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Integrability of Pseudomonotone Differentiable Maps and the Revealed Preference Problem

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The problem considered is as follows: given $C \subset \mathbb{R}^n$ and $F : C \to \mathbb{R}^n$ differentiable, find $f : C \to \mathbb{R}$ differentiable such that $\|\nabla f(x)\|^{-1}\nabla f(x) = \|F(x)\|^{-1}F(x)$ for all $x \in C$. Conditions for f to be pseudoconvex or convex are given. The results are applied to the differentiable case of the revealed preference problem.

Keywords: Generalized convexity, generalized monotonicity, consumer theory, direct and indirect utility functions, revealed preference theory

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1. Introduction and notation

Given a convex subset C of $\mathbb{R}^n$ and a continuously differentiable monotone map $F : C \to \mathbb{R}^n$, then a twice continuously differentiable convex function $f : C \to \mathbb{R}$ exists such that $F = \nabla f$ if and only if the matrix $F'(x)$ is symmetric and positive semidefinite for all $x \in C$. The function f is uniquely defined up to an additive constant. The vector $(\nabla f(x), -1)$ of $\mathbb{R}^{n+1}$ generates the normal cone at the point $(x, f(x))$ to the epigraph of f. Moreover, the vector $\nabla f(x)$ of $\mathbb{R}^n$ generates the normal cone at x to the level set $S(x) = \{x' \in C \,:\, f(x') \leq f(x)\}$. Actually, the symmetry of the matrices $F'(x)$ ensures the existence of f while the monotonicity of F gives the convexity of f and, thereby, its epigraph and its level sets.

The problem considered in the paper is as follows: given a convex subset C of $\mathbb{R}^n$ and a continuously differentiable pseudomonotone map $F : C \to \mathbb{R}^n$ not vanishing on C, find a differentiable function $f : C \to \mathbb{R}$ such that

$$\|\nabla f(x)\|^{-1}\nabla f(x) = \|F(x)\|^{-1}F(x) \quad \forall\, x \in C.$$

It is easy to see that if a function like this f exists, it is pseudoconvex and uniquely defined up to a scalarization: if f_1 and f_2 respond to the problem, then there exists $k : f_1(C) \to f_2(C)$ differentiable such that $k'(t) > 0$ for all $t \in f_1(C)$ and $f_2(x) = k(f_1(x))$ for all $x \in C$. Also, as in the previous case, $\nabla f(x)$ generates the normal cone at x to the convex level set $S(x)$, but there are no properties concerned with the epigraph. We shall see that a symmetry property is required on the matrices $F'(x)$, not on the whole space,

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but on the orthogonal subspace to $F(x)$. Similarly to the previous case, this symmetry property ensures the local existence of f while the pseudomonotonicity of F gives the pseudoconvexity of f and, thereby, the convexity of its level sets.

The paper is organized as follows. In Section 2, we give a brief background on generalized convexity and generalized monotonicity. Two local results are given in Section 3. Finally, in the last section, we apply the approach to the differentiable case of the revealed preferences problem in consumer theory.

Now, a few words on the notation in use in the paper.

The transposed matrix of a matrix A is denoted by A^t, a vector of $\mathbb{R}^n$ is considered as a column matrix, i.e., as a $n \times 1$ matrix. I denotes the $n \times n$ identity matrix. The inner product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $x^t y$ or $\langle x, y\rangle$, the euclidean norm of x by $\|x\|$. The gradient at a point x of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is denoted by the vector $\nabla f(x)$, the Hessian by the matrix $\nabla^2 f(x)$. For commodity, we write $\nabla^t f(x)$ for the $1 \times n$ matrix $[\nabla f(x)]^t$ and, if $F : \mathbb{R}^n \to \mathbb{R}$, $F^t(x)$ for the $1 \times n$ matrix $[F(x)]^t$.

Given two vectors $x, y \in \mathbb{R}^n$, $x \leq y$ ($x < y$) means $x_i \leq y_i$ ($x_i < y_i$) for all i. A real function u is said to be *nondecreasing* if $u(x) \leq u(y)$ whenever $x \leq y$ and *increasing* if $u(x) < u(y)$ whenever $x \leq y$ with $x \neq y$.

2. A background of generalized convexity

Given $C \subset \mathbb{R}^n$ and $f : C \to \mathbb{R}$, let us define

$$\mathrm{epi}(f) = \{(x, \lambda) \in C \times \mathbb{R} \; : \; f(x) \leq \lambda\}$$

and, for all $x \in C$,

$$S(x) = \{y \in C \; : \; f(y) \leq f(x)\}.$$

Assume that C is convex. f is said to be *convex* on C if its epigraph epi(f) is convex, *quasiconvex* on C if all the level sets $S(x)$, $x \in C$, are convex. Assume now that f is differentiable on C, then f is quasiconvex on C if and only if

$$[x, y \in C \text{ and } f(y) \leq f(x)] \Longrightarrow (y - x)^t \nabla f(x) \leq 0. \tag{1}$$

When f is quasiconvex, $\nabla f(a) = 0$ does not necessarily imply that f has a local (and, a fortiori, global) minimum at a. In order to remedy this deficiency, a slight modification of the condition leads to the following definition: a differentiable function f on C is said to be *pseudoconvex* on C if

$$[x, y \in C \text{ and } f(y) < f(x)] \Longrightarrow (y - x)^t \nabla f(x) < 0.$$

A differentiable convex function is pseudoconvex, a differentiable pseudoconvex function is quasiconvex. Conversely, it is known that if C is open, f differentiable and quasiconvex on C and ∇f does not vanish on C, then, f is pseudoconvex on C.

f is said to be *strictly quasiconvex* (*strictly pseudoconvex*) on C if quasiconvex (pseudoconvex) on C and

$$x, y \in C,\, x \neq y,\, t \in (0,1) \text{ and } f(x) = f(y) \Longrightarrow f(tx + (1-t)y) < f(x).$$

A second order necessary and sufficient condition for pseudoconvexity is as follows.

Theorem 2.1. *Assume that C is an open convex subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ is twice continuously differentiable on C. Assume, in addition, that f has a local minimum at any $x \in C$ such that $\nabla f(x) = 0$. Then, f is pseudoconvex on C if and only if*

$$x \in C, \;\; h \in \mathbb{R}^n, \;\; h^t \nabla f(x) = 0 \Longrightarrow h^t \nabla^2 f(x) h \geq 0.$$

Related to this result, we have the following sufficient condition for strict pseudoconvexity.

Corollary 2.2. *Assume that C is an open convex subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ is twice continuously differentiable on C. Assume that*

$$[x \in C, \;\; h \in \mathbb{R}^n, \;\; h \neq 0 \textit{ and } h^t \nabla f(x) = 0] \Longrightarrow h^t \nabla^2 f(x) h > 0.$$

Then, f is strictly pseudoconvex on C.

A function $f : C \to \mathbb{R}$, with $C \subset \mathbb{R}^n$ convex, is said to be *convexifiable* (*strongly convexifiable*) on C if there exists a continuous strictly increasing function $k : f(C) \to \mathbb{R}$ such that $k \circ f$ is convex (strongly convex). In connection with Theorem 2.1 and Corollary 2.2, we have the following sufficient condition for convexifiability.

Proposition 2.3. *Assume that f is twice continuously differentiable in a neighborhood of $\bar{x} \in \mathbb{R}^n$ and the following condition holds:*

$$[h \in \mathbb{R}^n, \;\; h \neq 0 \textit{ and } h^t \nabla f(\bar{x}) = 0] \Longrightarrow h^t \nabla^2 f(\bar{x}) h > 0.$$

Then, f is strongly convexifiable in a neighborhood of $\bar{x}$.

Proof. It follows from the Finsler-Debreu lemma [10, 9] that there exists $r > 0$ and $\alpha > 0$ such that

$$h^t [\nabla^2 f(\bar{x}) + r \nabla f(\bar{x}) \nabla^t f(\bar{x})] h \geq 2\alpha \|h\|^2, \quad \forall\, h \in \mathbb{R}^n.$$

Then, on a convex compact neighborhood V of $\bar{x}$, it holds that

$$h^t [\nabla^2 f(x) + r \nabla f(x) \nabla^t f(x)] h \geq \alpha \|h\|^2, \quad \forall\, h \in \mathbb{R}^n, \;\; \forall\, x \in V.$$

Set $g(x) = \exp(r f(x))$. Then,

$$\nabla^2 g(x) = r g(x) [\nabla^2 f(x) + r \nabla f(x) \nabla^t f(x)].$$

It follows that g is strongly convex on V. □

A map $F : C \to \mathbb{R}^n$ is said to be *monotone* on C if

$$\langle F(x_1), x_2 - x_1 \rangle \leq \langle F(x_2), x_2 - x_1 \rangle, \qquad \forall\, x_1, x_2 \in C$$

and *pseudomonotone* on C if

$$x_1, x_2 \in C \text{ and } 0 \leq \langle F(x_1), x_2 - x_1 \rangle \Longrightarrow 0 \leq \langle F(x_2), x_2 - x_1 \rangle.$$

Assume that F is continuously differentiable on the open convex set C. Then, F is monotone on C if and only if, for all $x \in C$, the matrix $F'(x)$ is positive semidefinite. If F does not vanish on C, it is pseudomonotone on C if and only if for all $x \in C$,

$$h \in \mathbb{R}^n, \;\; \langle F(x), h \rangle = 0 \Longrightarrow \langle F'(x) h, h \rangle \geq 0.$$

A differentiable function $f : C \to \mathbb{R}$ is convex (pseudoconvex) if and only if its gradient ∇f is monotone (pseudomonotone).

For a text book on generalized convexity and generalized monotonicity, see [2, 17]. Surveys on the first and second order characterizations of generalized convex functions and generalized monotonicity of maps can be found in [8, 9].

3. Two local results

In this section, $e \in \mathbb{R}^n$ is such that $\|e\| = 1$ and C is a compact convex neighborhood of $\bar{x} \in \mathbb{R}^n$.

Assume that f is a twice continuously differentiable function on C and $e^t\nabla f(x) > 0$ for all $x \in C$. Let us define for all $x \in C$

$$F(x) = \frac{\nabla f(x)}{e^t\nabla f(x)}. \tag{2}$$

Clearly $e^tF(x) = F^t(x)e = 1$.

This section adresses the inverse problem: given $F : C \to \mathbb{R}^n$ continuously differentiable such that $e^tF(x) = F^t(x)e = 1$ for all $x \in C$, find $f : C \to \mathbb{R}$ twice continuously differentiable such that (2) holds. In order to show the pertinence of the assumptions of our theorem, we look at some necessary assumptions. Assume that f exists, then

$$(I - F(x)e^t)^2 = I - F(x)e^t, \quad (I - eF^t(x))^2 = I - eF^t(x),$$

$$\text{and} \quad F'(x) = (I - F(x)e^t)\frac{\nabla^2 f(x)}{e^t\nabla f(x)}.$$

Hence,

$$\begin{aligned} F'(x)(I - eF^t(x)) &= (I - F(x)e^t)\frac{\nabla^2 f(x)}{e^t\nabla f(x)}(I - eF^t(x)) \\ &= (I - F(x)e^t)F'(x)(I - eF^t(x)). \end{aligned} \tag{3}$$

For simplicity, we set

$$\widetilde{F}(x) = F'(x)(I - eF^t(x)).$$

It follows that, for all $x \in C$, the matrix $\widetilde{F}(x)$ is symmetric. Also, for all $h \in \mathbb{R}^n$,

$$h^t\widetilde{F}(x)h = \frac{1}{e^t\nabla f(x)}k^t\nabla^2 f(x)k \;\text{ with }\; k = (I - eF^t(x))h.$$

Assume that f is pseudoconvex on C. Since by construction

$$\nabla^t f(x)k = \nabla^t f(x)(I - eF^t(x))h = 0,$$

Theorem 2.1 implies $h^t\widetilde{F}(x)h \geq 0$ for all $h \in \mathbb{R}^n$ and thereby the matrix $\widetilde{F}(x)$ is positive semidefinite. In line with these observations, our assumptions are:

$(F1)$ The map F is continuously differentiable on C and $e^tF(x) = 1$ for all $x \in C$.

$(F2)$ The matrix $\widetilde{F}(x)$ is symmetric for any $x \in C$.

If the pseudoconvexity of f is wished, the additional assumption is:

$(F3)$ The matrix $\widetilde{F}(x)$ is positive semidefinite for any $x \in C$.

Finally, if we wish the convexity of f, $(F3)$ is strengthened in:

$(F4)$ The matrix $\widetilde{F}(x)$ is positive definite for any $x \in C$.

Assumption $(F2)$ $((F3), (F4))$ means that the matrix $F'(x)$ is symmetric (positive semidefinite, positive definite) on the linear subspace orthogonal to the vector $F(x)$.

Our first theorem is as follows.

Theorem 3.1. *Assume that assumptions $(F1)$ and $(F2)$ hold. Then, there exists a neighborhood D of $\bar{x}$ contained in C and a continuously differentiable $f : D \to \mathbb{R}$ such that $F(x) = (e^t\nabla f(x))^{-1}\nabla f(x)$ for all $x \in D$. If, in addition, $(F3)$ holds, then f is pseudoconvex on a convex neighborhood of $\bar{x}$.*

Proof. a) Outline: the proof being rather long and technical, we outline the main ideas behind it: Assume that f exists and we are given some $n \times (n-1)$ matrix A such that the $n \times n$ matrix (A, e) is orthogonal, i.e., $(A, e)(A, e)^t = I$. For any vector $x \in \mathbb{R}^n$, there exists a uniquely defined vector $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that $x = Ay - te$. Let $\widehat{x} \in C$ and $\lambda = f(\widehat{x})$. Then, by the implicit function theorem, there exists a neighborhood $V = Y \times T$ of $\widehat{x} = A\widehat{y} - \widehat{t}e$ such that, for $x \in V$, $f(x) = \lambda$ if and only if x is of the form $x = Ay - g_\lambda(y)e$ where g_λ is a continuously differentiable function on Y such that, for all $y \in Y$,

$$\nabla g_\lambda(y) = \frac{A^t\nabla f(x)}{e^t\nabla f(x)} = A^tF(x).$$

If, in addition, f is twice differentiable, then so is g_λ and

$$\nabla^2 g_\lambda(y) = A^tF'(x)(I - eF^t(x))A = A^t\widetilde{F}(x)A.$$

It follows that g_λ is convex when f is pseudoconvex.

The proof consists in building such functions g_λ, next in constructing f finally in proving that f solves the problem. This approach is borrowed from Samuelson [31] who considered the case $n = 2$, a quite more easier case.

b) Preliminaries: Without loss of generality, we assume that $\bar{x} = 0$. Set

$$\widetilde{F}(x) = F'(x)(I - eF^t(x)) = (I - F(x)e^t)F'(x)(I - eF^t(x)).$$

Then, it follows from $(F2)$ that, for any $x \in C$, $\widetilde{F}(x)$ is symmetric and, if $(F3)$ holds, $\widetilde{F}(x)$ is also positive semidefinite. Since C is compact and F is continuously differentiable on C, the following constants are well defined

$$\begin{aligned}
M &= \sup[\ \|x\| \ : \ x \in C\ ],\\
K_0 &= \sup[\ \|F(x)\| \ : \ x \in C\ ],\\
K_1 &= \sup[\ \|x_2 - x_1\|^{-1}\|F(x_2) - F(x_1)\| \ : \ x_1, x_2 \in C,\ \ x_2 \neq x_1\ ],\\
K_2 &= \sup[\ \langle h, F'(x)k\rangle \ : \ x \in C,\ \ h, k \in \mathbb{R}^n,\ \ \|h\| = \|k\| = 1\ ].
\end{aligned}$$

For $x_1, x_2 \in C$, we define

$$\varepsilon(x_1, x_2) = \begin{cases} \|x_2 - x_1\|^{-1}[F(x_2) - F(x_1) - F'(x_1)(x_2 - x_1)] & \text{if} \quad x_1 \neq x_2,\\ 0 & \text{if} \quad x_1 = x_2. \end{cases}$$

This function ε is continuous on $C \times C$. For $r > 0$, define

$$\widetilde{\varepsilon}(r) = \sup[\ \|\varepsilon(x_1, x_2)\|\ :\ x_1, x_2 \in C,\ \ \|x_2 - x_1\| \leq r\ ].$$

Then $\widetilde{\varepsilon}(r) \to 0$ when $r \to 0$.

It can be found a neighborhood $\Lambda = [-\bar{\lambda}, \bar{\lambda}] \subset \mathbb{R}$ of 0 and a convex compact neighborhood $Y \subset \mathbb{R}^{n-1}$ of 0 such that

$$(y, \lambda) \in Y \times \Lambda \text{ and } |t| \leq K_0 \|Ay\| \Longrightarrow Ay - (\lambda + t)e \in C. \tag{4}$$

c) In this part of the proof, we construct an auxiliary function g. Given $(y, \lambda) \in Y \times \Lambda$, we consider the classical ordinary differential equation problem:

$$\text{Find } g_{y,\lambda} : [0,1] \to \mathbb{R} \text{ differentiable such that } g_{y,\lambda}(0) = \lambda \quad \text{and}$$

$$g'_{y,\lambda}(t) = (Ay)^t F(tAy - g_{y,\lambda}(t)e) \quad \text{ for } t > 0. \tag{5}$$

Since $tAy - g_{y,\lambda}(t)e = -\lambda e \in int(C)$ when $t = 0$ and the map F is continuous on C, $g_{y,\lambda}(t)$ is well defined and $(tAy - g_{y,\lambda}(t)e) \in C$ for small positive values of t. Moreover,

$$|g'_{y,\lambda}(t)| \leq \|F(tAy - g_{y,\lambda}(t)e)\|\, \|Ay\| \leq K_0\, \|Ay\|.$$

Hence, we deduce from (4) that, for all $(t, y, \lambda) \in [0,1] \times Y \times \Lambda$, $g_{y,\lambda}(t)$ is well defined and $tAy - g_{y,\lambda}(t)e \in C$. Set

$$g(t, y, \lambda) = g_{y,\lambda}(t), \quad \text{and } \ \widetilde{g}(y, \lambda) = g(1, y, \lambda). \tag{6}$$

By assumption, F is continuously differentiable, hence a classical result in the theory of ordinary differential equation problems says that g is continuously differentiable on $[0,1] \times Y \times \Lambda$ and thereby $\widetilde{g}$ is continuously differentiable on $Y \times \Lambda$. In the next two steps, we shall give a constructive proof of the differentiability of $\widetilde{g}$ and we shall compute its gradient. Let us remark, beforehand, that

$$\begin{aligned} g''_{y,\lambda}(t) &= (Ay)^t F'(tAy - g_{y,\lambda}(t)e)(Ay - e g'_{y,\lambda}(t)) \\ &= \langle Ay, \widetilde{F}(tAy - g_{y,\lambda}(t)e)Ay \rangle. \end{aligned}$$

d) We start with a Lipschitz property of g.

Let $y, y + z \in Y$ and $\lambda, \lambda + \mu \in \Lambda$. For $t \in [0,1]$, let us define

$$\alpha(t) = g(t, y + z, \lambda + \mu) - g(t, y, \lambda) - \mu.$$

Then, in view of (5), $\alpha(0) = 0$ and

$$\alpha'(t) = \langle Ay, F(x_2) - F(x_1) \rangle + \langle Az, F(x_2) \rangle,$$

where, for simplification,

$$x_2 = tA(y + z) - g(t, y + z, \lambda + \mu)e \text{ and } \ x_1 = tAy - g(t, y, \lambda)e.$$

Then,

$$\begin{aligned}
|\alpha'(t)| &\leq MK_1\,\|x_2 - x_1\| + K_0\|Az\|,\\
|\alpha'(t)| &\leq MK_1\,\|tAz - \alpha(t)e - \mu e\| + K_0\|Az\|,\\
|\alpha'(t)| &\leq (MK_1 + K_0)[\,\|Az\| + |\mu|] + MK_1|\alpha(t)|.
\end{aligned}$$

Let us consider the following associate ordinary differential equation problem:

Find $u : [0,1] \to \mathbb{R}$ such that $u(0) = 0$ and for all $t \in [0,1]$

$$u'(t) = (MK_1 + K_0)(\|Az\| + |\mu|) + MK_1 u(t).$$

Then, for all $t \in [0,1]$,

$$|\alpha(t)| \leq u(t) \leq \frac{MK_1 + K_0}{MK_1}(\exp(MK_1) - 1)(\|Az\| + |\mu|).$$

It follows that there exists $L > 0$ such that for all $t \in [0,1]$, $y, y+z \in Y$ and $\lambda, \lambda+\mu \in \Lambda$,

$$|g(t, y+z, \lambda+\mu) - g(t, y, \lambda)| \leq u(t) + |\mu| \leq L(\|Az\| + |\mu|). \tag{L}$$

e) In this step, we prove that $\widetilde{g}$ is differentiable on $Y \times \Lambda$.

Let $y, y+z \in Y$ and $\lambda, \lambda+\mu \in \Lambda$. For $t \in [0,1]$, let us define

$$\beta(t) = g(t, y+z, \lambda+\mu) - g(t, y, \lambda) - t\langle Az, F(tAy - g(t, y, \lambda)e)\rangle.$$

Then $\beta(0) = \mu$ and

$$\begin{aligned}
\beta'(t) &= \langle A(y+z), F(x_2) - F(x_1)\rangle - t\langle Az, F'(x_1)[Ay - g'_{y,\lambda}(t)e]\rangle,\\
&= \langle Ay, F(x_2) - F(x_1)\rangle + \langle Az, F(x_2) - F(x_1)\rangle - t\langle Az, \widetilde{F}(x_1)Ay\rangle,
\end{aligned}$$

where, as previously,

$$x_2 = tA(y+z) - g(t, y+z, \lambda+\mu)e \text{ and } x_1 = tAy - g(t, y, \lambda)e.$$

By assumption $(F2)$, the matrix $\widetilde{F}(x_1)$ is symmetric, hence

$$\begin{aligned}
\langle Az, \widetilde{F}(x_1)Ay\rangle &= \langle Ay, \widetilde{F}(x_1)Az\rangle\\
&= \langle Ay, F'(x_1)Az\rangle - \langle Ay, F'(x_1)e\rangle\langle F(x_1), Az\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
x_2 - x_1 &= tAz - g(t, y+z, \lambda+\mu)e + g(t, y, \lambda)e\\
&= tAz - t\langle Az, F(x_1)\rangle e - \beta(t)e
\end{aligned}$$

and

$$\begin{aligned}
F(x_2) - F(x_1) &= F'(x_1)(x_2 - x_1) + \|x_2 - x_1\|\varepsilon(x_1, x_2)\\
&= F'(x_1)[tAz - t\langle Az, F(x_1)\rangle e - \beta(t)e] + \|x_2 - x_1\|\varepsilon(x_1, x_2).
\end{aligned}$$

It follows that

$$\beta'(t) = -\beta(t)\langle Ay, F'(x_1)e\rangle + \xi(t, y, \lambda, z, \mu),$$

where

$$\xi(t, y, \lambda, z, \mu) = \|x_2 - x_1\|\langle Ay, \varepsilon(x_1, x_2)\rangle + \langle Az, F(x_2) - F(x_1)\rangle.$$

Let us consider the ordinary differential equation problem:

Find $\gamma : [0, 1] \to \mathbb{R}$ such that $\gamma(0) = \mu$ and for all $t \in [0, 1]$

$$\gamma'(t) = -\gamma(t)\langle Ay, F'(tAy - g(t, y, \lambda)e)e\rangle.$$

Then,

$$\gamma(t) = \mu \exp(h(t, y, \lambda)), \quad \text{where} \quad h(t, y, \lambda) = -\int_0^t \langle Ay, F'(sAy - g(s, y, \lambda)e)e\rangle ds.$$

It follows that $\beta(0) - \gamma(0) = 0$ and

$$\beta'(t) - \gamma'(t) = (\beta(t) - \gamma(t))\langle Ay, F'(tAy - g(t, y, \lambda)e)e\rangle + \xi(t, y, \lambda, z, \mu).$$

There exists $K_3 > 0$ such that for all $(t, y, \lambda) \in [0, 1] \times Y \times \Lambda$,

$$|\langle Ay, F'(tAy - g(t, y, \lambda)e)e\rangle| \leq K_3.$$

On the other hand,

$$\begin{aligned} x_2 - x_1 &= tAz - g(t, y + z, \lambda + \mu)e + g(t, y, \lambda)e, \\ \|x_2 - x_1\| &\leq \|Az\| + L(\|Az\| + |\mu|), \\ \|x_2 - x_1\| &\leq (1 + L)(\|Az\| + |\mu|). \end{aligned}$$

It results that there exists a function $\hat{\varepsilon}$ such that $\hat{\varepsilon}(s) \to 0$ when $s \to 0$ and for all $(t, y, \lambda) \in [0, 1] \times Y \times \Lambda$

$$|\xi(t, y, \lambda, z, \mu)| \leq (\|Az\| + |\mu|)\hat{\varepsilon}(\|Az\| + |\mu|).$$

Let us consider the ordinary differential equation problem: Find $\delta : [0, 1] \to \mathbb{R}$ such that $\delta(0) = 0$ and for all $t \in [0, 1]$

$$\delta'(t) = K_3\delta(t) + (\|Az\| + |\mu|)\hat{\varepsilon}(\|Az\| + |\mu|).$$

Then, $|\beta(t) - \gamma(t)| \leq \delta(t)$ for all $t \in [0, 1]$. Hence,

$$|\beta(t) - \gamma(t)| \leq \delta(t) \leq \delta(1) = \frac{\exp(K_3) - 1}{K_3}(\|Az\| + |\mu|)\hat{\varepsilon}(\|Az\| + |\mu|).$$

In particular, in view of (6), for $t = 1$,

$$|\widetilde{g}(y + z, \lambda + \mu) - \widetilde{g}(y, \lambda) - \langle G, (z, \mu)\rangle| \leq \frac{\exp(K_3)}{K_3}(\|Az\| + |\mu|)\hat{\varepsilon}(\|Az\| + |\mu|),$$

where

$$G = (A^t F(Ay - \widetilde{g}(y, \lambda)e), \exp(-\int_0^1 \langle Ay, F'(sAy - g(s, y, \lambda)e)e\rangle ds))^t \in \mathbb{R}^n.$$

It follows that $\widetilde{g}$ is continuously differentiable on $Y \times \Lambda$, its gradient is

$$\nabla\widetilde{g}(y,\lambda) = (A^t F(Ay - \widetilde{g}(y,\lambda)e), \exp(-\int_0^1 \langle Ay, F'(sAy - g(s,y,\lambda)e)e\rangle ds))^t.$$

For a fixed $\lambda \in \Lambda$, let us define

$$\widetilde{g}_\lambda(y) = \widetilde{g}(y,\lambda) \quad \forall\, y \in Y.$$

Then $\widetilde{g}_\lambda$ is twice continuously differentiable on Y and

$$\nabla^2\widetilde{g}_\lambda(y) = A^t \widetilde{F}(Ay - \widetilde{g}(y,\lambda)e)A. \tag{7}$$

Hence, if $(F3)$ holds, $\widetilde{g}_\lambda$ is convex on Y.

f) In the last step, we construct the function f. Beforehand, let us define

$$D = \{x = Ay - \mu e \in \mathbb{R}^n \,:\, y \in Y,\ \widetilde{g}(y, -\bar{\lambda}) \leq \mu \leq \widetilde{g}(y, \bar{\lambda})\}.$$

Then, D is a neighborhood of 0. Let us define $H : (Y \times \mathbb{R}) \times \mathbb{R} \to Y \times \mathbb{R}$ by

$$H(y,\mu,\lambda) = \widetilde{g}(y,\lambda) - \mu, \qquad (y,\lambda) \in Y \times \Lambda,\ \mu \in \mathbb{R}.$$

Remark that $H'_\lambda(y,\lambda,\mu) = \widetilde{g}'_\lambda(y,\lambda) > 0$. Hence, in view of the implicit function theorem, there exists a continuously differentiable function f defined on D such that

$$[f(x) = \lambda,\ x = Ay - \mu e \in D] \Longleftrightarrow [\widetilde{g}(y,\lambda) = \mu, \quad \lambda \in \Lambda].$$

It is clear that

$$\frac{\nabla f(x)}{e^t \nabla f(x)} = F(x) \qquad \forall\, x \in D.$$

We have seen that if assumption $(F3)$ holds, $\widetilde{g}_\lambda$ is convex. Hence, since

$$\widetilde{g}_\lambda(y) = \widetilde{g}(y,\lambda) \leq \mu \Longleftrightarrow f(Ay - \mu e) \leq \lambda,$$

f is quasiconvex on some convex neighborhood $\widetilde{D} \subset D$ of 0. Next, because ∇f does not vanish on this neighborhood, f is also pseudoconvex on $\widetilde{D}$. □

Remark 3.2. The theorem is local in the sense that the function f has been defined not on the whole set C but on a neighborhood D of $\bar{x}$ contained in C. Indeed, in order that the points $tAy - g(t, y\lambda)e,\ t \in [0,1]$ stay in C, C has been reduced a first time with the introduction of the set $Y \times \Lambda$ in part b) of the proof, a second time in part f) with the introduction of D.

Corollary 3.3. *Assume that assumptions $(F1)$ to $(F4)$ hold and F is twice continuously differentiable. Then, there exists a convex neighborhood $\widetilde{D}$ of $\bar{x}$ and a strongly convex function $\widetilde{f}$ such that $F(x) = (e^t\nabla\widetilde{f}(x))^{-1}\nabla\widetilde{f}(x)$ holds on $\widetilde{D}$.*

Proof. Take f and $\widetilde{D}$ as constructed in the theorem. Then, $\nabla\widetilde{g}$ and ∇f are continuously differentiable. Let $x = Ay - \mu e \in \widetilde{D}$ and $\lambda = f(x)$. Then,

$$h^t \nabla f(x) = 0 \Longleftrightarrow (I - eF^t(x))h = h.$$

It follows from relation (3) and assumption $(F4)$ that

$$h^t \nabla f(x) = 0 \Longrightarrow h^t \nabla^2 f(x) h > 0 \quad \forall\, h \neq 0.$$

Proposition 2.3 implies that, for $r > 0$ large enough, the function $\widetilde{f}(x) = \exp(rf(x))$ is strongly convex on a convex neighborhood of $\bar{x}$. □

Remark 3.4. Since $g_{y,\lambda}$ is the unique solution of problem (5), then it is also the unique solution of the following problem:

$$\text{Find } f_{y,\mu} : [0,1] \to \mathbb{R} \text{ differentiable such that } f_{y,\mu}(1) = \mu \quad \text{and}$$
$$f'_{y,\mu}(t) = (Ay)^t F(tAy - f_{y,\mu}(t)e) \quad \text{for } t \in [0,1], \tag{8}$$

where $\mu = g_\lambda(y) = g_{y,\lambda}(1)$. It results that $f_{y,\mu}(0) = \lambda$. Hence,

$$f(Ay - \mu e) = f_{y,\mu}(0).$$

Next, we consider the case where we are given a continuously differentiable map N on a compact neighborhood of $\bar{x}$ with $\|N(x)\| = 1$ for all $x \in C$ and our problem consists in finding a (convex) neighborhood $D \subset C$ of $\bar{x}$ and a (pseudoconvex) differentiable function $f : D \to \mathbb{R}$ such that

$$N(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|}, \quad \forall\, x \in D. \tag{9}$$

Let us define the matrix

$$\widetilde{N}(x) = N'(x)(I - N(x)N^t(x)).$$

Then, we consider the following assumptions:

$(H1)$ The map N is continuously differentiable on C.
$(H2)$ The matrix $\widetilde{N}(x)$ is symmetric for any $x \in C$.
$(H3)$ The matrix $\widetilde{N}(x)$ is positive semidefinite for any $x \in C$.
$(H4)$ The matrix $\widetilde{N}(x)$ is positive definite for any $x \in C$.

The result, derived from Theorem 3.1, is as follows.

Theorem 3.5. *Assume that assumptions $(H1)$ and $(H2)$ hold. Then, there exists a neighborhood D of $\bar{x}$ and a continuously differentiable $f : D \to \mathbb{R}$ such that $N(x) = \|\nabla f(x)\|^{-1} \nabla f(x)$ for all $x \in D$. If, in addition, $(H3)$ holds, then f is pseudoconvex on a convex neighborhood of $\bar{x}$ and if $(H1)$ to $(H4)$ hold, then f can be chosen convex on an appropriate neighborhood.*

Proof. Take $e = N(\bar{x})$ and reduce C to a compact neighborhood of $\bar{x}$ on which $e^t N(x) > 0$ for all x in the neighborhood. Without loss of generality, we denote by C this restriction. Let us define

$$F(x) = \frac{N(x)}{e^t N(x)}.$$

Then assumption (F1) holds. Next,

$$F'(x) = (I - F(x)e^t)\, \frac{N'(x)}{e^t N(x)}.$$

Thus,

$$\widetilde{F}(x) = (I - F(x)e^t)\, \frac{N'(x)}{e^t N(x)}\, (I - eF^t(x)).$$

It is easily seen that

$$I - eF^t(x) = (I - N(x)N^t(x))(I - eF^t(x)).$$

From what we deduce that

$$\widetilde{F}(x) = (I - F(x)e^t)\, \frac{\widetilde{N}(x)}{e^t N(x)}\, (I - eF^t(x)).$$

Then, assumption (H2) implies assumption (F2), assumption (H3) implies assumption (F3) and finally assumption (H4) implies assumption (F4). Apply Theorem 3.1 to obtain the function f. □

4. The problem of revealed preferences

In economics, the situation when the behaviour of the consumer is described through a utility function u is very convenient: the consumer determines his choice x by maximizing $u(x)$ on the commodity set G, subject to a budget constraint. Here, we assume that $G = \mathbb{R}^n_+$. Let $\hat{p} \in \mathbb{R}^n_+$, $\hat{p} \neq 0$ be the vector of prices and $w > 0$ be the income of the consumer. Then the problem of the consumer consists in

$$\text{maximize } u(x) \text{ subject to } x \geq 0 \text{ and } \hat{p}^t x \leq w.$$

Set $p = w^{-1}\hat{p}$, then the problem becomes

$$v(p) = \max\,[\, u(x) \ : \ x \geq 0,\ p^t x \leq 1].$$

Denote the set of optimal solutions of this problem by $X(p)$. The multivalued map X is called the *demand correspondence*, the function v the *indirect utility function* associated to u. Under some reasonable conditions, u can be recovered from v via the minimization problem:

$$u(x) = \min\,[\, v(p) \ : \ p \geq 0,\ p^t x \leq 1].$$

In fact, the concept of utility is rather theorical since observations on the behaviour of a consumer allow to know his choices (i.e., the demand correspondence $X(p)$), but not a representation in terms of a utility function. Furthermore, if one utility function exists, it is not uniquely defined since, given a utility function u and $k : u(G) \to \mathbb{R}$ strictly increasing, the function $\hat{u}(x) = k(u(x))$ describes the behaviour of the consumer as well.

Consequently, an important problem consists in constructing, when it is possible, a utility function from the knowledge of the demand correspondence. This problem, known as the problem of *revealed preferences*, has been given a special attention since the early beginning of the theory of consumer. See, for instance, the pioneering works of Samuelson [28, 29], Houthakker [18], Little [15].

With reference to the previous sections, we consider the case when the demand correspondence X is single-valued and continuously differentiable (the case when X is multivalued is the subject of another work [13]). Consequently, we write $X(p) = \{x(p)\}$. Note that the problem is completely solved when $n = 2$ (see for instance Samuelson [31]), indeed the problem reduces to a classical ordinary differential equation problem) and, for $n > 2$, when the demand function x is analytic, with a proof based on exterior differential calculus (see, e.g., Chiappori–Ekeland [3] and Ekeland [14]). The approach that we propose here, close to that of [31], consists in constructing an indirect utility function associated to the demand. Next, by duality, a direct utility function can be obtained.

Duality between direct and indirect utility functions has been actively investigated. See, e.g., Roy [25], Lau [21], Diewert [11, 12], Sakai [27], Crouzeix [4]. Recent references are Crouzeix [6] where the problem of differentiability of direct and indirect utility functions is analyzed and Martinez-Legaz [22, 23] where a duality scheme is given with minimal assumptions. Many other references can be found in the list of references of these papers. For illustration, we have the following result due to Diewert [11]. Here K is the positive orthant of $\mathbb{R}^n$, i.e., $K = \{x \in \mathbb{R}^n \, : \, x_i > 0, \quad \forall\, i\}$.

Theorem 4.1. *Let $u : cl(K) \to \mathbb{R}$ be such that u is finite, continuous, quasiconcave and nondecreasing on $cl(K)$.*

For $p \in K$, define

$$v(p) = \sup[\, u(x) \; : \; x \geq 0, \; p^t x \leq 1].$$

Then, v is finite, continuous, quasiconvex and nonincreasing on K. Furthermore,

$$u(x) = \inf[\, v(p) \; : \; p \geq 0, \; p^t x \leq 1].$$

Furthermore, in the differentiable case, we have the following result proved in [6].

Theorem 4.2. *Let $u : K \to \mathbb{R}$. For all $p \in K$, let us define*

$$v(p) = \sup[\, u(x) \; : \; x \geq 0, \; p^t x \leq 1]. \tag{10}$$

Denote the set of the optimal solutions of this problem by $X(p)$. Assume that:

i) *u is finite, strictly quasiconcave and continuously differentiable on K,*
ii) *$\nabla u(x) > 0$ for all $x \in K$,*
iii) *$X(p) \cap K \neq \emptyset$ for all $p \in K$.*

Then for all $x \in K$

$$u(x) = \min[\, v(p) \; : \; p \geq 0, \; p^t x \leq 1]. \tag{11}$$

Furthermore, denoting the set of the optimal solutions of this problem by $P(x)$,

iv) *v is finite, strictly quasiconvex and continuously differentiable on K,*
v) *$\nabla v(p) < 0$ for all $p \in K$,*
vi) *$P(x) \cap K \neq \emptyset$ for all $x \in K$.*

In this result, the duality is quite symmetric: under the assumptions u is strictly pseudoconcave on K and v is strictly pseudoconvex on K. For $p \in K$, $X(p)$ is single-valued and similarly, for $x \in K$, $P(x)$ is single-valued. Furthermore, we have the following implications.

$$[p \in K \text{ and } x \in X(p)] \Leftrightarrow [u(x) = v(p) \text{ and } p^t x = 1] \Leftrightarrow [x \in K \text{ and } p \in P(x)].$$

Hence, since $x(p)$ is the optimal solution of the differentiable optimization problem (10) and $p^t x(p) = 1$,

$$x(p) = \frac{\nabla v(p)}{p^t \nabla v(p)},$$

and if, in addition, v is twice differentiable,

$$x'(p) = [I - x(p)p^t]\frac{\nabla^2 v(p)}{p^t \nabla v(p)} - x(p)x^t(p),$$
$$x'(p)[I - px^t(p)] = [I - x(p)p^t]\frac{\nabla^2 v(p)}{p^t \nabla v(p)}[I - px^t(p)],$$
$$x'(p)[I - px^t(p)] = [I - x(p)p^t]x'(p)[I - px^t(p)].$$

Clearly $p^t \nabla v(p) < 0$. Hence, since v is pseudoconvex, the matrix $x'(p)$ is symmetric and negative semidefinite on the orthogonal subspace to $x(p)$, these properties are known as the Slutsky conditions [32, 1].

We return to the problem of reconstructing an indirect utility function v from a single-valued differentiable demand. Assume that the following assumptions hold on $x : K \to K$:

$(D1)$ The map x is continuously differentiable on K;
$(D2)$ The condition $p^t x(p) = 1$ holds for all $p \in K$;
$(D3)$ For all $p \in K$, the matrix $x'(p)$ is symmetric and negative semidefinite on the orthogonal subspace to $x(p)$.

The theorem is as follows.

Theorem 4.3. *Assume that assumptions $(D1)$, $(D2)$ and $(D3)$ hold. Then there exists a continuously differentiable indirect utility v on K such that*

$$x(p) = \frac{\nabla v(p)}{p^t \nabla v(p)}$$

for all $p \in K$. This function v is pseudoconvex and decreasing on K.

Proof. a) Preliminaries: Let $e = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^t \in \mathbb{R}^n$ and A be an $n \times (n-1)$ matrix such that $(A, e)(A, e)^t = I$. Any $p \in K$ can be written as $p = Ay + \mu e$, where $y = A^t p \in \mathbb{R}^{n-1}$ and $\mu = e^t p > 0$. Set

$$F(p) = \frac{-x(p)}{e^t x(p)}.$$

Assumption $(D3)$ implies that $F'(p)$ is symmetric and positive semidefinite on the orthogonal subspace to $F(p)$.

Fix $p = Ay + \mu e \in K$ and, in line with Remark 3.4, consider the ordinary differential equation problem:

Find $h(., y, \mu) : [\bar{t}, 1] \to \mathbb{R}$ differentiable such that $h(1, y, \mu) = \mu$ and

$$h'_t(t, y, \mu) = -\frac{(Ay)^t x(tAy + h(t, y, \mu)e)}{e^t x(tAy + h(t, y, \mu)e)} \quad \text{for } t \in [\bar{t}, 1], \tag{12}$$

where $\bar{t} \in [0,1]$ is taken in such a way that the vector $tAy + h(t,y,\mu)e$ stays positive on $[\bar{t},1]$.

b) In this part of the proof, we shall prove that $\bar{t}$ can be taken equal to 1. For simplicity, y and μ staying fixed, set

$$\theta(t) = h(t,y,\mu), \quad p(t) = tAy + \theta(t)e \;\text{ and }\; x(t) = x(p(t)).$$

Then, for $t \in [\bar{t},1]$,

$$\theta''(t) = -(Ay)^t[I - \frac{x(t)e^t}{e^tx(t)}]\,\frac{x'(t)}{e^tx(t)}\,[I - \frac{e[x(t)]^t}{e^tx(t)}]Ay.$$

The vector $[I - \frac{e[x(t)]^t}{e^tx(t)}]Ay$ is orthogonal to $x(p)$, thereby $(D3)$ implies that θ is convex on $[\bar{t},1]$. Thus,

$$\theta(t) \geq \theta(1) + (t-1)\theta'(1) = \mu + (1-t)\frac{y^tA^tx(1)}{e^tx(1)},$$

and since

$$1 = [p(1)]^tx(1) = (Ay)^tx(1) + \mu e^tx(1),$$

we obtain finally,

$$\theta(t) \geq \mu t + \frac{(1-t)}{e^tx(1)}.$$

From what, we deduce that there exists a positive vector $x_-(y,\mu)$ not depending on $\bar{t}$ such that for all $t \in [\bar{t},1]$,

$$x(t) \geq t(Ay + \mu e) + \frac{1-t}{e^tx(1)}e \geq x_-(y,\mu) > 0. \tag{13}$$

On the other hand, for all vector $z \in \mathbb{R}^n_+$, we have the inequalities,

$$e^tz \leq \|z\| \leq e^tz\,\sqrt{n}.$$

Because the vector $x(t)$ is assumed to be positive, we deduce from (12)

$$|\theta'(t)| \leq \|Ay\|\,\sqrt{n} = \|y\|\,\sqrt{n},$$

which implies

$$\theta(t) \leq \mu + (1-t)\|y\|\,\sqrt{n}.$$

Thus, there exists a positive vector $x_+(y,\mu)$ not depending on $\bar{t}$ such that for all $t \in [\bar{t},1]$

$$x(t) \leq t(Ay + \mu e) + (1-t)\sqrt{n}\|y\|\;e \leq x_+(y,\mu). \tag{14}$$

We deduce that the function $h(.,y,\mu)$ is well defined on $[0,1]$.

c) In this part, we introduce the function v on K by

$$v(p) = -h(0,y,\mu) \;\text{ where }\; p = Ay + \mu e.$$

This function v is well defined on K. Set $\lambda = -v(p)$ and let us consider the ordinary differential equation problem:

Find $k(., y, \lambda) : [\bar{t}, 1] \to \mathbb{R}$ differentiable such that $k(0, y, \lambda) = \lambda$ and

$$k'_t(t, y, \lambda) = -\frac{(Ay)^t x(tAy + k(t, y, \lambda)e)}{e^t x(tAy + k(t, y, \lambda)e)} \quad \text{for } t \in [0, 1].$$

Then, $\mu = k(1, y, \lambda)$.

Let $\bar{p} = A\bar{y} + \bar{\mu}e \in K$. In view of (13) and (14), there exists a convex compact neighborhood C of $\bar{p}$, $C \subset K$, such that for any $p = Ay + \mu e \in \mathrm{int}(C)$ and any $t \in [0, 1]$ the vector $tAy + h(t, y, \mu)e \in \mathrm{int}(C)$.

The remaining of the proof is, mutatis mutandis, similar to the proof of Theorem 3.1, but unlike in this proof, it is no more necessary to restrict C. It is proved that v is differentiable on C and

$$\frac{x(p)}{\|x(p)\|} = -\frac{\nabla v(p)}{\|\nabla v(p)\|},$$

next that v is pseudoconvex on C. Finally it is observed that v is the unique differentiable indirect utility function associated to the demand map x such that $v(te) = -t$. □

Remark 4.4. The theorem, unlike Theorems 3.1 and 3.5 is global since, under the assumptions, v is defined on the whole set K.

References

[1] S. N. Afriat: Demand Functions and the Slutsky Matrix, Princeton University Press (1982).

[2] M. Avriel, W. E. Diewert, S. Schaible, I. Zang: Generalized Concavity, Plenum Publishing Corporation, New York (1998).

[3] P. A. Chiappori, I. Ekeland: Aggregation and market demand: An exterior differential calculus viewpoint, Econometrica 67 (1999) 1435–1458.

[4] J.-P. Crouzeix: Contributions à l'Étude des Fonctions Quasiconvexes, Thèse de Doctorat es Sciences, Université de Clermont 2 (1977).

[5] J.-P. Crouzeix, J. A. Ferland: Criteria for quasiconvexity and pseudoconvexity: Relationships and comparisons, Math. Programming 23 (1982) 193–205.

[6] J.-P. Crouzeix: Duality between direct and indirect utility functions, differentiability properties, J. Math. Econ. 12 (1983) 149–165.

[7] J.-P. Crouzeix, J. A. Ferland: Criteria for differentiable generalized monotone maps, Math. Programming 75 (1996) 399–406.

[8] J.-P. Crouzeix: Characterizations of generalized convexity and generalized monotonicity, a survey, in: Generalized Convexity and Generalized Monotonicity, J.-P. Crouzeix, J.-E. Martinez-Legaz, M. Volle (eds.), Nonconvex Optim. Appl. 27, Kluwer Academic Publishers (1998) 237–256.

[9] J.-P. Crouzeix: Criteria for generalized convexity and generalized monotonicity in the differentiable case, in: Handbook on Generalized Convexity and Generalized Monotonicity, Chapter 2, Kluwer Academic Publishers (2004) 87–117.

[10] G. Debreu: Definite and semi-definite quadratic forms, Econometrica 20 (1952) 295–300.

[11] W. E. Diewert: Applications of duality theory, in: Frontiers of Quantitative Economics, Vol. 2, M. D. Intriligator, D. A. Kendrick (eds.), North-Holland, Amsterdam (1974) 106–171.

[12] W. E. Diewert: Duality approaches to microeconomics theory, in: Handbook of Mathematical Economics, Vol. 2, K. J. Arrow, M. D. Intrilligator (eds.), North-Holland, Amsterdam (1982) 535–599.

[13] A. Eberhard, J.-P. Crouzeix: Integration of a normal cone relation generated by the level sets of pseudoconvex functions, in process.

[14] I. Ekeland: Exterior differential calculus and applications to economy theory, Scuola Normale Superiore di Pisa Quaderni, Scuola Normale Superiore, Pisa (1998).

[15] I. M. D. Little: A reformulation of the theory of consumer's behaviour, Oxford Economic Papers 1 (1949) 90–99.

[16] J. Green, A. Mas-Colell, M. Whinston: Microeconomics Theory, Oxford University Press (1995).

[17] N. Hadjisavvas, S. Komlosi, S. Schaible (eds.): Handbook on Generalized Convexity and Generalized Monotonicity, Kluwer Academic Publishers (2004).

[18] H. S. Houthakker: Revealed preference and the utility function, Economica 17 (1950) 159–174.

[19] L. Hurwicz, H. Uzawa: On the integrability of demand functions, in: Preferences, Utility and Demand, Chapter 6, J. Chipman et al. (eds.), Harcourt Brace Jovanovich (1971) 114–148.

[20] R. Kihlstrom, A. Mas Colell, H. Sonnenscheim: The demand theory of the weak axiom of revealed preference, Econometrica 44 (1976) 971–978.

[21] L. J. Lau: Duality and the structure of utility functions, J. Econ. Theory 1 (1970) 374–396.

[22] J.-E. Martínez-Legaz: Duality between direct and indirect utility functions under minimal hypotheses, J. Math. Econ. 20 (1991) 199–209.

[23] J.-E. Martínez-Legaz: Generalized convex duality and its economic applications, in: Handbook on Generalized Convexity and Generalized Monotonicity, Chapter 6, N. Hadjisavvas, S. Komlosi, S. Schaible (eds.), Kluwer Academic Publishers (2004) 237–292.

[24] T. Rapcsák: An unsolved problem of Fenchel, J. Glob. Optim. 11 (1997) 207–217.

[25] R. Roy: De l'Utilité, Hermann, Paris (1942).

[26] Y. Sakai: Equivalence of the weak and strong axioms of revealed preference without demand continuity assumptions: A "regularity condition" approach, J. Econ. Theory 8 (1974) 292–304.

[27] Y. Sakai: Revealed favorability, indirect utility and direct utility, J. Econ. Theory 14 (1977) 113–129.

[28] P. A. Samuelson: A note on the pure theory of consumer's behaviour, Economica 5 (1938) 61–71.

[29] P. A. Samuelson: The empirical implications of utility analysis, Economica 6 (1938) 344–356.

[30] P. A. Samuelson: Foundations of Economic Analysis, Harvard University Press, Cambridge (1947).

[31] P. A. Samuelson: Consumption theory in terms of revealed preference, Economica 15 (1948) 243–253; also in: Utility Theory: A Book of Readings, A. N. Page (ed.), Wiley, New York (1968) 149–159.

[32] E. Slutsky: Sulla theoria del bilancio del consumatore, Giornali degli economisti 51 (1915) 1–26.