Journal of Convex Analysis
Volume 12 (2005), No. 2, 465–476

# Boundary Behaviour of Equilibria for Linear Exchange Economies

**Jean-Marc Bonnisseau**
*CERMSEM, Université de Paris 1,*
*106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France*
*jmbonni@univ-paris1.fr*

**Michael Florig**
*CERMSEM, Université de Paris 1,*
*106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France*
*florig@univ-paris1.fr*

Received April 5, 2004
Revised manuscript received January 4, 2005

We study the Walras equilibrium price function of a linear exchange economy. We show that it is locally Lipschitz with respect to utility functions and initial endowments on the relative interior of the set of parameters where the price is unique. We also prove that it is never locally Lipschitz on the boundary. This extend previous result of Bonnisseau, Florig and Jofré [2], [3].

*Keywords:* Walras equilibrium, linear utility functions, sensitivity, Lipschitz

*1991 Mathematics Subject Classification:* 90C31, 91B42, 91B50

## 1. Introduction

In a linear exchange economy, agents are characterized by an initial endowment of commodities and a linear function, which evaluates the utility from consumption of these goods. A Walras equilibrium is a price and a reallocation of the goods such that every consumer's allocation maximizes the utility function over the feasible points which cost less than the initial endowment. This special case of the Arrow-Debreu model has been extensively studied in the literature (cf. [2], [3], [7], [8], [9], [11], [13], [14], [15]).

Cornet [7] proved that an equilibrium is a solution of a convex program with the utility vectors and the initial endowments appearing as parameters. Eaves [8] proposes an algorithm which computes, in a finite number of steps, an equilibrium or a reduction of the economy if no equilibrium exists.

Although the assumption of linear preferences is quite strong, this model is helpful for several reasons. It is well adapted as a first approach of complex problems such as imperfect competition (Bonnisseau and Florig [1]), indivisible goods (Florig [10]) or the effect of taxes on asset prices (Bottazzi and de Meyer [5]). Indeed, equilibria of linear exchanges economies exhibit interesting properties like the uniqueness of the utility levels even in the presence of multiple equilibrium prices. For example, the definition of an oligopoly equilibrium à la Gabszewicz-Michel [12] can be stated without additional assumptions since the multiplicity of equilibria has no influence on the payoff of the consumers. Furthermore, an economy with general smooth preferences can be approximated by a linear

ISSN 0944-6532 / $ 2.50 

exchange economy, where the approximate utility functions are defined by the gradient vectors of the true utility functions. So the results obtained for linear economies may be interpreted in the original economy, when the agents are engaged in small trades around their initial endowments. The same idea is exploited to study short sighted exchange processes (Champsaur and Cornet [6], Bottazzi [4]). Finally, linear exchange economies naturally appear in the markets with limit price orders (Mertens [16]).

The purpose of this paper is the sensitivity of the equilibrium price of a linear exchange economy with respect to the utility functions and the initial endowments. The sensitivity analysis tools coming from optimization can not be applied. At present the results assume some strong second-order information and sometimes uniqueness of the optimal solution, which are not satisfied in the case of linear exchange economies. We extend previous results of Bonnisseau, Florig and Jofré [2], [3], who have proved that the Walras equilibrium price function of a linear exchange economy is Lipschitz continuous on the interior of the parameter space. These results have been extended to a continuum of agents by Florenzano-Moreno-Garcia [9].

Interiority of the initial endowments means that every consumer owns initially a strictly positive quantity of every commodity. Most consumers however own initially a single commodity - their labor. Interiority is thus a rather strong assumption. It does not fit with the analysis of strategic equilibria of the Cournot-Walras type (see Bonnisseau and Florig [1]). Moreover, analyzing the effect of potential entrants, is equivalent to study an economy with some agents having zero endowment, which is increased if they decide to enter the market.

It is thus important to study the case where the initial endowments lie on the boundary of the parameter space. Let $\mathcal{U}$ be the set of parameters (utility vectors and initial endowments) where the normalized equilibrium price is unique. We prove that the normalized equilibrium price is locally Lipschitz continuous on the (relative) interior of $\mathcal{U}$ and only on it.

The following simple example shows that the normalized equilibrium price is not locally Lipschitz continuous around an economy with several equilibrium prices. Nevertheless, in this simple case, $\mathcal{U}$ is open. A more interesting example, but with 4 consumers and 4 commodities, is given in Section 4, where the equilibrium price is not locally Lipschitz continuous around an economy on the relative boundary of $\mathcal{U}$.

We consider a linear economy with two consumers and two commodities. The utility functions are $u_1(a,b) = a + b$ and $u_2(a,b) = 2a + b$. If the initial endowments are $(0,1)$ and $(1,0)$, the equilibrium price set is the cone generated by $(1, \frac{1}{2})$ and $(1,1)$. For $\varepsilon > 0$ small enough, if the initial endowments are $(\varepsilon, 1-\varepsilon)$ and $(1-\varepsilon, \varepsilon)$, then the unique normalized equilibrium price is $(1,1)$. If the initial endowments are $(\varepsilon, 1-2\varepsilon)$ and $(1-\varepsilon, 2\varepsilon)$, then the unique normalized equilibrium price is $(1, \frac{1}{2})$. For a small value of $\varepsilon$, the initial endowments are close but the equilibrium prices remain far one from the other.

The proof uses a decomposition of the space by considering the connected components of a graph associated to the equilibrium. Then, we prove the local Lipschitz behavior for each element of the decomposition, and, we prove that the initial price function obtained as a selection of the previous collection of functions is locally Lipschitz.

## 2. The model

We consider a linear exchange economy with a finite set $L = \{1, \dots, \ell\}$ of commodities and $I = \{1, \dots, m\}$ of consumers. The consumption set of consumer $i$ is $R^\ell_+$ and his utility function $u_i : R^\ell_+ \to R$ is defined by $u_i(x_i) = b_i \cdot x_i$ for a given vector $b_i \in R^\ell_+$. His initial endowment is a vector $\omega_i \in R^\ell_+$. For each $(b, \omega) \in ((R^\ell_+)^m)^2$, $\mathcal{L}(b, \omega)$ denotes the linear exchange economy associated with the parameters $b$ and $\omega$. Throughout the paper we will make the two following assumptions:

(A1) $(\sum_{i=1}^m b_i, \sum_{i=1}^m \omega_i) \in R^\ell_{++} \times R^\ell_{++}$;

(A2) for every $i$, $b_i \neq 0$.

These assumptions mean that a positive quantity of each commodity is available, each commodity is desired and each consumer desires at least one commodity. Note that the set of elements $(b, \omega)$, which satisfies these two assumptions, is open in $((R^\ell_+)^m)^2$.

**Definition 2.1.**

(i) If $p \in R^\ell_+$ is a price vector, the demand of consumer $i$, denoted $d(b_i, p, p \cdot \omega_i)$, is the set of solutions of the following maximization problem:

$$\begin{cases} \max u_i(x_i) = b_i \cdot x_i \\ p \cdot x_i \leq p \cdot \omega_i \\ x_i \geq 0 \end{cases}$$

(ii) A Walras equilibrium of $\mathcal{L}(b, \omega)$ is an element $(x, p) \in (R^\ell_+)^m \times R^\ell_+$ such that:
 (a) for every $i$, $x_i \in d(b_i, p, p \cdot \omega_i)$;
 (b) $\sum\limits_{i=1}^m x_i = \sum\limits_{i=1}^m \omega_i$.

(iii) A proper subset $I'$ of $I$ is called self sufficient in $\mathcal{L}(b, \omega)$ if for all $h \in L$, $\sum_{i \in I'} b_{ih} > 0$ implies $\sum_{i \in I \setminus I'} \omega_{ih} = 0$.

For every $(b, \omega) \in (R^\ell_+)^m \times (R^\ell_+)^m$, we note $P(b, \omega)$ the set of Walrasian equilibrium price vectors in $R^\ell_+$. Note that (A1) implies $P(b, \omega) \subset R^\ell_{++}$. For each $p \in R^\ell_{++}$, for each $i \in I$,

$$\delta(b_i, p) = \{h \in L | p_h b_{ik} \leq p_k b_{ih}, \ \forall k \in L\}.$$

$\delta(b_i, p)$ is the set of commodities that the consumer wishes to consume if the price vector is $p$, since the ratio between the marginal utility and the price is maximal for these commodities. For each $p \in R^\ell_{++}$,let

$$G(b, p) = \{(i, h) \in I \times L \mid h \in \delta(b_i, p)\}.$$

Note that $G(b, p)$ may be seen as a graph where the set of vertices is $I \cup L$ and there exists an edge between the vertices $i \in I$ and $h \in L$ if and only if $(i, h) \in G(b, p)$.

We note $\mathcal{W}$ the set of couples $(b, \omega)$ such that $P(b, \omega)$ is nonempty and by $\mathcal{U}$ the set of couples $(b, \omega)$ such that $P(b, \omega)$ is an half line, which means that the equilibrium price is unique up to multiplication. In this case, we note without ambiguity $G(b, \omega) = G(b, p)$ for $p \in P(b, \omega)$.

We recall a characterization of the elements of $\mathcal{U}$ (see, Proposition 4.1. in [2]).

**Proposition 2.2.** *Let $(b,\omega)$ satisfying Assumptions (A1) and (A2) and let $p \in P(b,\omega)$. Then $(b,\omega)$ is in $\mathcal{U}$, if and only if, the economy $\mathcal{L}(c,\omega)$ with $c_{ih} = b_{ih}$ if $(i,h) \in G(b,p)$ and 0 if not, has no self sufficient subset.*

## 3. Local Explicit Formula

In this section, we show that on $\mathcal{U}$, the equilibrium price vector can be obtained by an explicit local formula, which depends on the utility functions and the initial endowments. Actually, we exhibit a finite number of algebraic mappings and the equilibrium price vector is always given by one of these mappings. To choose the right one, it suffices to know the graph $G(b,\omega)$ or, more precisely, the connected components of this graph. By Eaves [8], it is possible to compute an equilibrium in a finite number of steps, and therefore also the right graph.

Let $\mathcal{C}$ be the finite set of the correspondences from $I$ to $L$, which are onto and nonempty valued. For all $C \in \mathcal{C}$, let $G^C$ be the graph associated to $C$ with vertices $I \cup L$ and an edge between the vertices $i \in I$ and $h \in L$ if and only if $h \in C(i)$.

With each economy $(b,\omega) \in \mathcal{U}$, we associate the correspondence $C$ such that $G^C = G(b,\omega)$. Conversely, for each $C \in \mathcal{C}$, let

$$\Omega^C = \{(b,\omega) \in \mathcal{U} \mid G^C = G(b,\omega)\}.$$

Note that $(\Omega^C)_{C \in \mathcal{C}}$ is a finite partition of $\mathcal{U}$. It is also easy to check that for all $C \in \mathcal{C}$, $\Omega^C \neq \emptyset$.

For every $C \in \mathcal{C}$, we denote by $G_1^C, \ldots, G_n^C$ the connected components of $G^C$ and by $\mathcal{I}_1^C, \ldots, \mathcal{I}_n^C$ (resp. $\mathcal{H}_1^C, \ldots, \mathcal{H}_n^C$) the elements of $G_1^C, \ldots, G_n^C$ in $I$ (resp. $L$). Since $C$ is onto and nonempty valued, $\mathcal{I}_1^C, \ldots, \mathcal{I}_n^C$ (resp. $\mathcal{H}_1^C, \ldots, \mathcal{H}_n^C$) is a partition of $I$ (resp. $L$). For each $z \in R^\ell$ and for each $\nu \in \{1, \ldots, n\}$, we denote by $z^\nu$ the restriction of $z$ to the components in $\mathcal{H}_\nu^C$.

We obtain the formula in two steps. The following lemma shows that, if two economies are associated with the same graph, then the restrictions of the equilibrium price vectors to each subset of the partition $(\mathcal{H}_\nu^C)$ are proportional. For all $C \in \mathcal{C}$, let

$$\mathcal{B}^C = \{b \in (R_+^\ell)^m | h \in C(i) \Rightarrow b_{ih} > 0\}.$$

Note that $b \in \mathcal{B}^C$ if $(b,\omega) \in \Omega^C$ since $(i,h) \in G(b,\omega) = G^C$ implies $b_{ih} > 0$.

**Lemma 3.1.** *For all $C \in \mathcal{C}$, there exists an algebraic mapping $\pi^C$ from $\mathcal{B}^C$ to $R_{++}^\ell$ such that for all $(b,\omega) \in \Omega^C$, for all $\nu = 1, \ldots, n$, $p^\nu(b,\omega)$ is proportional to $\pi^{C\nu}(b)$.*

**Proof.** For each $\nu = 1, \ldots, n$, let $h^\nu \in \mathcal{H}_\nu^C$. From the definition of a connected component, for each $h \in \mathcal{H}_\nu^C$, there exists $q$ consumers, $i_1, \ldots, i_q$ and $q-1$ goods, $h_1, \ldots, h_{q-1}$ such that for each $k = 1, \ldots, q$,

$$h_{k-1} \text{ and } h_k \in C(i_k)$$

where $h_0 = h^\nu$ and $h_q = h$.

We now define the mapping $\pi^C$ as follows. For each $b \in \mathcal{B}^C$, for each $\nu = 1, \dots, n$, for each $h \in \mathcal{H}^C_\nu$,

$$(\pi^C(b))_h = \begin{cases} 1 & \text{if } h = h^\nu \\ \prod_{k=1}^q \frac{b_{i_k h_k}}{b_{i_k h_{k-1}}} & \text{if } h \neq h^\nu. \end{cases}$$

Note that the above formula is well defined since $b \in \mathcal{B}^C$ and $h_{k-1} \in C(i_k)$ implies $b_{i_k h_{k-1}} > 0$.

Let $(b, \omega) \in \Omega^C$. Recalling the fact that $G^C = G(b, \omega)$ and the definition of $\delta(b_i, p)$, for each $k = 1, \dots, q$:

$$\frac{b_{i_k h_{k-1}}}{p_{h_{k-1}}(b, \omega)} = \frac{b_{i_k h_k}}{p_{h_k}(b, \omega)}.$$

Thus, for all $\nu = 1, \dots, n$,

$$p^\nu(b, \omega) = p_{h^\nu}(b, \omega) \pi^{C\nu}(b)$$

which ends the proof of the lemma. $\square$

We now define a matrix, which will play a key role in the computation of the equilibrium price.

**Definition 3.2.** For all $C \in \mathcal{C}$, for all $(b, \omega) \in \mathcal{B}^C \times (R^\ell_+)^m$, the $n \times n$ matrix $T^C(b, \omega)$ is defined by:

$$t^C_{\nu\mu}(b, \omega) = \begin{cases} \pi^{C\nu}(b) \cdot (\sum\limits_{i \notin \mathcal{I}^C_\nu} \omega^\nu_i) & \text{if } \nu = \mu \\ -\pi^{C\mu}(b) \cdot (\sum\limits_{i \in \mathcal{I}^C_\nu} \omega^\mu_i) & \text{if } \nu \neq \mu. \end{cases}$$

For all $\bar{\nu} \in \{1, \dots, n\}$ the matrix $T^C_{\bar{\nu}}(b, \omega)$ is the submatrix of $T^C(b, \omega)$ obtained by suppressing the $\bar{\nu}$-th column and the $\bar{\nu}$-th row.

**Lemma 3.3.** *For all $C \in \mathcal{C}$, for all $(b, \omega) \in \Omega^C$, for all $\bar{\nu} \in \{1, \dots, n\}$ the matrix $T^C_{\bar{\nu}}(b, \omega)$ is of full rank, that is $n - 1$. Moreover, all the elements of the matrix $\left(T^C_{\bar{\nu}}(b, \omega)\right)^{-1}$ are non-negative.*

The proof of this lemma use the following technical lemma, which is proved in Appendix.

**Lemma 3.4.** *Let $T$ be a $n \times n$ matrix such that for all $i$, $t_{ii} \geq 0$, for all $i, j$, $i \neq j$, $t_{ij} \leq 0$, and, for all $j$, $\sum_{i=1}^n t_{ij} = 0$. We assume that for all $N \subset \{1, \dots, n\}$, $N \neq \{1, \dots, n\}$, there exist $i \notin N$ and $j \in N$, such that $t_{ij} < 0$. Then for all $i$, the sub-matrix $T^i$ of $T$ obtained by suppressing the $i$th column and the $i$th row is regular. Moreover, the elements of the inverse of the matrix $T^i$ are non-negative.*

**Proof of Lemma 3.3.** Let $C \in \mathcal{C}$, and $(b, \omega) \in \Omega^C$. Since $(b, \omega)$ is in $\mathcal{U}$, Proposition 2.2 implies that the economy $\mathcal{L}(c, \omega)$ with $c_{ih} = b_{ih}$ if $h \in \delta(b_i, p(b, \omega)) = C(i)$ and 0 if not, has no self sufficient subset. Note that if $i \in \mathcal{I}^C_\nu$, $c_{ih} > 0$ implies that $h \in \mathcal{H}^C_\nu$. Note also that $t^C_{\nu\mu}(b, \omega) = 0$ if and only if $\sum_{i \in \mathcal{I}^C_\nu} \omega^\mu_i = 0$.

It suffices to show that $T^C(b, \omega)$ satisfies the assumptions of Lemma 3.4. One easily checks that $T^C(b, \omega)$ satisfies the sign conditions of Lemma 3.4 on their elements. Furthermore,

$\sum_{\nu=1}^{n} t^C_{\nu\mu}(b,\omega) = \pi^{C\mu}(b) \cdot (-\sum_{\nu \neq \mu} \sum_{i \in \mathcal{I}^C_\nu} \omega^\mu_i + \sum_{i \notin \mathcal{I}^C_\mu} \omega^\mu_i) = 0$ since $(I^C_\nu)_{\nu=1}^n$ is a partition of $I$.

Let $N \subset \{1,\dots,n\}$, $N \neq \{1,\dots,n\}$. Since $\cup_{\nu \in N} \mathcal{I}^C_\nu$ is not self-sufficient for $\mathcal{L}(c,\omega)$, there exists $i \notin \cup_{\nu \in N} \mathcal{I}^C_\nu$ and $h \in \cup_{\nu \in N} \mathcal{H}^C_\nu$ such that $\omega_{ih} > 0$. Consequently, there exists $\mu \notin N$ such that $i \in \mathcal{I}^C_\mu$ and $\nu \in N$ such that $h \in H^C_\nu$. One easily checks that $t^C_{\mu\nu} < 0$. □

We now extend the previous result to the elements on the boundary of $\Omega^C$ but in the interior for $(R^\ell_+)^2$ of $\mathcal{U}$ denoted int $\mathcal{U}$.

**Lemma 3.5.** *For all $C \in \mathcal{C}$, for all $(b,\omega) \in \bar{\Omega}^C \cap int\, U$, for all $\bar{\nu} \in \{1,\dots,n\}$, the matrix $T^C_{\bar{\nu}}(b,\omega)$ is of full rank, that is $n-1$. Moreover, all the elements of the matrix $\left(T^C_{\bar{\nu}}(b,\omega)\right)^{-1}$ are non-negative.*

**Proof.** Let $(b,\omega) \in \bar{\Omega}^C \cap \text{int}\, U$. Let $(b^q,\omega^q)$ be a sequence of $\Omega^C$, which converges to $(b,\omega)$. Let $x^q$ an equilibrium allocation of the economy $\mathcal{L}(b^q,\omega^q)$. Since the normalized equilibrium price is continuous on $\mathcal{U}$ (See, Bonnisseau, Florig and Jofré [2]), and $\delta(b^q_i, p(b^q,\omega^q))$ is constant equal to $C(i)$, one deduces that $C(i) \subset \delta(b_i, p(b,\omega))$ for all $i$. These inclusions imply that $b \in \mathcal{B}^C$.

Since $(\omega^q)$ converges to $\omega$, the sequence $(x^q)$ is bounded. Hence, it has a converging subsequence, which converges to $x \in (R^\ell_+)^m$. From Bonnisseau, Florig and Jofré [2], $x$ is an equilibrium allocation of $\mathcal{L}(b,\omega)$. From the definition of the demand, $x^q_{ih} = 0$ if $h \notin \delta(b^q_i, p(b^q,\omega^q)) = C(i)$. Consequently, $x_{ih} = 0$ if $h \notin \delta(b^q_i, p(b^q,\omega^q)) = C(i)$. We now consider the auxiliary economy $\mathcal{L}(b^t,\omega)$ where $b^t$ is defined as follows. $b^t_{ih} = b_{ih} - t$ if $h \in \delta(b_i, p(b,\omega)) \setminus C(i)$ and $b^t_{ih} = b_{ih}$ otherwise. For $t$ small enough, $(b^t,\omega)$ satisfies our basic assumptions (A1) and (A2). Furthermore, one remarks that $x$ is an equilibrium allocation of $\mathcal{L}(b^t,\omega)$ associated to the price $p(b,\omega)$ since $x_{ih} = 0$ for the commodities in $\delta(b_i, p(b,\omega)) \setminus C(i)$. Hence, since $\delta(b^t_i, p(b,\omega)) = C(i)$ for all $i$ and $(b,\omega) \in \text{int}\, \mathcal{U}$, one gets $(b^t,\omega) \in \Omega^C$ for $t$ small enough. Since the mapping $\pi^{C\nu}$ only depends on the element $b_{ih}$ with $h \in C(i)$, one deduces that $T^C(b,\omega) = T^C(b^t,\omega)$. Hence the previous lemma leads to the conclusion. □

Now, we can state the main result of this section.

**Proposition 3.6.** *For all $C \in \mathcal{C}$, for all $(b,\omega) \in \Omega^C$, $p$ is an equilibrium price vector of $\mathcal{L}(b,\omega)$ if and only if there exists $\lambda \in R^n_{++}$ in the kernel of $T^C(b,\omega)$ such that for all $\nu \in \{1,\dots,n\}$, $p^\nu = \lambda_\nu \pi^{C\nu}(b)$.*

This result gives an explicit formula to compute the equilibrium price vector. Actually, when we have no information about the equilibrium price vector, this result gives a finite number of possibilities for the equilibrium price vector. This number may be large since it is of the same level than the number of correspondences from $I$ to $L$. Nevertheless, if we know the correspondence $C$ which is associated with the economy $\mathcal{L}(b,\omega)$, that is we know the commodities desired by the consumers at equilibrium, then one has a unique formula.

**Proof of Proposition 3.6.** Let $(b,\omega) \in \Omega^C$. From Lemma 3.1, we deduce that there exists $\lambda(b,\omega) \in R^n_{++}$ such that $p^\nu(b,\omega) = \lambda_\nu(b,\omega)\pi^{C\nu}(b)$ for all $\nu$. We now show that

$\lambda(b,\omega)$ belongs to the kernel of $T^C(b,\omega)$. For this, we use Walras law, that is the value of the equilibrium allocation of each consumer is equal to the value of his initial endowment and the market clearing equation, that is, the sum of the equilibrium allocations is equal to the sum of the initial endowments. Furthermore, we use the fact that the equilibrium allocation of a consumer in the component $\mathcal{I}^C_\nu$ is positive only for the commodities in $\mathcal{H}^C_\nu$.

Let $(x, p(b,\omega))$ be an equilibrium of $\mathcal{L}(b,\omega)$. For all $i \in I$, $p(b,\omega) \cdot x_i = p(b,\omega) \cdot \omega_i$. Furthermore, recalling the fact that $G^C = G(b,\omega)$, one has, for all $\nu = 1,\dots,n$ and all $h \in \mathcal{H}^C_\nu$, $x_{ih} = 0$ if $i \notin \mathcal{I}^C_\nu$. Consequently, $\sum_{i\in\mathcal{I}^C_\nu} x_{ih} = \sum_{i=1}^m \omega_{ih}$. For all $h \notin \mathcal{H}^C_\nu$, $\sum_{i\in\mathcal{I}^C_\nu} x_{ih} = 0$. One then deduces that

$$p(b,\omega) \cdot \sum_{i\in\mathcal{I}^C_\nu} x_i = p(b,\omega) \cdot \sum_{i\in\mathcal{I}^C_\nu} \omega_i = \sum_{\mu=1}^n \lambda_\mu(b,\omega)\pi^{C\mu}(b) \cdot \left(\sum_{i\in\mathcal{I}^C_\nu} \omega_i^\mu\right).$$

On the other hand,

$$\begin{aligned} p(b,\omega) \cdot \sum_{i\in\mathcal{I}^C_\nu} x_i &= p^\nu(b,\omega) \cdot \sum_{i\in\mathcal{I}^C_\nu} x_i^\nu \\ &= p^\nu(b,\omega) \cdot \sum_{i=1}^m \omega_i^\nu \\ &= \lambda_\nu(b,\omega)\pi^{C\nu}(b) \cdot \sum_{i=1}^m \omega_i^\nu. \end{aligned}$$

From the above equalities, one deduces that for all $\nu = 1,\dots,n$,

$$\lambda_\nu(b,\omega)\pi^{C\nu}(b) \cdot \sum_{i=1}^m \omega_i^\nu = \sum_{\mu=1}^n \lambda_\mu(b,\omega)\pi^{C\mu}(b) \cdot \left(\sum_{i\in\mathcal{I}^C_\nu} \omega_i^\mu\right)$$

or equivalently,

$$-\sum_{\mu\neq\nu} \lambda_\mu(b,\omega)\pi^{C\mu}(b) \cdot \left(\sum_{i\in\mathcal{I}^C_\nu} \omega_i^\mu\right) + \lambda_\nu(b,\omega)\pi^{C\nu}(b) \cdot \left(\sum_{i=1}^m \omega_i^\nu - \sum_{i\in\mathcal{I}^C_\nu} \omega_i^\nu\right) = 0.$$

Since $\sum_{i=1}^m \omega_i^\nu - \sum_{i\in\mathcal{I}^C_\nu} \omega_i^\nu = \sum_{i\notin\mathcal{I}^C_\nu} \omega_i^\nu$, one has:

$$T^C(b,\omega)\lambda(b,\omega) = 0.$$

Conversely, we remark that the rank of $T^C(b,\omega)$ is $n-1$ since the sum of the columns is zero and the matrix $T^C_{\bar{\nu}}(b,\omega)$ is regular by Lemma 3.3. Therefore, the kernel of $T^C(b,\omega)$ is a one dimensional subspace of $R^n$. If $\lambda$ is a positive element of the kernel of $T^C(b,\omega)$, then $\lambda$ is positively proportional to $\lambda(b,\omega)$. Hence, the vector $p$ defined by $p^\nu = \lambda^\nu\pi^{C\nu}(b)$ is positively proportional to $p(b,\omega)$. Consequently, $p$ is an equilibrium price vector of $\mathcal{L}(b,\omega)$. □

In the following corollary, we give an explicit formula when a good is chosen as numéraire.

**Corollary 3.7.** *Let $h \in \{1, \dots, \ell\}$ be the commodity chosen as numéraire. For all $C \in \mathcal{C}$, let $\bar{\nu} \in \{1, \dots, n\}$ such that $h \in \mathcal{H}^C_{\bar{\nu}}$. Then, for all $(b, \omega) \in \Omega^C$, the equilibrium price vector $p(b, \omega)$ of $\mathcal{L}(b, \omega)$ which satisfies $p_h(b, \omega) = 1$, is given by the following formula:*

$$p^{\bar{\nu}}(b, \omega) = \frac{1}{\pi^C_h(b)} \pi^{C\bar{\nu}}(b)$$

*and for all $\nu \neq \bar{\nu}$,*

$$p^{\nu}(\omega) = \lambda_\nu(b, \omega) \pi^{C\nu}(b),$$

*where $(\lambda_\nu(b, \omega))_{\nu \neq \bar{\nu}}$ is defined by:*

$$(\lambda_\nu(b, \omega))_{\nu \neq \bar{\nu}} = -\frac{1}{\pi^C_h(b)} \left(T^C_{\bar{\nu}}(b, \omega)\right)^{-1} (t^C_{\nu\bar{\nu}}(b, \omega))_{\nu \neq \bar{\nu}} \ .$$

When a commodity is chosen as numéraire, the mapping which gives the unique equilibrium price vector, is algebraic since it is an algebraic combination of $\pi^C$ which is algebraic and $\lambda_\nu$ which is also algebraic since the matrix $T^C$ (hence $(T^C_{\bar{\nu}})^{-1}$) is algebraic. This can be deduced from general results of algebraic geometry. However, the explicit formula is not a byproduct of this general approach.

In the following corollary, we extend the above result to $\bar{\Omega}^C \cap \operatorname{int} \mathcal{U}$ since the formula given in Corollary 3.7 is well defined on this set as it is shown in Lemma 3.5.

**Corollary 3.8.** *Let $h \in \{1, \dots, \ell\}$ and for all $C \in \mathcal{C}$, let $\bar{\nu} \in \{1, \dots, n\}$ such that $h \in \mathcal{H}^C_{\bar{\nu}}$. For all $C \in \mathcal{C}$, for all $(b, \omega) \in \bar{\Omega}^C \cap \operatorname{int} \mathcal{U}$, the equilibrium price vector $p(b, \omega)$ of $\mathcal{L}(b, \omega)$, which satisfies $p_h(b, \omega) = 1$ is given by the formula of Corollary 3.7.*

**Proof of Corollaries 3.7 and 3.8.** The proofs of Corollaries 3.7 and 3.8 are the same as the one of Corollaries 3.1 and 3.2 in Bonnisseau, Florig and Jofré ([3]). Note that they are a direct consequence of Proposition 3.6, the continuity of the equilibrium price vector on $\mathcal{U}$ and the fact that $T^C_{\bar{\nu}}(b, \omega)$ is of full rank on $\bar{\Omega}^C \cap \operatorname{int} \mathcal{U}$. □

## 4. Lipschitz behavior of equilibrium prices

In this section, we apply the results of the previous section to get two complementary results on the behavior of the equilibrium prices. First, we show that they are locally Lipschitz continuous on the interior of $\mathcal{U}$ and then, we prove that they are never locally Lipschitz continuous on the boundary.

**Proposition 4.1.** *When a commodity is chosen as numéraire, the normalized equilibrium price mapping $p(.,.)$ is locally Lipschitz continuous on $\operatorname{int} \mathcal{U}$.*

**Proof.** From Corollary 3.8, the mapping $p(.,.)$ is locally Lipschitz continuous on $\bar{\Omega}^C \cap \operatorname{int} \mathcal{U}$ as the restriction of a non degenerated algebraic mapping and $(\bar{\Omega}^C)_{C \in \bar{\mathcal{C}}} \cap \operatorname{int} \mathcal{U}$ is a finite closed covering of $\operatorname{int} \mathcal{U}$.

Therefore, Proposition 4.1 is a direct consequence of the following result. Its proof is left to the reader. □

**Lemma 4.2.** *Let $U$ be a locally convex subset of a finite dimensional Euclidean space $E$, which means that each element of $U$ has a convex neighborhood. Let $F_1, \dots, F_n$, $n$ closed (for the topology of $U$) subsets of $U$ such that $U = \cup_{k=1}^n F_k$. For all $k = 1, \dots, n$, let $f_k$ be a locally Lipschitz continuous mapping from $F_k$ to a finite Euclidean space $G$. For all $x \in U$, let $K(x) = \{k \in \{1, \dots, n\} \mid x \in F_k\}$. We assume that for all $x \in U$, for all $k, k' \in K(x)$, $f_k(x) = f_{k'}(x)$. Let $f$ be the mapping from $U$ to $G$, defined by $f(x) = f_k(x)$ for some $k \in K(x)$. Then, $f$ is locally Lipschitz continuous on $U$.*

The end of the section is devoted to the study of the price at the elements on the boundary of $\mathcal{U}$. We first give an example, which shows that $\mathcal{U}$ is not always open. Actually, this example shows a stronger fact. For fixed utilities, the set of endowments with a unique equilibrium price vector is not always open. The example given in Introduction shows that $\mathcal{U}$ may not be closed. We consider an economy with four consumers and four commodities, $b_1 = (1,0,0,0)$, $\omega_1 = (0,1,0,0)$, $b_2 = (0,1,1,0)$, $\omega_2 = (1,0,0,0)$, $b_3 = (1,0,1,0)$, $\omega_3 = (0,0,0,1)$, $b_4 = (0,0,0,1)$, $\omega_4 = (0,0,1,0)$. Note that $\{t(1,1,1,1) | t > 0\} \in P(b,\omega)$ and therefore by Proposition 2.2, $(b,\omega) \in \mathcal{U}$.

For $\lambda > 0$, let $\omega^\lambda$ with $\omega_1^\lambda = (0, 1+\lambda, 0, 0)$, $\omega_2^\lambda = \omega_2$, $\omega_3^\lambda = \omega_3$ and $\omega_4^\lambda = \omega_4$. Let $p^\lambda = (1+\lambda, 1, 1, 1)$, $q^\lambda = (1+\lambda, 1, 1+\lambda, 1+\lambda)$ For all $\lambda > 0$, $p^\lambda$ and $q^\lambda$ are equilibrium price of $\mathcal{L}(b, \omega^\lambda)$, and therefore $(b, \omega^\lambda) \notin \mathcal{U}$.

We now show that the equilibrium price is not locally Lipschitz continuous at $(b,\omega)$. The following construction illustrates the general proof given below. Let $\varepsilon > 0$ small enough. Let $\bar{\omega}^{\lambda\varepsilon}$ with $\bar{\omega}_1^{\lambda\varepsilon} = (\varepsilon, 1+\lambda-(3+\lambda)\varepsilon, \varepsilon, \varepsilon)$, $\bar{\omega}_2^{\lambda\varepsilon} = (1-\varepsilon, (1+\lambda)\varepsilon, 0, 0)$, $\bar{\omega}_3^{\lambda\varepsilon} = (0, \varepsilon, 0, 1-\varepsilon)$ and $\bar{\omega}_4^{\lambda\varepsilon} = (0, \varepsilon, 1-\varepsilon, 0)$. Since $p^\lambda \cdot (\omega_i^\lambda - \bar{\omega}_i^{\lambda\varepsilon}) = 0$ for all $i$, $\sum_{i=1}^4 \omega_i^\lambda - \bar{\omega}_i^{\lambda\varepsilon} = 0$, and $\bar{\omega}_1^{\lambda\varepsilon} \gg 0$, $p^\lambda$ is the unique equilibrium price vector of $(b, \bar{\omega}^{\lambda\varepsilon})$.

Let now $\tilde{\omega}^{\lambda\varepsilon}$ with $\tilde{\omega}_1^{\lambda\varepsilon} = (\varepsilon, 1+\lambda-3(1+\lambda)\varepsilon, \varepsilon, \varepsilon)$, $\tilde{\omega}_2^{\lambda\varepsilon} = (1-\varepsilon, (1+\lambda)\varepsilon, 0, 0)$, $\tilde{\omega}_3^{\lambda\varepsilon} = (0, (1+\lambda)\varepsilon, 0, 1-\varepsilon)$ and $\tilde{\omega}_4^{\lambda\varepsilon} = (0, (1+\lambda)\varepsilon, 1-\varepsilon, 0)$. Since $q^\lambda \cdot (\omega_i^\lambda - \tilde{\omega}_i^{\lambda\varepsilon}) = 0$ for all $i$, $\sum_{i=1}^4 \omega_i^\lambda - \tilde{\omega}_i^{\lambda\varepsilon} = 0$, and $\tilde{\omega}_1^{\lambda\varepsilon} \gg 0$, $q^\lambda$ is the unique equilibrium price vector of $(b, \tilde{\omega}^{\lambda\varepsilon})$.

Taken a neighbourhood of $(b,\omega)$, we can choose $\lambda$ and $\varepsilon$ small enough to get $(b, \bar{\omega}^{\lambda\varepsilon})$ and $(b, \tilde{\omega}^{\lambda\varepsilon})$ in this neighbourhood. Now, by reducing $\varepsilon$, we can obtain $\|(b, \bar{\omega}^{\lambda\varepsilon}) - (b, \tilde{\omega}^{\lambda\varepsilon})\|$ as small as we want, whereas the equilibrium prices remains constant equal to $p^\lambda$ and $q^\lambda$. So, the equilibrium price is not Lipschitz continuous in this neighborhood of $(b,\omega)$.

We now prove two lemmata, from which one easily deduces that the equilibrium price is not locally Lipschitz continuous around an economy in $\mathcal{U}$, which is not in the interior.

**Lemma 4.3.** *Let $(b,\omega) \in \mathcal{U}$. Then, there exists $r > 0$ such that $(B((b,\omega), r) \cap (R_+^\ell)^{2m}) \subset \mathcal{W}$.*

**Proof.** We first remark that there exists $\bar{r} > 0$ such that for all $(b', \omega') \in B((b,\omega), \bar{r}) \cap (R_+^\ell)^{2m})$, the two basic Assumptions (A1) and (A2) are satisfied. Then, if the result is not true, there exists a sequence of economy $(b^q, \omega^q)$ with no equilibrium and which satisfies Assumptions (A1) and (A2). From Gale [15], for all $q$, there exists a super self sufficient subset $I^q$ in $\mathcal{L}(b^q, \omega^q)$, that is a self sufficient subset such that there exists $h \in L$ with $\sum_{i \in I^q} b_{ih} = 0$ and $\sum_{i \in I^q} \omega_{ih} > 0$. Since the set of subsets of $\{1, \dots, m\}$ is finite, taking a subsequence allows us to assume that $I^q$ is constant equal to $J$. For all $h \in \{1, \dots, \ell\}$, if $\sum_{i \in J} b_{ih} > 0$, then $\sum_{i \in J} b_{ih}^q > 0$ for $q$ large enough. Since $J$ is self sufficient, $\sum_{i \notin J} \omega_{ih}^q = 0$ and at the limit, $\sum_{i \notin J} \omega_{ih} = 0$. Hence $J$ is self sufficient for the economy $\mathcal{L}(b,\omega)$. This implies that $J$ is self sufficient for the economy $\mathcal{L}(c,\omega)$ with $c_{ih} = b_{ih}$ if $h \in \delta(b_i, p(b,\omega))$ and 0 otherwise. From Proposition 2.2, this contradicts the fact that $(b,\omega) \in \mathcal{U}$. □

**Lemma 4.4.** *Let $(b,\omega) \in \mathcal{W} \setminus \mathcal{U}$. Then, for all equilibrium price $p \in R^\ell_{++}$, there exists a transfer $\tau \in (R^\ell)^m$ such that:*

*for all $t > 0$ small enough, $\omega + t\tau \in (R^\ell_+)^m$ and there exists $i_0$ such that $\omega_{i_0} + t\tau_{i_0} \in R^\ell_{++}$;*

*$\sum_{i=1}^m \tau_i = 0$ and for all $i$, $p \cdot \tau_i = 0$;*

*Consequently, for all $t > 0$ small enough, $p$ is the unique equilibrium of the economy $\mathcal{L}(b, \omega + t\tau)$.*

**Proof.** When an economy has at least one equilibrium, Proposition 2.2 implies that the equilibrium price is unique (up to multiplication) if one consumer has a strictly positive endowment. Indeed, a self-sufficient set must contain this consumer. Furthermore, since this set is not super self-sufficient and since the endowment of the set is strictly positive, this set must own the whole endowment hence every consumer belongs to this set, which shows that there is no proper self-sufficient set whatever are the preferences.

Let $(b,\omega) \in \mathcal{W} \setminus \mathcal{U}$. From our basic assumption (A1), we assume without any loss of generality that $\omega_1 \neq 0$. From the previous remark, there exists $h$ such that $\omega_{1h} = 0$, and, since $\omega_1 \neq 0$, there exists a commodity $h_1$ such that $\omega_{1h_1} > 0$. Using our basic assumption (A1), for each commodity $h$ such that $\omega_{1h} = 0$, there exists a consumer $i_h \neq 1$ such that $\omega_{i_h h} > 0$. We denote by $\hat{I} = \{i \mid \exists h, i = i_h\}$ and for all $i \in \hat{I}$, $H(i) = \{h \mid i = i_h\}$. For all $t > 0$, we now consider the initial endowment $\omega^t$ defined as follows.

$\omega^t_{1h} = \omega_{1h}$ if $\omega_{1h} > 0$ and $h \neq h_1$;

$\omega^t_{1h} = t$ if $\omega_{1h} = 0$;

$\omega^t_{1h_1} = \omega_{1h_1} - \frac{t\sum_{h|\omega_{1h}=0} p_h}{p_{h_1}}$;

for all $i > 1$, for all $h \neq h_1$, $\omega^t_{ih} = \omega_{ih}$ if $i \neq i_h$; For all $h$ such that $\omega_{1h} = 0$, $\omega^t_{i_h h} = \omega_{i_h h} - t$.

For all $i \notin \hat{I}$, $\omega^t_{ih_1} = \omega_{ih_1}$.

For all $i \in \hat{I}$, $\omega^t_{ih_1} = \omega_{i_h h_1} + \frac{t\sum_{h\in H(i)} p_h}{p_{h_1}}$.

For all $t > 0$ such that $t < \omega_{i_h h}$ for all $h$ satisfying $\omega_{1h} = 0$ and $t < \frac{p_{h_1}}{\sum_{h|\omega_{1h}=0} p_h}\omega_{1h_1}$, $\omega^t \in (R^\ell_+)^m$, $\sum_{i=1}^m \omega^t_i = \sum_{i=1}^m \omega_i$ and for all $i$, $p \cdot \omega^t_i = p \cdot \omega_i$ and finally, $\omega^t_1 \in R^\ell_{++}$. Thus, $p$ is an equilibrium price vector of $\mathcal{L}(b,\omega^t)$ and it is unique since one consumer has a strictly positive endowment. □

The following result concludes our study.

**Proposition 4.5.** *Let $(b,\omega) \in \mathcal{U} \setminus \operatorname{int}\mathcal{U}$. Then, the equilibrium price is not locally Lipschitz continuous at $(b,\omega)$.*

**Proof.** The goods $h$ is chosen as numéraire. Let $(b,\omega) \in \mathcal{U} \setminus \operatorname{int}\mathcal{U}$. Then there exists a sequence $(b^\nu,\omega^\nu) \in \mathcal{W} \setminus \mathcal{U}$, which converges to $(b,\omega)$. Since $(b^\nu,\omega^\nu) \notin \mathcal{U}$, the economy has at least two different normalized equilibrium price vectors $p^\nu$ and $q^\nu$. From the previous lemma, for all $\nu$, there exists $\bar{\omega}^\nu$ and $\tilde{\omega}^\nu$ such that $(b^\nu,\bar{\omega}^\nu)$ and $(b^\nu,\tilde{\omega}^\nu)$ belong to $\mathcal{U}$, $p^\nu$

is the normalized equilibrium price of $\mathcal{L}(b^\nu, \bar{\omega}^\nu)$, $q^\nu$ is the normalized equilibrium price of $\mathcal{L}(b^\nu, \tilde{\omega}^\nu)$, and $\|\bar{\omega}^\nu - \omega^\nu\| \leq \|p^\nu - q^\nu\|/\nu$ and $\|\tilde{\omega}^\nu - \omega^\nu\| \leq \|p^\nu - q^\nu\|/\nu$. Since $(b^\nu, \bar{\omega}^\nu)$ and $(b^\nu, \tilde{\omega}^\nu)$ converge to $(b, \omega)$, this implies that the equilibrium price is not locally Lipschitz continuous since $\|p^\nu - q^\nu\| \geq (\nu/2)\|\bar{\omega}^\nu - \tilde{\omega}^\nu\|$. □

## Appendix

We give the proof of Lemma 3.4. Let $T^i$ the sub-matrix of $T$ obtained by suppressing the $i$th column and $i$th row. Let us assume that it is non regular. Then, there exists $(\lambda_k)_{k \neq i} \neq 0$ in the kernel of the transpose of $T^i$. For all $j \neq i$, $\sum_{k \neq i} \lambda_k t_{kj} = 0$. By considering $\lambda$ or $-\lambda$, we can assume that $\bar{\lambda} = \max_{k \neq i}\{\lambda_k\} > 0$. Let $N = \{j \neq i \mid \lambda_j = \bar{\lambda}\}$.

Let $j \in N$. We now prove that for all $k \notin N$, $t_{kj} = 0$. Indeed, if $k \notin N$ and $k \neq i$, $t_{kj} \leq 0$ and $\lambda_k < \lambda_j = \bar{\lambda}$. Consequently, $\lambda_k t_{kj} \geq \bar{\lambda} t_{kj}$ with a strict inequality if $t_{kj} < 0$. Thus, $0 = \sum_{k \neq i} \lambda_k t_{kj} \geq \bar{\lambda} \sum_{k \neq i} t_{kj} = -\bar{\lambda} t_{ij} \geq 0$. The last equality comes from the fact that $\sum_{k=1}^n t_{kj} = 0$. This implies that the two inequalities are actually equalities and, thus, $t_{kj} = 0$ for all $k \notin N$ and $k \neq i$ and $t_{ij} = 0$.

But this property contradicts the assumption on $T$ since the above property is true for all $j \in N$. Hence $T^i$ is regular.

It remains to prove that $(T^i)^{-1}$ has non-negative elements. Note that for any $\varepsilon > 0$, the matrix obtained by adding $\varepsilon$ to the elements on the diagonal of $T^i$ is strictly diagonal dominant. Thus, the elements of its inverse are non-negative (see, for example, Varga [17]). Going to the limit, one deduces the desired result. □

## References


[1] J. M. Bonnisseau, M. Florig: Existence and optimality of oligopoly equilibria in linear exchange economies, Econ. Theory 22 (2003) 727–741.

[2] J. M. Bonnisseau, M. Florig, A. Jofré: Continuity and uniqueness of equilibria for linear exchange economies, J. Optimization Theory Appl. 109(2) (2001) 237–263.

[3] J. M. Bonnisseau, M. Florig, A. Jofré: Differentiability of equilibria for linear exchange economies, J. Optimization Theory Appl. 109(2) (2001) 264–288.

[4] J. M. Bottazzi: Accessibility of Pareto optima by Walrasian exchange processes, J. Math. Econ. 23 (1994) 585–603.

[5] J. M. Bottazzi, B. de Meyer: A market game for assets and taxed investors, J. Math. Econ. 39 (2003) 657–675.

[6] P. Champsaur, B. Cornet: Walrasian exchange processes, in: Economic Decision-Making: Games, Econometrics and Optimization, J. J. Gabszewicz, J. F. Richard, L. A. Wolsey (eds.), Elsevier (1990) 3–13.

[7] B. Cornet: Linear Exchange Economies, Cahier Eco-Math, CERMSEM, Université de Paris 1 (1989).

[8] B. C. Eaves: A finite algorithm for the linear exchange model, J. Math. Econ. 3 (1976) 197–203.

[9] M. Florenzano, E. Moreno-Garcia: Linear exchange economies with a continuum of agents, Spanish Economic Review 3 (2001) 253–272.

[10] M. Florig: Arbitrary small indivisibilities, Econ. Theory 22 (2003) 831–843.

[11] M. Florig: Equilibrium correspondence of linear exchange economies, J. Optimization Theory Appl. 120(1) (2004) 97–109.

[12] J. J. Gabszewicz, P. Michel: Oligopoly equilibria in exchange economies, in: Trade, Technology and Economics. Essays in Honour of Richard G. Lipsey, B. C. Eaton, R. G. Harris (eds.), Edward Elgar, Cheltenham (1997) 217–240.

[13] D. Gale: Price equilibrium for linear models of exchange, Technical Report P-1156, The Rand Corporation (1957).

[14] D. Gale: The Theory of Linear Economic Models, Academic Press (1960).

[15] D. Gale: The linear exchange model, J. Math. Econ. 3 (1976) 205–209.

[16] J. F. Mertens: The limit-price mechanism, J. Math. Econ. 39 (2003) 433–528.

[17] R. S. Varga: Matrix Iterative Analysis, Prentice Hall, Englewood (1962).