| \cdot \Big\|\lambda y + \sum_{k=1}^n Te_k\Big|L_p\Big\| - \lambda\|y'\|.\end{aligned}$$

Using the strong martingale type p property of T, we get

$$\Big\|\lambda y + \sum_{k=1}^n Te_k\Big|L_p\Big\| \le \Big(\lambda^p\|y\|^p + c^p\sum_{k=1}^n \|e_k|L_p\|^p\Big)^{1/p} \le (1+\epsilon)\big(c^p+\lambda^p\big)^{1/p}.$$

Our choice of λ now ensures that

$$\Big\|\sum_{k=0}^n d_k\Big|L_{p'}\Big\|(c^p+\lambda^p)^{1/p} = c\bigg(\Big\|\sum_{k=0}^n d_k\Big|L_{p'}\Big\|^{p'} - \|y'\|^{p'}\bigg)^{1/p'} + \lambda\|y'\|.$$

This proves that

$$\begin{aligned}\Big(\sum_{k=1}^n \|T'd_k|L_{p'}\|^{p'}\Big)^{1/p'} &\le \Big\|\sum_{k=0}^n d_k\Big|L_{p'}\Big\|(c^p+\lambda^p)^{1/p}(1+\epsilon) - \lambda\|y'\| \\ &= (1+\epsilon)c\bigg(\Big\|\sum_{k=0}^n d_k\Big|L_{p'}\Big\|^{p'} - \|y'\|^{p'}\bigg)^{1/p'} + \epsilon\lambda\|y'\|.\end{aligned}$$

Letting $\epsilon \to 0$ proves that T' has strong martingale cotype with the same constant.

The second part follows analogously. □

Proposition 3.4. *An operator T is uniformly p-smooth if and only if its dual T' is uniformly p'-convex. An operator T is uniformly q-convex if and only if its dual T' is uniformly q'-smooth.*

Proof. This was well known for Banach spaces, see Beauzamy [1, pp. 311–312] and has been proved for linear operators in Pietsch/Wenzel [4, 7.9.6]. □

We have now an alternative proof of Theorem 3.2 using duality.

Proof of Theorem 3.2 (alternative version). If T is uniformly p-smooth, then its dual T' is uniformly p'-convex, which by Theorem 3.1 happens if and only if T' has strong martingale cotype p' and by Proposition 3.3 this is equivalent to T having strong martingale type p. □

As our last project, we want to discuss the relation of martingale type/cotype and strong martingale type/cotype. It is due to Pisier [5] that whenever T has martingale cotype q, then there exists an equivalent norm on X, such that T becomes uniformly q-convex when considered as an operator from X equipped with this new norm into Y. From this it then follows that T has strong martingale cotype q. By duality the same works in the type case. However, we also want to give a direct argument similar to the one used for cotype.

Since we think that our approach makes more clear, why the respective proofs work and since we haven't seen the direct proof for the type case in print yet, we want to include a proof for both cases here.

We first provide a technical lemma, that allows us to define equivalent norms.

Lemma 3.5. *Let $|||\cdot||| : [X, \|\cdot\|] \to [0, \infty)$ be a continous, positively homogeneous functional. Assume that for $|||x_\pm||| \le 1$ it follows that $|||x_+ + x_-||| \le 2$. Then*

$$|||x_+ + x_-||| \le |||x_+||| + |||x_-|||$$

for all $x_\pm \in X$.

Proof. Assume first that $|||x_\pm||| \le 1$. By assumption

$$|||\lambda x_+ + (1-\lambda)x_-||| \le 1 \tag{9}$$

for $\lambda = 1/2$. It follows by induction over n, that the same holds for all $\lambda = k/2^n$, where $n = 0, 1, 2, \dots$ and $0 \le k \le 2^n$.

To see this, write

$$x := \frac{k}{2^{n-1}} x_+ + \Big(1 - \frac{k}{2^{n-1}}\Big) x_-$$

and note that $|||x||| \le 1$ by the induction hypothesis, and

$$\lambda x_+ + (1-\lambda)x_- = \frac{1}{2} x + \frac{1}{2} x_-.$$

It now follows by continuity that (9) holds in fact for all $\lambda \in [0, 1]$.

Finally, for arbitrary $x_\pm$ let

$$\lambda = \frac{|||x_+|||}{|||x_+||| + |||x_-|||}.$$

Then from (9) we get

$$\Big|\Big|\Big|\lambda \frac{x_+}{|||x_+|||} + (1-\lambda)\frac{x_-}{|||x_-|||}\Big|\Big|\Big| \le 1$$

which in turn implies the triangle inequality. $\square$

Theorem 3.6. *If an operator $T : X \to Y$ has martingale cotype q then there exists an equivalent norm on X, such that T considered as an operator from X equipped with the new norm into Y is uniformly q-convex.*

Proof. Assume that (3) holds. Let

$$\{x\} := \inf\left(\left\|x + \sum_{k=1}^{n} d_k \Big| L_q\right\|^q - \frac{1}{c^q}\sum_{k=1}^{n} \|Td_k|L_q\|^q\right)^{1/q},$$

where the infimum is taken over all sequences of X-valued differences $d_1, \dots, d_n$ of martingales adapted to the dyadic filtration $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n$.

Letting $d_1 \equiv \cdots \equiv d_n \equiv 0$ yields

$$\{x\} \le \|x\|.$$

Conversely, it follows from (3) that

$$\{x\} \ge \frac{1}{c}\|Tx\|.$$

Trivially with this expression (which need not yet be a norm) we have

$$\left(\{x\}^q + \frac{1}{c^q}\sum_{k=1}^{n} \|Td_k|L_q\|^q\right)^{1/q} \le \left\|x + \sum_{k=1}^{n} d_k \Big| L_q\right\|$$

which unfortunately is not yet strong martingale cotype q, since we would have to use $\{\cdot\}$ also on the right hand side.

Therefore we next show that this expression satisfies (1). Given x_+ and x_- choose differences $d_1^{\pm}, \dots, d_n^{\pm}$ of dyadic martingales such that

$$\left\|x_{\pm} + \sum_{k=1}^{n} d_k^{\pm} \Big| L_q\right\|^q - \frac{1}{c^q}\sum_{k=1}^{n} \|Td_k^{\pm}|L_q\|^q \le \{x_{\pm}\}^q + \epsilon.$$

Glueing together two differences as

$$d_{k+1}(t) := \begin{cases} d_k^+(2t) & \text{if } 0 \le t < 1/2, \\ d_k^-(2t-1) & \text{if } 1/2 \le t < 1, \end{cases}$$

we get a new sequence of X-valued differences $d_2, \dots, d_{n+1}$ of a dyadic martingale, which however is now adapted to $\mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_{n+1}$, so letting

$$d_1(t) := \begin{cases} \frac{x_+ - x_-}{2} & \text{if } 0 \le t < 1/2, \\ \frac{x_- - x_+}{2} & \text{if } 1/2 \le t < 1, \end{cases}$$

yields a sequence of differences of a dyadic martingale adapted to $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n+1}$.

This is, by the way, the point, where we cannot just use Rademacher functions, since the function equal to $x_k^+ r_k(2t)$ for $0 \le t < 1/2$ and to $x_k^- r_k(2t-1)$ for $1/2 \le t < 1$ will no longer be a multiple of a Rademacher function, but at best a martingale difference as soon as $x_k^+ \ne x_k^-$.

Continuing with our considerations, we now have consequently

$$\{x\}^q \le \left\|x + \sum_{k=1}^{n+1} d_k \Big| L_q\right\|^q - \frac{1}{c^q}\sum_{k=1}^{n+1} \|Td_k|L_q\|^q$$

for all $x \in X$, in particular for $(x_+ + x_-)/2$.

It is now clear that

$$\|Td_1|L_q\|^q = \Big\|\frac{Tx_+ - Tx_-}{2}\Big\|^q, \quad \|Td_{k+1}|L_q\|^q = \frac{\|Td_k^+|L_q\|^q + \|Td_k^-|L_q\|^q}{2},$$

and

$$\Big\|\frac{x_+ + x_-}{2} + \sum_{k=1}^{n+1} d_k\Big|L_q\Big\|^q = \frac{1}{2}\Big\|x_+ + \sum_{k=1}^{n} d_k^+\Big|L_q\Big\|^q + \frac{1}{2}\Big\|x_- + \sum_{k=1}^{n} d_k^-\Big|L_q\Big\|^q.$$

Therefore it follows from the definition of $d_k^{\pm}$ that

$$\Big\{\frac{x_+ + x_-}{2}\Big\}^q \le \frac{\{x_+\}^q + \{x_-\}^q}{2} + \epsilon - \frac{1}{c^q}\Big\|\frac{Tx_+ - Tx_-}{2}\Big\|^q.$$

Letting $\epsilon \longrightarrow 0$ shows

$$\Big\{\frac{x_+ + x_-}{2}\Big\}^q \le \frac{\{x_+\}^q + \{x_-\}^q}{2} - \frac{1}{c^q}\Big\|\frac{Tx_+ - Tx_-}{2}\Big\|^q. \tag{10}$$

However, as was mentioned earlier, the expression $\{\cdot\}$ need not be a norm on X. But it is positively homogeneous and continous so from (10) it follows that Lemma 3.5 applies and it also satisfies the triangle inequality. To get an *equivalent* norm, we define

$$|||x||| := \big(\|x\|^q + \{x\}^q\big)^{1/q}.$$

Adding (10) to

$$\Big\|\frac{x_+ + x_-}{2}\Big\|^q \le \frac{\|x_+\|^q + \|x_-\|^q}{2}$$

we finally get the uniform q-convexity for T considered as an operator from X equipped with $|||\cdot|||$ to Y. □

Theorem 3.7. *If an operator $T : X \to Y$ has martingale type p then there exists an equivalent norm on Y, such that T considered as an operator from X into Y equipped with this new norm is uniformly p-smooth.*

Proof. Assume that (6) holds. For $x \in X$ let

$$\{x\} := \sup\bigg(\Big\|Tx + \sum_{k=1}^{n} Td_k\Big|L_p\Big\|^p - c^p\sum_{k=1}^{n}\|d_k|L_p\|^p\bigg)^{1/p},$$

where the supremum is taken over all sequences of X-valued differences $d_1, \dots, d_n$ of martingales adapted to the dyadic filtration $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n$.

Letting $d_1 \equiv \cdots \equiv d_n \equiv 0$ yields

$$\{x\} \ge \|Tx\|.$$

Conversely, it follows from (6) that

$$\{x\} \le c\|x\|.$$

We next show, that the expression $\{\,\cdot\,\}$ satisfies

$$\Big\{\frac{x_+ + x_-}{2}\Big\}^p \ge \frac{\{x_+\}^p + \{x_-\}^p}{2} - c^p \Big\|\frac{x_+ - x_-}{2}\Big\|^p.$$

Given x_+ and x_- choose differences $d_1^\pm, \dots, d_n^\pm$ of dyadic martingales such that

$$\Big\|Tx_\pm + \sum_{k=1}^n Td_k^\pm \Big| L_p\Big\|^p - c^p \sum_{k=1}^n \|d_k^\pm | L_p\|^p \ge \{x_\pm\} - \epsilon.$$

Glueing together two differences as

$$d_{k+1}(t) := \begin{cases} d_k^+(2t) & \text{if } 0 \le t < 1/2, \\ d_k^-(2t-1) & \text{if } 1/2 \le t < 1, \end{cases}$$

we get a new sequence of X-valued differences $d_2, \dots, d_{n+1}$ of a dyadic martingale, which however is now adapted to $\mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_{n+1}$, so letting

$$d_1(t) := \begin{cases} \frac{x_+ - x_-}{2} & \text{if } 0 \le t < 1/2, \\ \frac{x_- - x_+}{2} & \text{if } 1/2 \le t < 1, \end{cases}$$

yields a sequence of differences of a dyadic martingale adapted to $\mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_{n+1}$.

Consequently

$$\{x\}^p \ge \Big\|Tx + \sum_{k=1}^{n+1} Td_k \Big| L_p\Big\|^p - c^p \sum_{k=1}^{n+1} \|d_k | L_p\|^p$$

for all $x \in X$, in particular for $(x_+ + x_-)/2$. It is now clear that

$$\|d_1 | L_p\|^p = \Big\|\frac{x_+ - x_-}{2}\Big\|^p, \quad \|d_k | L_p\|^p = \frac{\|d_k^+ | L_p\|^p + \|d_k^- | L_p\|^p}{2},$$

and

$$\Big\|\frac{Tx_+ + Tx_-}{2} + \sum_{k=1}^{n+1} Td_k \Big| L_p\Big\|^p = \frac{1}{2}\Big\|Tx_+ + \sum_{k=1}^n Td_k^+ \Big| L_p\Big\|^p + \frac{1}{2}\Big\|Tx_- + \sum_{k=1}^n Td_k^- \Big| L_p\Big\|^p.$$

Therefore it follows from the definition of $d_k^\pm$ that

$$\Big\{\frac{x_+ + x_-}{2}\Big\}^p \ge \frac{\{x_+\}^p + \{x_-\}^p}{2} - \epsilon - c^p \Big\|\frac{x_+ - x_-}{2}\Big\|^p.$$

Letting $\epsilon \to 0$ shows

$$\Big\{\frac{x_+ + x_-}{2}\Big\}^p \ge \frac{\{x_+\}^p + \{x_-\}^p}{2} - c^p \Big\|\frac{x_+ - x_-}{2}\Big\|^p. \tag{11}$$

Finally defining

$$|||y||| := \inf\Big(\frac{1}{2^n} \sum_{k=1}^{2^n} \big(\|y_k - Tx_k\|^p + \{x_k\}^p\big)\Big)^{1/p}$$

where the infimum is taken over all decompositions of y as $y = \sum_{k=1}^{2^n} y_k/2^n$ and all elements $x_k \in X$, we we get an equivalent norm on Y for which T becomes uniformly p-smooth. Indeed clearly $|||\cdot|||$ is positively homogeneous and choosing $n = 0$ and $x_0 = 0$ we get

$$|||y||| \leq \|y\|.$$

On the other hand

$$\begin{aligned}\Big(\frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x\}^p\big)\Big)^{1/p} &\geq \Big(\frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \|Tx_k\|^p\big)\Big)^{1/p} \\ &\geq \Big(\frac{1}{2^n}\sum_{k=1}^{2^n} 2^{1-p}\big(\|y_k - Tx_k\| + \|Tx_k\|\big)^p\Big)^{1/p} \\ &\geq 2^{1/p-1}\Big(\frac{1}{2^n}\sum_{k=1}^{2^n}\|y_k\|^p\Big)^{1/p} \geq 2^{1/p-1}\|y\|\end{aligned}$$

whenever $y = \sum_{k=1}^{2^n} /2^n$, so

$$|||y||| \geq 2^{1/p-1}\|y\|.$$

Next, given $y_\pm$ we choose $y_k^\pm$ and $x_k^\pm \in X$ such that

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k^\pm - Tx_k^\pm\|^p + \{x_k^\pm\}^p\big) \leq |||y_\pm|||^p + \epsilon \quad \text{and} \quad y_\pm = \frac{1}{2^n}\sum_{k=1}^{2^n} y_k^\pm.$$

Note that we can assume that for both y_+ and y_- the same n works, since if $y = \frac{1}{2^n}\sum_{k=1}^{2^n} y_k$ then also

$$y = \frac{1}{2^{n+1}}\Big(\sum_{k=1}^{2^n} y_k + \sum_{k=1}^{2^n} y_k\Big)$$

and moreover, for $x_1, \ldots, x_{2^n} \in X$ we have

$$\begin{aligned}&\Big(\frac{1}{2^{n+1}}\Big(\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x_k\}^p\big) + \sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x_k\}^p\big)\Big)\Big)^{1/p} \\ &\qquad = \Big(\frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x_k\}^p\big)\Big)^{1/p}.\end{aligned}$$

It now follows that

$$y_+ + y_- = \frac{1}{2^{n+1}}\Big(\sum_{k=1}^{2^n}(2y_k^+) + \sum_{k=1}^{2^n}(2y_k^-)\Big)$$

hence

$$
\begin{aligned}
&|||y_+ + y_-|||^p \\
&\leq \frac{1}{2^{n+1}}\Big(\sum_{k=1}^{2^n}\big(\|2y_k^+ - 2Tx_k^+\|^p + \{2x_k^+\}^p\big) + \sum_{k=1}^{2^n}\big(\|2y_k^- - 2Tx_k^-\|^p + \{2x_k^-\}^p\big)\Big) \\
&= 2^{p-1}\Big(\frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k^+ - Tx_k^+\|^p + \{x_k^+\}^p\big) + \frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k^- - Tx_k^-\|^p + \{x_k^-\}^p\big)\Big) \\
&\leq 2^{p-1}\big(|||y_+|||^p + |||y_-|||^p + 2\epsilon\big),
\end{aligned}
$$

which proves that

$$\Big|\Big|\Big|\frac{y_+ + y_-}{2}\Big|\Big|\Big| \leq \Big(\frac{|||y_+|||^p + |||y_-|||^p}{2}\Big)^{1/p}.$$

It now follows from Lemma 3.5 that $|||\cdot|||$ is actually a norm.

To see the p-smoothness property of T for this norm, given $y \in Y$, choose y_k and $x_k \in X$ such that

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x_k\}^p\big) \leq |||y|||^p + \epsilon \quad \text{and} \quad y = \frac{1}{2^n}\sum_{k=1}^{2^n} y_k.$$

Then for $x \in X$

$$
\begin{aligned}
|||y \pm Tx|||^p &\leq \frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k \pm Tx - T(x_k \pm x)\|^p + \{x_k \pm x\}^p\big) \\
&= \frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x_k \pm x\}^p\big).
\end{aligned}
$$

But by (11) we have for each k

$$\frac{\{x_k + x\}^p + \{x_k - x\}^p}{2} \leq \{x_k\}^p + c^p\|x\|^p$$

so that

$$
\begin{aligned}
\frac{|||y + Tx|||^p + |||y - Tx|||^p}{2} &\leq \frac{1}{2^n}\sum_{k=1}^{2^n}\big(\|y_k - Tx_k\|^p + \{x_k\}^p + c^p\|x\|^p\big) \\
&\leq |||y|||^p + \epsilon + c^p\|x\|^p
\end{aligned}
$$

which proves the uniform p-smoothness of T. $\square$

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