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Exceptional Sets in Convex Domains

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Assume that Ω is a strongly convex domain, balanced with boundary of class C^1. Fix number $p \geq 1$. For any set E which is circular and of type G_δ in $\partial\Omega$ we find a holomorphic function $f \in \mathbb{O}(\Omega)$ such that

$$E = E^p_\Omega(f) = \left\{ z \in \partial\Omega : \int_{|\lambda|<1} |f(\lambda z)|^p \, d\mathfrak{L}^2(\lambda) = \infty \right\}.$$

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1. Preface

The main topic of this paper is centered around the question:

What features of the function f can be recovered from a given collection of line integrals of f? This mathematical problem is encountered in a growing number of diverse settings in medicine, science, and technology ranging from the famous application in diagnostic radiology to research in quantum optics. Especially this issue is often discussed in computed tomography.

This paper deals with domain Ω which is balanced with boundary of class C^1.

Definition. We define p-exceptional set for the holomorphic function $f \in \mathbb{O}(\Omega)$ as:

$$E^p_\Omega(f) = \left\{ z \in \partial\Omega : \int_{|\lambda|<1} |f(\lambda z)|^p \, d\mathfrak{L}^2(\lambda) = \infty \right\}.$$

The above definition was inspired by the questions posed by Peter Pflug and Jacques Chaumat.

In the 1980s Peter Pflug posed a question whether there exists a domain $\Omega \subset \mathbb{C}^n$, a complex subspace M in $\mathbb{C}^n$ and a function f holomorphic in Ω, square-integrable such that $f|_{M\cap\Omega}$ is not square-integrable.

A similar question was posed by Jaques Chaumat in the late 1980s; he wondered whether there exists a function f holomorphic in the unit ball $\mathbb{B}^n$ such that for any subspace which is linear and complex M in $\mathbb{C}^n$ the function $f|_{M\cap\mathbb{B}^n}$ is not square-integrable.

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We can find in the literature many papers [1, 2, 3, 4, 5, 6, 7] inspired by the above questions. Papers [2, 3] deal with the domains $\Omega \subset \mathbb{C}^{n+m}$ and holomorphic functions $f \in \mathbb{O}(\Omega) \cap L^2(\Omega)$, non-integrable along directions of the form

$$\Omega_w := \{z \in \mathbb{C}^n : (z, w) \in \Omega\}.$$

In this case we define exceptional set $\widetilde{E}(\Omega, f)$ as

$$\widetilde{E}(\Omega, f) = \{w \in \mathbb{C}^m : f|_{\Omega_w} \notin L^2(\Omega_w)\}.$$

It is also possible to consider the functions which are non-integrable along complex lines containing 0. Among the papers dealing with this problem the following [1, 5, 4, 6, 7] should be mentioned. Due to [1, 5] we know that for a convex domain Ω with a boundary of class C^1 it is possible to construct a holomorphic function f, which is not square-integrable along any real manifold M of class C^1 intersecting the boundary of Ω transversally. Constructions of the holomorphic functions non-integrable along selected in advance set of complex directions of type G_δ and F_σ with some additional properties were presented in the paper [7] . In [7] we used some properties of Wojtaszczyk polynomials. As we know no similar construction for the domains different than a ball, then in this paper we use a different method.

Definition. Let $D \subset \partial\Omega$. We say that the domain Ω is strongly convex[1] according to D iff for every point $y \in D$ there exists a real hyperplane Θ_y such that $\Theta_y \cap \overline{\Omega} = \{y\}$. Additionally Ω is strongly convex iff Ω is strongly convex according to $\partial\Omega$.

The most important result obtained in our paper is the following:

Theorem 2.4. Let Ω be a strongly convex domain according to $D = \overline{D} \subset \partial\Omega$, whose boundary is of class C^1 in a neighbourhood of D. Then there exists a function $f \in \mathbb{O}(\Omega) \cap C(\overline{\Omega} \setminus D)$ such that for any k-dimensional, complex submanifold $M \subset \mathbb{C}^n$ intersecting $\partial\Omega$ transversally: one has $\int_{M\cap\Omega} |f| d\mathfrak{L}^{2k}_{M\cap\Omega} = \infty$ whenever $M \cap D \neq \emptyset$.

As a consequence of that theorem we have:

Theorem 3.2. If Ω is a strongly convex domain with boundary of class C^1 and E is a circular set of type G_δ in $\partial\Omega$, then there exists a holomorphic function $f \in \mathbb{O}(\Omega)$ such that $E = E^p_\Omega(f)$.

2. Convex domains

For $z, w \in \mathbb{C}^n$ and for a non-empty set D such that $D \subset \mathbb{C}^n$, we put

$$\begin{aligned} L_{z,w} &:= \{tz + (1-t)w : t \in \mathbb{R}\}, \\ d_D(z) &:= \inf_{w \in D} \|z - w\|. \end{aligned}$$

[1]In the literature the following definition of strong convexity can be found:
If $\Omega \subset\subset \mathbb{R}^n$ is a domain with a boundary of class C^2 and ρ is a defining function for Ω such that in the point $P \in \partial\Omega$ we have $\sum \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k > 0$ for $0 \neq w \in T_P(\partial\Omega)$, then we say that Ω is strongly convex in the point P.
It can be proved that if Ω is strongly convex in the point P, then there exists an open set U_P such that $T_P(\partial\Omega) \cap U_P \cap \overline{\Omega} = \{P\}$. Considering this property we introduce a geometric definition of strong convexity that can be more easily used in our paper.

For any $\varepsilon > 0$, any non-empty open set $\Omega \subset \mathbb{C}^n$, and for any non-empty closed set $D \subset \mathbb{C}^n$, we denote

$$\begin{aligned}\Omega_\varepsilon &:= \{z \in \Omega : \varepsilon \|z\| < 1, d_{\partial\Omega}(z) > \varepsilon\}\\ K(D;\varepsilon) &:= \{z \in \mathbb{C}^n : d_D(z) < \varepsilon\},\\ K(\emptyset;\varepsilon) &:= \emptyset.\end{aligned}$$

Moreover for $D, T \subset \overline{\Omega}$, let $\Gamma(D,T)$ be the set of all continuous functions $\gamma \in C\left([0,1],\overline{\Omega}\right)$ such that $\gamma(0) \in D$, $\gamma(1) \in T$ and $L_{\gamma(s),\gamma(t)} \cap T \neq \emptyset$ for all $0 \leq s < t \leq 1$.

Proposition 2.1. *Let $\varepsilon > 0$. If $D \subset \partial\Omega$ is a compact set and Ω is a strongly convex according to D, then there exists a finite family $\{\Theta_i\}_{i=1,\ldots,m}$ of real hyperplanes in $\mathbb{C}^n$ and compact sets $K_i \subset \Theta_i \cap \Omega$ such that:*

(1) $\Theta_i \cap \overline{\Omega} \subset K(D;\varepsilon)$ *for all* $i = 1, ..., m$*;*
(2) $\bigcup_{i=1}^m K_i \cap \gamma([0,1]) \neq \emptyset$ *for any* $\gamma \in \Gamma(D, \overline{\Omega}_\varepsilon)$.

Proof. For any $x \in \partial\Omega$ and $r > 0$, put

$$S_{x,r} := \{z \in \partial\Omega : \|z - x\| \leq r\}.$$

We proved in two steps. In the first step, we show that for any $y \in D$ there exists a real hyperplane Q_y, a number $\eta_y > 0$ and a compact set T_y such that:

(i) $Q_y \cap \overline{\Omega} \subset K(D,\varepsilon)$
(ii) $T_y \subset Q_y \cap \Omega$,
(iii) if $\gamma \in \Gamma(S_{y,\eta_y}, \overline{\Omega}_\varepsilon)$, then $\gamma([0,1]) \cap T_y \neq \emptyset$.

Choose a real hyperplane H such that $H \cap \overline{\Omega} = \{y\}$ by strong convexity of Ω and choose a vector $\overrightarrow{w}$ normal to H with a beginning in the point y and pointed outside Ω. Let $H_\delta := H - \delta\overrightarrow{w}$ and

$$K_{\delta,\eta} := H_\delta \cap \bigcup \Gamma(S_{y,\eta}, \overline{\Omega}_\varepsilon)$$

for $\delta, \eta > 0$. As $S_{y,\eta}$ and $\overline{\Omega}_\varepsilon$ are compact sets, $K_{\delta,\eta}$ is also a compact set. Due to the equality $H_0 \cap \overline{\Omega} = \{y\}$, a number δ may be selected so small that $H_\delta \cap \overline{\Omega} \subset K(D,\varepsilon)$ and $H_\delta \cap \overline{\Omega}_\varepsilon = \emptyset$. Additionally, as for small δ we have

$$H_\delta \cap \bigcup_{w \in \overline{\Omega}_\varepsilon} L_{y,w} \subset \Omega,$$

if necessary we can select smaller $\delta > 0$ and a number $\eta > 0$ so small that

$$H_\delta \cap \bigcup_{(x,w) \in S_{y,\eta} \times \overline{\Omega}_\varepsilon} L_{x,w} \subset \Omega \tag{1}$$

and the sets $S_{y,\eta}$, $\overline{\Omega}_\varepsilon$ lie on the opposite sides of the hyperplane H_δ.

Let $Q_y := H_\delta$, $T_y := K_{\delta,\eta}$, $\eta_y = \eta$. We observe that

$$Q_y \cap \overline{\Omega} = H_\delta \cap \overline{\Omega} \subset K(D,\varepsilon),$$

which yields the property (i).

It can be easily observed that

$$\bigcup \Gamma(S_{y,\eta}, \overline{\Omega}_\varepsilon) \subset \bigcup_{(x,w)\in S_{y,\eta}\times\overline{\Omega}_\varepsilon} L_{x,w}.$$

In particular, due to (1), we have the property (ii):

$$T_y = Q_y \cap \bigcup \Gamma(S_{y,\eta_y}, \overline{\Omega}_\varepsilon) \subset H_\delta \cap \bigcup_{(x,w)\in S_{y,\eta_y}\times\overline{\Omega}_\varepsilon} L_{x,w} \subset H_\delta \cap \Omega = Q_y \cap \Omega.$$

Let $\gamma \in \Gamma(S_{y,\eta_y}, \overline{\Omega}_\varepsilon)$. As the sets S_{y,η_y} and $\overline{\Omega}_\varepsilon$ lie on the opposite sides of the hyperplane Q_y, we have $\gamma([0,1]) \cap Q_y \neq \emptyset$. In particular, the property (iii) follows from:

$$\gamma([0,1]) \cap Q_y \subset Q_y \cap \bigcup \Gamma(S_{y,\eta_y}, \overline{\Omega}_\varepsilon) = T_y.$$

In the second step, we proceed to prove the existence of hyperplanes Θ_i and compact sets K_i such that the properties (1)-(2) are fulfilled.

Observe that

$$D \subset \bigcup_{x\in D} \{z \in \partial\Omega : \ \|z - x\| < \eta_x\},$$

where the number η_x was defined in the first step. As D is a compact set, there exist points $x_1, ..., x_m$ such that

$$D \subset \bigcup_{i=1,...,m} S_{x_i,\eta_{x_i}}.$$

It suffices to define $\Theta_i := Q_{x_i}$ and $K_i := T_{x_i}$. Obviously $K_i \subset \Theta_i \cap \Omega$ and $\Theta_i \cap \overline{\Omega} \subset K(D; \varepsilon)$. Moreover, if $\gamma \in \Gamma(D, \overline{\Omega}_\varepsilon)$, then $\gamma(0) \in D$. Therefore, there exists $j \in \{1, ..., m\}$ such that $\gamma(0) \in S_{x_j,\eta_{x_j}}$. In particular $\gamma \in \Gamma(S_{x_j,\eta_{x_j}}, \overline{\Omega}_\varepsilon)$ and $\gamma([0,1]) \cap K_j \neq \emptyset$. $\square$

Lemma 2.2. *Let $\varepsilon > 0$. Let also $\{\Theta_i\}_{i=1,...,m}$ - be a finite family of real hyperplanes in $\mathbb{C}^n$ such that $\Theta_i \cap \Omega_\varepsilon = \emptyset$. Let $U \subset \Omega$ be an open set. If $\{K_i\}_{i=1,...,m}$ is a finite family of compact sets such that $K_i \subset U \cap \Theta_i$ for $i = 1, ..., m$, then there exists a finite family of compact sets $\{T_i\}_{i=1,...,m}$ and a number $\eta > 0$ such that*

(1) *$K_i \subset T_i \subset U \cap \Theta_i$ for $i = 1, ..., m$;*
(2) *if $\gamma \in \Gamma(\partial\Omega, \overline{\Omega}_{2\varepsilon})$ is such that $\gamma([0,1]) \cap T_i \neq \emptyset$ and $\gamma([0,1]) \cap T_j = \emptyset$ for some $i, j \in \{1, ..., m\}$ with $i < j$, then for every $x \in \gamma([0,1]) \cap T_i$ the inequality $d_{\Theta_j}(x) \geq \eta$ holds.*

Proof. Fix any number $\mu > 0$ and define $T_1 := K_1$ and

$$S_1 := \overline{K(T_1 \cap \Theta_1; \mu)}.$$

Having defined T_i, S_i we define

$$\begin{aligned} S_{i+1} &:= \overline{K\left(\bigcup_{j=1}^{i} (T_j \cap \Theta_{i+1}); \mu\right)} \\ T_{i+1} &:= (S_{i+1} \cup K_{i+1}) \cap \Theta_{i+1}. \end{aligned}$$

Observe that the number μ can be so small that S_i, $T_i \subset U$. It can be easily seen that $K_i \subset T_i \subset U \cap \Theta_i$ for $i = 1, ..., m$.

Fix $i \in \{1, ..., m-1\}$ and $j \in \{i+1, ..., m\}$.

Assume that we have a sequence of curves $\{\gamma_n\}_{n\in\mathbb{N}}$ and points $\{x_n\}_{n\in\mathbb{N}}$ such that:

- $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma(\partial\Omega, \overline{\Omega}_{2\varepsilon})$,
- $\gamma_n([0,1]) \cap T_j = \emptyset$,
- $x_n \in \gamma_n([0,1]) \cap T_i$.

Let $\widetilde{\Theta}_n$ be a hyperplane parallel to Θ_j such that $x_n \in \widetilde{\Theta}_n$. It suffices to show that $d_{\Theta_j}(x_n) > \eta > 0$ for some constant η independent of the sequence selection $\{\gamma_n\}_{n\in\mathbb{N}}$.

We show this fact by considering proper cases. First we show the following fact:

(A) If $\lim_{n\to\infty} d_{\Theta_j}(x_n) = 0$, then $\gamma_n([0,1]) \cap \Theta_j = \emptyset$ for n sufficiently large.

Assume that there exists a sequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ such that $z_{n_k} \in \gamma_{n_k}([0,1]) \cap \Theta_j$. We may assume[2] that $x_{n_k} \to x_0 \in T_i \cap \Theta_j$. From the definition of the sets $T_1, ..., T_m$ it follows that $\overline{K(T_i;\mu)} \cap \Theta_j \subset T_j$. As $\overline{K(\{x_0\};\mu)} \cap \Theta_j \subset T_j$ and $\gamma_n([0,1]) \cap T_j = \emptyset$ one has $\|z_{n_k} - x_0\| \geq \mu > 0$. On the other hand, due to the equality $\Theta_j \cap \Omega_\varepsilon = \emptyset$, we must have

$$L_{z_{n_k}, x_{n_k}} \cap \overline{\Omega}_{2\varepsilon} = \emptyset$$

for k sufficiently large, which is inconsistent with the choice of γ_n.

Now let us show the second fact:

(B) If $\lim_{n\to\infty} d_{\Theta_j}(x_n) = 0$, then the points $\gamma_n(0)$, $\gamma_n(1)$ lie on different sides of the hyperplane[3] $\widetilde{\Theta}_n$ for n sufficiently large.

Assume that $\gamma_{n_k}(1)$ lie on the same side of $\widetilde{\Theta}_{n_k}$. We may assume that $\{n_k\}_{k\in\mathbb{N}} = \mathbb{N}$ and due to (A): $\gamma_n([0,1]) \cap \Theta_j = \emptyset$ for $n \in \mathbb{N}$.

There exist then different points z_n, w_n such that $z_n, w_n \in \gamma_n([0,1])$,

$$\lim_{n\to\infty} d_{\Theta_j}(z_n) = \lim_{n\to\infty} d_{\Theta_j}(w_n) = 0$$

and the line L_{z_n,w_n} is parallel to the hyperplane Θ_j. As $\Theta_j \cap \Omega_\varepsilon = \emptyset$, we have $L_{z_n,w_n} \cap \overline{\Omega}_{2\varepsilon} = \emptyset$ for n sufficiently large, which is inconsistent with the choice of γ_n.

Now let us show the following fact:

(C) If $\lim_{n\to\infty} d_{\Theta_j}(x_n) = 0$, then the points $\gamma_n(0)$, $\gamma_n(1)$ lie on the same side of the hyperplane[4] $\widetilde{\Theta}_n$ for n sufficiently large.

Assume that $\gamma_{n_k}(0)$ and $\gamma_{n_k}(1)$ lie on opposite sides of the hyperplane $\widetilde{\Theta}_{n_k}$. We may assume that $\{n_k\}_{k\in\mathbb{N}} = \mathbb{N}$ and due to (A): $\gamma_n([0,1]) \cap \Theta_j = \emptyset$ for $n \in \mathbb{N}$.

Observe that if the point z_n lies between the hyperplanes Θ_j and $\widetilde{\Theta}_n$, then

$$0 \leq \lim_{n\to\infty} d_{\Theta_j}(z_n) \leq \lim_{n\to\infty} d_{\Theta_j}(x_n) = 0.$$

[2] We select a subsequence if necessary.

[3] Obviously, in this case $\gamma_n(0) \notin \widetilde{\Theta}_n$ and $\gamma_n(1) \notin \widetilde{\Theta}_n$.

[4] In particular, we can have $\gamma_n(0) \in \widetilde{\Theta}_n$ or $\gamma_n(1) \in \widetilde{\Theta}_n$.

Therefore, as $\Theta_j \cap \Omega_\varepsilon = \emptyset$ and $\gamma_n(1) \in \overline{\Omega}_{2\varepsilon} \subset \Omega_\varepsilon$, $\gamma_n(1)$ cannot lie between the hyperplanes Θ_j and $\widetilde{\Theta}_n$ for n sufficiently large. As $\gamma_n([0,1]) \cap \Theta_j = \emptyset$ the points $\gamma_n(0)$ and $\gamma_n(1)$ lie on the same side of the hyperplane Θ_j. In particular, $\gamma_n(0)$ has to lie between the hyperplanes Θ_j and $\widetilde{\Theta}_n$ for n sufficiently large. Therefore

$$0 \leq \lim_{n\to\infty} d_{\Theta_j}(\gamma_n(0)) \leq \lim_{n\to\infty} d_{\Theta_j}(x_n) = 0.$$

As $x_0 \in \Theta_j \cap \Omega$ and $\gamma_n(0) \in \partial\Omega$, for n sufficiently large we have $L_{\gamma_n(0),x_n} \cap \overline{\Omega}_{2\varepsilon} = \emptyset$, which is inconsistent with the choice of γ_n.

It remains to establish the last fact:

(D) There exists a constant $\eta > 0$ such that if $\gamma \in \Gamma(\partial\Omega, \overline{\Omega}_{2\varepsilon})$, $\gamma([0,1]) \cap T_i \neq \emptyset$, $\gamma([0,1]) \cap T_j = \emptyset$, and $x \in \gamma([0,1]) \cap T_i$, then $d_{\Theta_j}(x) \geq \eta$.

Assume that such a constant η does not exist. We can then select a sequence of curves $\{\gamma_n\}_{n\in\mathbb{N}}$ and points $\{x_n\}_{n\in\mathbb{N}}$ such that:

- $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma(\partial\Omega, \overline{\Omega}_{2\varepsilon})$,
- $\gamma_n([0,1]) \cap T_j = \emptyset$,
- $x_n \in \gamma_n([0,1]) \cap T_i$,

and $\lim_{n\to\infty} d_{\Theta_j}(x_n) = 0$. By (B) and (C) above, we obtain a construction. □

Theorem 2.3. *Let D be a compact subset of $\partial\Omega$. Assume that Ω is strongly convex according to D. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all m, $M > 0$, one can select a holomorphic function $h \in \mathbb{O}(\mathbb{C}^n)$ with the following properties:*

(1) *$|h(z)| \leq m$ for all $z \in \overline{\Omega} \setminus K(D;\varepsilon)$.*
(2) *If $\gamma \in \Gamma(D, \overline{\Omega}_{2\varepsilon})$, then there exists $x \in \gamma([0,1])$ such that $d_{\partial\Omega}(x) \geq \delta$ and $|h(x)| \geq M$.*

Proof. On the basis of Proposition 2.1 and Lemma 2.2, we can select a finite family of real hyperplanes $\{\Theta_i\}_{i=1,...,k}$ in $\mathbb{C}^n$, compact sets T_i and a number $\eta > 0$ with the following properties:

(1) $T_i \subset \Theta_i \cap \Omega$;
(2) $\Theta_i \cap \overline{\Omega} \subset K(D;\varepsilon)$ for all $i = 1, ..., k$;
(3) $\bigcup_{i=1}^k T_i \cap \gamma \neq \emptyset$ for all $\gamma \in \Gamma(D, \overline{\Omega}_\varepsilon)$;
(4) if $\gamma \in \Gamma(D, \overline{\Omega}_{2\varepsilon})$ is such that $\gamma([0,1]) \cap T_i \neq \emptyset$ and $\gamma([0,1]) \cap T_j = \emptyset$ for some $i, j \in \{1, ..., k\}$ such that $i < j$, then for every $x \in \gamma([0,1]) \cap T_i$ we have the inequality $d_{\Theta_j}(x) \geq \eta$.

We may assume that ε is so small that $0 \in \Omega_{2\varepsilon}$. We may also assume that $\eta \in (0,1)$. Let $S = \bigcup \Gamma(D, \overline{\Omega}_{2\varepsilon})$. Because D and $\overline{\Omega_{2\varepsilon}}$ are compact sets, S is a compact set.

The real hyperplane Θ_i can be represented in the form of

$$\Theta_i := \{z \in \mathbb{C}^n : \operatorname{Re}(\langle z - z_i, z_i\rangle) = 0\} \tag{2}$$

for some $z_i \in \mathbb{C}^n \setminus \{0\}$. For any $\delta \in (0,1)$ we define

$$V_{i,\delta} := \left\{z \in \mathbb{C}^n : \operatorname{Re}(\langle z - z_i, z_i\rangle) \leq -\delta \left\|z_i\right\|^2\right\}. \tag{3}$$

Let us consider $\delta \in (0,1)$ so small that $0 < \delta \|z_i\| < \eta$ for $i = 1, ..., k$ and $d_{\partial\Omega}(z) \geq \delta$ for all $z \in \bigcup_{i=1}^k T_i$. As $\Theta_i \cap \overline{\Omega} \subset K(D, \varepsilon)$, making $\delta > 0$ smaller, we may assume that $\overline{\Omega} \setminus K(D, \varepsilon) \subset V_{i,\delta}$.

A) We have

$$V_{i,\delta} = \{z \in \mathbb{C}^n : \operatorname{Re}(\langle z - z_i, z_i \rangle) \leq 0, d_{\Theta_i}(z) \geq \delta \|z_i\|\}.$$

Indeed let

$$Q_{i,\delta} = \{z \in \mathbb{C}^n : \operatorname{Re}(\langle z - (1-\delta) z_i, z_i \rangle) = 0\}.$$

Observe that $V_{i,\delta} = \{z \in \mathbb{C}^n : \operatorname{Re}(\langle z - (1-\delta) z_i, z_i \rangle) \leq 0\}$. Then it is enough to note that distance between Θ_i and $Q_{i,\delta}$ is equal to $\|z_i - (1-\delta) z_i\| = \delta \|z_i\|$.

Now we claim that:

B) For all $m, M > 0$ there exists a function $h \in \mathbb{O}(\mathbb{C}^n)$ with the following properties:

(1) $|h(z)| \leq m$ for all $z \in \overline{\Omega} \setminus K(D; \varepsilon)$.
(2) If $\gamma \in \Gamma(D, \overline{\Omega}_{2\varepsilon})$, then there exists $x \in \gamma([0,1])$ such that $d_{\partial\Omega}(x) \geq \delta$ and $|h(x)| \geq M$.

Let us consider function

$$h_i(z) := b_i \exp(a_i \langle z - z_i, z_i \rangle)$$

for $z \in \mathbb{C}^n$ and $a_i, b_i > 0$. Due to (2), we have $|h_i(z)| = b_i$ for $z \in \Theta_i$.

We select a_i and b_i by means of induction. Let $b_1 = M + m$. If we have already defined b_i, then due to (3) we can define a_i as a positive number so large that

$$|h_i(z)| \leq \frac{m}{2^i} \text{ for all } z \in V_{i,\delta}.$$

Number b_{i+1} is defined by the formula:

$$b_{i+1} := M + m + \sum_{j=1}^{i} \sup_{z \in S} |h_j(z)|$$

for all $i = 1, ..., k$.

We define

$$h(z) := \sum_{i=1}^{k} h_i(z).$$

As $\overline{\Omega} \setminus K(D; \varepsilon) \subset V_{i,\delta}$, we have $|h(z)| \leq m$ for all $z \in \overline{\Omega} \setminus K(D; \varepsilon)$.

Let $\gamma \in \Gamma(D, \overline{\Omega}_{2\varepsilon})$. There exists $i \in \{1, ..., k\}$ such that $\gamma([0,1]) \cap T_i \neq \emptyset$. After having increased the index i, we may assume that $\gamma([0,1]) \cap T_j = \emptyset$ for all $j = i+1, ..., k$. Fix $x \in \gamma([0,1]) \cap T_i$. We have $d_{\Theta_j}(x) \geq \eta$ for all $j = i+1, ..., k$. As $x \in T_i \subset \Theta_i$, we have $\operatorname{Re}(\langle x - z_i, z_i \rangle) = 0$ and $d_{\partial\Omega}(x) \geq \delta$ (δ is so small that $d_{\partial\Omega}(z) \geq \delta$ for all$z \in \bigcup_{j=1}^k T_j$) . Moreover we can select maximally possible $\nu \in \{i, i+1, ..., k\}$ such that

$$\operatorname{Re}(\langle x - z_\nu, z_\nu \rangle) \geq 0.$$

If $v = k$, then (note that $x \in S = \bigcup \Gamma(D, \overline{\Omega}_{2\varepsilon})$):

$$\begin{aligned}|h(x)| &\geq |h_k(x)| - \sum_{j=1}^{k-1} |h_j(x)| \\ &\geq m + M + \sum_{j=1}^{k-1} \sup_{z\in S} |h_j(z)| - \sum_{j=1}^{k-1} |h_j(x)| \\ &\geq m + M.\end{aligned}$$

If $v < k$, then $\mathrm{Re}(\langle x - z_j, z_j\rangle) < 0$ and $d_{\Theta_j}(x) \geq \eta \geq \delta \|z_j\|$ for $j = \nu + 1, ..., k$. Therefore due to (A) we have $x \in V_{j,\delta}$ for all $j = \nu + 1, ..., k$. Finally we can estimate:

$$\begin{aligned}|h(x)| &\geq |h_\nu(x)| - \sum_{j=1}^{\nu-1} |h_j(x)| - \sum_{j=\nu+1}^{k} |h_j(x)| \\ &\geq m + M + \sum_{j=1}^{\nu-1} \sup_{z\in S} |h_j(z)| - \sum_{j=1}^{\nu-1} |h_j(x)| - \sum_{j=\nu+1}^{k} |h_j(x)| \\ &\geq m + M - \sum_{j=\nu+1}^{k} \frac{m}{2^j} \geq M.\end{aligned}$$

□

Theorem 2.4. *Let Ω be a strongly convex domain according to $D = \overline{D} \subset \partial\Omega$, whose boundary is of class C^1 in a neighbourhood of D. Then there exists a function $f \in \mathbb{O}(\Omega) \cap C(\overline{\Omega} \setminus D)$ such that for any k-dimensional, complex submanifold $M \subset \mathbb{C}^n$ intersecting $\partial\Omega$ transversally: one has $\int_{M\cap\Omega} |f| d\mathfrak{L}^{2k}_{M\cap\Omega} = \infty$ whenever $M \cap D \neq \emptyset$.*

Proof. Let $D = \bigcup_{i\in\mathbb{N}} D_i$, where $\{D_i\}_{i\in\mathbb{N}}$ is a sequence of compact sets such that $D_i \subset D_{i+1}$. If $f \in L^1(M \cap \Omega)$, then we denote a proper norm as $\|f\|_{M,1} = \int_{M\cap\Omega} |f| d\mathfrak{L}^{2k}_{M\cap\Omega}$. We present the proof in a few steps.

First we show:

(A) Let $P \in M \cap \partial\Omega$. There exists a constant $C_M > 0$ and an open set U_P such that $P \in U_P$ with the property: if $f \in \mathbb{O}(\Omega)$ and $\int_{M\cap\Omega} |f| d\mathfrak{L}^{2k}_{M\cap\Omega} < \infty$, then

$$|f(z)| \leq \frac{C_M \|f\|_{M,1}}{d_{\partial\Omega}(z)^k}$$

for $z \in M \cap \Omega \cap U_P$.

Observe that M is locally the graph of a holomorphic function. Without loss of generality, we may assume that $P = 0$. In particular:

$$M \cap U = \{(w, g(w)) :\ w \in V\}$$

where V, U are open sets such that $\overline{U}$, $\overline{V}$ are compact, $0 \in V \subset \mathbb{C}^k$, $0 \in U \subset \mathbb{C}^n$ and g is a holomorphic function on V with the values in $\mathbb{C}^{n-k}$, such that $g(0) = 0$. Let

$$\Omega_V := \{w \in V :\ (w, g(w)) \in \Omega\}.$$

Shrinking the sets V, U, we may assume that there exists a constant C_1 such that for any $\eta, \xi \in V$:

$$\|(\eta, g(\eta)) - (\xi, g(\xi))\| \leq C_1 \|\eta - \xi\|.$$

There exists an open set U_0 such that $0 \in U_0 \subset U$ and

$$\inf_{\eta \in \partial\Omega_V} \|(\eta, g(\eta)) - z\| \geq d_{\partial\Omega}(z)$$

for $z \in M \cap U_0$.

Using the Cauchy integral formula on polydisc for the function

$$\psi : V \ni w \to f(w, g(w)) \in \mathbb{C}^n,$$

we have the inequality:

$$|f(w, g(w))| \leq \frac{C_2 \|f\|_{M,1}}{d_{\partial\Omega_V}(w)^k}$$

for $w \in \Omega_V$, where the constant C_2 depends only on the manifold M.

In particular, if $z \in M \cap \Omega \cap U_0$, then there exists $w \in \Omega_V$ such that $z = (w, g(w))$. It follows that for all $z \in M \cap \Omega \cap U_0$ one has:

$$|f(z)| \leq \frac{C_2 \|f\|_{M,1}}{d_{\partial\Omega_V}(w)^k} \leq \frac{C_1^k C_2 \|f\|_{M,1}}{\inf_{\eta \in \partial\Omega_V} \|(\eta, g(\eta)) - z\|^k} \leq \frac{C_1^k C_2 \|f\|_{M,1}}{d_{\partial\Omega}(z)^k}.$$

Next we prove:

(B) There exists a sequence of holomorphic functions $\{f_i\}_{i\in\mathbb{N}} \subset \mathbb{O}(\mathbb{C}^n)$ and a sequence of positive, real numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that:
 (1) $0 < 2\varepsilon_{i+1} < \varepsilon_i$;
 (2) $|f_i(z)| \leq \frac{1}{2^i}$ for all $z \in \overline{\Omega} \setminus K(D_i; \varepsilon_i)$;
 (3) If $\gamma \in \Gamma(D_i, \overline{\Omega}_{2\varepsilon_i})$, then there exists $x \in \gamma([0,1])$ such that $d_{\partial\Omega}(x) \geq \varepsilon_{i+1}$ and

$$|f_i(x)| \geq \left(\frac{1}{\varepsilon_{i+1}}\right)^i + 1 + \sum_{j=1}^{i-1} \sup_{z\in S_i} |f_j(z)|,$$

 where $S_i := \bigcup \Gamma\left(D_i, \overline{\Omega}_{2\varepsilon_i}\right)$.

The sequences $\{f_i\}_{i\in\mathbb{N}}$, $\{\varepsilon_i\}_{i\in\mathbb{N}}$ are defined inductively. Let $\varepsilon_1 = 1$. If we have already defined a number ε_i, then on the basis of Theorem 2.3 we can select a number δ for $\varepsilon = \frac{1}{2}\varepsilon_i$. In this case we define $\varepsilon_{i+1} := \delta$. Let $m := 2^{-i}$ and

$$M := \left(\frac{1}{\varepsilon_{i+1}}\right)^i + 1 + \sum_{j=1}^{i-1} \sup_{z\in S_i} |f_j(z)|.$$

Use again Theorem 2.3 for the numbers m, M. In particular, there exists a holomorphic function $f_i \in \mathbb{O}(\mathbb{C}^n)$ such that the properties (2)-(3) are fulfilled.

Finally, we prove:

(C) Function $f(z) := \sum_{i=1}^{\infty} f_i(z)$, where a sequence $\{f_i\}_{i\in\mathbb{N}}$ was selected in (B) has the following properties:
  (1) $f \in \mathbb{O}(\Omega) \cap C(\overline{\Omega} \setminus D)$;
  (2) if M is a compact submanifold $\mathbb{C}^n$, intersecting $\partial\Omega$ transversally and $M \cap D \neq \emptyset$, then $\|f\|_{M,1} = \infty$.

Let K be a compact set in $\overline{\Omega} \setminus D$. As $0 < 2\varepsilon_{i+1} < \varepsilon_i$, one has $\lim_{i\to\infty} \varepsilon_i = 0$. Therefore, there exists a constant i_0 so large that $K \subset \overline{\Omega}_0 \setminus K(D; \varepsilon_j)$ for all $j > i_0$. As $f_i \in \mathbb{O}(\mathbb{C}^n)$ and

$$\sup_{z\in K} \sum_{i=j}^{\infty} |f_i(z)| \leq \sum_{i=j}^{\infty} \frac{1}{2^i} \leq \frac{2}{2^j}$$

for $j > i_0$, one has $f \in \mathbb{O}(\Omega) \cap C(\overline{\Omega} \setminus D)$.

Let M be a k-dimensional, compact submanifold $\mathbb{C}^n$. Assume that $M \cap D \neq \emptyset$ and

$$\|f\|_{M,1} = \int_{M\cap\Omega} |f| d\mathfrak{L}^{2k}_{M\cap\Omega} < \infty.$$

Let $P \in M \cap D$. As M intersects the boundary of Ω transversally in the point P, there exist an indicator i_1 and a curve $\gamma \in \Gamma\left(D_{i_1}, \overline{\Omega}_{2\varepsilon_{i_1}}\right)$ such that $\gamma(0) = P$ and $\gamma([0,1]) \subset M$. Observe that $\gamma \in \Gamma\left(D_i, \overline{\Omega}_{2\varepsilon_i}\right)$ for $i \geq i_1$. Due to the property (B)(3), there exists then a sequence of points $x_i \in \gamma([0,1])$ such that $d_{\partial\Omega}(x_i) \geq \varepsilon_{i+1}$ and

$$|f_i(x_i)| \geq \left(\frac{1}{\varepsilon_{i+1}}\right)^i + 1 + \sum_{j=1}^{i-1} \sup_{z\in S_i} |f_j(z)|$$

for all $i > i_1$. As $x_i \in \overline{\Omega} \setminus K(D; \varepsilon_j)$ for all $j \geq i+1$, one has

$$|f_j(x_i)| \leq \frac{1}{2^j}$$

for $j \geq i + 1 > i_1 + 1$. Observe that $x_i \in S_i$. Therefore the following estimation follows:

$$\begin{aligned}
|f(x_i)| &\geq |f_i(x_i)| - \sum_{j=1}^{i-1} |f_j(x_i)| - \sum_{j=i+1}^{\infty} |f_j(x_i)| \\
&\geq \left(\frac{1}{\varepsilon_{i+1}}\right)^i + 1 + \sum_{j=1}^{i-1} \sup_{z\in S_i} |f_j(z)| - \sum_{j=1}^{i-1} |f_j(x_i)| - \sum_{j=i+1}^{\infty} |f_j(x_i)| \\
&\geq \left(\frac{1}{\varepsilon_{i+1}}\right)^i + 1 - \sum_{j=i+1}^{\infty} \frac{1}{2^j} \geq \left(\frac{1}{d_{\partial\Omega}(x_i)}\right)^i.
\end{aligned}$$

Because of (A), we have an obvious contradiction.

We must therefore have $\int_{M\cap\Omega} |f| d\mathfrak{L}^{2k}_{M\cap\Omega} = \infty$. □

3. Exceptional sets

We present how to describe exceptional sets using Theorem 2.4.

To each pair (i,j) we assign a positive integer $\lfloor i,j \rfloor$ so that:

$$\lfloor i,j \rfloor < \lfloor k,l \rfloor \Leftrightarrow \begin{cases} i+j<k+l & \text{if} \quad i+j \neq k+l \\ i<k & \text{if} \quad i+j=k+l \end{cases}.$$

For $z \in \partial\Omega$ we denote

$$\mathbb{D}z = \{\lambda z : \; |\lambda| < 1\}.$$

Lemma 3.1. *If $E = \bigcap_{i\in\mathbb{N}} U_i \subset ... \subset U_{i+1} \subset U_i \subset ... \subset \partial\Omega$, where $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of open, circular sets in $\partial\Omega$, then there exist sequences of closed, circular sets $\{T_{i,j}\}_{i,j\in\mathbb{N}}$, $\{D_{i,j}\}_{i,j\in\mathbb{N}}$ with $T_{i,j} \cup D_{i,j} \subset \partial\Omega$ such that*

(1) $U_i = \bigcup_{j\in\mathbb{N}} T_{i,j}$ *for* $i \in \mathbb{N}$*;*
(2) $T_{i,j} \cap D_{i,j} = \emptyset$ *for* $i,j \in \mathbb{N}$*;*
(3) $\partial\Omega \setminus E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\lfloor i,j \rfloor \geq n} D_{i,j}$.

Proof. We denote:

$$\begin{aligned} T_{i,j} &:= \left\{ z \in U_i : \; \frac{1}{j+2} \leq \inf_{w\in\partial U_i} \|z-w\| \leq \frac{1}{j+1} \right\}, \\ D_{i,j} &:= \left\{ z \in \partial\Omega : \; \frac{1}{(j+3)^2} \leq \inf_{w\in T_{i,j}} \|z-w\| \right\}. \end{aligned}$$

Obviously the sets $T_{i,j}$ and $D_{i,j}$ are circular. Conditions (1) and (2) result directly from the definition. Moreover, it can also be easily noted that $\partial\Omega \setminus U_i \subset D_{i,j}$.

A) Note that $\|z-w\| \geq \frac{1}{(j+3)^2}$ for $z \in T_{i,j}$ and $w \in T_{i,k}$ when $k-j \geq 2$.

Assume that $z \in T_{i,j}$, $w \in T_{i,k}$ and $\|z-w\| < \frac{1}{(j+3)^2}$. In this case there exists $u \in \partial U_i$ such that $\|u-w\| \leq \frac{1}{k+1} \leq \frac{1}{j+3}$.

We can estimate

$$\frac{1}{j+2} \leq \|u-z\| \leq \|u-w\| + \|w-z\| < \frac{1}{j+3} + \frac{1}{(j+3)^2} \leq \frac{1}{j+2},$$

which is impossible.

B) Note that $T_{i,j} \subset D_{i,k}$ when $k-j \geq 2$.

Let $k \geq j+2$. If $x \in T_{i,j} \setminus D_{i,k}$, then there exists a point $y \in T_{i,k}$ such that $\|x-y\| < \frac{1}{(k+3)^2} \leq \frac{1}{(j+3)^2}$, which is impossible in reference to A).

C) We have the following property $\partial\Omega \setminus E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\lfloor i,j \rfloor \geq n} D_{i,j}$.

Indeed, fix $z \in \partial\Omega \setminus E$. If $z \notin U_0$, then $z \in D_{i,j}$ for any $i,j \in \mathbb{N}$, as $\partial\Omega \setminus U_i \subset D_{i,j}$ and $U_{i+1} \subset U_i$. If $z \in U_0$, then there exists $m \in \mathbb{N}$ such that $z \notin U_i$ for $i \geq m$ and $z \in U_i$ for $i < m$. Moreover, there exist numbers k_i for $i < m$ such that $z \in T_{i,k_i}$ for $i < m$. Let now $n = 2 + m + \max\{k_1, ..., k_m\}$. Due to B) we conclude that $z \in D_{i,j}$, when $i+j > n$. If $\lfloor i,j \rfloor > \lfloor n,1 \rfloor$, then $i+j \geq n+1$. Therefore $z \in \bigcup_{m\in\mathbb{N}} \bigcap_{\lfloor i,j \rfloor > \lfloor m,1 \rfloor} D_{i,j}$, which finishes the proof. $\square$

Theorem 3.2. *If Ω is a strongly convex domain with boundary of class C^1 and if E is a circular subset of type G_δ in $\partial\Omega$, then there exists a function $f \in \mathbb{O}(\Omega)$ such that $E = E^p_\Omega(f)$.*

Proof. There exists $\{U_i\}_{i\in\mathbb{N}}$ a sequence of open circular sets in $\partial\Omega$ such that $E = \bigcap_{i\in\mathbb{N}} U_i \subset ... \subset U_{i+1} \subset U_i \subset ....$ On the basis of Lemma 3.1 there exist sequences of compact[5] circular sets $\{T_{i,j}\}_{i,j\in\mathbb{N}}$, $\{D_{i,j}\}_{i,j\in\mathbb{N}}$ with $T_{i,j} \cup D_{i,j} \subset \partial\Omega$ such that

- $U_i = \bigcup_{j\in\mathbb{N}} T_{i,j}$;
- $T_{i,j} \cap D_{i,j} = \emptyset$;
- $\partial\Omega \setminus E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\lfloor i,j\rfloor \geq n} D_{i,j}$.

By Theorem 2.4 for every i, j there exists a function $g_{i,j}$ such that $\int_{\mathbb{D}z} |g_{i,j}| d\mathfrak{L}^2_{\mathbb{D}z} = \infty$ for $z \in T_{i,j}$ and $g_{i,j} \in C(\overline{\Omega} \setminus T_{i,j})$.

We denote

$$h_{i,j}(z) := \int_0^{2\pi} \left|g_{i,j}\left(ze^{i\theta}\right)\right|^p d\theta.$$

Observe that

$$\int_0^1 t h_{i,j}(zt) dt = \infty$$

when $z \in T_{i,j}$. Moreover $h_{i,j} \in C(\overline{\Omega} \setminus T_{i,j})$.

We select constants $a_{i,j}, b_{i,j}, \varepsilon_{i,j}$ such that $0 < \varepsilon_{i,j} < 1$, $0 < a_{i,j} < b_{i,j} < a_{k,l} < 1$ when $\lfloor i,j\rfloor < \lfloor k,l\rfloor$ and $2(1 - b_{i,j}) < 1 - b_{k,l}$ when $\lfloor k,l\rfloor < \lfloor i,j\rfloor$. Moreover:

(1)

$$\left(\int_{a_{i,j}}^{b_{i,j}} \varepsilon^p_{i,j} h_{i,j}(b_{i,j} z t) dt\right)^{\frac{1}{p}} > 1 + \sum_{\lfloor k,l\rfloor < \lfloor i,j\rfloor} \left(\int_{a_{i,j}}^1 \varepsilon^p_{k,l} h_{k,l}(b_{k,l} z t) dt\right)^{\frac{1}{p}}$$

for $z \in T_{i,j}$;

(2) $\varepsilon_{i,j} |g_{i,j}(z)| \leq 2^{-\lfloor i,j\rfloor}$ for $z \in [0,1]D_{i,j} \cup [0, a_{i,j}]\overline{\Omega}$.

Let $a_{1,1} = \frac{1}{2}$. Number $\varepsilon_{1,1}$ is selected so that the condition (2) is fulfilled. As $\int_{a_1}^1 \varepsilon^p_{1,1} h_{1,1}(z) dt = \infty$ for $z \in T_{1,1}$ and $h_{1,1} \in C(\overline{\Omega} \setminus T_{1,1})$, we can select $b_{1,1}$ close to 1, so that $a_{1,1} < b_{1,1} < 1$ and

$$\left(b^{-1}_{1,1} \int_{a_{1,1}b_{1,1}}^{b^2_{1,1}} \varepsilon^p_{1,1} h_{1,1}(zt) dt\right)^{\frac{1}{p}} > 1$$

for $z \in T_{1,1}$. In particular

$$\left(\int_{a_{1,1}}^{b_{1,1}} \varepsilon^p_{1,1} h_{1,1}(b_{1,1} z t) dt\right)^{\frac{1}{p}} = \left(b^{-1}_{1,1} \int_{a_{1,1}b_{1,1}}^{b^2_{1,1}} \varepsilon^p_{1,1} h_{1,1}(zt) dt\right)^{\frac{1}{p}} > 1$$

[5]If $\partial\Omega$ is not compact it is enough to modify slightly the sets $T_{i,j}$, $D_{i,j}$ as follows $\widetilde{T}_{i,j} := T_{i,j} \cap F_{\lfloor i,j\rfloor}$ and $\widetilde{D}_{i,j} := D_{i,j} \cap F_{\lfloor i,j\rfloor}$, where $\{F_i\}_{i\in\mathbb{N}}$ is an increasing sequence of compact, circular sets so that $\bigcup_{i\in\mathbb{N}} F_i = \partial\Omega$.

for each $z \in T_{1,1}$. Assume now that we have already selected the numbers $a_{k,l}$, $b_{k,l}$, $\varepsilon_{k,l}$ for $\lfloor k,l \rfloor < \lfloor i,j \rfloor$. The number $a_{i,j}$ is selected so that $b_{k,l} < a_{i,j} < 1$ when $\lfloor k,l \rfloor < \lfloor i,j \rfloor$. Note that the number $\varepsilon_{i,j}$ is selected so that the condition (2) is fulfilled. As $\int_{a_{i,j}}^1 \varepsilon_{i,j}^p h_{i,j}(zt)dt = \infty$ for $z \in T_{i,j}$ and $h_{i,j} \in C(\overline{\Omega} \setminus T_{i,j})$, we can select $b_{i,j}$ close to 1, so that $b_{k,l} < a_{i,j} < b_{i,j} < 1$, $2(1 - b_{i,j}) < 1 - b_{k,l}$ when $\lfloor k,l \rfloor < \lfloor i,j \rfloor$ and

$$\left(b_{i,j}^{-1} \int_{a_{i,j}b_{i,j}}^{b_{i,j}^2} \varepsilon_{i,j}^2 h_{i,j}(zt)dt \right)^{\frac{1}{p}} > 1 + \sum_{\lfloor k,l \rfloor < \lfloor i,j \rfloor} \left(\int_{a_{i,j}}^1 \varepsilon_{k,l}^p h_{k,l}(b_{k,l}z)dt \right)^{\frac{1}{p}}$$

for $z \in T_{i,j}$. In particular

$$\begin{aligned} \left(\int_{a_{i,j}}^{b_{i,j}} \varepsilon_{i,j}^p h_{i,j}(b_{i,j}zt)dt \right)^{\frac{1}{p}} &= \left(b_{i,j}^{-1} \int_{a_{i,j}b_{i,j}}^{b_{i,j}^2} \varepsilon_{i,j}^p h_{i,j}(zt)dt \right)^{\frac{1}{p}} \\ &> 1 + \sum_{\lfloor k,l \rfloor < \lfloor i,j \rfloor} \left(\int_{a_{i,j}}^1 \varepsilon_{k,l}^p h_{k,l}(b_{k,l}z)dt \right)^{\frac{1}{p}} \end{aligned}$$

for each $z \in T_{i,j}$.

We define a function f by

$$f(z) = \sum_{i,j \in \mathbb{N}} \varepsilon_{i,j} g_{i,j}(b_{i,j}z).$$

Due to the property (2) the function f is holomorphic on Ω.

If $z \in \partial\Omega \setminus E$, then there exists $n \in \mathbb{N}$ such that $z \in D_{i,j}$ when $\lfloor i,j \rfloor \geq n$. Note that due to (2):

$$\begin{aligned} \left(\int_{z\mathbb{D}} |f|^p d\mathfrak{L}_{z\mathbb{D}}^2 \right)^{\frac{1}{p}} &\leq \sum_{\lfloor i,j \rfloor < n} \left(\int_0^1 \varepsilon_{i,j}^p h_{i,j}(b_{i,j}zt)dt \right)^{\frac{1}{p}} + \\ &\quad + \sum_{\lfloor i,j \rfloor \geq n} \left(\int_0^1 \varepsilon_{i,j}^p h_{i,j}(b_{i,j}zt)dt \right)^{\frac{1}{p}} \\ &\leq \sum_{\lfloor i,j \rfloor < n} \left(\int_0^1 \varepsilon_{i,j}^p h_{i,j}(b_{i,j}zt)dt \right)^{\frac{1}{p}} + \sum_{\lfloor i,j \rfloor \geq n} 2^{-\lfloor i,j \rfloor} < \infty. \end{aligned}$$

Denote $Q(z, i.j) = \{\lambda z : a_{i,j} \leq |\lambda| \leq b_{i,j}\}$. When $z \in T_{i,j}$, then on the basis of the properties (1) − (2):

$$\begin{aligned}\left(\int_{Q(z,i,j)}|f|^p d\mathfrak{L}^2_{\mathbb{D}z}\right)^{\frac{1}{p}} &\geq \left(\int_{a_{i,j}}^{b_{i,j}}\varepsilon^p_{i,j}h_{i,j}(b_{i,j}zt)dt\right)^{\frac{1}{p}} - \\ &\quad - \sum_{\lfloor k,l\rfloor<\lfloor i,j\rfloor}\left(\int_{a_{i,j}}^{b_{i,j}}\varepsilon^p_{k,l}h_{k,l}(b_{k,l}zt)dt\right)^{\frac{1}{p}} \\ &\quad - \sum_{\lfloor k,l\rfloor>\lfloor i,j\rfloor}\left(\int_{a_{i,j}}^{b_{i,j}}\varepsilon^p_{k,l}h_{k,l}(b_{k,l}zt)dt\right)^{\frac{1}{p}} \\ &\geq 1-\sum_{\lfloor k,l\rfloor>\lfloor i,j\rfloor}\left(\int_{a_{i,j}}^{b_{i,j}}\varepsilon^p_{k,l}h_{k,l}(b_{k,l}zt)dt\right)^{\frac{1}{p}} \\ &\geq 1-\sum_{\lfloor k,l\rfloor>\lfloor i,j\rfloor}2^{-\lfloor k,l\rfloor} = 1-2^{-\lfloor i,j\rfloor}.\end{aligned}$$

If now $z \in E$, then there exists a sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers such that $z \in T_{i,k_i}$. In particular, we can estimate

$$\begin{aligned}\int_{\mathbb{D}z}|f|^p d\mathfrak{L}^2_{\mathbb{D}z} &\geq \sum_{i=1}^{\infty}\int_{Q(z,i,k_i)}|f|^p d\mathfrak{L}^2_{z\mathbb{D}} \\ &\geq \sum_{i=1}^{\infty}\left(1-2^{-\lfloor i,k_i\rfloor}\right)^p = \infty.\end{aligned}$$

The proof is then complete. □

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