Journal of Convex Analysis
Volume 12 (2005), No. 2, 331–349

A Turnpike Property for a Class of Variational Problems

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Received May 17, 2004
Revised manuscript received October 04, 2004

In this paper we examine the structure of extremals of variational problems with continuous integrands $f : R^n \times R^n \to R^1$ which belong to a complete metric space of functions. Our results deal with the turnpike properties of variational problems. To have this property means that the solutions of the problems are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions.

Keywords: Complete metric space, good function, integrand, turnpike property

1991 Mathematics Subject Classification: 49J99, 54E52

1. Introduction

The study of turnpike properties of variational and optimal control problems has recently been a rapidly growing area of research. These problems arise in engineering [16, 17], in models of economic growth [1, 2, 4, 5, 6, 7] and in the theory of thermodynamical equilibrium for materials [13]. In this paper we analyse the structure of solutions of the variational problems

$$\int_{T_1}^{T_2} f(z(t), z'(t))dt \to \min,\ z(T_1) = x,\ z(T_2) = y, \tag{P}$$

$$z : [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$ and $f : R^n \times R^n \to R^1$ belongs to a space of integrands described below.

The main results in this paper deal with the so-called turnpike property of the variational problems (P). To have this property means, roughly speaking, that the solutions of the problems (P) are determined mainly by the integrand (cost function), and are essentially independent of the choice of interval and endpoint conditions.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [11]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, [1, 2, 4-10, 15] and the references mentioned there). In control theory turnpike properties were studied in [16, 17] for linear control systems with convex integrands.

ISSN 0944-6532 / \$ 2.50

Let us now define the space of integrands. Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$.

Denote by $\mathfrak{A}$ the set of all continuous functions $f : R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) for each $x \in R^n$ the function $f(x, \cdot) : R^n \to R^1$ is convex;

A(ii) $f(x,u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x,u) \in R^n \times R^n$;

A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1,u_1) - f(x_2,u_2)| \leq \epsilon \max\{f(x_1,u_1), f(x_2,u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \leq M, \ |u_i| \geq \Gamma \ (i = 1,2), \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta.$$

It is an elementary exercise to show that an integrand $f = f(x,u) \in C^1(R^{2n})$ belongs to $\mathfrak{A}$ if f satisfies assumptions A(i), A(ii) and there exists an increasing function $\psi_0 : [0,\infty) \to [0,\infty)$ such that for each $x, u \in R^n$

$$\max\{|\partial f/\partial x(x,u)|, \ |\partial f/\partial u(x,u)|\} \leq \psi_0(|x|)(1 + \psi(|u|)|u|).$$

For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base:

$$\begin{aligned} E(N,\epsilon,\lambda) = \{&(f,g) \in \mathfrak{A} \times \mathfrak{A} : \ |f(x,u) - g(x,u)| \leq \epsilon \\ &\text{for each } x, u \in R^n \text{ satisfying } |x|, |u| \leq N \\ &\text{and } (|f(x,u)| + 1)(|g(x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \\ &\text{for each } x, u \in R^n \text{ satisfying } |x| \leq N\}, \end{aligned}$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$ [3]. The space of integrands $\mathfrak{A}$ was introduced in [12].

Clearly, the uniform space $\mathfrak{A}$ is Hausdorff and has a countable base. Therefore $\mathfrak{A}$ is metrizable (by a metric ρ). It was shown in [14, Proposition 2.2] that the uniform space $\mathfrak{A}$ is complete. The metric ρ induces in $\mathcal{M}$ a topology.

Let $f \in \mathfrak{A}$. We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt \tag{1}$$

where $-\infty < T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For each $y, z \in R^n$ and each pair of numbers $T_1, T_2 \in R^1$ satisfying $T_1 < T_2$ we set

$$\begin{aligned} U^f(T_1, T_2, y, z) = \inf\{&I^f(T_1, T_2, x) : \ x : [T_1, T_2] \to R^n \\ &\text{is an a.c. function satisfying } x(T_1) = y, \ x(T_2) = z\}. \end{aligned} \tag{2}$$

It is not difficult to see that $-\infty < U^f(T_1, T_2, y, z) < \infty$ for each $y, z \in R^n$ and each $T_1 \in R^1$, $T_2 > T_1$.

For any a.c. function $x : [0, \infty) \to R^n$ we set

$$J(x) = \liminf_{T\to\infty} T^{-1} I^f(0, T, x). \tag{3}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{J(x) : \ x : [0, \infty) \to R^n \text{ is an a.c. function}\}. \tag{4}$$

Clearly $-\infty < \mu(f) < \infty$.

For each $x \in R^n$, $A \subset R^n$ set

$$d(x, A) = \inf\{|x - y| : \ y \in A\}.$$

Denote by $\mathcal{M}(f)$ the set of all locally absolutely continuous (a.c.) functions $v : R^1 \to R^n$ such that

$$\sup\{|v(t)| : \ t \in R^1\} < \infty \tag{5}$$

and that for each $T_1 \in R^1$, $T_2 > T_1$,

$$I^f(T_1, T_2, v) = U^f(T_1, T_2, v(T_1), v(T_2)). \tag{6}$$

Set

$$\mathcal{D}(f) = \cup\{v(R^1) : \ v \in \mathcal{M}(f)\}. \tag{7}$$

In [18, Theorem 1.1] we established the following result.

Theorem 1.1. *Let $f \in \mathfrak{A}$. Then $\mathcal{M}(f) \neq \emptyset$ and $\mathcal{D}(f)$ is a bounded closed subset of R^n.*

If a function $f : R^{2n} \to R^1$ is strictly convex, differentiable and satisfies a growth condition, and $\bar{y} \in R^n$ is a unique solution of the minimization problem $f(z, 0) \to \min$, $z \in R^n$, then optimal solutions of problem (P) spend most of time in a neighborhood of $\bar{y}$. Following the tradition, the point $\bar{y}$ is called the turnpike. If the function f is nonconvex, then the problem is more difficult. In [12] we studied the structure of extremals of nonconvex variational problems (P) with integrands $f \in \mathfrak{A}$. More precisely, in [12] we proved the existence of a subset $\mathcal{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ such that each integrand $f \in \mathcal{F}$ has a turnpike property which is an analog of the classical turnpike property for convex problems. We showed that for a generic integrand $f \in \mathfrak{A}$ there exists a nonempty compact set $H(f) \subset R^n$ such that the following property holds:

For each $\epsilon > 0$ there exists a constant $L > 0$ such that if v is a solution of problem (P), then for most of $t \in [T_1, T_2]$ the set $v([t, t + L])$ is equal to $H(f)$ up to ϵ with respect to the Hausdorff metric.

Note that for a generic integrand $f \in \mathfrak{A}$ the turnpike $H(f)$ is a nonempty compact subset of R^n which is not necessarily a singleton. It should be mentioned that there exists $f \in \mathfrak{A}$ which do not has this turnpike property. In [18] for any $f \in \mathfrak{A}$ we showed that approximate solutions of problem (P) spend most of time in a neighborhood of $\mathcal{D}(f)$.

Namely, we established the following result.

Theorem 1.2 (18, Theorem 1.2). *Let $f \in \mathfrak{A}$ and $M, \epsilon > 0$. Then there exist $\delta, L > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}$, each number $T \geq 2L$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq M, \; I^g(0,T,v) \leq U^g(0,T,v(0),v(T)) + \delta$$

the inequality $d(v(t), \mathcal{D}(f)) \leq \epsilon$ holds for all $t \in [L, T-L]$.

In this paper with any $f \in \mathfrak{A}$ we associate a nonempty closed subset $\mathcal{K}(f) \subset \mathcal{D}(f)$ and show that approximate solutions of problem (P) spend most of time in a neighborhood of $\mathcal{K}(f)$. We construct an example of $f \in \mathfrak{A}$ such that $\mathcal{K}(f) \neq \mathcal{D}(f)$. Thus the results of the present paper are improvements of the main results of [18].

In [12, Proposition 1.1] we obtained the following useful result.

Proposition 1.3. *For any a.c. function $x : [0,\infty) \to R^n$ either*

$$I^f(0,T,x) - T\mu(f) \to \infty \text{ as } T \to \infty$$

or

$$\sup\{|I^f(0,T,x) - T\mu(f)| : \; T \in (0,\infty)\} < \infty. \tag{8}$$

If an a.c. function $x : [0,\infty) \to R^n$ satisfies (8), then x is called an (f)-good function. It was shown in [14, Theorem 1.1] that for each $z \in R^n$ there exists an (f)-good function $Z : [0,\infty) \to R^n$ satisfying $Z(0) = z$.

For every $x \in R^n$ set

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - \mu(f)T]\colon v : [0,\infty) \to R^n \tag{9}$$
$$\text{is an a.c. function satisfying } v(0) = x.\}.$$

It follows from Theorems 8.1 and 8.2 and Proposition 7.3 of [12] that $\pi^f : R^n \to R^1$ is a continuous function such that

$$\lim_{|x|\to\infty} \pi^f(x) = \infty. \tag{10}$$

In view of Theorems 8.1 and 8.2 of [12]

$$U^f(0,T,x,y) - T\mu(f) - \pi^f(x) + \pi^f(y) \geq 0 \tag{11}$$
$$\text{for each } T > 0 \text{ and each } x, y \in R^n.$$

This inequality implies that for each interval $D = [T_1, T_2]$ where $T_1 \in R^1$ and $T_2 > T_1$ and each a.c. function $v : D \to R^n$

$$\Lambda^f(D,v) := I^f(T_1,T_2,v) - (T_2 - T_1)\mu(f) - \pi^f(v(T_1)) + \pi^f(v(T_2)) \geq 0. \tag{12}$$

Denote by $\mathcal{N}(f)$ the set of all a.c. functions $v : R^1 \to R^n$ such that

$$\sup\{|v(t)| : \; t \in R^1\} < \infty \tag{13}$$

and that

$$I^f(T_1,T_2,v) = (T_2 - T_1)\mu(f) + \pi^f(v(T_1)) - \pi^f(v(T_2)) \tag{14}$$

for each $T_1 \in R^1$, $T_2 > T_1$. Set

$$\mathcal{K}(f) = \cup\{v(R^1) : \ v \in \mathcal{N}(f)\}. \tag{15}$$

It follows from (12)-(14) that

$$\mathcal{N}(f) \subset \mathcal{M}(f), \ \mathcal{K}(f) \subset \mathcal{D}(f). \tag{16}$$

The next two theorems are our main results.

Theorem 1.4. *Let $f \in \mathfrak{A}$. Then the set $\mathcal{N}(f) \neq \emptyset$ and $\mathcal{K}(f)$ is a closed subset of R^n.*

Theorem 1.5. *Let $f \in \mathfrak{A}$ and let M_0, M_1, ϵ be positive numbers. Then there exist a positive number l and a natural number q such that for each $T \geq l$ and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq M_0, \ I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + M_1$$

there is a finite number of closed intervals $[b_i, c_i]$, $i = 1, \dots, p$ such that

$$p \leq q \text{ and } c_i \leq b_{i+1} \text{ for all integers } i \text{ satisfying } 1 \leq i < p,$$
$$0 \leq c_i - b_i \leq l, \ i = 1, \dots, p$$

and

$$d(v(t), \mathcal{K}(f)) \leq \epsilon \text{ for all } t \in [0, T] \setminus \cup_{i=1}^{p} [b_i, c_i].$$

Note that Theorems 1.2 and 1.5 and some other auxiliary results in the sequel are stated for intervals $[T_1, T_2]$ with $T_1 = 0$. Since integrands $f \in \mathfrak{A}$ do not depend on t these results are valid without this assumption.

The paper is organized as follows. In the second section we present several auxiliary results. In the third section we prove Theorem 1.4. Theorem 1.5 is proved in Section 4. In Section 5 we consider an example of $f \in \mathfrak{A}$ such that $\mathcal{K}(f) \neq \mathcal{D}(f)$.

2. Preliminary results

We denote by $\mathrm{mes}(E)$ the Lebesgue measure of a Lebesgue measurable set $E \subset R^q$ and by $\mathrm{Card}(A)$ the cardinality of a set A.

We need the following results.

Proposition 2.1 (12, Theorem 8.3). *Let $f \in \mathfrak{A}$. Then for every $x \in R^n$ there exists an (f)-good function $v : [0, \infty) \to R^n$ such that $v(0) = x$ and the equality*

$$I^f(T_1, T_2, v) = (T_2 - T_1)\mu(f) + \pi^f(v(T_1)) - \pi^f(v(T_2))$$

is true for each $T_1 \geq 0$, $T_2 > T_1$.

Proposition 2.2 (14, Proposition 2.5). *Assume that $f \in \mathfrak{A}$, $M_1 > 0$, $-\infty < T_1 < T_2 < \infty$, $x_i : \ [T_1, T_2] \to R^n$, $i = 1, 2, \dots$ is a sequence of a.c. functions such that*

$$I^f(T_1, T_2, x_i) \leq M_1, \ i = 1, 2, \dots$$

Then there exist a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ *and an a.c. function* $x : [T_1, T_2] \to R^n$ *such that*

$$I^f(T_1, T_2, x) \le M_1,\ x_{i_k}(t) \to x(t) \text{ as } k \to \infty \text{ uniformly in } [T_1, T_2] \text{ and}$$
$$x'_{i_k} \to x' \text{ as } k \to \infty \text{ weakly in } L^1(R^n; (T_1, T_2)).$$

Proposition 2.3 (14, Theorem 1.3). *Let* $f \in \mathfrak{A}$ *and* M_1, M_2, c *be positive numbers. Then there exists* $S > 0$ *such that for each* $T_1 \in R^1$, $T_2 \in [T_1+c, \infty)$ *and each a.c. function* $v : [T_1, T_2] \to R^n$ *satisfying*

$$|v(T_1)|,\ |v(T_2)| \le M_1,\ I^f(T_1, T_2, v) \le U^f(T_1, T_2, v(T_1), v(T_2)) + M_2$$

the following relation holds:

$$|v(t)| \le S,\ t \in [T_1, T_2].$$

Proposition 2.4 (12, Theorem 6.1). *Assume that* $f \in \mathfrak{A}$*. Then the mapping* $(T_1, T_2, x, y) \to U^f(T_1, T_2, x, y)$ *is continuous for* $T_1 \in R^1$, $T_2 \in (T_1, \infty)$, $x, y \in R^n$.

Proposition 2.5 (14, Corollary 2.1). *For each* $f \in \mathfrak{A}$*, each pair of numbers* T_1, T_2 *satisfying* $T_1 < T_2$ *and each* $z_1, z_2 \in R^n$ *there exists an a.c. function* $x : [T_1, T_2] \to R^n$ *such that* $x(T_i) = z_i,\ i = 1, 2$ *and* $I^f(T_1, T_2, x) = U^f(T_1, T_2, z_1, z_2)$.

3. Proof of Theorem 1.4

First we show that $\mathcal{N}(f) \neq \emptyset$. By Proposition 2.1 there exists an (f)-good function $v : [0, \infty) \to R^n$ such that

$$I^f(0, T, v) = T\mu(f) + \pi^f(v(0)) - \pi^f(v(T)) \text{ for each } T > 0. \tag{17}$$

Since v is an (f)-good function the inequality (8) is valid. In view of (17) and (8)

$$\sup\{|\pi^f(v(T))| :\ T \in (0, \infty)\} < \infty. \tag{18}$$

Combined with (10) the inequality (18) implies that

$$\sup\{|v(T)| :\ T \in [0, \infty)\} < \infty. \tag{19}$$

For each natural number i define

$$v_i(t) = v(t + i),\ t \in [-i, \infty). \tag{20}$$

Let k be a natural number. It follows from (20), (17) and (19) that the sequence $\{I^f(-k, k, v_i\}_{i=k}^{\infty}$ is bounded. Combined with Proposition 2.2 this implies that there exist a subsequence $\{v_{i_q}\}_{q=1}^{\infty}$ of $\{v_i\}_{i=1}^{\infty}$ and an a.c. function $w : R^1 \to R^n$ such that for each natural number k

$$v_{i_q} \to w \text{ as } q \to \infty \text{ uniformly on } [-k, k], \tag{21}$$
$$v'_{i_q} \to w' \text{ as } q \to \infty \text{ weakly in } L^1(R^n, (-k, k)),$$
$$I^f(-k, k, w) \le \liminf_{q\to\infty} I^f(-k, k, v_{i_q}).$$

Let k be a natural number. In view of (21), (20), (17) and the continuity of π^f

$$\begin{aligned} I^f(-k,k,w) &\le \liminf_{q\to\infty} I^f(-k,k,v_{i_q}) \\ &\le \liminf_{q\to\infty}[2k\mu(f)+\pi^f(v_{i_q}(-k))-\pi^f(v_{i_q}(k))] \\ &= 2k\mu(f)+\pi^f(w(-k))-\pi^f(w(k)). \end{aligned}$$

Together with (12) this inequality implies that

$$I^f(-k,k,w) = 2k\mu(f)+\pi^f(w(-k))-\pi^f(w(k)) \quad (22)$$

for each integer $k \ge 1$.

Since $D \to \Lambda^f(D,w)$ is an additive nonnegative set function it follows from (22) and (12) that

$$I^f(T_1,T_2,w) = (T_2-T_1)\mu(f)+\pi^f(w(T_1))-\pi^f(w(T_2)) \quad (23)$$

for each $T_1 \in R^1$, $T_2 > T_1$. (19), (20) and (21) imply that

$$\sup\{|w(t)| : \ t \in R^1\} < \infty.$$

Combined with (23) this inequality implies that $w \in \mathcal{N}(f)$. Therefore $\mathcal{N}(f) \ne \emptyset$.

Let us show that the set $\mathcal{K}(f)$ is closed. Since the set $\mathcal{D}(f)$ is bounded it follows from (16) that there is $d_0 > 0$ such that

$$|z| \le d_0 \text{ for all } z \in \mathcal{K}(f). \quad (24)$$

Assume that

$$\{z_i\}_{i=1}^\infty \subset \mathcal{K}(f), \ z = \lim_{i\to\infty} z_i. \quad (25)$$

For each integer $i \ge 1$ there is $u_i \in \mathcal{N}(f)$ such that $z_i \in u_i(R^1)$. We may assume without loss of generality that

$$u_i(0) = z_i, \ i = 1,2,\dots \quad (26)$$

Inequality (24) implies that

$$|u_i(t)| \le d_0 \text{ for all } t \in R^1 \text{ and each integer } i \ge 1. \quad (27)$$

Since $u_i \in \mathcal{N}(f)$ for all integers $i \ge 1$ we have that for each pair of natural numbers i,k

$$I^f(-k,k,u_i) = 2k\mu(f)+\pi^f(u_i(-k))-\pi^f(u_i(k)). \quad (28)$$

It follows from (28) and (27) that for each natural number k the sequence $\{I^f(-k,k,u_i)\}_{i=1}^\infty$ is bounded. Combined with Proposition 2.2 this implies that there exist a subsequence $\{u_{i_q}\}_{q=1}^\infty$ of $\{u_i\}_{i=1}^\infty$ and an a.c. function $h : R^1 \to R^n$ such that for each natural number k

$$\begin{gathered} u_{i_q} \to h \text{ as } q \to \infty \text{ uniformly in } [-k,k], \quad (29) \\ u'_{i_q} \to h' \text{ as } q \to \infty \text{ weakly in } L^1(R^n;(-k,k)), \\ I^f(-k,k,h) \le \liminf_{q\to\infty} I^f(-k,k,u_{i_q}). \end{gathered}$$

By (29), (26) and (25)

$$h(0) = \lim_{q\to\infty} u_{i_q}(0) = \lim_{q\to\infty} z_{i_q} = z. \tag{30}$$

Relations (29) and (27) imply that

$$|h(t)| \le d_0 \text{ for all } t \in R^1. \tag{31}$$

In view of (29), (14) and the continuity of π^f for each natural number k

$$\begin{aligned} I^f(-k,k,h) &\le \liminf_{q\to\infty} I^f(-k,k,u_{i_q}) \\ &= \liminf_{q\to\infty}[2k\mu(f) + \pi^f(u_{i_q}(-k)) - \pi^f(u_{i_q}(k))] \\ &= 2k\mu(f) + \pi^f(h(-k)) - \pi^f(h(k)). \end{aligned}$$

Together with (12) this relation implies that

$$I^f(-k,k,h) = 2k\mu(f) + \pi^f(h(-k)) - \pi^f(h(k)) \text{ for all integers } k \ge 1. \tag{32}$$

Since $D \to \Lambda^f(D,h)$ is an additive nonnegative set function it follows from (32) and (12) that

$$I^f(T_1,T_2,h) = (T_2 - T_1)\mu(f) + \pi^f(h(T_1)) - \pi^f(h(T_2))$$

for each $T_1 \in R^1$, $T_2 > T_1$. Combined with (31) this equality implies that $h \in \mathcal{N}(f)$. By this inclusion and (30) $z \in \mathcal{K}(f)$. Therefore $\mathcal{K}(f)$ is closed. This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.5

Note that the set $\mathcal{K}(f)$ is bounded and choose a number d_0 such that

$$|z| \le d_0 \text{ for all } z \in \mathcal{K}(f). \tag{33}$$

Lemma 4.1. *Let $M_0, \epsilon > 0$. Then there exist $l_0, \delta > 0$ such that for each $T \ge 2l_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \le M_0, \tag{34}$$

$$I^f(0,T,v) \le T\mu(f) + \pi^f(v(0)) - \pi^f(v(T)) + \delta \tag{35}$$

the following inequality holds:

$$d(v(t), \mathcal{K}(f)) \le \epsilon \textit{ for all } t \in [l_0, T - l_0]. \tag{36}$$

Proof. Let us assume the converse. Then for each natural number i there exist $T_i \ge 2i$, an a.c. function $v_i : [0,T_i] \to R^n$ which satisfies

$$|v_i(0)|, |v_i(T_i)| \le M_0, \tag{37}$$

$$I^f(0,T_i,v_i) \le T_i\mu(f) + \pi^f(v_i(0)) - \pi^f(v_i(T_i)) + i^{-1} \tag{38}$$

and

$$t_i \in [i, T_i - i] \tag{39}$$

such that

$$d(v_i(t_i), \mathcal{K}(f)) > \epsilon. \tag{40}$$

Relations (38) and (11) imply that for all integers $i \geq 1$

$$I^f(0, T_i, v_i) \leq U^f(0, T_i, v_i(0), v_i(T_i)) + i^{-1}. \tag{41}$$

In view of (41), (37) and Proposition 2.3 there exists $S_0 > 0$ such that

$$|v_i(t)| \leq S_0 \text{ for each } t \in [0, T_i] \text{ and each integer } i \geq 1. \tag{42}$$

For each natural number i define

$$u_i(t) = v_i(t + t_i),\ t \in [-t_i, T_i - t_i]. \tag{43}$$

(39) implies that for each integer $i \geq 1$

$$-t_i \leq -i < i \leq T_i - t_i. \tag{44}$$

By (43) and (38) for each integer $i \geq 1$

$$\begin{aligned} & I^f(-t_i, T_i - t_i, u_i) - T_i\mu(f) - \pi^f(u_i(-t_i)) + \pi^f(u_i(T_i - t_i)) \\ = \ & I^f(0, T_i, v_i) - T_i\mu(f) - \pi^f(v_i(0)) + \pi^f(v_i(T_i)) \leq i^{-1}. \end{aligned} \tag{45}$$

It follows from (43) and (42) that

$$|u_i(t)| \leq S_0 \text{ for each } t \in [-t_i, T_i - t_i] \text{ and each integer } i \geq 1. \tag{46}$$

Relations (43) and (40) imply that

$$d(u_i(0), \mathcal{K}(f)) > \epsilon. \tag{47}$$

Let k be a natural number. By (45), (12), (46) and (44) the sequence $\{I^f(-k, k, u_i)\}_{i=k}^{\infty}$ is bounded. Combined with Proposition 2.2 this implies that there exist a subsequence $\{u_{i_q}\}_{q=1}^{\infty}$ of $\{u_i\}_{i=1}^{\infty}$ and an a.c. function $u : R^1 \to R^n$ such that for each natural natural number k

$$\begin{gathered} u_{i_q} \to u \text{ as } q \to \infty \text{ uniformly in } [-k, k], \\ u'_{i_q} \to u' \text{ as } q \to \infty \text{ weakly in } L^1(R^n; (-k, k)), \\ I^f(-k, k, u) \leq \liminf_{q \to \infty} I^f(-k, k, u_{i_q}). \end{gathered} \tag{48}$$

Relations (48) and (46) imply that

$$|u(t)| \leq S_0 \text{ for all } t \in R^1. \tag{49}$$

In view of (48), (45) and (12) for each natural number k

$$\begin{aligned} I^f(-k, k, u) &\leq \liminf_{q \to \infty} I^f(0, k, u_{i_q}) \\ &\leq \liminf_{q \to \infty} [2k\mu(f) + \pi^f(u_{i_q}(-k)) - \pi^f(u_{i_q}(k)) + i_q^{-1}] \\ &= 2k\mu(f) + \pi^f(u(-k)) - \pi^f(u(k)). \end{aligned}$$

Combined with (12) this implies that

$$I^f(-k,k,u) = 2k\mu(f) + \pi^f(u(-k)) - \pi^f(u(k)) \tag{50}$$

for each natural number k. Since $D \to \Lambda^f(D,u)$ is an additive nonnegative set function it follows from (50) and (12) that

$$I^f(T_1,T_2,u) = (T_2 - T_1)\mu(f) + \pi^f(u(T_1)) - \pi^f(u(T_2))$$

for each $T_1 \in R^1, T_2 > T_1$. Together with (49) this implies that $u \in \mathcal{N}(f)$ and $u(0) \in \mathcal{K}(f)$. On the other hand it follows from (48) and (47) that $d(u(0),\mathcal{K}(f)) \geq \epsilon$. The contradiction we have reached proves Letmma 4.1. □

Lemma 4.2. *Let $M_0, M_1 > 0$. Then there exists $M_2 > 0$ such that for each $T \geq 4$ and each a.c. function $v : [0,T] \to R^n$ satisfying*

$$|v(0)|, |v(T)| \leq M_0, \; I^f(0,T,v) \leq U^f(0,T,v(0),v(T)) + M_1 \tag{51}$$

the inequality

$$I^f(0,T,v) - T\mu(f) - \pi^f(v(0)) + \pi^f(v(T)) \leq M_2 \tag{52}$$

holds.

Proof. Choose $w \in \mathcal{N}(f)$. By Proposition 2.4 there is $D_0 > 0$ such that

$$|U^f(0,1,z_1,z_2)| \leq D_0 \text{ for each } z_1, z_2 \in R^n \text{ satisfying } |z_1|, |z_2| \leq d_0 + M_0. \tag{53}$$

There is $D_1 > 0$ such that

$$|\pi^f(z)| \leq D_1 \text{ for each } z \in R^n \text{ satisfying } |z| \leq d_0 + M_0. \tag{54}$$

Assume that $T \geq 4$ and an a.c. function $v : [0,T] \to R^n$ satisfies (51). Consider an a.c. function $u : [0,T] \to R^n$ such that

$$\begin{gathered} u(0) = v(0), \; u(T) = v(T), \; u(t) = w(t), \; t \in [1, T-1], \\ I^f(0,1,u) \leq U^f(0,1,v(0),w(1)) + 1, \; I^f(T-1,T,u) \leq U^f(T-1,T,w(T-1),v(T)) + 1. \end{gathered} \tag{55}$$

It follows from (55) and (51) that

$$\begin{aligned} & I^f(0,T,u) - T\mu(f) - \pi^f(u(0)) + \pi^f(u(T)) \\ & \quad -[I^f(0,T,v) - T\mu(f) - \pi^f(v(0)) + \pi^f(v(T))] \\ = \; & I^f(0,T,u) - I^f(0,T,v) \geq U^f(0,T,v(0),v(T)) - I^f(0,T,v) \geq -M_1. \end{aligned} \tag{56}$$

In view of the relation $w \in \mathcal{N}(f)$, (55), (51), (33) and (54)

$$\begin{aligned} & I^f(0,T,u) - T\mu(f) - \pi^f(u(0)) + \pi^f(u(T)) \\ = \; & I^f(0,T,u) - T\mu(f) - \pi^f(u(0)) + \pi^f(u(T)) \\ & \quad -[I^f(0,T,w) - T\mu(f) - \pi^f(w(0)) + \pi^f(w(T))] \\ \leq \; & I^f(0,T,u) - I^f(0,T,w) + 4D_1 \\ = \; & I^f(0,1,u) + I^f(1,T-1,w) + I^f(T-1,T,u) - I^f0,T,w) + 4D_1 \\ \leq \; & U^f(0,1,v(0),w(1)) + 1 + U^f(T-1,T,w(T-1),v(T)) + 1 \\ & \quad -U^f(0,1,w(0),w(1)) - U^f(T-1,T,w(T-1),w(T)) + 4D_1. \end{aligned} \tag{57}$$

Relations (33), (51), (53) and (57) imply that

$$I^f(0,T,u) - T\mu(f) - \pi^f(u(0)) + \pi^f(u(T)) \leq 4D_0 + 4D_1 + 2. \tag{58}$$

By (56) and (58)

$$\begin{aligned} & I^f(0,T,v) - T\mu(f) - \pi^f(v(0)) + \pi^f(v(T)) \\ \leq \ & M_1 + I^f(0,T,u) - T\mu(f) - \pi^f(u(0)) + \pi^f(u(T)) \\ \leq \ & M_1 + 4D_0 + 4D_1 + 2. \end{aligned}$$

Thus (52) holds with

$$M_2 = M_1 + 4D_0 + 4D_1 + 2.$$

Lemma 4.2 is proved. □

Completion of the proof of Theorem 1.5 By Proposition 2.3 there is $S_0 > 0$ such that the following property holds:

(P1) For each $T \geq 1$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq M_0, \ I^f(0,T,v) \leq U^f(0,T,v(0),v(T)) + M_1 \tag{59}$$

the following inequality holds:

$$|v(t)| \leq S_0, \ t \in [0,T]. \tag{60}$$

By Lemma 4.1 there exist $l_0 > 4$, $\delta_0 \in (0,1)$ such that the following property holds:

(P2) For each $T \geq 2l_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq S_0,$$

$$I^f(0,T,v) \leq T\mu(f) + \pi^f(v(0)) - \pi^f(v(T)) + \delta_0 \tag{61}$$

the following inequality holds:

$$d(v(t), \mathcal{K}(f)) \leq \epsilon \text{ for all } t \in [l_0, T - l_0]. \tag{62}$$

By Lemma 4.2 there exists $M_2 > 1$ such that the following property holds:

(P3) For each $T \geq 4$ and each a.c. function $v : [0,T] \to R^n$ satisfying (51) the inequality (52) holds.

Choose a natural number

$$q > 3(2 + \delta_0^{-1}(M_2 + 1)) \tag{63}$$

and set

$$l = 2l_0. \tag{64}$$

Assume that $T \geq l$ and an a.c. function $v : [0,T] \to R^n$ satisfies

$$|v(0)|, |v(T)| \leq M_0,$$

$$I^f(0,T,v) \leq U^f(0,T,v(0),v(T)) + M_1. \tag{65}$$

By property (P1), (64) and (65) the inequality (60) holds. In view of property (P3), (64), the choice of l_0, (65), (51) and (52)

$$I^f(0,T,v) - T\mu(f) - \pi^f(v(0)) + \pi^f(v(T)) \le M_2. \tag{66}$$

By induction we can construct a finite sequence of numbers $\{t_i\}_{i=0}^p \subset [0,T]$ such that

$$t_0 = 0,\ t_p = T,\ t_i < t_{i+1} \text{ for each integer } i \text{ satisfying } 0 \le i \le p-1$$

and that for each integer i satisfying $0 \le i < p-1$ the following relations hold:

$$I^f(t_i,t_{i+1},v) - (t_{i+1}-t_i)\mu(f) - \pi^f(v(t_i)) + \pi^f(v(t_{i+1})) = \delta_0, \tag{67}$$

$$I^f(t_{p-1},t_p,v) - (t_p-t_{p-1})\mu(f) - \pi^f(v(t_{p-1})) + \pi^f(v(t_p)) \le \delta_0.$$

It follows from (66), (12) and (67) that

$$\begin{aligned} M_2 &\ge I^f(0,T,v) - T\mu(f) - \pi^f(v(0)) + \pi^f(v(T)) \\ &= \sum_{i=0}^{p-1}[I^f(t_i,t_{i+1},v) - (t_{i+1}-t_i)\mu(f) - \pi^f(v(t_i)) + \pi^f(v(t_{i+1}))] \ge \delta_0(p-1) \end{aligned}$$

and

$$p-1 \le M_2/\delta_0,\ p \le 1 + M_2\delta_0^{-1}. \tag{68}$$

Set

$$\begin{aligned} \mathcal{A} = \{&[t_i,t_{i+1}]:\ 0 \le i \le p-1 \text{ and } t_{i+1}-t_i \le 2l_0\} \\ \cup\{&[t_i,t_i+l_0],\ [t_{i+1}-l_0,t_{i+1}]:\ 0 \le i \le p-1 \text{ and } t_{i+1}-t_i > 2l_0\}. \end{aligned} \tag{69}$$

In view of (69), (68) and (63)

$$\mathrm{Card}(\mathcal{A}) \le 3p \le 3(1 + M_2\delta_0^{-1}) \le q. \tag{70}$$

Clearly if a set $e \in \mathcal{A}$, then

$$\mathrm{mes}(e) \le 2l_0 = l. \tag{71}$$

Assume that

$$\tau \in [0,T] \setminus \cup\{e:\ e \in \mathcal{A}\}. \tag{72}$$

Relations (72) and (69) imply that there is $j \in \{0,\dots,p-1\}$ such that

$$\tau \in [t_j + l_0, t_{j+1} - l_0]. \tag{73}$$

It follows from (73), (67), (60) and property (P2) with the restriction of v to the interval $[t_j,t_{j+1}]$ that

$$d(v(\tau),\mathcal{K}(f)) \le \epsilon.$$

This completes the proof of Theorem 1.5.

5. An example

Let $\psi(t) = t$, $t \in [0, \infty)$. Define

$$f(x,u) = |x|^2|x-e|^2 + |u|^2, \; x, u \in R^n \tag{74}$$

where $e = (1, 1, \dots, 1) \in R^n$. It is easy to see that $f \in \mathfrak{A}$ with some positive constant a. Clearly

$$\mu(f) = 0, \; \pi^f(0) = \pi^f(e) = 0. \tag{75}$$

In this section we will prove the following result.

Theorem 5.1. $\mathcal{D}(f) \neq \mathcal{K}(f) = \{0, e\}$.

We preface the proof of Theorem 5.1 by several auxiliary results.

Lemma 5.2. *Let $w \in \mathcal{N}(f)$. Then either $w(t) = 0$ for all $t \in R^1$ or $w(t) = e$ for all $t \in R^1$.*

Proof. By the definition of $\mathcal{N}(f)$

$$\sup\{|w(t)| : \; t \in R^1\} < \infty, \tag{76}$$

$$I^f(T_1, T_2, w) = (T_2 - T_1)\mu(f) + \pi^f(w(T_1)) - \pi^f(w(T_2)) \tag{77}$$

for each $T_1 \in R^1$, $T_2 > T_1$.

First we show that there exist $z_1 \in \{0, e\}$ and a strictly increasing sequence $\{t_i\}_{i=0}^\infty$ of real numbers such that

$$\lim_{i\to\infty} t_i = \infty \text{ and } \lim_{i\to\infty} w(t_i) = z_1. \tag{78}$$

Let us assume the converse. Then there are $\epsilon > 0$ and $t_0 \in R^1$ such that

$$|w(t)|, \; |w(t) - e| \geq \epsilon \text{ for all } t \geq t_0. \tag{79}$$

Inequality (79) implies that

$$f(w(t), w'(t)) \geq \epsilon^4 \text{ for all } t \geq t_0.$$

In view of this inequality, (75) and (76) for each $t \geq t_0$

$$\begin{aligned} & I^f(t_0, t, w) - (t - t_0)\mu(f) - \pi^f(w(t_0)) + \pi^f(w(t)) \\ \geq \; & \epsilon^4(t - t_0) - \pi^f(w(t_0)) + \pi^f(w(t)) \to \infty \text{ as } t \to \infty. \end{aligned}$$

This contradicts (77). The contradiction we have reached shows that there exists a strictly increasing sequence $\{t_i\}_{i=0}^\infty$ of real numbers and $z_1 \in \{0, e\}$ such that (78) is valid.

Analogously we can show that there exist a strictly decreasing sequence $\{\tau_i\}_{i=0}^\infty$ of real numbers and $z_2 \in \{0, e\}$ such that

$$\lim_{i\to\infty} \tau_i = -\infty, \; \lim_{i\to\infty} w(\tau_i) = z_2. \tag{80}$$

We may assume without loss of generality that

$$t_i > 0, \; \tau_i < 0 \text{ for all integers } i \geq 1.$$

By (77), (75), (80), (78) and continuity of π^f for each integer $i \ge 1$

$$I^f(\tau_i, t_i, w) = (t_i - \tau_i)\mu(f) + \pi^f(w(\tau_i)) - \pi^f(w(t_i))$$
$$= \pi^f(w(\tau_i)) - \pi^f(w(t_i)) \to 0 \text{ as } i \to \infty.$$

Together with (74) this implies that $f(w(t), w'(t)) = 0$ for almost every $t \in R^1$. This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let $\epsilon \in (0, 4^{-1})$, $l > 0$ and $M_0 > 0$. Then there exists $L > l$ such that for each $T \ge L$, each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \le M_0, \; I^f(0, T, v) \le U^f(0, T, v(0), v(T)) + 1 \tag{81}$$

and each $s \in [0, T - L]$ there are

$$\tau \in [s, s + L - l] \text{ and } z \in \{0, e\} \tag{82}$$

such that

$$|v(t) - z| \le \epsilon \text{ for all } t \in [\tau, \tau + l]. \tag{83}$$

Proof. By Theorem 1.5 and Lemma 5.2 there exist a natural number q and a real number $l_0 > 0$ such that for each $T \ge l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies (81) the following property holds:

(P4) there exists a finite number of intervals $[b_i, c_i]$, $i = 1, \dots, p$ where $p \le q$ such that

$$c_i \le b_{i+1} \text{ for each integer } i \text{ satisfying } 1 \le i < p,$$

$$0 \le c_i - b_i \le l_0, \; i = 1, \dots, p,$$
$$d(v(t), \{0, e\}) \le \epsilon \text{ for all } t \in [0, T] \setminus \cup_{i=1}^p [b_i, c_i].$$

Set

$$L = (l_0 + 3l + 1)q. \tag{84}$$

Assume that $T \ge L$, an a.c. function $v : [0, T] \to R^n$ satisfies (81) and $s \in [0, T - L]$. Let a sequence of intervals $[b_i, c_i]$, $i = 1, \dots, p$ be as guaranteed by property (P4). We show that there is $\tau \in [s, s + L - l]$ such that

$$[\tau, \tau + l] \cap [b_i, c_i] = \emptyset, \; i = 1, \dots, p.$$

Let us assume the converse. Then for each $t \in [s, s + L - l]$ there is $i \in \{1, \dots, p\}$ such that

$$[t, t + l] \cap [b_i, c_i] \ne \emptyset$$

and $t \in [b_i - l, c_i + l]$. Therefore

$$[s, s + L - l] \subset \cup_{i=1}^p [b_i - l, c_i + l]$$

and

$$L - l \le \sum_{i=1}^p [c_i - b_i + 2l]. \tag{85}$$

Inequality (85) and property (P4) imply that

$$L \le l + q(2l + l_0).$$

This contradicts (84). The contradiction we have reached proves that there is

$$\tau \in [s, s + L - l] \tag{86}$$

for which

$$[\tau, \tau + l] \cap [b_i, c_i] = \emptyset,\ i = 1, \dots, p. \tag{87}$$

By property (P4), (87) and (86)

$$d(v(t), \{0, e\}) \le \epsilon \text{ for all } t \in [\tau, \tau + l]. \tag{88}$$

We show that there is $z \in \{0, e\}$ such that

$$|v(t) - z| \le \epsilon \text{ for all } t \in [\tau, \tau + l]. \tag{89}$$

Let us assume the converse. Then it follows from (88) and the inequality $\epsilon < 4^{-1}$ that there are

$$t_1, t_2 \in [\tau, \tau + l],\ z_1, z_2 \in \{0, e\} \tag{90}$$

such that

$$t_1 < t_2,\ z_1 \ne z_2,\ |v(t_i) - z_i| < \epsilon,\ i = 1, 2. \tag{91}$$

By the mean-value theorem, (91) and the inequality $\epsilon < 1/4$ there is $\xi \in (t_1, t_2)$ such that

$$|v(\xi) - z_1| = 1/2. \tag{92}$$

Relations (92) and (88) imply that

$$|v(\xi) - z_2| \le \epsilon < 4^{-1}. \tag{93}$$

In view of (83)

$$|v(\xi) - z_2| \ge |z_2 - z_1| - |z_1 - v(\xi)| \ge n^{1/2} - 1/2 \ge 1/2,$$

a contradiction. The contradiction we have reached proves that there is $z \in \{0, e\}$ such that (89) holds. Lemma 5.3 is proved. □

Lemma 5.4. *Let i be a natural number. Then there exist $T \ge 2i$, an a.c. function $v : [0, T] \to R^n$ and a number τ such that*

$$i \le \tau \le T - i, \tag{94}$$

$$|v(0)|, |v(T)| \le 2n + 1, \tag{95}$$

$$I^f(0, T, v) \le U^f(0, T, v(0), v(T)) + i^{-1}, \tag{96}$$

$$|v(\tau)|, |v(\tau) - e| \ge 2^{-1}. \tag{97}$$

Proof. There is

$$\delta \in (0, (4i)^{-1}) \tag{98}$$

such that

$$|U^f(0, 1, y_1, y_2)| \le (8i)^{-1} \tag{99}$$

for each $y_1 \in R^n$ satisfying $d(y_1, \{0, e\}) \le 4\delta$ and each $y_2 \in R^n$ satisfying $|y_2 - y_1| \le 4\delta$.

By Lemma 5.3 there exists

$$L > 4(i + 2) \tag{100}$$

such that the following property holds:

(P5) For each $T \ge L$, each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \le 2n + 1,\ I^f(0, T, v) \le U^f(0, T, v(0), v(T)) + 1 \tag{101}$$

and each $S \in [0, T - L]$ there is $\tau \in [S, S + L - 4(i + 2)]$ and $z \in \{0, e\}$ such that

$$|v(t) - z| \le \delta,\ t \in [\tau, \tau + 4(i + 2)]. \tag{102}$$

Fix

$$T > 8L. \tag{103}$$

By Proposition 2.5 there exists an a.c. function $v : [0, T] \to R^n$ such that

$$v(0) = 0,\ v(T) = e,\ I^f(0, T, v) = U^f(0, T, 0, e). \tag{104}$$

Relation (104) and the mean-value theorem imply that there is $T_0 \in (0, T)$ such that

$$|v(T_0)| = 2^{-1}. \tag{105}$$

In view of (105)

$$|v(T_0) - e| \ge |e - 0| - |0 - v(T_0)| \ge n^{1/2} - 2^{-1} \ge 2^{-1}. \tag{106}$$

There are three cases:

$$i \le T_0 \le T - i; \tag{107}$$

$$T_0 < i; \tag{108}$$

$$T_0 > T - i. \tag{109}$$

If (107) is valid, then by (104), (110), (106) and (107) the assertion of the lemma holds.

Assume that (108) is valid. It follows from (103), (104) and property (P5) that there exist

$$\tau \in [T - L, T - 4(i + 2)],\ z \in \{0, e\} \tag{110}$$

such that

$$|v(t) - z| \le \delta,\ t \in [\tau, \tau + 4(i + 2)]. \tag{111}$$

By (104), (100) and Propositon 2.5 there exists an a.c. function $u : [0, T] \to R^n$ such that

$$\begin{gathered} u(t) = 0,\ t \in [0, i + 1],\ u(i + 1 + t) = v(t),\ t \in (0, \tau], \\ u(t) = v(t),\ t \in [i + 2 + \tau, T], \\ I^f(i + 1 + \tau, i + 2 + \tau, u) = U^f(i + 1 + \tau, i + 2 + \tau, v(\tau), v(i + 2 + \tau)). \end{gathered} \tag{112}$$

Relations (112) and (104) imply that

$$u(0) = 0, \; u(T) = e. \tag{113}$$

In view of (108), (100) and (103)

$$i < T_0 + i + 1 < 2i + 1 < L - i < T - i. \tag{114}$$

By (103), (100) and (108)

$$T - L > 7L > 4(i + 2) > T_0. \tag{115}$$

It follows from (112), (115), (105), (110) and (106) that

$$u(T_0 + i + 1) = v(T_0),$$

$$|u(T_0 + i + 1)|, |u(T_0 + i + 1) - e| \geq 2^{-1},$$
$$|u(T_0 + i + 1) - z| \geq 2^{-1}. \tag{116}$$

By (112), (111), (110) and the choice of δ (see (99))

$$I^f(i + 1 + \tau, i + 2 + \tau, u) = U^f(i + 1 + \tau, i + 2 + \tau, v(\tau), v(i + 2 + \tau)) \leq (8i)^{-1}. \tag{117}$$

Relations (112), (74) and (114) imply that

$$\begin{aligned} I^f(0, T, u) - I^f(0, T, v) &= I^f(0, i + 1, u) + I^f(i + 1, \tau + i + 1, u) \\ &\quad + I^f(i + 1 + \tau, i + 2 + \tau, u) + I^f(i + 2 + \tau, T, u) - I^f(0, T, v) \\ \leq &\; I^f(0, \tau, v) + (8i)^{-1} + I^f(i + 2 + \tau, T, v) - I^f(0, T, v) \\ \leq &\; (8i)^{-1} - I^f(\tau, i + 2 + \tau, v) \leq (8i)^{-1} \end{aligned}$$

and

$$I^f(0, T, u) \leq I^f(0, T, v) + (8i)^{-1}.$$

Combined with (104) and (112) this inequality implies that

$$I^f(0, T, u) \leq U^f(0, T, u(0), u(T)) + (8i)^{-1}.$$

Therefore if (108) holds, then the assertion of the lemma holds with $v = u$ and $\tau = T_0 + i + 1$ (see (113), (100), (116) and (114)). Analogously we can show that if (109) is true, then the assertion of the lemma holds. This completes the proof of Lemma 5.4. □

Completion of the proof of Theorem 5.1 In order to prove the theorem it is sufficient to show the existence of $v \in \mathcal{M}(f)$ such that

$$v(R^1) \neq \{0, e\}.$$

By Lemma 5.4 for each natural number i there exists $T_i \geq 2i$, an a.c. function $v_i : [0, T_i] \to R^n$ and

$$\tau_i \in [i, T_i - i] \tag{118}$$

such that

$$|v_i(0)|, |v_i(T_i)| \leq 2n + 1, \tag{119}$$

$$I^f(0, T_i, v_i) \le U^f(0, T_i, v_i(0), v_i(T_i)) + i^{-1}, \tag{120}$$

$$|v_i(\tau_i)|, |v_i(\tau_i) - e| \ge 2^{-1}. \tag{121}$$

It follows from (119), (120) and Proposition 2.3 that there exists $S > 0$ such that

$$|v_i(t)| \le S \text{ for all } t \in [0, T_i] \text{ and all integers } i \ge 1. \tag{122}$$

Let $i \ge 1$ be an integer. The inclusion (118) implies that

$$[-i, i] \subset [-\tau_i, T_i - \tau_i]. \tag{123}$$

Define

$$u_i(t) = v_i(t + \tau_i),\ t \in [-\tau_i, T_i - \tau_i]. \tag{124}$$

In view of (124), (122) and (120)

$$|u_i(t)| \le S,\ t \in [-\tau_i, T_i - \tau_i], \tag{125}$$

$$I^f(-\tau_i, T_i - \tau_i, u_i) \le U^f(-\tau_i, T_i - \tau_i, u_i(-\tau_i), u_i(-\tau_i + T_i)) + i^{-1}. \tag{126}$$

It follows from (125) and (126) that for each natural number k the sequence $\{I^f(-k, k, u_i)\}_{i=k}^{\infty}$ is bounded. Combined with Proposition 2.2 this implies that there exist a subsequence $\{u_{i_q}\}_{q=1}^{\infty}$ and an a.c. function $u : R^1 \to R^n$ such that for each natural number k

$$u_{i_q} \to u \text{ as } q \to \infty \text{ uniformly on } [-k, k], \tag{127}$$
$$u'_{i_q} \to u' \text{ as } q \to \infty \text{ weakly in } L^1(R^n, (-k, k)),$$
$$I^f(-k, k, u) \le \liminf_{q\to\infty} I^f(-k, k, u_{i_q}).$$

Relations (121), (124), (127) and (125) imply that

$$|u(0)|, |u(0) - e| \ge 2^{-1}, \tag{128}$$

$$|u(t)| \le S,\ t \in R^1. \tag{129}$$

By (127) and (126) for each natural number k

$$\begin{aligned} I^f(-k, k, u) &\le \liminf_{q\to\infty} I^f(-k, k, u_{i_q}) \\ &\le \liminf_{q\to\infty} (U^f(-k, k, u_{i_q}(-k), u_{i_q}(k)) + i^{-1}) \\ &= U^f(-k, k, u(-k), u(k)). \end{aligned}$$

Together with (129) this relation implies that $u \in \mathcal{M}(f)$. In view of (124) $u(0) \notin \{0, e\}$. Theorem 5.1 is proved.

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