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# Remarks on Γ-Convergence and the Monge-Kantorovich Mass Transfer Problem

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By looking at the Γ-convergence of some easy functionals in terms of underlying Young measures, we emphasize the connection of this issue with the Monge-Kantorovich mass transfer problem. After exploring this relationship, we study a typical example in periodic homogenization to realize the difference between the Monge-Kantorovich problem and the usual cell-problem. We also state a version of the Monge-Kantorovich problem for gradients.

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## 1. Introduction

Γ-convergence is the main concept for variational convergence of functionals. Since its introduction ([5], [6]), it has found many important applications in various fields: singularly perturbed problems, models of thin structures via dimension reduction, homogenization of functionals, shape optimization, to name a few important cases. See [2], [4] for a complete treatment.

Our motivation here is to discuss briefly Γ-convergence of functionals when we consider the Γ-limit defined on Young measures rather than on functions. Since the Young measure associated with a sequence of functions carries more information than its weak limit, we expect that such a Γ-limit will provide more information on the asymptotic structure of a sequence of minimizers for the sequence of functionals we are considering. But on the other hand, and because of this same reason, its computation in specific examples should be much more involved. We would like to present some simple examples to show that this is indeed so. In fact, the computation of such Γ-limit in these cases will lead to consider the classical Monge-Kantorovich mass transfer problem ([7]). We will be able to understand such Γ-limit in those situations where we can use fine results for this classical problem.

Consider a sequence of functionals determined by a certain sequence of functions

$$I_j(u) = \int_\Omega \varphi(v_j(x), u(x))\, dx$$

where $\Omega \subset \mathbf{R}^N$ is a regular, bounded domain,

$$v_j : \Omega \to \mathbf{R}^d, \quad u : \Omega \to \mathbf{R}^m, \quad \varphi : \mathbf{R}^d \times \mathbf{R}^m \to \mathbf{R}.$$

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For simplicity, we will assume that $v_j$ takes on values on a given cube $Z$ in $\mathbf{R}^d$ and, as part of admissibility, competing functions $u$ should also take values on another cube $Q$ in $\mathbf{R}^m$. In this simplified framework, a Young measure associated with a sequence $\{u_j\}$ is nothing more than a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ on $\mathbf{R}^m$ depending measurably on $x \in \Omega$ ([12]) so that whenever the sequence $\{g(u_j)\}$ converges weakly, the weak limit can be represented by the integral of $g$ against the Young measure

$$\overline{g}(x) = \int_{\mathbf{R}^m} g(\lambda)\, d\nu_x(\lambda).$$

We would like to explore

$$\tilde{I}(\nu) = \liminf_{j\to\infty} \{I_j(u_j) : \{u_j\} \text{ generates } \nu\}.$$

This sort of Γ-convergence of functionals defined in terms of Young measures was explicitly explored in [13] for a particular situation in magnetostriction. Notice that the usual (weak) Γ-limit can be recovered from $\tilde{I}$ by putting

$$I^w(u) = \inf\left\{\tilde{I}(\nu) : \int_Q \lambda\, d\nu_x(\lambda) = u(x)\right\};$$

and the same is true for the strong Γ-limit

$$I^s(u) = \tilde{I}(\delta_u),$$

where $\delta_u$ stands for the Young measure $\{\delta_{u(x)}\}_{x\in\Omega}$.

To simplify even further the situation, we will assume that the Young measure associated with $\{v_j\}$ is $\sigma$, homogeneous, and compute $\tilde{I}(\nu)$ for $\nu$ homogeneous too, so that the Young measure associated with $\{u_j\}$ is $\nu$ (no dependence on $x$). Since

$$I_j(u_j) = \int_\Omega \varphi(v_j(x), u_j(x))\, dx,$$

the limit behavior of these integrals can be grasped by looking at their joint Young measure $\mu = \{\mu_x\}_{x\in\Omega}$ whose support is contained in $Z \times Q$. Indeed,

$$\lim_{j\to\infty} I_j(u_j) = \int_\Omega \int_{Z\times Q} \varphi(\lambda_1, \lambda_2)\, d\mu_x(\lambda_1, \lambda_2)\, dx.$$

Note that the sequence $\{\varphi(u_j, v_j)\}$ converges (or rather an appropriate subsequence) weak * in $L^\infty(\Omega)$. What do we know about each $\mu_x$? Their marginals must be $\sigma$ and $\nu$ because they are generated by $\{v_j\}$ and $\{u_j\}$, respectively. Thus, we are led to consider

$$\tilde{I}(\nu) = \inf_\mu \left\{\int_{Z\times Q} \varphi(\lambda_1, \lambda_2)\, d\mu(\lambda_1, \lambda_2) : \pi_1\mu = \sigma, \pi_2\mu = \nu\right\},$$

where $\pi_i\mu$, $i = 1, 2$, designates the two marginals. Notice how we can drop the $x$-dependence on $\mu$. This is exactly the relaxed formulation of the Monge-Kantorovich mass transfer problem ([7]) with cost given by $\varphi$. The classical formulation by Monge

([11]) consists in finding an optimal map $a : Z \to \mathbf{R}^m$ such that it "pushes $\sigma$ forward to $\nu$" and it minimizes the cost functional

$$\int_Z \varphi(\lambda, a(\lambda))\, d\sigma(\lambda)$$

among all such maps, where the integrand $\varphi : Z \times \mathbf{R}^m \to \mathbf{R}$ is given. The sentence "$a$ pushes $\sigma$ forward to $\nu$" exactly means

$$\nu(E) = \sigma\left(a^{-1}(E)\right)$$

for all Borel sets $E$. Hence, we see that the computation of the Γ-limit $\tilde{I}(\nu)$ is closely related to the Monge-Kantorovich problem. To establish this connection clearly was one of the main motivations of this note.

In Section 2, we will summarize some of the known results on the Monge-Kantorovich problem. We will closely follow [7] for this. We pretend to translate these facts to our Γ-convergence framework. We will do this in Section 3. Finally, Section 4 is devoted to examining one typical situation in which we become interested in finding the Γ-limit of

$$I_j(u) = \int_Z h(\langle jx\rangle - u(x))\, dx$$

where $Z = (0, 1)^N$, and $\langle z\rangle$ is the fractional part of the vector $z$ componentwise. We apply some of the ideas of the Monge-Kantorovich problem, and finish stating a kind of Monge-Kantorovich problem for gradients. Namely, given two probability measures $\nu, \sigma$ with $\operatorname{supp} \sigma \subset Q \subset \mathbf{R}^N$, find $a : Q \to \mathbf{R}$, $a \in W^{1,1}(Q)$, such that $\nabla a$ pushes $\sigma$ forward to $\nu$ and it minimizes the cost functional

$$\int_Q \varphi(\lambda, \nabla a(\lambda))\, d\sigma(\lambda)$$

among all such $a$'s.

## 2. The Monge-Kantorovich Mass Transfer Problem

The original Monge problem [11] was considered in economics. Suppose two sets $Z \subset \mathbf{R}^d$ and $Q \subset \mathbf{R}^m$ are given, and $Z$ is filled uniformly with mass. Let $\varphi(\lambda_1, \lambda_2)$ be the cost (per unit of mass) for transporting material from $\lambda_1 \in Z$ to $\lambda_2 \in Q$. The question is to find the optimal map $a : Z \to Q$ that minimizes the cost of redistributing the mass from $Z$ to $Q$. The history of this problem is quite remarkable and several people made important contributions. See all the references cited in [7] for details. In more precise terms, we seek to

$$\text{Minimize in } a : \quad \int_Z \varphi(\lambda, a(\lambda))\, d\sigma(\lambda) \tag{1}$$

where $a$ is restricted by "pushing $\sigma$ forward to $\nu$". Here $\sigma$ and $\nu$ are probability measures supported on $Z$ and $Q$, respectively, and the relationship between the map $a$, and $\sigma$ and $\nu$ is given by

$$\nu(E) = \sigma(a^{-1}(E)), \quad E \subset Q;$$

or alternatively, we can say that

$$\int_Q \psi(\lambda)\, d\nu(\lambda) = \int_Z \psi(a(\lambda))\, d\sigma(\lambda) \tag{2}$$

for every continuous $\psi$.

(1) looks like a rather typical minimization problem in the map $a$. The great difficulty is that the restriction of pushing $\sigma$ forward to $\nu$ is not preserved under weak convergence so that a direct approach cannot furnish the existence of optimal maps even if $\varphi$ is convex in its second variable. A previous issue is whether the set of such maps for given $\sigma$ and $\nu$ is non-empty. This is not completely trivial, and indeed imposes some mutual restrictions on $\sigma$ and $\nu$. See again [7] for further comments.

Suppose that the set of admissible maps is non-empty. Whenever we have to face a variational problem and we cannot apply the direct method to show existence of optimal solutions, we can setup the problem in terms of Young measures associated with sequences of maps $a_j : Z \to Q$. Let $\{a_j\}$ be minimizing in (1) and let $\Lambda = \{\Lambda_\lambda\}_{\lambda\in Z}$ be its corresponding Young measure with respect to $\sigma$. The relevant issue is how the main constraint on the maps $a_j$ translates into $\Lambda$. By (2), we have

$$\int_Q \psi(\lambda)\, d\nu(\lambda) = \int_Z \psi(a_j(\lambda))\, d\sigma(\lambda) \quad \text{for all } j.$$

This in particular implies, by the main property of the Young measure ([12]), that

$$\int_Q \psi(\lambda)\, d\nu(\lambda) = \int_Z \int_Q \psi(\xi)\, d\Lambda_\lambda(\xi)\, d\sigma(\lambda). \tag{3}$$

If we put $\mu = \Lambda_\lambda \otimes \sigma$, a probability measure over $Z \times Q$ (in fact the full Young measure), then identity (3) simply means that the second marginal of $\mu$ is $\nu$ while the first one is $\sigma$. In addition, the cost functional can be written

$$\int_Z \int_Q \varphi(\lambda, \xi)\, d\Lambda_\lambda(\xi)\, d\sigma(\lambda) = \int_{Z\times Q} \varphi(\lambda, \xi)\, d\mu(\lambda, \xi).$$

Therefore a relaxed formulation of the problem is to

$$\text{Minimize in } \mu : \quad \int_{Z\times Q} \varphi(\lambda_1, \lambda_2)\, d\mu(\lambda_1, \lambda_2)$$

under $\pi_1\mu = \sigma$, $\pi_2\mu = \nu$. This relaxed formulation was indeed the main contribution of Kantorovich ([9], [10]). The minimization problem on measures with prescribed marginals can be taken, under precise requirements on the integrand $\varphi$, as a metric in the space of measures. For the particular case $\varphi(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|^2$ the resulting metric is known as the Wasserstein metric. It has been recently used in the numerical approximation of the Fokker-Planck equation and in dynamical problems involving dissipation or diffusion (see [8] for instance). By setting this relaxed formulation, it is relatively easy to show that there exists an optimal measure $\mu$, and the question is to decide if this measure $\mu$ is ever supported on the graph of an admissible map $a$. Further developments permitted to put hypotheses on the cost $\varphi$ so that optimal maps could be found ([7]).

In order to use these results in our Γ-convergence scenario, we state here one of the typical results explicitly taken from [7]. We will consider costs of the form $\varphi(\lambda_1, \lambda_2) = h(\lambda_1 - \lambda_2)$ with the main condition on $h$ of being a strictly convex function. More restrictions are needed to allow probability measures with unbounded support.

**Theorem 2.1 ([7]).** *For costs $\varphi$ as indicated above, and probability measures $\sigma$, $\nu$ on $\mathbf{R}^d$, if $\sigma$ is absolutely continuous with respect to the Lebesgue measure, then there is a unique optimal map $a$.*

There is a number of other results relaxing the requirements on $h$ and/or on $\sigma$ ([7]).

## 3. Γ-Limits of Functionals

We go back to the computations of Γ-limits in terms of underlying Young measures. We will place ourselves in a situation

$$I_j(u) = \int_\Omega \varphi(x, v_j(x), u(x))\, dx$$

where $\Omega \subset \mathbf{R}^N$ is a regular bounded domain, and

$$v_j : \Omega \to \mathbf{R}^d, \quad u : \Omega \to \mathbf{R}^m, \quad \varphi : \Omega \times \mathbf{R}^d \times \mathbf{R}^m \to \mathbf{R}.$$

The integrand $\varphi$ is assumed to be a Carathodory integrand (measurable in $x$ and continuous in the other variables). An important restriction relates to this integrand $\varphi$ and to the sequence $\{v_j\}$ defining the functionals $I_j$. We will suppose that for a.e. $x \in \Omega$, the Monge-Kantorovich problem corresponding to $\varphi(x, \cdot, \cdot)$ and to the probability measure $\sigma_x$, where $\sigma = \{\sigma_x\}_{x\in\Omega}$ is the Young measure for $\{v_j\}$, is solvable for any target measure $\nu_x$, a.e. $x \in \Omega$. Specifically, if we want to compute $\tilde{I}(\nu)$ for $\nu = \{\nu_x\}_{x\in\Omega}$ the Young measure associated with $\{u_j\}$, we will assume that there is a map

$$a_\nu : \Omega \times \mathbf{R}^d \to \mathbf{R}^m$$

such that for a.e. $x \in \Omega$, $a_\nu(x, \cdot)$ is an optimal map for the problem

$$\text{Minimize in } a : \quad \int_{\mathbf{R}^d} \varphi(x, \lambda, a(x, \lambda))\, d\sigma_x(\lambda)$$

subject to the fact that $a$ must push $\sigma_x$ forward to $\nu_x$. We will not be more explicit here and then we will study some concrete situations where we know these facts about the Monge-Kantorovich problem are true.

It is then immediate that

$$\tilde{I}(\nu) = \int_\Omega \int_{\mathbf{R}^d} \varphi(x, \lambda, a_\nu(x, \lambda))\, d\sigma_x(\lambda)\, dx, \tag{4}$$

and we would have a full description of $\tilde{I}(\nu)$ as soon as the mapping

$$\nu_x \mapsto a_\nu(x, \cdot)$$

is determined. For a situation in which the Monge-Kantorovich problem is not solvable, we would have

$$\tilde{I}(\nu) = \int_\Omega \int_{\mathbf{R}^d \times \mathbf{R}^m} \varphi(x, \lambda_1, \lambda_2)\, d\mu_x^\nu(\lambda_1, \lambda_2)\, dx$$

where the probability measure $\mu_x^\nu$ is the optimal solution of the problem

$$\text{Minimize in } \mu : \quad \int_{\mathbf{R}^d \times \mathbf{R}^m} \varphi(x, \lambda_1, \lambda_2)\, d\mu(\lambda_1, \lambda_2)$$

subject to the fact that the marginals of $\mu$ should be $\sigma_x$ and $\nu_x$, respectively.

In many situations, all the detailed information recorded in the computation of $\tilde{I}(\nu)$ for any admissible family of probability measures $\nu$ cannot be explicitly found. Thinking about the parallelism with the convexification of variational problems in terms of Young measures ([12]), all we might be interested in is the computation of $I^w(u)$ and, at least, one optimal $\nu$ associated with $u$. Recall that

$$I^w(u) = \inf\left\{ \tilde{I}(\nu) : \int_Q \lambda\, d\nu(\lambda) = u \right\}.$$

What we are saying is that, somehow, the important information is contained in the computation of $I^w(u)$ and the determination of $\nu^u$ such that

$$I^w(u) = I(\nu^u).$$

Though the computation of $\tilde{I}(\nu)$ in this full generality for any $\nu$ may be almost impossible to achieve, yet $I^w(u)$ and $\nu^u$ may be more easily computed.

Let us go back to (4), and try to compute $I^w(u)$. For this, we have to treat the overall optimization problem

$$\text{Minimize in } a : \quad \int_\Omega \int_{\mathbf{R}^d} \varphi(x, \lambda, a(\lambda))\, d\sigma_x(\lambda)\, dx$$

subject to

$$u(x) = \int_{\mathbf{R}^d} a(\lambda)\, d\sigma_x(\lambda).$$

Notice that this constraint comes from the fact that $a$ should push $\sigma_x$ forward to $\nu_x$. If we define a new integrand $C\varphi$ by putting

$$C\varphi(x, u) = \inf_a \left\{ \int_{\mathbf{R}^d} \varphi(x, \lambda, a(\lambda))\, d\sigma_x(\lambda) : u = \int_{\mathbf{R}^d} a(\lambda)\, d\sigma_x(\lambda) \right\}, \tag{5}$$

then it is evident that

$$I^w(u) = \int_\Omega C\varphi(x, u(x))\, dx.$$

In addition, an optimal map $a$ in (5) will furnish an optimal measure $\nu_x^u$. Almost the same observations apply even if the associated Monge-Kantorovich problem is not solvable.

## 4. Some Examples

One typical situation where we can use the ideas steming from the Monge-Kantorovich problem corresponds to taking

$$I_j(u) = \int_Z h(\langle jx\rangle - u(x))\,dx$$

where $Z = (0,1)^N$, $\langle z\rangle$ is the fractional part of $z$ componentwise, $u : Z \to \mathbf{R}^N$ belongs to appropriate Lebesgue spaces and $h : \mathbf{R}^N \to \mathbf{R}$ satisfies appropriate hypotheses so that Theorem 2.1 can be applied. This essentially amounts to taking $h$ strictly convex and $h(0) = 0$.

As a consequence of the well-known Riemann-Lebesgue lemma ([3]) the Young measure associated with $\langle jx\rangle$ is the Lebesgue measure restricted to $Z$, homogeneous. Thus $\sigma_x = dz$ for all $x \in Z$. By applying Theorem 2.1, we conclude that

$$\tilde{I}(\nu) = \int_Z h(z - a_\nu(z))\,dz$$

where the map $a_\nu : Z \to \mathbf{R}^N$ is uniquely determined as the optimal solution of the problem

$$\text{Minimize in } a: \quad \int_Z h(z - a(z))\,dz$$

subject to $\nu = \overline{\delta_a}$. This last expression is a short way of saying

$$\int_Z \psi(z)\,d\nu(z) = \int_Z \psi(a(z))\,dz$$

where by $\overline{\delta_a}$ we mean the "average" of the trivial Young measure $\delta_a$. See [12]. We can also say

$$\tilde{I}(\nu) = \min_a \left\{ \int_Z h(z - a(z))\,dz : \nu = \overline{\delta_a} \right\}.$$

In addition, an optimal sequence to achieve this value is $a_\nu(\langle jx\rangle)$ in the sense that

$$\int_Z h(\langle jx\rangle - a_\nu(\langle jx\rangle))\,dx \to \tilde{I}(\nu). \tag{6}$$

From here, we can also give in a more-or-less explicit fashion an integral form for $I^w(u)$. Indeed

$$\begin{aligned} I^w(u) &= \inf_\nu \left\{ \tilde{I}(\nu) : \nu = \{\nu_x\}_{x\in\Omega},\, \int_{\mathbf{R}^N} \lambda\,d\nu_x(\lambda) = u(x) \right\} \\ &= \inf_\nu \min_a \left\{ \int_Z h(z - a(z))\,dz : \nu_x = \overline{\delta_a}, \int_{\mathbf{R}^N} \lambda\,d\nu_x(\lambda) = u(x) \right\} \\ &= \int_Z \left[ \min_a \left\{ \int_Z h(z - a(z))\,dz : \int_Z a(z)\,dz = u(x) \right\} \right] dx. \end{aligned}$$

If we define the integrand $\psi : \mathbf{R}^N \to \mathbf{R}$ by putting

$$\psi(\lambda) = \min_a \left\{ \int_Z h(z - a(z))\, dz : \int_Z a(z)\, dz = \lambda \right\}$$

then

$$I^w(u) = \int_Z \psi(u(x))\, dx.$$

For each $x \in Z$, there exists an optimal map $a(x, z)$ such that

$$u(x) = \int_Z a(x, z)\, dz \quad \text{a.e. } x \in \Omega$$

and the optimal measure $\nu^u = \{\nu_x^u\}_{x \in Z}$ associated with $u$ so that $I^w(u) = \tilde{I}(\nu^u)$ is the push-forward of $dz$ by the map $a(x, \cdot)$. The definition of $\psi$ above involves the typical cell-problem in periodic homogenization ([1]).

We can now realize the distinction between the Γ-limit in terms of Young measures and in term of functions (their first moments). In the simplified framework of our example, the first amounts to understanding the Monge-Kantorovich problem

$$\text{Minimize in } a : \quad \int_Z h(z - a(z))\, dz$$

subject to the fact that $a$ pushes $dz$ forward to a given $\nu$. This is a rather difficult and delicate problem. The second leads to consider

$$\text{Minimize in } a : \quad \int_Z h(z - a(z))\, dz$$

subject to $\int_Z a(z)\, dz = u$ for given $u$. We know that this last problem admits optimal maps in much more generality. This is so because the integral constraint in $a$ is preserved under weak convergence while the property of pushing $dz$ forward to $\nu$ is not.

A more interesting situation occurs when we consider

$$I_j(u) = \int_Z h(\langle jx \rangle - \nabla u(x))\, dx$$

where now $u : Z \to \mathbf{R}$. The situation is, however, formally the same. Because there is no formal restriction on local gradient Young measures in the scalar case ([12]), the analysis carried out above is still valid here. The only difficulty we encounter relates to (6). For $\nu$ given (no dependence on $x$), an optimal sequence yielding the value $\tilde{I}(\nu)$ is $\{a_\nu(\langle jx \rangle)\}$. But this is not a gradient unless the field $a_\nu$ itself is a gradient.

To resolve this difficulty, we would have to go back to the Monge-Kantorovich mass transfer problem and add the fundamental ingredient that the admissible maps $a$ be gradients. Bearing in mind the framework at the beginning of Section 2, we would have to examine the problem

$$\text{Minimize in } a : \quad \int_Z \varphi(\lambda, \nabla a(\lambda))\, d\sigma(\lambda)$$

subject to $\nu = \overline{\delta_{\nabla a}}$, for $a : Z \to \mathbf{R}$. Without the gradient requirement, the optimal map has been explicitly given under the hypotheses of Theorem 2.1 ([7]). Its explicit form is

$$s(x) = x - (\nabla h)^{-1}(\nabla g(x))$$

for a certain (c-concave) function $g$. If this map $s$ turns out to be a gradient, this would be the solution when we incorporate the gradient requirement. If it is not, we would have to study the Monge-Kantorovich mass transfer problem for gradients as stated at the end of the Introduction.

The example in this section is closely related to periodic homogenization of integrals ([1]). See [14] for an alternative way to studying Γ-convergence by means of Young measures from a different, yet related, perspective.

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