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Continuity of the Optimal Value Function under some Hyperspace Topologies

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In this paper we characterize UC, boundedly UC and reflexive spaces in terms of continuity of the optimal value function of abstract minimization problems with respect to some hyperspace topologies defined on the classes of nonempty closed and closed convex sets.

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1. Introduction

Given a metric space X, $A(\neq \emptyset) \subseteq X$ and $f : X \longrightarrow \mathbb{R}$, the optimal value or simply the value of the minimization problem (f, A) minimizing f over A is $v(f, A) = \inf\{f(x) : x \in A\}$. For many practical and theoretical reasons it is often of interest to study the continuous dependence of the optimal value of a minimization problem under various perturbations of the constraint set. By varying the constraint set continuously with respect to various hyperspace topologies, several authors have studied the continuity of the value function and their results were obtained mainly by assuming some nice properties on the objective function [3, 8, 11, 12, 13, 17, 18, 21-23]. Our major objective in this paper is to establish the equivalence of continuity of the value function with respect to some hyperspace topologies and certain properties of the space on which the minimization problem is defined.

The necessary and sufficient conditions for upper semicontinuity of the value function have already been observed in [22, Theorem 3.3], and these conditions do not involve any special properties of the space. In Section 3, we provide necessary and sufficient conditions for lower semicontinuity of the value function in terms of some properties of the space. Sufficient conditions for continuity of the value function then follow from the results of Section 3 and Theorem 3.3 of [22]. However, characterizing the space through continuity of the value function is more involved, which is discussed in Section 4.

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2. Notations and Preliminaries

Let X be a metric space. If $x \in X$, $A(\neq \emptyset) \subseteq X$, $B(\neq \emptyset) \subseteq X$ and $\varepsilon > 0$, we write $d(x, A)$ for $\inf\{d(x, a) : a \in A\}$, $d(A, B)$ for $\inf\{d(a, b) : a \in A,\ b \in B\}$, $e(A, B)$ for $\sup\{d(a, B) : a \in A\}$, $B_\varepsilon(A)$ for $\{x \in X : d(x, A) < \varepsilon\}$ and $\overline{B}_\varepsilon(A)$ for $\{x \in X : d(x, A) \leq \varepsilon\}$. $C(X)$ (resp. $CB(X)$) denotes the class of all nonempty closed (resp. nonempty closed bounded) subsets of X. In case X is a normed linear space, we denote by $CC(X)$ (resp. $CCB(X)$, $WK(X)$) the class of all nonempty closed convex (resp. nonempty closed convex bounded, nonempty weakly compact) subsets of X. For a normed linear space X, the closed unit ball $\{x \in X : \|x\| \leq 1\}$ is denoted by B_X and the set $\{f \in X^* : \|f\| = 1\}$ is denoted by S_{X^*}, where X^* is the dual of X. In the sequel, X will denote a metric space unless we consider convex subsets or weakly compact subsets of X, in which case X will be a normed linear space.

We now recall various notions of convergence of nonempty closed and nonempty closed convex sets [5, 22]. Let $\mathcal{X}_1(\neq \emptyset) \subseteq C(X)$ and $\mathcal{X}_2(\neq \emptyset) \subseteq C(X)$. We define the topological convergence on $\mathcal{X}_1$ induced by $\mathcal{X}_2$ (and hence a hyperspace topology $\tau(\mathcal{X}_2)$ on $\mathcal{X}_1$ induced by $\mathcal{X}_2$) as follows: Given $A \in \mathcal{X}_1$ and a net $(A_i)_{i \in I}$ in $\mathcal{X}_1$, where I is a directed set,

$$A_i \to A \text{ with respect to } \tau(\mathcal{X}_2) \text{ iff } d(A_i, C) \to d(A, C) \text{ for all } C \in \mathcal{X}_2.$$

By taking $\mathcal{X}_2$ as $C(X)$ (resp. $CB(X)$), we get the convergence in the proximal (resp. bounded proximal) topology on $\mathcal{X}_1$. If $\mathcal{X}_1 \subseteq CC(X)$, then by taking $\mathcal{X}_2$ as $CC(X)$ (resp. $CCB(X)$), we get the convergence in the linear (resp. slice) topology on $\mathcal{X}_1$.

We also recall [5] that the Mosco topology on $CC(X)$ is defined as the topology having the following family of sets as its subbase:

$$\{V^- : V \text{ norm open }\}, \qquad \{(K^c)^+ : K \text{ weakly compact }\},$$

where

$$V^- = \{E \in CC(X) : E \cap V \neq \emptyset\},$$
$$(K^c)^+ = \{E \in CC(X) : E \subseteq K^c\},$$

and K^c denotes the complement of K in X.
For any normed linear space X, the Mosco topology on $CC(X)$ is compatible [5, Theorem 5.4.6] with a fundamental notion of sequential convergence introduced by U. Mosco [19]. Moreover if X is reflexive, the Mosco topology is obtained by taking $\mathcal{X}_1 \subseteq CC(X)$ and $\mathcal{X}_2 = WK(X)$ in the above presentation of topological convergences [5, Proposition 5.4.15].

The Attouch-Wets topology on $C(X)$ is the topology that $C(X)$ inherits from $C(X, \mathbb{R})$, the class of all real-valued continuous functions on X, equipped with the topology of uniform convergence on bounded subsets of X, under the identification $A \leftrightarrow d(\cdot, A)$.

The sequential convergence under the Attouch-Wets topology can be described in the following convenient way [2], [5, Corollary 3.1.8]. Given A, $A_n \in C(X)$ ($n \in \mathbb{N} := \{1, 2, ...\}$), $A_n \to A$ in the Attouch-Wets topology iff for each $\rho > 0$, $\text{haus}_\rho(A_n, A) \to 0$, where for all C, $D \in C(X)$ and for all $\rho > 0$,

$$\text{haus}_\rho(C, D) = \max\{e(C \cap \rho B_X, D), e(D \cap \rho B_X, C)\}.$$

The following facts comparing some of the hyperspace topologies are well-known [5, 22] and are recorded here for later use. The Attouch-Wets topology on $C(X)$ is stronger than the bounded proximal topology and on $CC(X)$, the bounded proximal topology is stronger than the slice topology which in turn is stronger than the Mosco topology. The slice topology coincides with the Mosco topology on $CC(X)$ iff X is reflexive. But even if X is reflexive, the Attouch-Wets topology on $CC(X)$ may be strictly stronger than the bounded proximal topology and the bounded proximal topology on $CC(X)$ may be strictly stronger than the slice topology. However, if X happens to be finite-dimensional, all the four topologies mentioned above coincide on $CC(X)$.

The following condition on $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) will be referred to as the enlargement condition on $\mathcal{X}_1$ (resp. $\mathcal{X}_2$):

$$\text{Whenever } A \in \mathcal{X}_1 \text{ (resp. } A \in \mathcal{X}_2), \overline{B}_\varepsilon(A) \in \mathcal{X}_1 \text{ (resp. } \overline{B}_\varepsilon(A) \in \mathcal{X}_2) \text{ for all } \varepsilon > 0.$$

For a given function $f : X \longrightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$, the sublevel set of f at height α is $\operatorname{lev}(f,\alpha) = \{x \in X : f(x) \leq \alpha\}$. The function f is said to be inf-bounded (resp. weakly inf-compact, quasi-convex) if $\operatorname{lev}(f,\alpha)$ is bounded (resp. weakly compact, convex) for each $\alpha \in \mathbb{R}$.

3. Lower Semicontinuity of the Value Function

The following result concerning the upper semicontinuity of the value function is in Theorem 3.3 of [22].

Theorem 3.1. *Let X be a metric space and $(A_i)_{i\in I} \subseteq C(X)$, $A \in C(X)$. The following assertions are equivalent.*

(*i*) $\limsup d(A_i, C) \leq d(A, C)$ *for all* $C \in C(X)$.
(*ii*) $\limsup d(x, A_i) \leq d(x, A)$ *for all* $x \in X$.
(*iii*) $\limsup v(f, A_i) \leq v(f, A)$ *for every upper semicontinuous function* $f : X \longrightarrow \mathbb{R}$.

We note here that the upper semicontinuity of f in the above mentioned result is essential. In fact, given $f : X \longrightarrow \mathbb{R}$, each of the following two statements implies that f is upper semicontinuous.

(i) $\limsup v(f, A_i) \leq v(f, A)$ for every net $(A_i)_{i\in I} \subseteq C(X)$ and for every $A \in C(X)$ satisfying $\limsup d(A_i, C) \leq d(A, C)$ for all $C \in C(X)$.
(ii) $\limsup v(f, A_i) \leq v(f, A)$ for every net $(A_i)_{i\in I} \subseteq C(X)$ and for every $A \in C(X)$ satisfying $\limsup d(x, A_i) \leq d(x, A)$ for all $x \in X$.

On the contrary, if f is not upper semicontinuous at $x_0 \in X$, then there exist $\alpha \in \mathbb{R}$ and a sequence (x_n) in X converging to x_0 such that $f(x_0) < \alpha$ and $f(x_n) \geq \alpha$ for all $n \in \mathbb{N}$. Considering $A_n = \{x_n\}$ $(n \in \mathbb{N})$ and $A = \{x_0\}$, we find that $d(A_n, C) \to d(A, C)$ for all $C \in C(X)$ and $d(x, A_n) \to d(x, A)$ for all $x \in X$. However $v(f, A) < \alpha \leq v(f, A_n)$ for all $n \in \mathbb{N}$.

Theorem 3.1 deals with the upper semicontinuity of the value function for upper semicontinuous functions. The following result is crucial for obtaining conditions for lower semicontinuity of the value function for lower semicontinuous functions. (As in the case of upper semicontinuity mentioned above, we have to consider lower semicontiuous functions in order to obtain lower semicontinuity of the value function.)

Theorem 3.2. *Let $\mathcal{X}_1(\neq \emptyset) \subseteq C(X)$ and $\mathcal{X}_2(\neq \emptyset) \subseteq C(X)$ satisfy the enlargement condition. Then the following conditions are equivalent.*

(*i*) *Whenever $A \in \mathcal{X}_1$, $E \in \mathcal{X}_2$ and $d(A, E) = 0$, then $A \cap E \neq \emptyset$.*

(*ii*) *$v(f, \cdot) : (\mathcal{X}_1, \tau(\mathcal{X}_2)) \longrightarrow [-\infty, \infty)$ is lower semicontinuous for every $f : X \longrightarrow \mathbb{R}$ with $lev(f, \alpha) \in \mathcal{X}_2$ for all $\alpha > v(f, X)$.*

Proof. (*i*) $\Rightarrow$ (*ii*): Let $f : X \longrightarrow \mathbb{R}$ be such that $\mathrm{lev}\,(f, \alpha) \in \mathcal{X}_2$ for all $\alpha > v(f, X)$ and let $A_0 \in \mathcal{X}_1$. If $v(f, A_0) = -\infty$, then $v(f, \cdot)$ is lower semicontinuous at A_0. Suppose that $v(f, A_0) > -\infty$ and that $v(f, \cdot)$ is not lower semicontinuous at A_0. Then there exist α, $\beta \in \mathbb{R}$ and a net $(A_i)_{i\in I}$ in $\mathcal{X}_1$ converging to A_0 with respect to $\tau(\mathcal{X}_2)$ such that $v(f, A_i) \leq \alpha < \beta < v(f, A_0)$ for all $i \in I$. This implies that $\mathrm{lev}\,(f, \beta) \cap A_i \neq \emptyset$ for all $i \in I$. Hence $d(A_i, \mathrm{lev}\,(f, \beta)) = 0$ for all $i \in I$. So we get $d(A_0, \mathrm{lev}\,(f, \beta)) = 0$ and therefore by (*i*), $\mathrm{lev}\,(f, \beta) \cap A_0 \neq \emptyset$, which is a contradiction.

(*ii*) $\Rightarrow$ (*i*): If (*i*) is not true, there exist $A \in \mathcal{X}_1$, $E \in \mathcal{X}_2$ such that $d(A, E) = 0$ and $A \cap E = \emptyset$. By the enlargement condition on $\mathcal{X}_2$, $\overline{B}_n(E) \in \mathcal{X}_2$ for all $n \in \mathbb{N}$. Define $f : X \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in E, \\ n & \text{if } x \in \overline{B}_n(E) \setminus \overline{B}_{n-1}(E),\ n = 1, 2, ... \end{cases}$$

Clearly $\mathrm{lev}\,(f, \alpha) \in \mathcal{X}_2$ for all $\alpha > v(f, X) = 0$. If $A_n = \overline{B}_{\frac{1}{n}}(A)$ for $n = 1, 2, ...$, then $A_n \in \mathcal{X}_1$ and $A_n \to A$ with respect to $\tau(\mathcal{X}_2)$. (In fact, $A_n \to A$ in the proximal topology on $C(X)$.)

However, $v(f, A) > 0$ and $v(f, A_n) = 0$ for all $n \geq 2$. Hence $v(f, \cdot)$ is not lower semicontinuous. This completes the proof. □

Applying the previous result to several known hyperspace topologies, we will obtain characterizations of some spaces in terms of lower semicontinuity of the value function. These are mentioned in the Corollaries 3.3, 3.4 and 3.5 below.

We need the following definitions.

A metric space is called a UC space [1, 5] if each continuous function on it with values in an arbitrary metric space is uniformly continuous. It can be shown [1, 5] that a metric space X is a UC space iff whenever A and B are disjoint nonempty closed subsets of X, then there exists $\varepsilon > 0$ such that $B_\varepsilon(A) \cap B_\varepsilon(B) = \emptyset$.

A metric space such that every continuous function on it with values in an arbitrary metric space is uniformly continuous on bounded sets is called a boundedly UC space [6, 14]. A metric space X is a boundedly UC space iff whenever A and B are disjoint nonempty closed subsets of X with one of them bounded, then there exists $\varepsilon > 0$ such that $B_\varepsilon(A) \cap B_\varepsilon(B) = \emptyset$ [6].

We refer to [1, 5, 6, 14, 20] for several characterizations of UC and boundedly UC spaces.

Corollary 3.3. *The following statements are equivalent.*

(*i*) *X is a UC space.*

(*ii*) *$v(f, \cdot)$ is a lower semicontinuous function in the proximal topology on $C(X)$ for every lower semicontinuous function $f : X \longrightarrow \mathbb{R}$.*

Proof. Take $\mathcal{X}_1 = \mathcal{X}_2 = C(X)$ in Theorem 3.2. □

Corollary 3.4. *The following assertions are equivalent.*

(*i*) *X is a boundedly UC space.*
(*ii*) *$v(f,\cdot)$ is a lower semicontinuous function in the proximal topology on $CB(X)$ for every lower semicontinuous function $f : X \longrightarrow \mathbb{R}$.*
(*iii*) *$v(f,\cdot)$ is a lower semicontinuous function in the bounded proximal topology on $C(X)$ for every lower semicontinuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

Proof. Take $\mathcal{X}_1$ as $CB(X)$ (resp. $C(X)$) and $\mathcal{X}_2$ as $C(X)$ (resp. $CB(X)$) in Theorem 3.2. □

Corollary 3.5. *For a Banach space X, the following statements are equivalent.*

(*i*) *X is reflexive.*
(*ii*) *$v(f,\cdot)$ is a lower semicontinuous function in the slice topology on $CC(X)$ for every quasi-convex lower semicontinuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*
(*iii*) *$v(f,\cdot)$ is a lower semicontinuous function in the linear topology on $CCB(X)$ for every quasi-convex lower semicontinuous function $f : X \longrightarrow \mathbb{R}$.*

Proof. Take $\mathcal{X}_1$ as $CC(X)$ (resp. $CCB(X)$) and $\mathcal{X}_2$ as $CCB(X)$ (resp. $CC(X)$) in Theorem 3.2 and use the fact [15, p. 161] that a Banach space X is reflexive iff whenever A and B are disjoint nonempty closed convex subsets of X with one of them bounded, then there exists $\varepsilon > 0$ such that $B_\varepsilon(A) \cap B_\varepsilon(B) = \emptyset$. □

4. Continuity of the Value Function

The following two programs related to the continuity of the value function on a given subclass of $C(X)$ for a given class of functions arise naturally.

(I) Characterizing certain property of the space X in terms of continuity of $v(f,\cdot)$ with respect to a given hyperspace topology.
(II) For a given X, characterizing the hyperspace topology in terms of continuity of $v(f,\cdot)$.

Such programs are not completely independent of each other and are not entirely original. For example, we refer to [10] for (I) for the Mosco topology and to [8, 21, 22] for (II) for several known hyperspace topologies. The main aim of this section is to discuss (I) for some of the important hyperspace topologies, like the Attouch-Wets topology, the slice topology and the bounded proximal topology. We also briefly discuss (II) in this section.

The following result giving continuity of the value function in reflexive spaces follows as a direct consequence of Theorem 3.1 of [21].

Theorem 4.1. *If X is a reflexive space, then $v(f,\cdot)$ is continuous in the Mosco topology on $CC(X)$ for each quasi-convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

We note here that the reflexivity of X in the above mentioned result is essential. In fact, if X is not reflexive, then there exist C_0, $C_n \in CC(X)$ $(n \in \mathbb{N})$ such that $C_n \to C_0$ in the Mosco topology, but $\lim d(x_0, C_n)$ fails to exist for some $x_0 \in X$ [10]. Consider the continuous convex inf-bounded function $f : X \longrightarrow \mathbb{R}$, where $f(x) = \|x - x_0\|$ for all $x \in X$. Then $v(f, C_n) = d(x_0, C_n)$ for all $n \in \mathbb{N}$, so $\lim v(f, C_n)$ does not exist.

The above argument shows that the following result, which is stronger than the converse of Theorem 4.1, is in fact true.

Theorem 4.2. *If $v(f,\cdot)$ is a continuous function in the Mosco topology on $CC(X)$ for every convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$, then X is reflexive.*

Theorem 4.1 and Theorem 4.2 together characterize the reflexivity of X in terms of the continuity of $v(f,\cdot)$ under the Mosco topology on $CC(X)$ for quasi-convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$. The following theorem, which is the main result of this paper, provides a similar characterization of reflexivity of a Banach space X for three important hyperspace topologies on $CC(X)$, namely, the slice topology, the bounded proximal topology and the Attouch-Wets topology. Although these three topologies may be distinct even in reflexive spaces (as noted in Section 2), this theorem shows that the continuity of the value function under each of them is equivalent to the reflexivity of the underlying space.

Theorem 4.3. *For a Banach space X, the following statements are equivalent.*

(*i*) *X is reflexive.*

(*ii*) *$v(f,\cdot)$ is a continuous function in the Mosco topology on $CC(X)$ for every quasi-convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

(*iii*) *$v(f,\cdot)$ is a continuous function in the slice topology on $CC(X)$ for every quasi-convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

(*iv*) *$v(f,\cdot)$ is a continuous function in the bounded proximal topology on $CC(X)$ for every quasi-convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

(*v*) *$v(f,\cdot)$ is a continuous function in the Attouch-Wets topology on $CC(X)$ for every quasi-convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

Proof. Although $(i) \Rightarrow (ii)$ is just the Theorem 4.1 mentioned above, we give here another argument using the results of Section 3. The upper semicontinuity of $v(f,\cdot)$ follows from Theorem 3.1 and the lower semicontinuity of $v(f,\cdot)$ follows from Corollary 3.5, since the slice topology coincides with the Mosco topology on $CC(X)$ if X is reflexive.

Using the comparisons of hyperspace topologies given in Section 2, the implications $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ follow immediately.

$(v) \Rightarrow (i)$: If X is not reflexive, then by the James theorem [16], there exists $f \in S_{X^*}$ such that f does not attain its norm on B_X. So $B_X \cap H = \emptyset$, where $H = \{x \in X : f(x) = 1\}$. Using the Bishop-Phelps theorem [9], we can choose $f_n \in S_{X^*}$ $(n \in \mathbb{N})$ such that $f_n^{-1}(1) \cap B_X \neq \emptyset$ for all $n \in \mathbb{N}$ and $\|f_n - f\| \to 0$. Clearly $f_n^{-1}(1)$, $f^{-1}(1) \in CC(X)$ for all $n \in \mathbb{N}$ and it is shown in [5, Theorem 3.4.1] that $f_n^{-1}(1) \to f^{-1}(1)$ in the Attouch-Wets topology.

We construct below a quasi-convex continuous inf-bounded function $h : X \longrightarrow \mathbb{R}$ such that $\lim v(h, f_n^{-1}(1)) \neq v(h, f^{-1}(1))$.

Define $h : X \longrightarrow \mathbb{R}$ by

$$h(x) = \max\{g(x), d(x, B_X)\}, \quad x \in X,$$

where for $x \in X$,

$$g(x) = \begin{cases} \frac{d(x,B_X)}{d(x,B_X)+d(x,H)} & \text{if } f(x) \leq 1, \\ 1 & \text{if } f(x) > 1. \end{cases}$$

Note that the function $g : X \longrightarrow \mathbb{R}$ is continuous, $g(x) = 0$ for all $x \in B_X$ and $g(x) = 1$ for all $x \in H$. We show that g is quasi-convex.

Let $0 \leq \alpha < 1$, x_1, $x_2 \in \operatorname{lev}(g, \alpha)$ and λ_1, $\lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$. Using Ascoli's formula [5, Theorem 1.1.2], we obtain

$$\begin{aligned} d(\lambda_1 x_1 + \lambda_2 x_2, H) &= 1 - f(\lambda_1 x_1 + \lambda_2 x_2) \\ &= \lambda_1[1 - f(x_1)] + \lambda_2[1 - f(x_2)] \\ &= \lambda_1 d(x_1, H) + \lambda_2 d(x_2, H). \end{aligned}$$

Given $t_0 > 0$, the function $t \longmapsto \frac{t}{t+t_0}$ $(t \geq 0)$ is increasing, so using the convexity of the function $x \longmapsto d(x, B_X)$ $(x \in X)$, we have

$$g(\lambda_1 x_1 + \lambda_2 x_2) \leq \frac{\lambda_1 d(x_1, B_X) + \lambda_2 d(x_2, B_X)}{\lambda_1 d(x_1, B_X) + \lambda_2 d(x_2, B_X) + \lambda_1 d(x_1, H) + \lambda_2 d(x_2, H)}.$$

Since

$$d(x_1, B_X) \leq \alpha[d(x_1, B_X) + d(x_1, H)]$$

and

$$d(x_2, B_X) \leq \alpha[d(x_2, B_X) + d(x_2, H)],$$

we get $g(\lambda_1 x_1 + \lambda_2 x_2) \leq \alpha$. In case $\alpha \geq 1$, we have $\operatorname{lev}(g, \alpha) = X$. Therefore g is quasi-convex.

It follows that the function h is quasi-convex continuous inf-bounded such that $h(x) = 0$ for all $x \in B_X$ and $h(x) \geq 1$ for all $x \in H$. Moreover $v(h, f_n^{-1}(1)) = 0$ for all $n \in \mathbb{N}$ and $v(h, f^{-1}(1)) = 1$. Consequently $v(h, \cdot)$ is not continuous in the Attouch-Wets topology on $CC(X)$. $\square$

We note that in the previous result the convexity conditions on the constraint sets and the objective function play a crucial role. In Corollary 4.6, we will show that without any convexity conditions on the constraint sets and the objective function, the continuity of the value function under the Attouch-Wets topology forces the underlying normed linear space to be finite-dimensional. On the other hand, it follows from Theorem 3.6 of [7] and Lemma 7.5.3 of [5] that for any normed linear space X, $v(f, \cdot)$ is a continuous function in the Attouch-Wets topology on $CC(X)$ for every convex continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$. Thus the quasi-convexity condition on the objective function in Theorem 4.3 is appropriate for characterizing reflexivity of the space.

The next two theorems characterize UC and boundedly UC spaces in terms of continuity of the value function.

Theorem 4.4. *The following statements are equivalent.*

(*i*) *X is UC (resp. boundedly UC).*

(*ii*) *$v(f, \cdot)$ is a continuous function in the proximal topology on $C(X)$ (resp. $CB(X)$) for every continuous function $f : X \longrightarrow \mathbb{R}$.*

Proof. (*ii*) $\Rightarrow$ (*i*): If X is not a UC space then there exist nonempty disjoint closed subsets A and B of X such that $B_{\frac{1}{n}}(A) \cap B_{\frac{1}{n}}(B) \neq \emptyset$ for $n = 1, 2, ...$ For $n = 1, 2, ...,$

choose $x_n \in B$ such that $d(x_n, A) < \frac{1}{n}$. The function $f : X \longrightarrow \mathbb{R}$ defined by

$$f(x) = \max\{g(x), d(x, A)\} \quad (x \in X),$$

where

$$g(x) = \frac{d(x, B)}{d(x, A) + d(x, B)} \quad (x \in X),$$

is continuous, $f(x) = 1$ for all $x \in A$ and $f(x_n) = d(x_n, A)$ for all $n \in \mathbb{N}$. Consider the closed sets $A_n = A \cup \{x_n\}$ for all $n \in \mathbb{N}$. Clearly $A_n \to A$ in the proximal topology. However $v(f, A) = 1$ and $\lim v(f, A_n) = 0$.

If X is not boundedly UC then we may choose the set A in above to be bounded and repeat the above argument.

The proof of $(i) \Rightarrow (ii)$ follows from Theorem 3.1, Corollary 3.3 and Corollary 3.4. □

Theorem 4.5. *The following assertions are equivalent.*

(i) *X is boundedly UC.*
(ii) *$v(f, \cdot)$ is a continuous function in the bounded proximal topology on $C(X)$ for every continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*
(iii) *$v(f, \cdot)$ is a continuous function in the Attouch-Wets topology on $C(X)$ for every continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

Proof. The proof of $(i) \Rightarrow (ii)$ follows from Theorem 3.1 and Corollary 3.4.

The implication $(iii) \Rightarrow (i)$ can be proved as in Theorem 4.4 by taking the set A to be bounded. In this case $A_n \to A$ in the Attouch-wets topology and f becomes inf-bounded. □

We remark that the implication $(i) \Rightarrow (ii)$ of Theorem 4.4 for the case of UC space also follows directly from Theorem 7.2 of [22] and the implication $(i) \Rightarrow (ii)$ of Theorem 4.5 also follows directly from Theorem 9.2 of [22]. The implication $(i) \Rightarrow (ii)$ of Theorem 4.4 for the case of UC space is also proved in Proposition 3.2 of [8] and the implication $(i) \Rightarrow (ii)$ of Theorem 4.5 can also be obtained from Proposition 3.5 of [8].

Since a normed linear space is boundedly UC iff it is finite-dimensional, the following corollary is immediate from Theorem 4.5.

Corollary 4.6. *For a normed linear space X, the following assertions are equivalent.*

(i) *X is finite-dimensional.*
(ii) *$v(f, \cdot)$ is a continuous function in the Attouch-Wets topology on $C(X)$ for every continuous inf-bounded function $f : X \longrightarrow \mathbb{R}$.*

We now discuss (II) mentioned at the beginning of this section.

For reflexive spaces the following characterization of the Mosco topology in terms of continuity of the value function is provided in Theorem 3.1 of [21].

Theorem 4.7. *If X is a reflexive space, then the Mosco topology $\tau(WK(X))$ is the weakest topology on $CC(X)$ for which*

$$v(f, \cdot) : (CC(X), \tau(WK(X))) \longrightarrow \mathbb{R}$$

is continuous for each continuous weakly inf-compact function $f : X \longrightarrow \mathbb{R}$.

The following result will provide similar characterizations for other topologies such as proximal and bounded proximal.

Theorem 4.8. *Let* $\mathcal{X}_1(\neq \emptyset) \subseteq C(X)$ *and* $\mathcal{X}_2(\neq \emptyset) \subseteq C(X)$ *satisfy the enlargement condition. Suppose that* $A \cap E \neq \emptyset$ *whenever* $A \in \mathcal{X}_1$, $E \in \mathcal{X}_2$ *and* $d(A, E) = 0$. *Then* $\tau(\mathcal{X}_2)$ *is the weakest topology on* $\mathcal{X}_1$ *such that* $v(f, \cdot)$ *is continuous on* $\mathcal{X}_1$ *for every continuous function* $f : X \longrightarrow \mathbb{R}$ *such that* $lev(f, \alpha) \in \mathcal{X}_2$ *for all* $\alpha > v(f, X)$.

Proof. The continuity of $v(f, \cdot)$ with respect to $\tau(\mathcal{X}_2)$ for every continuous function $f : X \longrightarrow \mathbb{R}$ with lev $(f, \alpha) \in \mathcal{X}_2$ for all $\alpha > v(f, X)$, follows from Theorem 3.1 together with $(i) \Rightarrow (ii)$ of Theorem 3.2.

To show that $\tau(\mathcal{X}_2)$ is the weakest such topology on $\mathcal{X}_1$, we note the following:

For a fixed $C \in \mathcal{X}_2$, define $f(x) = d(x, C)$ for all $x \in X$. Then f is continuous, lev $(f, \alpha) = \overline{B}_\alpha(C) \in \mathcal{X}_2$ for all $\alpha > v(f, X)$ and $v(f, A) = d(A, C)$ for all $A \in \mathcal{X}_1$. □

Taking suitable $\mathcal{X}_1$ and $\mathcal{X}_2$, we obtain the following corollary.

Corollary 4.9. *Let* X *be a UC space (resp. boundedly UC space). Then the proximal topology (resp. bounded proximal topology) is the weakest topology on* $C(X)$ *such that* $v(f, \cdot)$ *is continuous for every continuous (resp. continuous inf-bounded) function* $f : X \longrightarrow \mathbb{R}$.

We remark that Corollary 4.9 is not new and it follows from Theorem 7.2 and Theorem 9.2 of [22]. Corollary 4.9 can also be obtained from a unified result given in Proposition 3.5 of [8]. Moreover it is easy to see that Theorem 4.7 is also a consequence of Theorem 4.8. A stronger result than Theorem 4.7 is contained in Proposition 9.5 of [22]. In fact, such interesting characterizations of other known hyperspace topologies in terms of continuity of the value function can be found in [8, 22]. Similar characterizations of topologies on certain classes of functions have been explored in [4].

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