

A Dual Condition for the Convex Subdifferential Sum Formula with Applications

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In this paper we present a simple dual condition for the convex subdifferential sum formula. We show that if f and $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper lower semi-continuous convex functions then $\partial(f + g)(x) = \partial f(x) + \partial g(x)$, for each $x \in \text{dom } f \cap \text{dom } g$, whenever $\text{Epi } f^* + \text{Epi } g^*$ is weak* closed, where $\text{Epi } f^*$ denotes the epigraph of the conjugate function f^* of f . This dual closure condition, which is shown to be weaker than the well known primal interior point like conditions, is completely characterized by the subdifferential sum formula in the case where f and g are sublinear. It also provides a simple global condition for the strong conical hull intersection property (CHIP), which is a key regularity condition in the study of constrained interpolation and approximation problems. The subdifferential sum formula is then used to derive necessary and sufficient optimality conditions for a general cone-constrained convex optimization problem under a much weaker dual constraint qualification, and to obtain a generalized Clarke-Ekeland dual least action principle.

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1. Introduction

There is more to a subdifferential sum formula than just a rule of subdifferential calculus. The subdifferential sum formula for the two proper lower semi-continuous convex functions f and $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ states that

$$\partial(f + g)(x) = \partial f(x) + \partial g(x), \quad \forall x \in \text{dom } f \cap \text{dom } g, \quad (1)$$

whenever certain regularity condition on f and g holds. It is a key for characterizing optimal solutions of constrained convex optimization and approximation problems [9, 8, 16]. The study of a regularity condition which ensures the subdifferential sum formula, is also central to various other areas of convex optimization such as the duality theory for convex cones, and the existence of error bounds for systems of convex inequalities (see [3] and other references therein). Moreover, such a regularity condition provides a constraint qualification for convex optimization. When both f and g are replaced by the indicator

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functions of two closed and convex sets C and D , the sum formula yields the normal cone intersection formula [5]: for each $x \in C \cap D$, $N_{C \cap D}(x) = N_C(x) + N_D(x)$. This is popularly known as the strong conical hull intersection property (CHIP) (see [3, 4, 17]). The strong CHIP is the basic ingredient in the study of constrained interpolation and approximation problems [8], and in particular, it completely characterizes a strong duality relationship between a pair of optimization problems arising in constrained best approximation [9]. On the other hand, subdifferential sum formulas without any constraint qualification have been given by Hiriart-Urruty and Phelps in [11] in terms of approximate subdifferentials, and by Thibault in [20, 21] in terms of subdifferentials at nearby points.

In recent years various (primal) conditions for the subdifferential sum formula or the normal cone intersection formula have been presented in the literature (see [1, 3, 4, 9, 17, 18, 19]). However, these primal regularity conditions are either (global) interior-point type conditions [1, 18, 19] which frequently restrict applications, or are based on local conditions [3, 4]. The purpose of this paper is to examine global dual regularity conditions that are weaker than the interior-point type conditions for the subdifferential sum formula and then to derive general optimality and duality principles. We show that the sum formula (1) holds whenever $\text{Epi } f^* + \text{Epi } g^*$ is weak*closed, where $\text{Epi } f^*$ denotes the epigraph of the conjugate function f^* of f .

The significance of the closure condition is that it yields a simple global condition for strong CHIP, and it is completely characterized by the sum formula in the case where both f and g are sublinear. We then apply the subdifferential sum formula to derive necessary and sufficient optimality conditions for a general cone-constrained convex optimization problem under a dual closure constraint qualification, and to obtain a general Clarke-Ekeland dual least action principle.

The layout of the paper is as follows. In Section 2, we recall the basic definitions and present basic results on epigraphs of conjugate functions. In Section 3, we present the subdifferential sum formula under a regularity condition and discuss the links among various related regularity conditions in the literature. In Section 4, we present characterizations of optimal solutions of cone-constrained convex optimization problems under closure constraint qualifications. In Section 5, we obtain a generalized Clarke-Ekeland dual least action principle.

2. Epigraphs of Conjugate Functions

We begin by recalling some definitions and by fixing notations. Let X and Z be Banach spaces. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset X$, the *closure* of D will be denoted $\text{cl } D$. If a set $A \subset X'$, then $\text{cl } A$ will stand for the weak* closure. The *indicator function* δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The *support function* σ_D is defined by $\sigma_D(u) = \sup_{x \in D} u(x)$. The *normal cone* of D is given by $N_D(x) := \{v \in X' : \sigma_D(v) = v(x)\} = \{v \in X' : v(y - x) \leq 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) = \emptyset$ when $x \notin D$.

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the *conjugate function* of f , $f^* : X' \rightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \text{dom } f\}$$

where the domain of f , $\text{dom } f$, is given by

$$\text{dom } f = \{x \in X \mid f(x) < +\infty\}.$$

The epigraph of f , $\text{Epi } f$, is defined by

$$\text{Epi } f = \{(x, r) \in X \times \mathbb{R} \mid x \in \text{dom } f, f(x) \leq r\}.$$

The subdifferential of f , $\partial f : X \rightrightarrows X'$ is defined as

$$\partial f(x) = \{v \in X' \mid f(y) \geq f(x) + v(y - x) \forall y \in X\}.$$

For a closed and convex $D \subset X$, it follows from the definitions that $\partial \delta_D = N_D$. If $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous *sublinear* function, i.e. f is convex and positively homogeneous ($f(0) = 0$ and $f(\lambda x) = \lambda f(x)$, $\forall x \in X, \forall \lambda > 0$), then $\partial f(0)$ is non-empty and for each $x \in \text{dom } f$,

$$\partial f(x) = \{v \in \partial f(0) \mid v(x) = f(x)\}.$$

For a function $h : X' \rightarrow \mathbb{R} \cup \{+\infty\}$, the *weak* lower semicontinuous regularization* is defined as the unique function $\text{cl } h$ verifying

$$\text{Epi } (\text{cl } h) = \text{cl } \text{Epi } h,$$

where the closure in the right hand side is taken with respect to the weak* topology. The function $\text{cl } h$ is also characterized as the biggest function among all the weak* lower-semicontinuous minorants of h . For every convex proper function $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ it holds that $f \geq \text{cl } f \geq f^{**}$. If f is a proper convex function then Fenchel-Moreau Theorem gives us $\text{cl } f = f^{**}$. Therefore, whenever f is proper, convex and lower-semicontinuous, we have $f = \text{cl } f = f^{**}$. For details see [22, Theorem 6.18].

For the proper lower semi-continuous functions $f, g : Z \rightarrow \mathbb{R} \cup \{+\infty\}$, the *infimal convolution* of f with g is denoted by $f^* \oplus g^* : Z' \rightarrow \mathbb{R} \cup \{+\infty\}$ and is defined by

$$(f^* \oplus g^*)(z) := \inf_{z_1 + z_2 = z} \{f^*(z_1) + g^*(z_2)\}.$$

The infimal convolution of f^* with g^* is said to be *exact* provided the infimum above is achieved for every $z \in Z$. Note that $f^* \oplus g^*$ is exact at each $z \in Z$ where $(f^* \oplus g^*)(z) \in \mathbb{R}$ if and only if the equality

$$\text{Epi } (f^* \oplus g^*) = \text{Epi } f^* + \text{Epi } g^*, \tag{2}$$

holds (see, e.g [19, Theorem 2.2(c)]).

Applying the well-known Moreau-Rockafellar theorem (see e.g. Theorem 3.2, [19]), we see that

$$\text{Epi } (f + g)^* = \text{Epi } (\text{cl } (f^* \oplus g^*)).$$

But by Theorem 2.2 (e) of [19], we get that $\text{Epi } (\text{cl } (f^* \oplus g^*)) = \text{cl } (\text{Epi } f^* + \text{Epi } g^*)$. This gives us that

$$\text{Epi } (f + g)^* = \text{cl } (\text{Epi } f^* + \text{Epi } g^*). \tag{3}$$

If both f and g are proper lower semi-continuous and sublinear functions then it easily follows from (3) that

$$\partial(f + g)(0) = \text{cl}(\partial f(0) + \partial g(0)),$$

since in this case

$$\partial(f + g)(0) \times \mathbb{R}_+ = \text{Epi}(f + g)^* = \text{cl}(\text{Epi} f^* + \text{Epi} g^*) = \text{cl}(\partial f(0) \times \mathbb{R}_+ + \partial g(0) \times \mathbb{R}_+).$$

Remark 2.1. Let C and D be closed convex subsets of X . If $C \cap D \neq \emptyset$ then

$$\text{Epi} \sigma_{C \cap D} = \text{cl}(\text{Epi} \sigma_C + \text{Epi} \sigma_D).$$

Indeed, the functions $f := \delta_C$ and $g := \delta_D$ are proper lower semi-continuous functions and $(f + g) = \delta_{C \cap D}$. So,

$$\text{Epi} \sigma_{C \cap D} = \text{Epi}(f + g)^* = \text{cl}(\text{Epi} f^* + \text{Epi} g^*) = \text{cl}(\text{Epi} \sigma_C + \text{Epi} \sigma_D).$$

3. Subdifferential Sum Formula

In this section we establish the subdifferential sum formula for convex functions under a simple dual regularity condition. We then show that the dual condition is in fact completely characterized by the sum formula in the case where the functions involved in the formula are sublinear. We also illustrate how the regularity condition is related to strong CHIP and other interior-point type conditions.

Theorem 3.1. *Let $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous convex functions such that $\text{dom} f \cap \text{dom} g \neq \emptyset$. If $\text{Epi} f^* + \text{Epi} g^*$ is weak $*$ closed then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x), \quad \forall x \in \text{dom} f \cap \text{dom} g.$$

Proof. Let $x \in \text{dom} f \cap \text{dom} g$. Clearly, $\partial(f + g)(x) \supset \partial f(x) + \partial g(x)$. To prove the converse inclusion, let $v \in \partial(f + g)(x)$. Then $(f + g)^*(v) + (f + g)(x) = v(x)$; thus, $(f + g)^*(v) = v(x) - (f + g)(x)$. So,

$$(v, v(x) - (f + g)(x)) \in \text{Epi}(f + g)^* = \text{cl}(\text{Epi} f^* + \text{Epi} g^*) = \text{Epi} f^* + \text{Epi} g^*,$$

by the assumption. Now, we can find $(u, \alpha) \in \text{Epi} f^*$ and $(w, \beta) \in \text{Epi} g^*$ such that

$$v = u + w \text{ and } v(x) - (f + g)(x) = \alpha + \beta.$$

As $f^*(u) \leq \alpha$ and $g^*(w) \leq \beta$, we get

$$\begin{aligned} f^*(u) + g^*(w) &\leq \alpha + \beta \\ &= v(x) - (f + g)(x) \\ &= u(x) + w(x) - f(x) - g(x). \end{aligned}$$

On the other hand, $f^*(u) \geq u(x) - f(x)$ and $g^*(w) \geq w(x) - g(x)$, and so,

$$f^*(u) + g^*(w) \geq u(x) + w(x) - f(x) - g(x).$$

Hence,

$$f^*(u) + g^*(w) = u(x) + w(x) - f(x) - g(x).$$

This equality together with the definition of f^* gives us

$$\begin{aligned} 0 &\geq u(x) - f(x) - f^*(u) \\ &= g^*(w) + g(x) - w(x), \end{aligned}$$

which yields $w \in \partial g(x)$. Similarly, we can show that $u \in \partial f(x)$. Hence, $v = u + w \in \partial f(x) + \partial g(x)$. □

Proposition 3.2. *Let $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous convex functions such that $\text{dom } f \cap \text{dom } g \neq \emptyset$. If $\text{cone}(\text{dom } f - \text{dom } g)$ is a closed subspace then $\text{Epi } f^* + \text{Epi } g^*$ is weak* closed.*

Proof. Since $\text{cone}(\text{dom } f - \text{dom } g)$ is a closed subspace it follows from Theorem 1.1, [1] (see also Theorem 3.6, [19]) that $(f + g)^* = f^* \oplus g^*$ with exact infimal convolution. As a consequence of exactness, we have that

$$\text{cl}(\text{Epi } f^* + \text{Epi } g^*) = \text{Epi } (f + g)^* = \text{Epi } (f^* \oplus g^*) = \text{Epi } f^* + \text{Epi } g^*.$$

□

Observe that the classical interiority condition, $\text{dom } g \cap \text{dom } f \neq \emptyset$ implies $0 \in \text{core}(\text{dom } f - \text{dom } g)$, which in turn implies that $\text{cone}(\text{dom } f - \text{dom } g)$ is a closed subspace [1], where for a convex set A of X $\text{core}(A) := \{a \in A \mid (\forall x \in X)(\exists \varepsilon > 0) \text{ such that } (\forall \lambda \in [-\varepsilon, \varepsilon]) a + \lambda x \in A\}$ and $\text{int}(A)$ denotes the interior of A . The following example shows that the dual closure condition is much weaker than these interiority conditions.

Example 3.3. Let $f = \delta_{[0, \infty)}$ and $g = \delta_{(-\infty, 0]}$. Then $f^* = \sigma_{[0, \infty)}$, $g^* = \sigma_{(-\infty, 0]}$ and $\text{Epi } f^* + \text{Epi } g^* = \mathbb{R} \times \mathbb{R}_+$, which is a closed convex cone. However, $\text{int } \text{dom } g \cap \text{dom } f = \emptyset$, and $\text{cone}(\text{dom } f - \text{dom } g) = [0, \infty)$, which is not a subspace.

We now see that our dual condition is completely characterized by the subdifferential sum formula in the case where the functions involved in the formula are lower semi-continuous and sublinear.

Corollary 3.4. *Let $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous sublinear functions. Then the following conditions are equivalent.*

- (i) $\text{Epi } f^* + \text{Epi } g^*$ is weak* closed.
- (ii) $\partial(f + g)(x) = \partial f(x) + \partial g(x)$, $\forall x \in \text{dom } f \cap \text{dom } g$.

Proof. Note that $f(0) = g(0) = 0$ and hence $\text{dom } f \cap \text{dom } g \neq \emptyset$. Therefore, the equivalence of (i) and (ii) follows from the previous theorem if we show that (ii) implies (i). If (ii) holds then $\partial(f + g)(0) = \partial f(0) + \partial g(0)$. Since f and g are proper lower semi-continuous sublinear functions, $\partial(f + g)(0) = \text{cl}(\partial f(0) + \partial g(0))$. So, $\text{cl}(\partial f(0) + \partial g(0)) = \partial f(0) + \partial g(0)$; thus, $\partial f(0) + \partial g(0)$ is weak* closed. Hence,

$$\text{Epi } f^* + \text{Epi } g^* = \partial f(0) \times \mathbb{R}_+ + \partial g(0) \times \mathbb{R}_+ = (\partial f(0) + \partial g(0)) \times \mathbb{R}_+$$

is weak* closed. □

In the following Corollary, we see that our closure condition gives a global sufficient condition for the strong conical hull intersection property (CHIP) of two closed convex sets C and D . Recall that the pair $\{C, D\}$ satisfies strong CHIP at $x \in C \cap D$, if $N_{C \cap D}(x) = N_C(x) + N_D(x)$.

Corollary 3.5. *Let C and D be closed convex subsets of X with $C \cap D \neq \emptyset$. If $\text{Epi } \sigma_C + \text{Epi } \sigma_D$ is weak* closed then for each $x \in C \cap D$,*

$$N_{C \cap D}(x) = N_C(x) + N_D(x).$$

Proof. Let $f = \delta_C$ and let $g = \delta_D$. Then $(f + g) = \delta_{C \cap D}$ and so, from the previous Theorem we obtain that

$$N_{C \cap D}(x) = \partial \delta_{C \cap D}(x) = \partial \delta_C(x) + \partial \delta_D(x) = N_C(x) + N_D(x).$$

□

It is worth noting from the above Theorem that if C and D are two closed and convex subsets of X such that $C \cap D \neq \emptyset$ and if $\text{cone}(C - D)$ is a closed subspace then $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is weak* closed. Moreover, if X is a Euclidean space, C and D are closed convex cones and if the pair $\{C, D\}$ is boundedly linearly regular then $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is closed. For details see [5]. Recall that the pair $\{C, D\}$ is said to be boundedly linearly regular [3, 4] if for every bounded set S in X , there exists $\kappa_S > 0$ such that the distance to the sets C, D and $C \cap D$ are related by

$$d(x, C \cap D) \leq \kappa_S \max\{d(x, C), d(x, D)\},$$

for every $x \in S$, where $d(x, C) := \inf\{\|x - c\| \mid c \in C\}$ is the distance function.

It also worth observing from Corollary 3.5 that if C and D are two closed and convex subsets of X such that $0 \in C \cap D$ and the set $(\text{Epi } \sigma_C + \text{Epi } \sigma_D)$ is weak* closed then $(C \cap D)^+ = C^+ + D^+$. This follows from the fact that $N_C(0) = -C^+$ and $N_D(0) = -D^+$ and

$$-(C \cap D)^+ = N_{C \cap D}(0) = N_C(0) + N_D(0) = -(C^+) - (D^+) = -(C^+ + D^+).$$

4. Characterizing Optimal Solutions

In this section we derive the well known subdifferential characterizations of optimality of convex optimization problems under dual closure conditions in terms of epigraphs of conjugate functions. We first consider the convex optimization problem (P_A) :

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && x \in A \end{aligned}$$

where $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function and $A \subset X$ is a non-empty closed convex set with $A \cap \text{dom } f \neq \emptyset$.

The next theorem provides a more general regularity condition under which the basic subdifferential characterization of a minimizer of a convex function over a closed convex set holds.

Proposition 4.1. For the problem (P_A) , let $a \in A \cap \text{dom } f$. Assume that $(\text{Epi } f^* + \text{Epi } \delta_A^*)$ is weak* closed. Then a is a minimizer of (P_A) if and only if $0 \in \partial f(a) + N_A(a)$.

Proof. Let $g = \delta_A$. Then, g is a proper lower semi-continuous convex function. Now, the point $a \in A \cap \text{dom } f$ is a minimizer of (P_A) if and only if a is a minimizer of $(f + \delta_A)$, which means that $0 \in \partial(f + \delta_A)(a)$. Now, the previous theorem under the closure condition, that $(\text{Epi } f^* + \text{Epi } \delta_A^*)$ is weak* closed, gives us

$$0 \in \partial(f + \delta_A)(a) = \partial f(a) + \partial \delta_A(a) = \partial f(a) + N_A(a).$$

□

Now consider the convex optimization problem (P) :

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && x \in C, \quad -g(x) \in S, \end{aligned}$$

where $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function and $C \subset X$ is a closed convex set, $g : X \rightarrow Z$ is an S -convex continuous function, $S \subset Z$ is a closed convex cone, $A \cap \text{dom } f \neq \emptyset$, and $A = \{x \in X \mid x \in C, -g(x) \in S\} = C \cap g^{-1}(-S)$. The dual cone of S is given by $S^+ = \{\theta \in Z' \mid \theta(s) \geq 0, \forall s \in S\}$.

It is easy to show (see [14]) using the Hahn-Banach separation theorem that if C is a closed convex subset of X and $g : X \rightarrow Z$ is a continuous and S -convex mapping and if $C \cap g^{-1}(-S) \neq \emptyset$ then

$$\text{Epi } \sigma_{C \cap g^{-1}(-S)} = \text{cl} (\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*).$$

The following theorem provides necessary and sufficient conditions for optimality of (P) , extending the corresponding result in [14], where a closed cone regularity condition and a separation theorem were used to derive the optimality conditions in the case where f is a real valued function.

Theorem 4.2. For the problem (P) , assume that the sets $(\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ and $(\text{Epi } f^* + \cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ are weak* closed. If $a \in A \cap \text{dom } f$ is a minimizer of (P) if and only if there exists $\lambda \in S^+$ such that

$$0 \in \partial f(a) + \partial(\lambda \circ g)(a) + N_C(a) \quad \text{and} \quad (\lambda \circ g)(a) = 0.$$

Remark 4.3. If $(\text{Epi } f^* + \cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ is weak* closed then the set $(\text{Epi } f^* + \text{cl } \cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ is also weak* closed. Moreover, if $\text{int } S \neq \emptyset$ and if the generalized Slater's condition that there exists $x_0 \in C, -g(x_0) \in \text{int } S$, is satisfied then $(\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ is weak* closed. For other general sufficient conditions that ensure the set $(\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ is weak* closed, see [14]. Note also that if f is continuous at some $x_0 \in A \cap \text{dom } f$ (or more generally if $\text{cone}(\text{dom } f - A)$ is a closed subspace of X) then $(\text{Epi } f^* + \cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ is weak* closed, provided the set $(\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$ is weak* closed, since

$$\text{Epi } \sigma_{C \cap g^{-1}(-S)} = (\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*).$$

Proof of Theorem 4.2. $[\implies]$ Assume that a is a minimizer of (P) . Then, Proposition 4.1 gives us the point a is a minimizer of (P) if and only if $0 \in \partial f(a) + N_A(a)$. This is equivalent to the condition that there exists $u \in \partial f(a)$ such that $u(x) \geq u(a)$, for each $x \in A = C \cap g^{-1}(-S)$. By the definition of the epigraph of the support function $\text{Epi } \sigma_{C \cap g^{-1}(-S)}$ and by the hypothesis, we get that

$$\begin{aligned} [\forall x \in C \cap g^{-1}(-S), u(x) \geq u(a)] &\iff (-u, -u(a)) \in \text{Epi } \sigma_{C \cap g^{-1}(-S)} \\ &\iff (-u, -u(a)) \in \text{cl}(\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*) \\ &\iff (-u, -u(a)) \in (\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*). \end{aligned}$$

So, the point a is a minimizer of (P) if and only if there exists $u \in \partial f(a)$ such that $(-u, -u(a)) \in (\cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^*)$, which implies that there exists $\lambda \in S^+$ such that for each $x \in C$,

$$f(x) + (\lambda \circ g)(x) \geq f(a) + (\lambda \circ g)(a) \text{ and } (\lambda \circ g)(a) = 0.$$

This in turn gives us that

$$0 \in \partial f(a) + \partial(\lambda \circ g)(a) + N_C(a) \text{ and } (\lambda \circ g)(a) = 0.$$

$[\impliedby]$ This implication follows from the definitions of convexity and the subgradients of the functions involved. □

The following numerical example illustrates the case where the optimality conditions for a convex optimization problem hold, verifying the preceding theorem. However, the generalized Slater-type interior-point conditions fail to hold for the convex problem. A related example is discussed in [14] for comparing constraint qualifications.

Example 4.4. Consider the following simple one-dimensional convex problem

$$\begin{aligned} &\text{minimize } |x| \\ &\text{subject to } x \in C, g(x) \leq 0, \end{aligned}$$

where $C = [-1, 1]$, $f(x) = |x|$ and

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Clearly, the generalized Slater-type interior-point conditions do not hold as $g(C) + S = \mathbb{R}_+$. On the other hand, $\sigma_C(v) = |v|$ and so, $\text{Epi } \sigma_C = \text{Epi } |\cdot|$ and $\bigcup_{\lambda \geq 0} \text{Epi } (\lambda g)^* = \bigcup_{\lambda \geq 0} ([0, \lambda] \times \mathbb{R}_+) = \mathbb{R}_+^2$. Hence,

$$\bigcup_{\lambda \geq 0} \text{Epi } (\lambda g)^* + \text{Epi } \delta_C^* = \mathbb{R}_+^2 + \text{Epi } |\cdot|,$$

which is a closed convex cone. Moreover,

$$\begin{aligned} \text{Epi } f^* + \cup_{v \in S^+} \text{Epi } (v \circ g)^* + \text{Epi } \delta_C^* &= [-1, 1] \times \mathbb{R}_+ + \mathbb{R}_+^2 + \text{Epi } |\cdot| \\ &= \{(\alpha - \beta - 1, \beta) \mid \alpha \geq 0, \beta \geq 0\} \end{aligned}$$

is also closed. It is easy to verify that $0 \in \partial f(0) + \partial(\lambda \circ g)(0) + N_C(0)$ and $(\lambda \circ g)(0) = 0$ for $\lambda = 1$. □

5. Clarke-Ekeland Dual Least Action Principle

Hamilton Principle of least action in classical mechanics states that the evolution of a physical system is such that it minimizes a certain functional, called the *action*. This fact leads to the classical Hamiltonian equations. However, this action is usually indefinite, i.e., it admits no local maxima or minima. So, the solutions of the Hamiltonian equations cannot be obtained through the minimization of the action. It is known that a *dual action*, involving the Fenchel-Moreau conjugate of the Hamiltonian, can be shown to attain a minimum for certain classes of Hamiltonians. In [2], a duality principle was employed to obtain a dual action. For a system with a convex Hamiltonian, and for some P.D.E arising in the study of nonlinear wave equations, the dual action is well behaved and its critical points lead to solutions of the original Hamiltonian system [7].

In this section, we show that our subdifferential sum formula allows us to extend the class of Hamiltonians for which the dual least action principle can be applied.

Let H be a Hilbert space and $A : H \rightarrow H$ a (possibly unbounded) linear operator which is self-adjoint, i.e., $A = A^*$, with $A^* : H \rightarrow H$ defined by $\langle A^*x, y \rangle = \langle x, Ay \rangle$ for all $x, y \in H$. We assume that $R(A)$ is closed, so that H admits the orthogonal decomposition $H = R(A) \oplus \text{Ker}(A)$.

The operator A is not assumed to be positive semidefinite, and hence the associated quadratic form $Q_A : H \rightarrow \mathbb{R}$ given by

$$Q_A(x) := 1/2 \langle Ax, x \rangle,$$

is not convex in general. Let $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semicontinuous function. We pose the problem of finding $x \in H$ for which the inclusion

$$0 \in Ax + \partial f(x), \tag{4}$$

holds. Under suitable hypotheses, the solutions of problem (4) correspond to critical points of the functional $\Phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$\Phi(y) := 1/2 \langle Ay, y \rangle + f(y).$$

Since the spectrum of A may be unbounded, the functional Φ may be also unbounded both from below and above. This precludes in general to solve problem (4) by finding the critical points of Φ . In order to obtain the solutions of (4) as critical points of a better behaved functional, we reformulate the inclusion following classical duality principles (e.g.,[2]). By setting $y \in \partial f(x)$, (4) is equivalent to

$$0 \in A^{-1}y + \partial f^*(y), \tag{5}$$

where we are using the fact that $\partial f^{-1} = \partial f^*$. The operator A^{-1} is in general multivalued since $\text{Ker}(A)$ may not be zero. However, and using the orthogonal decomposition of H , we can define the point-to-point operator $A_0^{-1} : R(A) \rightarrow R(A)$ by setting $A_0^{-1}(z)$ as the unique $x \in R(A)$ such that $Ax = z$. With these definitions in mind, (5) becomes

$$0 \in A_0^{-1}y + \partial f^*(y) + \text{Ker}(A), \quad y \in R(A). \tag{6}$$

Note that $\text{Ker}(A) = \partial \delta_{R(A)}(y)$ and hence the above inclusion can be rewritten as

$$0 \in A_0^{-1}y + \partial f^*(y) + \partial \delta_{R(A)}(y). \tag{7}$$

Assuming that $\text{Epi } f + \text{Ker } (A) \times \mathbb{R}_+$ is closed, we conclude, by Theorem 3.1, that

$$\partial(f^* + \delta_{R(A)}) = \partial f^* + \partial \delta_{R(A)}.$$

We are using here that the weak* closure condition reduces to plain closedness by reflexivity and convexity. In this way, we can associate the solutions of (6) with the critical points of the functional $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$\phi(y) := 1/2 \langle A_0^{-1}y, y \rangle + f^*(y),$$

subject to the constraint $y \in R(A)$. More precisely, the critical points of ϕ , subject to the subspace $R(A)$ are the points $y \in H$ such that

$$\partial^\circ \phi(y) \cap \text{Ker } (A) \neq \emptyset,$$

where $\partial^\circ \phi$ stands for the generalized subgradient of ϕ [6, Chapter 2]. Using the fact that A_0^{-1} is point-to-point and strictly differentiable ([6, Section 2.2]), we have that (see [6, Corollary 1, Section 2.3])

$$\partial^\circ \phi(y) = A_0^{-1}y + \partial f^*(y).$$

Since f^* is convex, ∂f^* is the subgradient in the sense of convex analysis. This yields the condition

$$\text{Ker } (A) \cap (A_0^{-1}y + \partial f^*(y)) \neq \emptyset,$$

for describing the critical points of ϕ . Note that ϕ is better behaved than Φ , since A_0^{-1} is a bounded operator.

The preceding discussion has proved the following fact.

Theorem 5.1. *Assume that $\bar{y} \in R(A) \cap \text{dom } f^* \neq \emptyset$ and $\text{Epi } f + \text{Ker } (A) \times \mathbb{R}_+$ is closed. If \bar{y} is a critical point of ϕ , then there exists some $w \in \text{Ker } (A)$ such that $\bar{x} := A_0^{-1}(-\bar{y}) + w$ is a solution of (4).*

Remark 5.2. In [2, Theorem 4.4], the function f^* is required to be continuous and everywhere defined. Our closure condition allows us to relax this assumption on f^* . We proceed to prove that the condition $\text{dom } f^* = H$ implies that $\text{Epi } f + \text{Ker } (A) \times \mathbb{R}_+$ is closed. To see this, let $(y, \alpha) = \lim_n (x_n, \alpha_n) + (z_n, r_n)$, with $(x_n, \alpha_n) \in \text{Epi } f$ and $(z_n, r_n) \in \text{Ker } (A) \times \mathbb{R}_+$. We claim that $\{x_n\}$ must be bounded. Otherwise, using the fact that $\text{dom } f^* = H$ implies that f is coercive, i.e., $\liminf_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty$ (see Lemma 2.3 [15]), we see that

$$0 = \liminf_n \frac{\alpha + 1}{\|x_n\|} \geq \liminf_n \frac{\alpha_n + r_n}{\|x_n\|} \geq \liminf_n \frac{f(x_n)}{\|x_n\|} = +\infty,$$

a contradiction. This implies that the sequence $\{x_n\} \subset H$ is bounded and hence it has a subsequence $\{x_{n_j}\}_j$ weakly converging to some \bar{x} . By (weak) lower semicontinuity of f , we conclude that

$$f(\bar{x}) \leq \liminf_n f(x_n) \leq \liminf_n \alpha_n + r_n = \alpha$$

so that $(\bar{x}, \alpha) \in \text{Epi } f$. Since the sequence $\{x_{n_j} + z_{n_j}\}$ converges weakly to y , we conclude that $\{z_{n_j}\}$ converges weakly to $y - \bar{x}$. But the set $\text{Ker } (A)$ is (weakly) closed, which gives $y - \bar{x} \in \text{Ker } (A)$. Altogether, we get $(y, \alpha) = (\bar{x}, \alpha) + (y - \bar{x}, 0) \in \text{Epi } f + \text{Ker } (A) \times \mathbb{R}_+$.

It is also worth noting that the assumption $\text{dom } f^* = H$, of [2], precludes important choices of f , such as the norm of the space, or, more generally, a sublinear function. In this case, $f^* = \delta_{\partial f(0)}$ and hence the requirement on f^* may not be fulfilled. Our closure condition, however, allows one to consider sublinear functions in the study of problem (4).

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