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Conditional and Relative Weak Compactness in Vector-Valued Function Spaces

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Let E be an ideal of L^o over a σ-finite measure space (Ω, Σ, μ), and let $(X, \|\cdot\|_X)$ be a real Banach space. Let $E(X)$ be a subspace of the space $L^o(X)$ of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \longrightarrow X$ and consisting of all those $f \in L^o(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. Let $E(X)_n^\sim$ stand for the order continuous dual of $E(X)$. In this paper we characterize both conditionally $\sigma(E(X), I)$-compact and relatively $\sigma(E(X), I)$-sequentially compact subsets of $E(X)$ whenever I is an ideal of $E(X)_n^\sim$. As an application, we obtain a characterization of almost reflexivity and reflexivity of a Banach space X in terms of conditionally $\sigma(E(X), I)$-compact and relatively $\sigma(E(X), I)$-sequentially compact subsets of $E(X)$.

Keywords: Vector-valued function spaces, Köthe-Bochner spaces, conditional weak compactness, weak sequential compactness, weak compactness, weak sequential completeness, almost reflexivity, reflexivity, absolute weak topologies

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1. Introduction and preliminaries

Given a topological vector space (L, ξ) by $(L, \xi)^*$ we will denote its topological dual. We denote by $\sigma(L, K)$ the weak topology on L with respect to a dual pair $\langle L, K\rangle$. Recall that a subset Z of L is said to be *conditionally* $\sigma(L, K)$-*compact* (resp. *relatively* $\sigma(L, K)$-*sequentially compact*) whenever each sequence in Z contains a $\sigma(L, K)$-Cauchy subsequence (resp. each sequence in Z contains a subsequence which is $\sigma(L, K)$-convergent to some element of L).

The problem of characterizing of conditionally and relatively weakly compact subsets of Lebesgue-Bochner spaces $L^p(X)$, where $1 \leq p \leq \infty$ (in particular in $L^1(X)$) and X is supposed to satisfy some conditions has been considered by many authors (see [6], [5], [26], [29], [17], [11], [28], [3], [14], [19], [30]). Recently, H. Benabdellah and C. Castaing [4] and J. Diestel, W. M. Ruess, W. Schachermayer [15] have found criteria for conditional weak compactness and relative weak compactness in $L^1(X)$ (over a finite measure space) for a general Banach space X.

Assume that E is an ideal of L^o over a σ-finite measure space and let X be a Banach space. In Section 2 and Section 3 we characterize both conditionally $\sigma(E(X), I)$-compact and relatively $\sigma(E(X), I)$-sequentially compact subsets of $E(X)$ whenever I is an ideal of the order continuous dual $E(X)_n^\sim$ separating the points of $E(X)$ (see Theorem 2.2 and Theorem 3.3).

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Now we establish notation and terminology concerning function spaces (see [2], [20], [32]). Let (Ω, Σ, μ) be a complete σ-finite measure space and let $\Sigma_f = \{A \in \Sigma : \ \mu(A) < \infty\}$. Let L^o denote the space of μ-equivalence classes of all Σ-measurable real valued functions defined and finite a.e. on Ω. Let χ_A stand for the characteristic function of a set A, and let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all natural and real numbers.

Let E be an ideal of L^o with $\operatorname{supp} E = \Omega$, and let $E^\sim$ and $E_n^\sim$ stand for the order dual and the order continuous dual of E respectively. Let E' denote the Köthe dual of E. Throughout the paper we will assume that $\operatorname{supp} E' = \Omega$. Then $E_n^\sim$ can be identified with E' through the mapping: $E' \ni v \longmapsto \varphi_v \in E_n^\sim$, where $\varphi_v(u) = \int_\Omega u(\omega)\, v(\omega)\, \mathrm{d}\mu$ for all $u \in E$.

For a sequence (A_n) in Σ we write $A_n \searrow_\mu \emptyset$ whenever $A_n \downarrow$ and $\mu(\bigcap_{n=1}^\infty A_n) = 0$ (that is $A_n \downarrow$ and $\mu(A \cap A_n) \longrightarrow 0$ for each $A \in \Sigma_f$). It is known that a subset$Z$ of L^1 is *uniformly integrable* (i.e., $\sup_{u \in Z} \int_{A_n} |u(\omega)|\, \mathrm{d}\mu \xrightarrow[n]{} 0$ as $A_n \searrow_\mu \emptyset$) if and only if for each $\varepsilon > 0$ there exist $\Omega_o \in \Sigma_f$ and $\delta > 0$ such that $\sup_{u \in Z} \int_{\Omega \setminus \Omega_o} |u(\omega)|\, \mathrm{d}\mu \le \varepsilon$ and $\sup_{u \in Z} \int_A |u(\omega)|\, \mathrm{d}\mu \le \varepsilon$ whenever $\mu(A) \le \delta$.

Let M be an ideal of E' with $\operatorname{supp} M = \Omega$. Assume that Z is a $\sigma(E, M)$-bounded subset of E. Then Z is also $|\sigma|(E, M)$-bounded (see [2, Theorem 6.6]), so one can define a Riesz seminorm p_Z on M by

$$p_Z(v) = \sup_{u \in Z} \int_\Omega |u(\omega) v(\omega)|\, \mathrm{d}\mu.$$

Now we recall the concept of M-*equicontinuity* of a set Z in E (see [10, Definition 2.1]) which allows us to tie together the various characterizations of conditionally and relatively $\sigma(E, M)$-sequentially compact sets in E.

A subset Z of E is said to be M-*equicontinuous* whenever

$$\sup\{|\varphi_{v_n}(u)| : u \in Z\} \xrightarrow[n]{} 0$$

as $\varphi_{v_n} \downarrow 0$ in the ideal $\Phi_M (= \{\varphi_v : v \in M\})$ of $E_n^\sim$.

Let $S(Z)$ stand for the solid hull of Z in E. Note that for each $v \in M$ we have:

$$\begin{aligned} \sup\big\{|\varphi_v(u)| : \ u \in S(Z)\big\} &= \sup\big\{|\varphi_v|(|u|) : \ u \in Z\big\} \\ &= \sup\left\{\int_\Omega |u(\omega) v(\omega)|\, \mathrm{d}\mu : \ u \in Z\right\}. \end{aligned} \tag{+}$$

Observe that $v_n \downarrow 0$ in M if and only if $\varphi_{v_n} \downarrow 0$ in Φ_M because the mapping $M \ni v \mapsto \varphi_v \in \Phi_M$ is a Riesz isomorphism.

Now we are able to state a characterization of conditionally and relatively $\sigma(E, M)$-sequentially compact sets in E.

Proposition 1.1. *Let E be an ideal of L^o and M an ideal of E' such that* $\operatorname{supp} M = \Omega$. *Then for a subset Z of E the following statements are equivalent:*

(*i*) *Z is conditionally $\sigma(E, M)$-compact.*

(*ii*) *Z is $\sigma(E, M)$-bounded and the seminorm p_Z on M is order continuous.*

(*iii*) *Z is $\sigma(E,M)$-bounded and for each $v \in M$ the subset $\{uv : u \in Z\}$ of L^1 is uniformly integrable.*

Moreover, if E is $\sigma(E,M)$-sequentially complete, then the statements (i)–(iii) are equivalent to the following:

(*iv*) *Z is relatively $\sigma(E,M)$-sequentially compact.*

Proof. (*i*)⇒(*ii*) Assume that Z is conditionally $\sigma(E,M)$-compact. Then in view of [10, Theorem 3.4] the set Z is Φ_M-equicontinuous; hence, by [10, Proposition 2.2(b)] Z is $|\sigma|(E,M)$-bounded. Moreover, by [10, Proposition 2.2(a)] its solid hull $S(Z)$ is also M-equicontinuous. Making use of (+) we obtain that the seminorm p_Z on M is order continuous.

(*ii*)⇒(*i*) Assume that Z is $\sigma(E,M)$-bounded and the seminorm p_Z on M is order continuous. In view of (+) it follows that the set $S(Z)$ is M-equicontinuous, so Z is also M-equicontinuous. Hence, by [10, Theorem 3.4] Z is conditionally $\sigma(E,M)$-compact, as desired.

(*ii*)⇔(*iii*) Let Z be $\sigma(E,M)$-bounded. It is well known that the seminorm p_Z on M is order continuous if and only if it is absolutely continuous, i.e., for each $v \in M$, $p_Z(\chi_{A_n} v) \xrightarrow[n]{} 0$ for every sequence (A_n) in Σ with $A_n \searrow_\mu \emptyset$ (see [21, Theorem 2.1]). This means that for each $v \in M$ the set $\{uv : u \in Z\}$ in L^1 is uniformly integrable. □

Remark. It is well known the space L^1 is weakly sequentially complete and by the "Dunford's theorem" weakly compact sets in L^1 are uniformly integrable.

Recall that E is said to be *perfect* whenever $E'' = E$. The following characterization of the perfectness of E will be needed.

Proposition 1.2. *Let E be an ideal of L°. Then the following statements are equivalent:*

(*i*) *E is perfect.*
(*ii*) *$|\sigma|(E,E')$ has the Levy property.*
(*iii*) *$|\sigma|(E,E')$ has the σ-Levy property.*
(*iv*) *E is $\sigma(E,E')$-sequentially complete.*

Proof. (*i*)⟺(*ii*) See [2, Theorem 9.4].

(*ii*)⇒(*iii*) It is obvious; (*iii*)⟹(*iv*) See [2, Theorem 20.26].

(*iv*)⟹(*i*) Assume that (*iv*) holds and (*i*) fails, i.e., $E \subsetneq E''$ and let $0 \le u \in E'' \setminus E$. Since $\operatorname{supp} E = \Omega$, there exists a sequence (Ω_n) in Σ_f such that $\Omega_n \uparrow \Omega$ with $\chi_{\Omega_n} \in E$ for $n \in \mathbb{N}$ (see [32, Theorem 86.2]). For $n \in \mathbb{N}$ let us put

$$u_n(\omega) = \begin{cases} u(\omega) & \text{if } \omega \in \Omega_n \text{ and } u(\omega) \le n \ , \\ 0 & \text{elsewhere.} \end{cases}$$

Then $u_n \le n\, \chi_{\Omega_n}$ for $n \in \mathbb{N}$, so $u_n \in E$ and $u_n(\omega) \uparrow u(\omega)$ for $\omega \in \Omega$. In fact, we have directly $u_n \longrightarrow u$ for $\sigma(E'',E')$. Indeed, for $v \in E'$

$$\Big| \int_\Omega (u(\omega) - u_m(\omega)) v(\omega)\, \mathrm{d}\mu \Big| \le \int_\Omega (u(\omega) - u_m(\omega))\, |v(\omega)|\, \mathrm{d}\mu$$

and by the Lebesque dominated convergence theorem, we have

$$\int_\Omega (u(\omega) - u_m(\omega))\,|v(\omega)|\,\mathrm{d}\mu \longrightarrow 0 \text{ as } m \to \infty.$$

On the other hand, since (u_n) is a Cauchy sequence for $\sigma(E, E')$, in view of (iv) there is $z \in E$ such that $u_n \longrightarrow z$ for $\sigma(E, E')$, so $u_n \longrightarrow z$ for $\sigma(E'', E')$. Since $\operatorname{supp} E' = \Omega$, it follows that $z = u$, so $u \in E$ and this contradicts the choice of u. □

Now we establish terminology and prove some results concerning vector-valued function spaces (see [8], [9], [22]).

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let X^* stand for the Banach dual of X. By B_X and S_X we denote the unit ball and the unit sphere in X respectively. By $L^{\mathrm{o}}(X)$ we will denote the set of μ-equivalence classes of strongly Σ-measurable functions $f : \Omega \longrightarrow X$. For $f \in L^{\mathrm{o}}(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{\, f \in L^{\mathrm{o}}(X) : \widetilde{f} \in E \,\}.$$

Recall that the algebraic tensor product $E \otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u \otimes x$, $(u \otimes x)(\omega) = u(\omega)\,x$, where $u \in E$, $x \in X$.

From now on for a subset H of $E(X)$ and a set $A \in \Sigma$ we will write

$$\widetilde{H} = \{\, \widetilde{f} \in E : \; f \in H \,\} \quad \text{and} \quad H_A = \{\chi_A f : \; f \in H\}.$$

In particular, for a Banach function space $(E, \|\cdot\|_E)$, the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is usually called a *Köthe-Bochner space*.

For a linear functional F on $E(X)$ let us set

$$|F|(f) = \sup\{\, |F(h)| : \; h \in E(X),\; \widetilde{h} \le \widetilde{f} \,\} \quad \text{for all} \quad f \in E(X).$$

Then the set

$$E(X)^\sim = \{F \in E(X)^\# : \; |F|(f) < \infty \quad \text{for all} \quad f \in E(X)\}$$

will be called the *order dual* of $E(X)$ (here $E(X)^\#$ denotes the algebraic dual of $E(X)$) (see [9, §3], [22]).

For $F_1, F_2 \in E(X)^\sim$ we will write $|F_1| \le |F_2|$ whenever $|F_1|(f) \le |F_2|(f)$ for all $f \in E(X)$. A subset $A \in E(X)^\sim$ is said to be *solid* whenever $|F_1| \le |F_2|$ with $F_1 \in E(X)^\sim$ and $F_2 \in A$ imply $F_1 \in A$. A linear subspace I of $E(X)$ will be called an *ideal* of $E(X)^\sim$ whenever I is solid.

Let $F \in E(X)^\sim$ and $x_{\mathrm{o}} \in S_X$ be fixed. For every $u \in E^+$ let us set:

$$\varphi_F(u) = |F|(u \otimes x_{\mathrm{o}}) = \sup\{\, |F(h)| : \; h \in E(X),\; \widetilde{h} \le u \,\}.$$

Then $\varphi_F : E^+ \longrightarrow \mathbb{R}^+$ is an additive mapping and φ_F has a unique positive extension to a linear mapping from E to $\mathbb{R}$ (denoted by φ_F again) and given by

$$\varphi_F(u) := \varphi_F(u^+) - \varphi_F(u^-) \quad \text{for all} \quad u \in E.$$

A linear functional $F \in E(X)^\sim$ is said to be *order continuous* whenever for a net (f_α) in $E(X)$, $\widetilde{f_\alpha} \xrightarrow{(o)} 0$ in E implies $F(f_\alpha) \longrightarrow 0$. The set $E(X)^\sim_n$ consisting of all order continuous linear functionals on $E(X)$ is called the *order continuous dual of* $E(X)$.

Let $L^o(X^*, X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \longrightarrow X^*$. One can define the so called abstract norm

$$\vartheta : L^o(X^*, X) \longrightarrow L^o$$

by $\vartheta(g) = \sup\{\ |g_x| : x \in B_X\ \}$, where $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$ and $x \in X$. Note that $\vartheta(g)$ is well defined because L^o is order complete. Then for $f \in L^o(X)$ and $g \in L^o(X^*, X)$ the function $\langle f, g\rangle : \Omega \longrightarrow \mathbb{R}$ defined by $\langle f, g\rangle(\omega) := \langle f(\omega), g(\omega)\rangle$ is measurable, and $|\langle f, g\rangle| \leq \widetilde{f}\,\vartheta(g)$. Moreover, $\vartheta(g) = \widetilde{g}$ for $g \in L^o(X^*)$. For an ideal M of E' let

$$M(X^*, X) = \{g \in L^o(X^*, X) : \vartheta(g) \in M\}.$$

Due to A. V. Bukhvalov (see [8, Theorem 4.1]) $E(X)^\sim_n$ can be identified with $E'(X^*, X)$ through the mapping: $E'(X^*, X) \ni g \longmapsto F_g \in E(X)^\sim_n$, where

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\mu \quad \text{for all} \quad f \in E(X),$$

and moreover

$$|F_g|(f) = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu \quad \text{for all} \quad f \in E(X).$$

Hence for each $g \in E'(X^*, X)$ we get for $u \in E^+$

$$\varphi_{F_g}(u) = |F_g|(u \otimes x_o) = \int_\Omega u(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu = \varphi_{\vartheta(g)}(u).$$

It is known that in a weak*-equivalence class $g \in E'(X^*, X)$ there is a function $g(\cdot)$ such that the scalar function $\|g(\cdot)\|_{X^*}$ is measurable and its equivalance class in L^o belongs to E' (see [13, p. 279–280, Theorem 8; p. 213, Proposition 5]).

One can show that if M is an ideal of E' then the set

$$I_M = \{F_g :\ g \in M(X^*, X)\}$$

is an ideal of $E(X)^\sim_n$ (see [22, Theorem 2.6]). Conversely, if I is an ideal of $E(X)^\sim_n$, then the set

$$M_I = \big\{v \in E' :\ |v| \leq \vartheta(g) \ \text{ for some } \ g \in M(X^*, X) \ \text{ with } \ F_g \in I\big\}$$

is an ideal of E' and $I = I_{M_I}$ (see [22, Theorem 1.2, Theorem 2.6]). Moreover, the ideal I_M separates the points of $E(X)$ if and only if $\operatorname{supp} M = \Omega$ (see [22, Theorem 2.7]).

Let M be an ideal of E' with $\operatorname{supp} M = \Omega$. Recall that the absolute weak topology $|\sigma|(E, M)$ on E is generated by the family $\{p_v :\ v \in M\}$ of Riesz semi-norms, where $p_v(u) = \int_\Omega |u(\omega)v(\omega)|\,\mathrm{d}\mu$ for $u \in E$. Denote by $\overline{|\sigma|(E, M)}$ the locally convex topology on

$E(X)$ that is generated by the family $\{\bar{p}_v : v \in M\}$ of seminorms on $E(X)$, where for $v \in M$

$$\bar{p}_v(f) := p_v(\tilde{f}) \quad \text{for} \quad f \in E(X).$$

We denote by $|\sigma|(E(X), M(X^*, X))$ the *absolute weak topology* on $E(X)$, generated by the family $\{\varrho_g : g \in M(X^*, X)\}$ of seminorms on $E(X)$, where

$$\varrho_g(f) = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu \quad \text{for} \quad f \in E(X)$$

(see [22, § 4]). It is known that (see [22, Theorem 4.3, Theorem 4.4]):

$$|\sigma|(E(X), M(X^*, X)) = \overline{|\sigma|(E, M)} \tag{1}$$

and

$$(E(X), |\sigma|(E(X), M(X^*, X)))^* = \{F_g : g \in M(X^*, X)\}. \tag{2}$$

We shall need the following technical result.

Proposition 1.3. *Let E be an ideal of L° and let M be ideal of E' with* $\operatorname{supp} M = \Omega$. *Then for a subset H of $E(X)$ the following statements are equivalent:*

(*i*) *H is $\sigma(E(X), M(X^*, X))$-bounded.*

(*ii*) *H is $|\sigma|(E(X), M(X^*, X))$-bounded.*

(*iii*) *$\widetilde{H}$ is $|\sigma|(E, M)$-bounded.*

(*iv*) *$\widetilde{H}$ is $\sigma(E, M)$-bounded.*

Proof. (*i*)⇔(*ii*) In view of (2) it follows from the Mackey theorem (see [31, Theorem 8.4.1])

(*ii*)⇒(*iii*) Assume that H is $|\sigma|(E(X), M(X^*, X))$-bounded and let $v \in M$. Then for $g = v \otimes x_\circ^*$, where $x_\circ^* \in S_{X^*}$, we get

$$\sup_{f\in H} \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu = \sup_{f\in H} \int_\Omega \widetilde{f}(\omega)|v(\omega)|\,\mathrm{d}\mu < \infty.$$

This means that $\widetilde{H}$ is $|\sigma|(E, M)$-bounded.

(*iii*)⇒(*ii*) Assume that $\widetilde{H}$ is $\sigma(E, M)$-bounded, and let $g \in M(X^*, X)$. Then $\vartheta(g) \in M$, so

$$\sup_{f\in H} \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu < \infty.$$

(*iii*)⇔(*iv*) It is obvious, because $(E, |\sigma|(E, M))^* = (E, \sigma(E, M))^*$ (see [2, Theorem 6.6]). □

2. Conditional weak compactness in vector-valued function spaces

H. Benabdellah and C. Castaing [4] employing M. Talagrand's result about a "parametrized version" of H. P. Rosenthal's ℓ^1-theorem obtained the following characterization of conditionally weakly compact sets in $L^1(X)$.

Theorem 2.1 ([4, Theorem 2.2]). *Let (Ω, Σ, μ) be a finite measure space and X be a Banach space. For a norm bounded subset H of $L^1(X)$ the following statements are equivalent:*

(i) *H is conditionally weakly compact.*

(ii) (a) *$\widetilde{H}$ is uniformly integrable in L^1,*
(b) *given any sequence (f_n) in H there exists a sequence (h_n) with $h_n \in \operatorname{conv}\{f_k : k \geq n\}$ such that $(h_n(\omega))$ is weakly Cauchy in X for a.e. $\omega \in \Omega$.*

Now, by making use of Theorem 2.1 we are ready to state our main result.

Theorem 2.2. *Let E be an ideal of L°, M be an ideal of E' with $\operatorname{supp} M = \Omega$ and let X be a Banach space. Then for a subset H of $E(X)$ the following statements are equivalent:*

(i) *H is conditionally $\sigma(E(X), M(X^*, X))$-compact.*

(ii) $a)$ *The set $\widetilde{H}$ is conditionally $\sigma(E, M)$-compact.*
$b)$ *for each subset $A \in \Sigma_f$ with $\chi_A \in M$ and each sequence (f_n) in H there is a sequence (h_n^A) with $h_n^A \in \operatorname{conv}\{\chi_A f_k : k \geq n\}$ such that $(h_n^A(\omega))$ is weakly Cauchy in X for a.e. $\omega \in A$.*

Proof. $(i)\Rightarrow(ii)$ Assertion (a) follows from [24, Theorem 2.3]. It is easy to observe that H is $\sigma(E(X), M(X^*, X))$-bounded, so by Proposition 1.3 $\widetilde{H}$ is $\sigma(E, M)$-bounded. Let $A \in \Sigma_f$ with $\chi_A \in M$. Hence $\sup_{f\in H} \int_\Omega \widetilde{f}(\omega)\chi_A(\omega)\,\mathrm{d}\mu < \infty$. Thus H_A is a norm bounded subset of $L^1_A(X)$.

We shall now show that H_A is conditionally $\sigma(L^1_A(X), L^\infty_A(X^*, X))$-compact. Indeed, let (f_n) be a sequence in H. Then there is a $\sigma(E(X), M(X^*, X))$-Cauchy subsequence (f_{k_n}) of (f_n). Let $g \in L^\infty_A(X^*, X)$ and let $g'(\omega) = g(\omega)$ for $\omega \in A$ and $g'(\omega) = 0$ for $\omega \in \Omega \setminus A$. Choose a weak*-measurable function $g(\cdot)$ in g such that the function $\|g(\cdot)\|_{X^*}$ is measurable. Then $\|g(\omega)\|_{X^*} \leq c$ for some $c > 0$ and a.e. $\omega \in A$. Hence $\|g'(\omega)\|_{X^*} \leq c\chi_A(\omega)$ for a.e. $\omega \in \Omega$, so $\|g'(\cdot)\|_{X^*} \in M$, because $\chi_A \in M$. It means that $g' \in M(X^*, X)$. Since

$$\int_A \langle f_{k_n}(\omega), g(\omega)\rangle\,\mathrm{d}\mu = \int_\Omega \langle f_{k_n}(\omega), g'(\omega)\rangle\,\mathrm{d}\mu$$

and $\lim_n \int_\Omega \langle f_{k_n}(\omega), g'(\omega)\rangle\,\mathrm{d}\mu$ exists, we obtain that $(\chi_A f_{k_n})$ is a $\sigma(L^1_A(X), L^\infty_A(X^*, X))$-Cauchy sequence in $L^1_A(X)$. Thus in view of Theorem 2.1 there is a sequence (h_n^A) with $h_n^A \in \operatorname{conv}\{\chi_A f_k : k \geq n\}$ for $n \in \mathbb{N}$ such that $(h_n^A(\omega))$ is weakly Cauchy in X for a.e. $\omega \in A$.

$(ii)\Rightarrow(i)$ Since $\operatorname{supp} M = \Omega$, there is a sequence (Ω_m) in Σ_f such that $\Omega_m \uparrow \Omega$ and $\chi_{\Omega_m} \in M$ for all $m \in \mathbb{N}$ (see [32, Theorem 86.2]). Hence for $m \in \mathbb{N}$ we get $\sup_{f\in H} \int_{\Omega_m} \widetilde{f}(\omega)\,\mathrm{d}\mu = c_m < \infty$, so $\{\chi_{\Omega_m} f : f \in H\}$ is a norm bounded subset of $L^1_{\Omega_m}(X)$ and by (a) the set $\{\chi_{\Omega_m}\widetilde{f} : f \in H\}$ in $L^1_{\Omega_m}$ is uniformly integrable (see Proposition 1.1). Thus in view of (b) and making use of Theorem 2.1 we conclude that for each $m \in \mathbb{N}$, $\{\chi_{\Omega_m} f : f \in H\}$ is a conditionally $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*, X))$-compact subset of $L^1_{\Omega_m}(X)$.

Let (f_n) be a sequence in H. In view of the above observation there is a $\sigma(L^1_{\Omega_1}(X), L^\infty_{\Omega_1}(X^*, X))$-Cauchy subsequence $(\chi_{\Omega_1} f_{k_n^1})$ of $(\chi_{\Omega_1} f_n)$. Next, there is a $\sigma(L^1_{\Omega_2}(X), L^\infty_{\Omega_2}(X^*, X))$-

Cauchy subsequence $(\chi_{\Omega_2} f_{k_n^2})$ of $(\chi_{\Omega_2} f_{k_n^1})$. It follows that the diagonal sequence $(f_{k_n^n})$ has the property that for each $m \in \mathbb{N}$, $(\chi_{\Omega_m} f_{k_n^n})$ is a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*,X))$-Cauchy sequence.

Let $h_n = f_{k_n^n}$ for $n \in \mathbb{N}$. We shall now show that (h_n) is a $\sigma(E(X), M(X^*,X))$-Cauchy sequence. Indeed, let $g \in M(X^*,X)$. Choose a weak*-measurable function $g(\cdot)$ in g such that the function $\|g(\cdot)\|_{X^*}$ is measurable. For $n \in \mathbb{N}$ let

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in \Omega_n \text{ and } \|g(\omega)\|_{X^*} \le n, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly $g_n \in L^\infty_{\Omega_n}(X^*,X)$. By (a), given $\varepsilon > 0$ there exist $m_\circ \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_n \int_{\Omega \setminus \Omega_{m_\circ}} \widetilde{f}_n(\omega)\|g(\omega)\|_{X^*}\, d\mu \le \frac{\varepsilon}{8} \quad \text{and} \quad \sup_n \int_A \widetilde{f}_n(\omega)\|g(\omega)\|_{X^*}\, d\mu \le \frac{\varepsilon}{8}, \tag{3}$$

for each $A \in \Sigma$ with $\mu(A) \le \delta$. For $\eta = \varepsilon/(8c_{m_\circ})$ let

$$B_n = \big\{\omega \in \Omega_{m_\circ} : \ \big\|g(\omega) - g_n(\omega)\|_{X^*} \ge \eta\big\}.$$

It is easy to observe that $B_n \downarrow \emptyset$, so $\mu(B_n) \longrightarrow 0$. Choose $n_\circ \in \mathbb{N}$ with $n_\circ \ge m_\circ$ such that $\mu(B_{n_\circ}) \le \delta$. Then by (3) we get

$$\sup_n \int_{B_{n_\circ}} \widetilde{h}_n(\omega)\|g(\omega)\|_{X^*}\, d\mu \le \frac{\varepsilon}{8}. \tag{4}$$

Hence, by (4) for $n \in \mathbb{N}$ we get

$$\begin{aligned} &\Big|\int_{\Omega_{m_\circ}} \langle h_n(\omega), g(\omega) - g_{n_\circ}(\omega)\rangle\, d\mu\Big| \le \int_{\Omega_{m_\circ}} \widetilde{h}_n(\omega)\|g(\omega) - g_{n_\circ}(\omega)\|_{X^*}\, d\mu \\ \le\ & \int_{B_{n_\circ}} \widetilde{h}_n(\omega)\|g(\omega) - g_{n_\circ}(\omega)\|_{X^*}\, d\mu + \int_{\Omega_{m_\circ} \setminus B_{n_\circ}} \widetilde{h}_n(\omega)\|g(\omega) - g_{n_\circ}(\omega)\|_{X^*}\, d\mu \\ \le\ & \int_{B_{n_\circ}} \widetilde{h}_n(\omega)\|g(\omega)\|_{X^*}\, d\mu + \eta \int_{\Omega_{m_\circ}} \widetilde{h}_n(\omega)\, d\mu \le \frac{\varepsilon}{8} + \frac{\varepsilon}{8c_{m_\circ}} c_{m_\circ} = \frac{\varepsilon}{4}. \end{aligned} \tag{5}$$

Since $(\chi_{\Omega_{m_\circ}} h_n)$ is a $\sigma(L^1_{\Omega_{m_\circ}}(X), L^\infty_{\Omega_{m_\circ}}(X^*,X))$-Cauchy sequence and $\chi_{\Omega_{m_\circ}} g_{n_\circ} \in L^\infty_{\Omega_{m_\circ}}(X^*, X)$ there exists $n_1 \in \mathbb{N}$ such that for $n, n' \ge n_1$ we have:

$$\Big|\int_{\Omega_{m_\circ}} \langle h_n(\omega) - h_{n'}(\omega), g_{n_\circ}(\omega)\rangle\, d\mu\Big| \le \frac{\varepsilon}{4}. \tag{6}$$

For $n, n' \in \mathbb{N}$ we have

$$\begin{aligned} &\Big|\int_{\Omega} \langle h_n(\omega) - h_{n'}(\omega), g(\omega)\rangle\, d\mu\Big| \\ \le\ & \Big|\int_{\Omega \setminus \Omega_{m_\circ}} \langle h_n(\omega) - h_{n'}(\omega), g(\omega)\rangle\, d\mu\Big| + \Big|\int_{\Omega_{m_\circ}} \langle h_n(\omega) - h_{n'}(\omega), g(\omega)\rangle\, d\mu\Big| \end{aligned}$$

Using (3) for $n, n' \in \mathbb{N}$ we get:

$$\begin{aligned} & \Big| \int_{\Omega \smallsetminus \Omega_{m_o}} \langle h_n(\omega) - h_{n'}(\omega), g(\omega) \rangle \, \mathrm{d}\mu \Big| \\ \leq \; & \int_{\Omega \smallsetminus \Omega_{m_o}} \widetilde{h}_n(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu + \int_{\Omega \smallsetminus \Omega_{m_o}} \widetilde{h}_{n'}(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \\ \leq \; & \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}. \end{aligned} \tag{7}$$

Moreover, by (5) and (6) for $n, n' \geq n_1$ we get:

$$\begin{aligned} & \Big| \int_{\Omega_{m_o}} \langle h_n(\omega) - h_{n'}(\omega), g(\omega) \rangle \, \mathrm{d}\mu \Big| \\ \leq \; & \Big| \int_{\Omega_{m_o}} \langle h_n(\omega) - h_{n'}(\omega), g(\omega) - g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| + \Big| \int_{\Omega_{m_o}} \langle h_n(\omega) - h_{n'}(\omega), g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| \\ \leq \; & \Big| \int_{\Omega_{m_o}} \langle h_n(\omega), g(\omega) - g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| + \Big| \int_{\Omega_{m_o}} \langle h_{n'}(\omega), g(\omega) - g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| \\ + \; & \Big| \int_{\Omega_{m_o}} \langle h_n(\omega) - h_{n'}(\omega), g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3}{4}\varepsilon. \end{aligned} \tag{8}$$

At last, using (7) and (8) for $n, n' \geq n_1$ we have:

$$\Big| \int_{\Omega} \langle h_n(\omega) - h_{n'}(\omega), g(\omega) \rangle \, \mathrm{d}\mu \Big| \leq \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} = \varepsilon.$$

Thus H is conditionally $\sigma(E(X), M(X^*, X))$-compact, as desired. □

Recall that a normed space X is said to be *almost reflexive* if every norm-bounded subset of X is conditionally weakly compact (see [11]). The fundamental ℓ^1-Rosenthal theorem [27] says that a Banach space X is almost reflexive if and only if it contains no isomorphic copy of ℓ^1. J. Bourgain [5] and M. Talagrand [29] have studied the relationship between conditional weak compactness and uniform integrability in $L^1(X)$ in terms of the presence of isomorphic copies of ℓ^1 in X.

Now, using [5, Corollary 9] and Theorem 2.2 we are in position to present a characterization of the almost reflexivity of a Banach space X in terms of conditionally $\sigma(E(X), M(X^*, X))$-compact subsets of $E(X)$ (see [1, Theorem 2.6]).

Corollary 2.3. *Let E be an ideal of L^o, M an ideal of E' with* $\operatorname{supp} M = \Omega$, *and let X be a Banach space. Then the following statements are equivalent:*

- (i) *X is almost reflexive.*
- (ii) *Every subset H of $E(X)$ such that the set $\widetilde{H}$ is conditionally $\sigma(E, M)$-compact is conditionally $\sigma(E(X), M(X^*, X))$-compact.*

Proof. $(i) \Rightarrow (ii)$ Assume that X is almost reflexive, and let H be a subset of $E(X)$ such that $\widetilde{H}$ is a conditionally $\sigma(E, M)$-compact subset of E. Let $A \in \Sigma_f$ with $\chi_A \in M$ and (f_n) a sequence of H. The set $\{\chi_A \widetilde{f}_n : n \geq 1\}$ is bounded in L^1_A. By Levin theorem, there exist a sequence of convex combination $h_n = \sum_{i=n}^{N_n} \lambda_i^n \chi_A \widetilde{f_i}$ (with $\lambda_i^n \geq 0$, $\sum_{i=n}^{N_n} \lambda_i^n = 1$) such that $h_n(\omega)$ converges for almost every ω. Hence the sequence $\left\| \sum_{i=n}^{N_n} \lambda_i^n (\chi_A f_i)(\omega) \right\|$, $n \geq 1$, is bounded in X for almost every ω. Now we have to use almost reflexivity of X for this sequence and conclude by Theorem 2.2.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. We shall show that the unit ball B_X is conditionally weakly compact. Indeed, let (x_n) be a sequence in B_X. Given a fixed $u \in E^+ \setminus \{0\}$ let us put $h_n = u \otimes x_n$ for $n \in \mathbb{N}$. Let $H = \{u \otimes x : \ x \in B_X\}$. Making use of Proposition 1.1 it is easy to verify that the set $\widetilde{H}$ is conditionally $\sigma(E, M)$-compact. Hence by (ii) there exists a $\sigma(E(X), M(X^*, X))$-Cauchy subsequence (h_{k_n}) of (h_n). Choose $0 \leq v_o \in M$ such that $\int_\Omega u(\omega) v_o(\omega)\, d\mu = 1$. Then $v_o \otimes x^* \in M(X^*, X)$ for each $x^* \in X^*$, and

$$x^*(x_{k_n}) = \int_\Omega u(\omega) v_o(\omega) x^*(x_{k_n})\, d\mu = F_{v_o \otimes x^*}(u \otimes x_{k_n}) \longrightarrow a \in \mathbb{R}\,.$$

This means that (x_{k_n}) is a weakly Cauchy sequence in X. Thus X is almost reflexive. $\square$

3. Weak sequential compactness in vector-valued function spaces

A. Ülger [30] obtained a characterization of relatively weakly compact subsets H of $L^1(X)$ for a general Banach space X and a finite measure space (that are norm bounded in $L^\infty(X)$) in terms of "convex compactness" i.e., of drawing convex combinations out of "tails" of sequences (f_n) in H. Next, J. Diestel, W. Ruess and W. Schachermayer [15] remowed the restriction of $L^\infty(X)$-boundedness of a subset H of $L^1(X)$ and obtained the following result:

Theorem 3.1 ([15, Theorem 2.1]). *Let (Ω, Σ, μ) be a finite measure space and X be a Banach space. For a norm bounded subset H of $L^1(X)$ the following statements are equivalent:*

(i) *H is relatively weakly compact.*

(ii) (a) *$\widetilde{H}$ is uniformly integrable in L^1,*
(b) *given any sequence (f_n) in H there exists a sequence (h_n) with $h_n \in \operatorname{conv}\{f_k : k \geq n\}$ such that $(h_n(\omega))$ is weakly convergent in X for a.e. $\omega \in \Omega$.*

The present author [23] found a criterion for relative $\sigma(E(X), E(X)_n^\sim)$-compactness in a Köthe-Bochner space $E(X)$ whenever a Banach function space E (over a finite measure space) is such that $L^\infty \subset E \subset L^1$ and the inclusion maps are continuous.

In this section by making use of Theorem 3.1 we characterize relatively $\sigma(E(X), I)$-sequentially compact subsets of $E(X)$ whenever X is a general Banach space, E is an ideal of L^o (over a σ-finite measure space) and I is an ideal of $E(X)_n^\sim$ separating the points of $E(X)$.

The following version of the Eberlein-Smulian theorem for the locally convex space $(E(X), \sigma(E(X), M(X^*, X)))$ will be needed.

Theorem 3.2. *Let E be an ideal of L^o, M an ideal of E' with $\operatorname{supp} M = \Omega$, and let X be a Banach space. Then for a subset H of $E(X)$ the following statements are equivalent:*

(i) *H is relatively $\sigma(E(X), M(X^*, X))$-sequentially compact.*

(*ii*) *H is relatively* $\sigma(E(X), M(X^*, X))$*-countably compact.*

Moreover, if the absolute weak topology $|\sigma|(E, M)$ *has the Levy property, then the statements* (*i*)–(*ii*) *are equivalent to the following:*

(*iii*) *H is relatively* $\sigma(E(X), M(X^*, X))$*-compact.*

Proof. (*i*)⇔(*ii*) It follows from [22, Theorem 4.8].

(*ii*)⇔(*iii*) Since $|\sigma|(E, M)$ is a Lebesgue, Levy topology on E, the space $(E, |\sigma|(E, M))$ is complete (see [2, Theorem 22.2]). Making use of [7, Theorem 3] we obtain that the space $(E(X), \overline{|\sigma|(E, M)})$ is also complete. On the other hand, in view of (1) and (2) we see that $(E(X), \overline{|\sigma|(E, M)}\,)^* = (E(X), |\sigma|(E(X), M(X^*, X)))^* = I_M$. Hence in view of [18, Proposition 2] we conclude that (*ii*)⇔(*iii*) holds. □

Since $L^1(X)^* = \{F_g : \ g \in L^\infty(X^*, X)\}$, a set H in $L^1(X)$ is relatively weakly compact if and only if it is relatively $\sigma(L^1(X), L^\infty(X^*, X))$-sequentially compact. Moreover, we will need the following identity (see [25, Theorem 1.1]):

$$E''(X) = \Big\{ f \in L^{\mathrm{o}}(X) : \int_\Omega \langle f(\omega), g(\omega)\rangle \, \mathrm{d}\mu < \infty \quad \text{for all} \quad g \in E'(X^*) \Big\}. \tag{9}$$

Note that the absolute weak topology $|\sigma|(E, M)$ on E is a σ-Levy topology if and only if the space $(E, \sigma(E, M))$ is sequentially complete (see [2, Theorem 20.26]). It follows that then E is perfect, i.e., $E = E''$ (see [Proposition 1.2]).

It is known that for $|\sigma|(E, M)$ the σ-Levy property and the Levy property coincide (see [16, Proposition 3.2]).

Now we are is position to state our main result.

Theorem 3.3. *Let E be an ideal of* L^{o}, *M an ideal of* E' *with* supp $M = \Omega$, *and let X be a Banach space. Assume that the absolute weak topology* $|\sigma|(E, M)$ *on E has the* σ*-Levy property. Then for a subset H of* $E(X)$ *the following statements are equivalent:*

(*i*) *H is relatively* $\sigma(E(X), M(X^*, X))$*-sequentially compact.*

(*ii*) (*a*) $\widetilde{H}$ *is relatively* $\sigma(E, M)$*-sequentially compact.*
(*b*) *for each* $A \in \Sigma_f$ *with* $\chi_A \in M$ *and each sequence* (f_n) *in H there is a sequence* (h_n^A) *with* $h_n^A \in \operatorname{conv}\{f_k : \ k \geq n\}$ *such that* $(h_n^A(\omega))$ *is weakly convergent in X for a.e.* $\omega \in A$.

(*iii*) *H is relatively* $\sigma(E(X), M(X^*, X))$*-compact.*

(*iv*) *H is relatively* $\sigma(E(X), M(X^*, X))$*-countably compact.*

Proof. (*i*)⇒(*ii*) Assume that H is relatively $\sigma(E(X), M(X^*, X))$-sequentially compact. Since H is conditionally $\sigma(E(X), M(X^*, X))$-compact and the space $(E, \sigma(E, M))$ is sequentially complete, in view of Theorem 2.2 $\widetilde{H}$ is relatively $\sigma(E, M)$-sequentially compact, i.e., (*a*) holds.

Now assume that $A \in \Sigma_f$ with $\chi_A \in M$. Then $\sup_{f \in H} \int_A \widetilde{f}(\omega)\, \mathrm{d}\mu < \infty$, because $\widetilde{H}$ is $\sigma(E, M)$-bounded (see Proposition 1.2). One can observe that the subset $\{\chi_A f : f \in H\}$ of $L^1_A(X)$ is relatively $\sigma(L^1_A(X), L^\infty_A(X^*, X))$-sequentially compact. In fact, let $g \in L^\infty(X^*, X)$, and choose a function $g(\cdot)$ in g such that the scalar function $\|g(\cdot)\|_{X^*}$ is measurable. Hence $\chi_A(\omega)\,\|g(\omega)\|_{X^*} \le c\,\chi_A(\omega)$ μ-a.e. for some $c > 0$. It follows that $\chi_A g \in M(X^*, X)$, because $\chi_A \in M$. This means that the set $\{\chi_A f : f \in H\}$ is relatively weakly compact in $L^1_A(X)$, so by Theorem 3.2 condition (b) holds.

$(ii)\Longrightarrow(i)$ Assume that (iii) holds and let (f_n) be a sequence in H. Then there is a pairwise disjoint sequence (Ω_m) in Σ_f such that $\chi_{\Omega_m} \in M$ for $m \in \mathbb{N}$ and $\bigcup_{m=1}^\infty \Omega_m = \Omega$ (see [20, Corollary 4.3.2]). Given $m \in \mathbb{N}$ we have $\sup_n \int_{\Omega_m} \widetilde{f}_n(\omega)\,\mathrm{d}\mu < \infty$, so $\{\, \chi_{\Omega_m} f_n : n \in \mathbb{N} \,\}$ is a norm bounded subset of $L^1_{\Omega_m}(X)$. Since $\chi_{\Omega_m} \in M$, in view of (a) and Proposition 1.1 for each $m \in \mathbb{N}$ the subset $\{\, \chi_{\Omega_m} \widetilde{f}_n : n \in \mathbb{N} \,\}$ of $L^1_{\Omega_m}$ is uniformly integrable. Hence by Theorem 3.1 for each $m \in \mathbb{N}$ the set $\{\chi_{\Omega_m} f_n : n \in \mathbb{N}\}$ is relatively $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*, X))$-sequentially compact. Hence by the diagonal process we can extract a subsequence (h_n) of (f_n) such that for each $m \in \mathbb{N}$ there is $f_{\Omega_m} \in L^1_{\Omega_m}(X)$ such that

$$\chi_{\Omega_m} h_n \underset{n}{\longrightarrow} f_{\Omega_m} \quad \text{for} \quad \sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*, X)). \tag{10}$$

Define a function $f : \Omega \longrightarrow X$ by setting $f(\omega) = f_{\Omega_m}(\omega)$ for $\omega \in \Omega_m$, i.e., $\chi_{\Omega_m} f = f_{\Omega_m}$ for all $m \in \mathbb{N}$. Then $f \in L^\circ(X)$ and we shall now show that $f \in E(X)$ and $h_n \longrightarrow f$ for $\sigma(E(X), M(X^*, X))$.

To show that $f \in E(X)$, in view of (9) and the perfectness of E it is enough to prove that

$$\int_\Omega \langle f(\omega), g(\omega)\rangle\,\mathrm{d}\mu < \infty \quad \text{for all} \quad g \in E'(X^*).$$

Let $B_m = \bigcup_{i=1}^m \Omega_i$ for $m \in \mathbb{N}$. Then $B_m \uparrow \Omega$ and $\chi_{\Omega_m} \in M$. Then from (10) it easily follows that for each $m \in \mathbb{N}$

$$\chi_{B_m} h_n \underset{n}{\longrightarrow} \chi_{B_m} f \quad \text{for} \quad \sigma(L^1_{B_m}(X), L^\infty_{B_m}(X^*, X)). \tag{11}$$

Let $g \in M(X^*)$ and for $m \in \mathbb{N}$ let us put

$$g_m(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in B_m \text{ and } \|g(\omega)\|_{X^*} \le m, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\|g_m(\omega) - g(\omega)\|_{X^*} \longrightarrow 0$ μ-a.e. and $g_m \in M(X^*)$. Given $m \in \mathbb{N}$ by $\mathcal{S}_{B_m}(X^*)$ we denote the set of all simple functions of the form:

$$s = \sum_{i=1}^k \chi_{A_i} \otimes x_i^*,$$

where $A_i \in B_m \cap \Sigma$, $x_i^* \in X^*$, $k \in \mathbb{N}$. Making use of (11) one can see that for each $s \in \mathcal{S}_{B_m}(X^*)$ we have

$$\int_\Omega \langle h_n(\omega), s(\omega)\rangle\,\mathrm{d}\mu \longrightarrow \int_\Omega \langle f(\omega), s(\omega)\rangle\,\mathrm{d}\mu,$$

so

$$\left|\int_\Omega \langle f(\omega), s(\omega)\rangle\,\mathrm{d}\mu\right| = \lim_n \left|\int_\Omega \langle h_n(\omega), s(\omega)\rangle\,\mathrm{d}\mu\right| \le \sup_n \int_\Omega \widetilde{h}_n(\omega)\,\widetilde{s}(\omega)\,\mathrm{d}\mu < \infty.$$

For $s \in \mathcal{S}_{B_m}(X^*)$ define a measure $\nu_s : \Sigma \longrightarrow \mathbb{R}$ by

$$\nu_s(A) := \int_A \langle f(\omega), s(\omega)\rangle \, \mathrm{d}\mu \quad \text{for all} \quad A \in \Sigma.$$

Then

$$\begin{aligned} |\nu_s|(\Omega) &= \int_\Omega |\langle f(\omega), s(\omega)\rangle| \, \mathrm{d}\mu \\ &\leq 4 \sup\nolimits_{A\in\Sigma} |\nu_s(A)| \leq 4 \sup\nolimits_{A\in\Sigma} (\sup\nolimits_n \int_A \widetilde{h}_n(\omega)\ \widetilde{s}(\omega) \, \mathrm{d}\mu) \\ &= 4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{s}(\omega) \, \mathrm{d}\mu < \infty. \end{aligned} \tag{12}$$

Given $m \in \mathbb{N}$ one can chose a sequence (s_k^m) in $\mathcal{S}_{B_m}(X^*)$ such that $\|s_k^m(\omega) - g_m(\omega)\|_{X^*} \underset{k}{\longrightarrow} 0$ and $\|s_k^m(\omega)\|_{X^*} \leq \|g_m(\omega)\|_{X^*}$ μ-a.e. on Ω. Moreover, since $\lim\limits_m \langle f(\omega), g_m(\omega)\rangle = \langle f(\omega), g(\omega)\rangle$ μ-a.e., by the Fatou lemma we get

$$\textstyle\int_\Omega |\langle f(\omega), g(\omega)\rangle| \, \mathrm{d}\mu \leq \liminf\limits_m \int_\Omega |\langle f(\omega), g_m(\omega)\rangle| \, \mathrm{d}\mu. \tag{13}$$

On the other hand, for each $m \in \mathbb{N}$, in view of the Fatou lemma and (12) we have

$$\begin{aligned} \int_\Omega |\langle f(\omega), g_m(\omega)\rangle| \, \mathrm{d}\mu &\leq \liminf_k \int_\Omega |\langle f(\omega), s_k^m(\omega)\rangle| \, \mathrm{d}\mu \\ &\leq \liminf_k \, (4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{s}_k^m(\omega) \, \mathrm{d}\mu\,) \\ &\leq \liminf_k \, (4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{g}_m(\omega) \, \mathrm{d}\mu\,) \\ &= 4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{g}_m(\omega) \, \mathrm{d}\mu. \end{aligned} \tag{14}$$

Hence, by (13) and (14) we get

$$\begin{aligned} \int_\Omega |\langle f(\omega), g(\omega)\rangle| \, \mathrm{d}\mu &\leq \liminf_m \, (4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{g}_m(\omega) \, \mathrm{d}\mu\,) \\ &\leq \liminf_m \, (4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{g}(\omega) \, \mathrm{d}\mu\,) \\ &= 4 \sup\nolimits_n \int_\Omega \widetilde{h}_n(\omega)\ \widetilde{g}(\omega) \, \mathrm{d}\mu < \infty\,, \end{aligned}$$

and this proves that $f \in E(X)$, as desired.

To show that $h_n \longrightarrow f$ for $\sigma(E(X), M(X^*, X))$, let $g \in M(X^*, X)$ be given. Choose a weak*-measurable function in $g(\cdot)$ in g such that the scalar function $\|g(\cdot)\|_{X^*}$ is measurable and its equivalence class in L^0 belongs to M. Setting $A_m = \Omega \smallsetminus B_m$ we see that

$A_m \searrow_\mu \emptyset$. Let $\varepsilon > 0$ be given. Then by (a) and Proposition 1.1 one can choose $m_o \in \mathbb{N}$ and $\delta > 0$ such that for each $A \in \Sigma$ with $\mu(A) \leq \delta$ we have

$$\sup_n \int_{\Omega \setminus B_{m_o}} \widetilde{h}_n(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \leq \frac{\varepsilon}{4} \quad \text{and} \quad \sup_n \int_A h_n(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \leq \frac{\varepsilon}{12}, \tag{15}$$

and

$$\int_{\Omega \setminus B_{m_o}} \widetilde{f}(\omega) \; \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \leq \frac{\varepsilon}{4} \quad \text{and} \quad \int_A \widetilde{f}(\omega) \; \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \leq \frac{\varepsilon}{12}. \tag{16}$$

For $n \in \mathbb{N}$ let us put

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in B_n \text{ and } \|g(\omega)\|_{X^*} \leq n \;, \\ 0 & \text{elsewhere.} \end{cases}$$

It is seen that $\|g_n(\omega)\|_{X^*} \leq n \chi_{B_m}(\omega)$ for $\omega \in \Omega$, so $g_n \in L^\infty_{B_m}(X^*, X))$. We have $\sup_n \int_{B_{m_o}} \widetilde{h}_n(\omega) \, \mathrm{d}\mu = d_{m_o}$ for some $d_{m_o} > 0$ and $\int_{B_{m_o}} \widetilde{f}(\omega) \, \mathrm{d}\mu = d$ for some $d > 0$. For $\eta = \varepsilon/(12 \max(d, d_{m_o}))$ and $n \in \mathbb{N}$ let us put

$$D_n = \{\omega \in B_{m_o} : \|g(\omega) - g_n(\omega)\|_{X^*} \geq \eta\} \, .$$

It is easy to see that $D_n \downarrow \emptyset$, so $\mu(D_n) \longrightarrow 0$. Choose $n_o \in \mathbb{N}$ with $n_o \geq m_o$ such that $\mu(D_{n_o}) \leq \delta$. Then by (14) and (15) we get

$$\sup_n \int_{D_{n_o}} \widetilde{h}_n(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \leq \frac{\varepsilon}{12} \quad \text{and} \quad \int_{D_{n_o}} \widetilde{f}(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu \leq \frac{\varepsilon}{12}. \tag{17}$$

Hence, by (17) we have

$$\begin{aligned} &\Big| \int_{B_{m_o}} \langle h_n(\omega), g(\omega) - g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| \leq \int_{B_{m_o}} \widetilde{h}_n(\omega) \; \|g(\omega) - g_{n_o}(\omega)\|_{X^*} \, \mathrm{d}\mu \\ \leq \;& \int_{D_{n_o}} \widetilde{h}_n(\omega) \|g(\omega) - g_{n_o}(\omega)\|_{X^*} \, \mathrm{d}\mu + \int_{B_{m_o} \setminus D_{n_o}} \widetilde{h}_n(\omega) \|g(\omega) - g_{n_o}(\omega)\|_{X^*} \, \mathrm{d}\mu \\ \leq \;& \int_{D_{n_o}} \widetilde{h}_n(\omega) \|g(\omega)\|_{X^*} \, \mathrm{d}\mu + \eta \int_{B_{m_o}} \widetilde{h}_n(\omega) \, \mathrm{d}\mu \leq \frac{\varepsilon}{12} + \frac{\varepsilon}{12 d_{m_o}} d_{m_o} = \frac{\varepsilon}{6}. \end{aligned} \tag{18}$$

Similarly, by (17) we get

$$\Big| \int_{B_{m_o}} \langle f(\omega), g(\omega) - g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| \leq \frac{\varepsilon}{6}. \tag{19}$$

Since $g_{n_o} \in L^\infty_{B_{m_o}}(X^*, X)$, using (11) one can choose $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$\Big| \textstyle\int_{B_{m_o}} \langle h_n(\omega) - f(\omega), g_{n_o}(\omega) \rangle \, \mathrm{d}\mu \Big| \leq \frac{\varepsilon}{6}. \tag{20}$$

Hence, using (18), (19) and (20) for $n \geq n_1$ we get

$$\begin{aligned}\Big|\int_{B_{m_o}} \langle h_n(\omega) - f(\omega), g(\omega)\rangle \,\mathrm{d}\mu\Big| &\leq \Big|\int_{B_{m_o}} \langle h_n(\omega) - f(\omega), g_{n_o}(\omega)\rangle \,\mathrm{d}\mu\Big| \\ &+\Big|\int_{B_{m_o}} \langle h_n(\omega), g(\omega) - g_{n_o}(\omega)\rangle \,\mathrm{d}\mu\Big| + \Big|\int_{B_{m_o}} \langle f(\omega), g(\omega) - g_{n_o}(\omega)\rangle \,\mathrm{d}\mu\Big| \\ \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.\end{aligned} \tag{21}$$

Moreover, using (15) and (16) for every $n \in \mathbb{N}$ we have

$$\begin{aligned}&\Big|\int_{\Omega\setminus B_{m_o}} \langle h_n(\omega) - f(\omega), g(\omega)\rangle \, d\mu\Big| \\ \leq\ &\int_{\Omega\setminus B_{m_o}} \widetilde{h}_n(\omega)\|g(\omega)\|_{X^*}\, d\mu + \int_{\Omega\setminus B_{m_o}} \widetilde{f}(\omega)\|g(\omega)\|_{X^*}\,\mathrm{d}\mu \\ \leq\ &\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned} \tag{22}$$

At last, using (21) and (22) for $n \geq n_1$ we get

$$\begin{aligned}\Big|\int_{\Omega} \langle h_n(\omega) - f(\omega), g(\omega)\rangle \,\mathrm{d}\mu\Big| &\leq \Big|\int_{\Omega\setminus B_{m_o}} \langle h_n(\omega) - f(\omega), g(\omega)\rangle \,\mathrm{d}\mu\,\Big| \\ &+\Big|\int_{B_{m_o}} \langle h_n(\omega) - f(\omega), g(\omega)\rangle \,\mathrm{d}\mu\Big| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus the proof is complete.

$(i)\Leftrightarrow(iii)\Leftrightarrow(iv)$ It follows from Theorem 3.2. □

As a consequence of Theorem 3.3 we obtain a characterization of reflexivity of a Banach space X in terms of relatively $\sigma(E(X), M(X^*, X))$-sequentially compact subsets of $E(X)$.

Corollary 3.4. *Let E be an ideal of L^o, M an ideal of E' with* $\operatorname{supp} M = \Omega$ *and let X be a Banach space. Assume that the absolute weak topology $|\sigma|(E, M)$ on E has the σ-Levy property. Then the following statements are equivalent:*

(*i*) *X is reflexive.*

(*ii*) *Every subset H of $E(X)$ such that the set $\widetilde{H}$ is relatively $\sigma(E, M)$-sequentially compact is relatively $\sigma(E(X), M(X^*, X))$-sequentially compact.*

Proof. $(i)\Rightarrow(ii)$ Assume that $X$is reflexive, i.e., X is almost reflexive and weakly sequentially complete. Let H be a subset of $E(X)$ such that $\widetilde{H}$ is relatively $\sigma(E, M)$-sequentially compact in E. Hence in view of Corollary 2.3 H is conditionally $\sigma(E(X), M(X^*, X))$-compact. Combining Theorem 2.2 and Theorem 3.3 we obtain that H is relatively $\sigma(E(X), M(X^*, X))$-sequentially compact.

$(ii)\Rightarrow(i)$ Assume that (ii) holds. It is enough to show that the unit ball B_X is weakly sequentially compact (see [31, Chap. 10.2]). Indeed, let (x_n) be a sequence in B_X. Given a fixed $u \in E^+ \setminus \{0\}$ let us put $h_n = u \otimes x_n$ for $n \in \mathbb{N}$. Let $H = \{u \otimes x_n :\ n \in \mathbb{N}\}$.

Using Proposition 1.1 we easily obtain that $\widetilde{H}$ is relatively $\sigma(E, M)$-sequentially compact. Hence by (ii) there exists a subsequence (h_{k_n}) of (h_n) and $h_{\rm o} \in E(X)$ such that $h_{k_n} \longrightarrow h_{\rm o}$ for $\sigma(E(X), M(X^*, X))$. Choose $0 \leq v_{\rm o} \in M$ such that $\int_\Omega u(\omega)v_{\rm o}(\omega)\,\mathrm{d}\mu = 1$. Then $v_{\rm o} \otimes x^* \in M(X^*, X)$ for each $x^* \in X^*$, and

$$\begin{aligned} x^*(x_{k_n}) &= \int_\Omega u(\omega)v_{\rm o}(\omega)x^*(x_{k_n})\,\mathrm{d}\mu \\ &= F_{v_{\rm o}\otimes x^*}(u \otimes x_{k_n}) \longrightarrow F_{v_{\rm o}\otimes x^*}(h_{\rm o}) = \int_\Omega \langle h_{\rm o}(\omega), v_{\rm o}(\omega)x^*\rangle\,\mathrm{d}\mu \\ &= \int_\Omega x^*(v_{\rm o}(\omega)h_{\rm o}(\omega))\,\mathrm{d}\mu = x^*\Big(\int_\Omega v_{\rm o}(\omega)h_{\rm o}(\omega)\,\mathrm{d}\mu\Big). \end{aligned}$$

Hence $x_{k_n} \longrightarrow x_{\rm o}$ for $\sigma(X, X^*)$, where $x_{\rm o} = \int_\Omega v_{\rm o}(\omega)h_{\rm o}(\omega)\,\mathrm{d}\mu \in B_X$. This means that X is reflexive. □

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