Journal of Convex Analysis
Volume 12 (2005), No. 1, 139–143

On Brickman's Theorem

Juan Enrique Martínez-Legaz*
CODE and Departament d'Economia i d'Història Econòmica,
Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
JuanEnrique.Martinez@uab.es

Received March 30, 2004

We give an elementary proof of a theorem of Brickman, which establishes the convexity of the image of the unit sphere of a space with dimension at least three under a vector mapping into $\mathbb{R}^2$ whose component functions are quadratic forms. We also discuss some consequences of this theorem regarding the range of that mapping.

Keywords: Quadratic forms, convexity

2000 Mathematics Subject Classification: 15A63, 52A10

1. Introduction

In [2], Brickman proved the following theorem:

Theorem 1.1 (Brickman). *Let $n \geq 3$ and let A and B be two symmetric $n \times n$ matrices. Then the set $C := \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ x \in \mathbb{R}^n, \ \|x\| = 1\}$ is convex.*

The proof given in [2] works with the $n-1$ dimensional projective space and considers a real projective hyperconic in it. The aim of this note is to provide a more elementary proof, not requiring other concepts than those of the Euclidean space setting, to which the statement belongs. In doing so we will actually prove, with no extra effort, that Brickman's theorem also holds in arbitrary real inner product spaces of (possibly infinite) dimension at least three. This will be carried out in Section 2. In Section 3 we will present some consequences of Brickman's Theorem regarding the cone $\{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ x \in \mathbb{R}^n\}$.

Brickman's theorem is just an instance of a class of results on convexity of images of quadratic mappings. For a survey of such results we refer the reader to [5].

2. An Elementary Proof of Brickman's Theorem

We will use the following lemma, whose easy proof we omit.

Lemma 2.1. *Let S be a 3×3 matrix, $\gamma \in \mathbb{R}$ and*

$$X := \left\{x \in \mathbb{R}^3 \ : \ \|x\| = 1, \ \langle Sx, x\rangle = \gamma\right\}.$$

Then there exists a connected set $Y \subset X$ such that $X = Y \cup (-Y)$.

*This work has been supported by the Spanish Ministry of Science and Technology, project BEC2002-00642, and by the Departament d'Universitats, Recerca i Societat de la Informació, Direcció General de Recerca de la Generalitat de Catalunya, project 2001SGR-00162. The author thanks the support of the Barcelona Economics Program of CREA.

ISSN 0944-6532 / $ 2.50

Theorem 2.2. *Let V be a real inner product space of dimension at least three and let A and B be two endomorphisms of V. Then the set*

$$C := \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ x \in V, \ \|x\| = 1\}$$

is convex.

Proof. Let us first consider the case when V has dimension three. We can identify V with $\mathbb{R}^3$ and the endomorphisms A and B with their matrices in the canonical basis. To prove that C is convex it suffices to prove that the intersection of C with any straight line L is connected. Let $\alpha s + \beta t = \gamma$ be the equation of an arbitrary straight line L in the st plane. Then $C \cap L = \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ \|x\| = 1, \ \langle S_{\alpha,\beta} x, x\rangle = \gamma\}$, with $S_{\alpha,\beta} := \alpha A + \beta B$. Thus if we set $X_{\alpha,\beta} := \{x \in \mathbb{R}^3 \ : \ \|x\| = 1, \ \langle S_{\alpha,\beta} x, x\rangle = \gamma\}$ and define $T : X \to \mathbb{R}^2$ by $T(x) := (\langle Ax, x\rangle, \langle Bx, x\rangle)$, we have $C \cap L = T(X_{\alpha,\beta})$. By Lemma 2.1 there exists a connected set $Y_{\alpha,\beta}$ such that $X_{\alpha,\beta} = Y_{\alpha,\beta} \cup (-Y_{\alpha,\beta})$. Since $T(X_{\alpha,\beta}) = T(Y_{\alpha,\beta})$, it follows that $C \cap L = T(Y_{\alpha,\beta})$ is connected.

Let now V have dimension at least three and $(\langle Ax^1, x^1\rangle, \langle Bx^1, x^1\rangle)$ and $(\langle Ax^2, x^2\rangle, \langle Bx^2, x^2\rangle)$, with $\|x^1\| = \|x^2\| = 1$, be any two points in C and $\lambda \in [0,1]$. Take any orthonormal basis $\{w^1, w^2\}$ of the subspace spanned by x^1 and x^2 (or of any twodimensional subspace containing x^1 if x^1 and x^2 are linearly dependent) and another vector w^3 orthogonal to w^1 and w^2 and such that $\|w^3\| = 1$. Let W be the subspace spanned by w^1, w^2 and w^3 and $\widetilde{A} := (a_{ij})$ and $\widetilde{B} := (b_{ij})$ be the 3×3 matrices with entries $a_{ij} := \langle Aw^i, w^j\rangle$ and $b_{ij} := \langle Bw^i, w^j\rangle$. Consider the linear isometry $\varphi : W \to \mathbb{R}^3$ assigning to w^i the i−th unit vector, for each $i = 1, 2, 3$. Clearly, $\langle Aw, w\rangle = \left\langle \widetilde{A}\varphi(w), \varphi(w)\right\rangle$ and $\langle Bw, w\rangle = \left\langle \widetilde{B}\varphi(w), \varphi(w)\right\rangle$ for every $w \in W$. Since the set $\left\{\left(\left\langle \widetilde{A}u, u\right\rangle, \left\langle \widetilde{B}u, u\right\rangle\right) \ : \ \|u\| = 1\right\}$ is convex, we have

$$\begin{aligned}
&(1-\lambda)\left(\left\langle \widetilde{A}\varphi(x^1), \varphi(x^1)\right\rangle, \left\langle \widetilde{B}\varphi(x^1), \varphi(x^1)\right\rangle\right) \\
&\quad + \lambda\left(\left\langle \widetilde{A}\varphi(x^2), \varphi(x^2)\right\rangle, \left\langle \widetilde{B}\varphi(x^2), \varphi(x^2)\right\rangle\right) = \left(\left\langle \widetilde{A}u, u\right\rangle, \left\langle \widetilde{B}u, u\right\rangle\right)
\end{aligned}$$

for some $u \in \mathbb{R}^3$ with $\|u\| = 1$. Hence

$$\begin{aligned}
&(1-\lambda)\left(\langle Ax^1, x^1\rangle, \langle Bx^1, x^1\rangle\right) + \lambda\left(\langle Ax^2, x^2\rangle, \langle Bx^2, x^2\rangle\right) \\
&= (1-\lambda)\left(\left\langle \widetilde{A}\varphi(x^1), \varphi(x^1)\right\rangle, \left\langle \widetilde{B}\varphi(x^1), \varphi(x^1)\right\rangle\right) \\
&\quad + \lambda\left(\left\langle \widetilde{A}\varphi(x^2), \varphi(x^2)\right\rangle, \left\langle \widetilde{B}\varphi(x^2), \varphi(x^2)\right\rangle\right) \\
&= \left(\left\langle \widetilde{A}u, u\right\rangle, \left\langle \widetilde{B}u, u\right\rangle\right) \\
&= \left(\left(\left\langle A\varphi^{-1}(u), \varphi^{-1}(u)\right\rangle, \left\langle B\varphi^{-1}(u), \varphi^{-1}(u)\right\rangle\right)\right) \in C.
\end{aligned}$$

□

3. Some Consequences of Brickman's Theorem

In this section we discuss some consequences of Brickman's Theorem regarding the cone $K := \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ x \in V\}$ in the case when V is finite dimensional. A first immediate consequence of Theorem 2.2 is that K, being the cone generated by the convex set C, is itself convex if V has (possibly infinite) dimension at least three. The next lemma says that K is also convex when V has dimension two. Notice that, on the other hand, in the one dimensional case K is obviously a closed halfline unless both A and B are 0, in which case K reduces to the origin.

Lemma 3.1. *Let A and B be two 2×2 real matrices. Then the set $K := \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ x \in \mathbb{R}^2\}$ is a convex cone.*

Proof. We will assume, without loss of generality, that A and B are symmetric and will prove that

$$K = \{(\langle\langle A, X\rangle\rangle, \langle\langle B, X\rangle\rangle) \ : \ X \text{ is a psd symmetric } 2\times 2 \text{ real matrix}\}; \tag{1}$$

$\langle\langle\cdot,\cdot\rangle\rangle$ denotes here the standard inner product in the space of symmetric 2×2 real matrices, defined as the trace of the product, and we use "psd" as an abbreviation of "positive semidefinite". The inclusion $\subseteq$ is obvious, as for any $x \in \mathbb{R}^2$ one has $(\langle Ax, x\rangle, \langle Bx, x\rangle) = \left(\langle\langle A, xx^T\rangle\rangle, \langle\langle B, xx^T\rangle\rangle\right)$ (interpreting x as a column vector and the superscript ^{T} denoting transpose). To prove the opposite inclusion, let (a, b) belong to the right hand side of (1). Then the solution set M of the linear system of equations $\langle\langle A, X\rangle\rangle = a$, $\langle\langle B, X\rangle\rangle = b$ in the space of symmetric 2×2 real matrices is an affine manifold of dimension at least one, which intersects the closed convex cone of psd matrices. Since this cone is pointed (i.e., it contains no straight line), M intersects its boundary. Given that all boundary points of this cone are of rank one, that is, of the type xx^T, with $x \in \mathbb{R}^2$, it follows that $(a, b) \in K$. This proves the inclusion $\supseteq$ in (1). □

Using this fact we will prove the following corollary:

Corollary 3.2. *Let A and B be two symmetric $n\times n$ real matrices,*

$$K := \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \ : \ x \in \mathbb{R}^n\},$$

and denote by clK the closure of K. Then

$$clK = \{(a, b) \ : \ \alpha a + \beta b \geq 0 \qquad \forall\ \alpha, \beta \in \mathbb{R} \text{ such that } \alpha A + \beta B \text{ is psd}\}.$$

Proof. Since K is a convex cone, one has $clK = (K^+)^+$, the superscript $^+$ denoting dual cone. So the statement follows by observing that $K^+ = \{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \alpha A + \beta B$ is positive semidefinite$\}$. □

Lemma 3.3. *Let C be a compact subset of $\mathbb{R}^2$. If $0 \notin C$ then the convex cone K generated by C is closed.*

Proof. Since K is the cone generated by the compact convex set coC, the convex hull of C, in case $0 \notin coC$ the closedness of K immediately follows. Assume now that $0 \in coC$. Then, since $0 \notin C$, 0 is not an extreme point of coC. This means that 0 belongs to the relative interior of a segment contained in coC, which clearly implies that K contains a straight line through the origin. It only remains to observe that a convex cone in $\mathbb{R}^2$ containing a straight line through the origin is either the line itself, or a closed halfplane, or the whole $\mathbb{R}^2$. □

We remark that the preceding lemma does not hold in higher dimensions, as shown by the compact set $\left\{(x,y,z)\in\mathbb{R}^3 \ : \ |x|=1,\ y^2+(z-1)^2=1\right\}$, which generates the convex cone $\left\{(x,y,z)\in\mathbb{R}^3 \ : \ \text{either } z>0 \text{ or } y^2+z^2=0\right\}$.

Corollary 3.4. *Let A and B be two symmetric $n\times n$ real matrices, $a,b\in\mathbb{R}$, and assume that the homogeneous system of quadratic equations $\langle Ax,x\rangle=0$, $\langle Bx,x\rangle=0$ has no solution other than $x=0$. Then the system $\langle Ax,x\rangle=a$, $\langle Bx,x\rangle=b$ has a solution if and only if $\alpha a+\beta b\geq 0$ for every $\alpha,\beta\in\mathbb{R}$ such that the matrix $\alpha A+\beta B$ is positive semidefinite.*

Proof. Let K be as in Corollary 3.2. The assumption on the homogeneous system means that the origin of $\mathbb{R}^2$ does not belong to the compact set $C:=\{(\langle Ax,x\rangle,\langle Bx,x\rangle) \ : \ x\in\mathbb{R}^n, \|x\|=1\}$; hence K, being the convex cone generated by C, is closed in view of Lemma 3.3. Therefore the statement follows from Corollary 3.2 together with the observation that the system $\langle Ax,x\rangle=a$, $\langle Bx,x\rangle=b$ has a solution if and only if $(a,b)\in K$. $\square$

To see that the assumption on the homogeneous system is not superfluous in the preceding corollary, consider the 2×2 matrices $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $B=\begin{pmatrix}2&-1\\-1&0\end{pmatrix}$. Then the matrix $\alpha A+\beta B=\begin{pmatrix}\alpha+2\beta&-\beta\\-\beta&-\alpha\end{pmatrix}$ is positive semidefinite if and only if $\alpha+\beta=0\leq\beta$. Thus $(a,b)=(1,1)$ satisfies the inequalities in the statement; however the system $\langle Ax,x\rangle=1$, $\langle Bx,x\rangle=1$ has no solution. Indeed, if $x_1,x_2\in\mathbb{R}$ are such that $x_1^2-x_2^2=1$ and $2x_1^2-2x_1x_2=1$, one has $x_1^2-x_2^2=2x_1^2-2x_1x_2$, so that $x_1=x_2$, which is impossible. Note that $x=(1,1)$ is a solution of the homogeneous system.

The condition that the homogeneous system has no nonzero solution was proved by Finsler [4] (see also [5, p. 173]) to be equivalent, in the case $n\geq 3$, to the existence of real numbers α and β such that $\alpha A+\beta B$ is positive definite.

The convexity of the cone $K:=\{(\langle Ax,x\rangle,\langle Bx,x\rangle) \ : \ x\in\mathbb{R}^n\}$, together with the fact that it is closed if the system $\langle Ax,x\rangle=0$, $\langle Bx,x\rangle=0$ has no solution other than $x=0$, was proved by Dines [3] (see also [5, p. 171]). Here we have obtained these results as direct consequences of Brickman's theorem. In [3] it was also proved that, in the case the homogeneous system above has no nonzero solution, K is either the whole $\mathbb{R}^2$ or a closed convex cone different from a halfplane. It can be easily seen that the above proof yields the same conclusion if $n\neq 2$.

An alternative proof of Brickman's theorem has been given by Beckermann [1].

Acknowledgements. I am grateful to J.-B. Hiriart-Urruty for his interesting comments and suggestions and to M. A. López for his remarks on an earlier version, which have been very helpful for improving the presentation.

References

[1] B. Beckermann: On the characterization of the numerical range, private communication from B. Beckermann to J.-B. Hiriart-Urruty (fall 2001).

[2] L. Brickman: On the field of values of a matrix, Proc. Am. Math. Soc. 12 (1961) 61–66.

[3] L. L. Dines: On the mapping of quadratic forms, Bull. Am. Math. Soc. 47 (1941) 494–498.

[4] P. Finsler: Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen, Comment. Math. Helv. 9 (1937) 188–192.

[5] J.-B. Hiriart-Urruty, M. Torki: Permanently going back and forth between the "quadratic world" and the "convexity world" in optimization, Appl. Math. Opt. 45(2) (2002) 169–184.