Journal of Convex Analysis
Volume 12 (2005), No. 2, 315–329

Filling the Gap Between Lower-C^1 and Lower-C^2 Functions

Aris Daniilidis
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Cerdanyola del Vallès), Spain
arisd@mat.uab.es, http://mat.uab.es/˜arisd

Jérôme Malick
INRIA, Rhône-Alpes, 655 Avenue de l'Europe, Montbonnot, St Martin, 38334 Saint Ismier, France
jerome.malick@inria.fr, http://www.inrialpes.fr/bipop/people/malick/

The classes of lower-$C^{1,\alpha}$ functions ($0 < \alpha \le 1$), that is, functions locally representable as a maximum of a compactly parametrized family of continuously differentiable functions with α-Hölder derivative, are hereby introduced. These classes form a strictly decreasing sequence from the larger class of lower-C^1 towards the smaller class of lower-C^2 functions, and can be analogously characterized via perturbed convex inequalities or via appropriate generalized monotonicity properties of their subdifferentials. Several examples are provided and a complete classification is given.

Keywords: Maximum function, lower-$C^{1,\alpha}$ function, α-weakly convex function, α-hypomonotone operator

1991 Mathematics Subject Classification: Primary 26B25; Secondary 49J52, 47H05

1. Introduction

Let U be an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}^*$. A function $f : U \to \mathbb{R}$ is called lower-C^k (for short, LC^k), if for every $x_0 \in U$ there exist $\delta > 0$, a compact topological space S, and a jointly continuous function $F : B(x_0, \delta) \times S \to \mathbb{R}$ satisfying

$$f(x) = \max_{s \in S} F(x, s), \qquad \text{for all } x \in B(x_0, \delta),$$

and such that all derivatives of F up to order k with respect to x exist and are jointly continuous. It is easily seen that every such function is locally Lipschitz. In particular, LC^k functions provide a robust extension of both convexity and smoothness. For their role in optimization we refer to the survey [8] and to [19]; see also [17] for extensions in Hilbert spaces.

The class of LC^1 functions is first introduced by Spingarn in [22]. In that work, Spingarn shows that these functions are (Mifflin) semi-smooth and Clarke regular, and that are characterized by a generalized monotonicity property of their subgradients, called submonotonicity. Recently, in [5, Corollary 3], it has been pointed out that the class of LC^1 functions coincides with the class of locally Lipschitz approximately convex functions. We recall that a function $f : U \to \mathbb{R}$ is called *approximately convex* on U if for every $x_0 \in U$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, $y \in B(x_0, \delta)$ and all $t \in [0, 1]$

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\|. \qquad (1)$$

ISSN 0944-6532 / \$ 2.50

The above notion (introduced in [14], [15]) corresponds to a first order relaxation of convexity and is strongly related to the notion of α-paraconvexity studied in [11], [21]. A more general class – corresponding to the case that the ε of the above definition is always bounded below away from 0 – is recently considered in [16] for functions on the real line: these functions (which are not Clarke regular in general) are characterized by their local decomposability into a sum of a convex and a Lipschitz function. We refer also to [9] and [7] for related notions.

Shortly after Spingarn's work, the (smaller) class of LC^2 functions has been introduced and studied by Rockafellar [19]. In that work the following important results are established:

– for every $k \geq 2$, the class of LC^k functions coincides with the class of LC^2 functions;
– LC^2 are exactly the locally Lipschitz weakly convex functions.

We recall that a function $f : U \to \mathbb{R}$ is called *weakly convex* on U if for every $x_0 \in U$, there exist $\sigma > 0$ and $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$ and $t \in (0, 1)$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \sigma t(1-t)||x-y||^2. \tag{2}$$

Let us note that LC^2 functions are characterized by the fact that they are locally decomposable into a sum of a convex continuous function and a concave quadratic function (see [23], [19], [10] e.g.). The existence of a similar decomposition for the class of LC^1 functions remains open (see also Remark 3.6).

Remark 1.1 (terminology issues). We wish to draw the attention of the reader on some terminology issues: speaking about locally Lipschitz functions, the classes of weakly convex functions [23], of prox-regular (or proximal retract) functions [2] and of prime-lower nice functions [18] all coincide with the class of LC^2 functions. See also [1], [4], [18] and references therein for related topics.

In this paper, we consider the class of lower-$C^{1,\alpha}$ functions (in short, $LC^{1,\alpha}$), where $0 < \alpha \leq 1$. Roughly speaking, these are LC^1 functions of the form $f(x) = \max_{s \in S} F(x, s)$ for which $\nabla_x F(., s)$ is α-Hölder (see exact definition in Section 2). We shall show that every such function is characterized by the α-hypomonotonicity (Definition 2.4) of its (Clarke) subdifferential and enjoys an alternative geometrical description as a $(1+\alpha)$-order perturbation of convexity (see Theorem 3.2). In particular, as the notation suggests, for $\alpha = 1$ we recover the class of LC^2 functions (see Remark 3.3).

2. Prerequisites and definitions

Let $f : U \to \mathbb{R}$ be a locally Lipschitz function defined in an open subset U of $\mathbb{R}^n$. For every $x_0 \in U$, the (Clarke) generalized derivative of f at x_0 is defined as follows:

$$f^o(x_0; d) := \limsup_{(y,t) \to (x_0, 0+)} \frac{f(y+td) - f(y)}{t}, \quad \text{for all } d \in \mathbb{R}^n.$$

It follows (see [3, Proposition 2.1.1], for example) that $d \mapsto f^o(x_0; d)$ is a continuous sublinear functional, so that the Clarke subdifferential $\partial f(x_0)$ of f, that is, the set

$$\partial f(x_0) = \{x^* \in \mathbb{R}^n : f^o(x_0; d) \geq \langle x^*, d \rangle, \forall d \in \mathbb{R}^n\} \tag{3}$$

is nonempty. In particular, the multivalued operator $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by (3) if $x \in U$ and being empty for $x \in \mathbb{R}^n \setminus U$ is called subdifferential of f. If f is a C^1 function then $\partial f(x) = \{\nabla f(x)\}$, for all $x \in U$. Natural operations in optimization (as for instance taking the maximum of an index family of differentiable functions) often lead to nonsmooth functions, in which case ∂f is used to substitute the derivative. We refer to the classical textbooks [3], [4] and [20] for details and applications to optimization.

In this work we study a particular class of maximum-type locally Lipschitz functions. Let us give the following definition.

Definition 2.1 (lower-$C^{1,\alpha}$ function). Let U be an open set of $\mathbb{R}^n$, and $0 < \alpha \leq 1$. A locally Lipschitz function $f : U \to \mathbb{R}$ is called lower-$C^{1,\alpha}$ at $x_0 \in U$, if there exist a non-empty compact set S, positive constants $\delta, \sigma > 0$ and a continuous function $F : B(x_0, \delta) \times S \to \mathbb{R}$ which is differentiable with respect to the x-variable, such that

$$f(x) \;=\; \max_{s \in S} F(x, s), \qquad \text{for all } x \in B(x_0, \delta),$$

where $\nabla_x F(x, s)$ is (jointly) continuous and

$$||\nabla_x F(y, s) - \nabla_x F(x, s)|| \;\leq\; \sigma ||y - x||^{\alpha}, \tag{4}$$

for all $x, y \in B(x_0, \delta)$ and all $s \in I(x) \cup I(y)$, where

$$I(x) = \{s^* \in S : f(x) = F(x, s^*)\}. \tag{5}$$

We say that f is lower-$C^{1,\alpha}$ on U (and we denote $f \in LC^{1,\alpha}$) if the above definition is fulfilled at every $x \in U$. Removing condition (4) from Definition 2.1 or setting $\alpha = 0$, we obtain the definition of the lower-C^1 function given in the introduction. Hence, the above definition is a strengthening of the lower-C^1 property. In Subsection 3.3 we provide an example of a LC^1 function that is not $LC^{1,\alpha}$ for any $\alpha > 0$ (see Proposition 3.7).

Similarly to Definition 2.1, the following notion strengthens the notion of approximate convexity defined in (1).

Definition 2.2 (α-weakly convex function). Let U be a nonempty open subset of $\mathbb{R}^n$ and $0 < \alpha \leq 1$. A locally Lipschitz function $f : U \to \mathbb{R}$ is called α-weakly convex at $x_0 \in U$, if there exist $\sigma > 0$ and $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$ and $t \in (0, 1)$

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \sigma t(1 - t)||x - y||^{1+\alpha}. \tag{6}$$

The function f is called α-weakly convex, if it is α-weakly convex at every $x \in U$.

Remark 2.3. Taking $\alpha = 1$ in the above definition corresponds to the notion of weak convexity, see (2). On the other hand, the value $\alpha = 0$ has no practical interest. It yields a notion which is strictly weaker than approximate convexity (since "for every $\varepsilon > 0$" has been replaced by "there exists $\sigma > 0$") and which does not ensure the Clarke regularity of the function.

Finally we need the notion of α-hypomonotone operator, which lies strictly between sub-monotonicity and hypomonotonicity.

Definition 2.4 (α-hypomonotone operator). Let U be a nonempty open subset of $\mathbb{R}^n$ and $0 < \alpha \leq 1$. A multivalued mapping $T : U \rightrightarrows \mathbb{R}^n$ is called α-hypomonotone at $x_0 \in U$, if there exist $\sigma > 0$ and $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ we have

$$\langle y^* - x^*, y - x \rangle \;\geq\; -\sigma ||y - x||^{1+\alpha}. \tag{7}$$

The operator T is called α-hypomonotone, if it is α-hypomonotone at every $x \in U$.

Remark 2.5. An analogous remark applies here. Setting $\alpha = 1$ we recover the notion of hypomonotonicity, while the value $\alpha = 0$ has no interest for our purposes.

3. Main results

In Subsection 3.1 we establish subdifferential and mixed characterizations of the class of lower-$C^{1,\alpha}$ functions, while in Subsection 3.2 we show the coincidence of that class with the class of locally Lipschitz α-weakly convex functions and give an epigraphical characterization. These results are in the spirit of [22], [5], [15] (for approximately convex functions) and of [19], [4], [2] (for weakly convex functions). We also quote [4] and [1] for a study of epigraphical properties of such functions.

In Subsection 3.3 we give a complete classification of the aforementioned classes and examples distinguishing them. We also present subclasses with a particular interest in optimization.

3.1. Subdifferential characterizations

The following result is an expected characterization of α-weak convexity.

Theorem 3.1 (characterizations). *Let U be an open set of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a locally Lipschitz function. The following statements are equivalent:*

(*i*) *f is α-weakly convex on U;*
(*ii*) *∂f is α-hypomonotone on U;*
(*iii*) *for all $x_0 \in U$, there exist $\sigma, \delta > 0$ such that for all $x \in B(x_0, \delta)$, $x^* \in \partial f(x)$, and $u \in \mathbb{R}^n$ with $x + u \in B(x_0, \delta)$,*

$$f(x + u) \;\geq\; f(x) + \langle x^*, u \rangle - \sigma ||u||^{1+\alpha}. \tag{8}$$

Proof. ***(i) ⇒ (iii)***. Fix $x_0 \in U$, $\sigma > 0$, $\delta > 0$ given by Definition 2.2. Let us consider any $x \in B(x_0, \delta)$ and $u \in \mathbb{R}^n$ such that $x + u \in B(x_0, \delta)$. Then for $z \in B(x_0, \delta)$ sufficiently closed to x and such that $z + u \in B(x_0, \delta)$, one has

$$f(z + t\,u) \leq t\, f(z + u) + (1 - t)f(z) + \sigma t(1 - t)\,||u||^{1+\alpha}$$

or equivalently

$$\frac{f(z + t\,u) - f(z)}{t} \;\leq\; f(z + u) - f(z) + \sigma(1 - t)\,||u||^{1+\alpha}$$

Taking the "limsup" when $z \to x$ and $t \to 0+$ in both sides, one gets

$$f^o(x; u) \;\leq\; f(x + u) - f(x) + \sigma ||u||^{1+\alpha}$$

which in view of (3) yields the result.

(iii) $\Rightarrow$ ***(ii)***. Fix $x_0 \in U$, $\sigma > 0$, $\delta > 0$ and take any $x, y \in B(x_0, \delta)$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$. Then one has

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - \sigma ||x - y||^{1+\alpha} \qquad \text{and} \qquad f(x) \geq f(y) + \langle y^*, x - y\rangle - \sigma ||x - y||^{1+\alpha}$$

which by addition yields

$$\langle x^* - y^*, x - y\rangle \geq -2\sigma ||x - y||^{1+\alpha}.$$

This shows the α-hypomonotonicity of ∂f.

(ii) $\Rightarrow$ ***(i)***. Suppose ∂f is α-hypomonotone and let $\sigma > 0$, $\delta > 0$ as in Definition 2.4. Fix $x_1, x_2 \in B(x_0, \delta)$ and for any $t \in (0, 1)$ set $x_t = tx_1 + (1 - t)x_2$ so that

$$x_t - x_1 = (1 - t)(x_2 - x_1) \quad \text{and} \quad x_t - x_2 = t(x_1 - x_2). \tag{9}$$

By the Lebourg mean value theorem (see [12] or [3, Theorem 2.3.7]), for every $i \in \{1, 2\}$ there exists $z_i \in [x_i, x_t[$ and $z_i^* \in \partial f(z_i)$ such that

$$f(x_t) = f(x_i) + \langle z_i^*, x_t - x_i\rangle. \tag{10}$$

Multiplying (10) respectively by t for $i = 1$ and by $(1 - t)$ for $i = 2$ and adding the resulting inequalities we conclude in view of (9) that

$$f(x_t) = tf(x_1) + (1 - t)f(x_2) - t(1 - t)\langle z_1^* - z_2^*, x_1 - x_2\rangle. \tag{11}$$

Since

$$\frac{x_1 - x_2}{||x_1 - x_2||} = \frac{z_1 - z_2}{||z_1 - z_2||},$$

the definition of α-hypomonotonicity implies

$$\langle z_1^* - z_2^*, x_1 - x_2\rangle \;\geq\; -\sigma ||z_1 - z_2||^{\alpha} ||x_1 - x_2|| \;\geq -\sigma\, ||x_1 - x_2||^{1+\alpha},$$

so (11) yields

$$f(x_t) \;\leq\; tf(x) + (1 - t)f(y) + \,\sigma\, t(1 - t)\, ||x - y||^{1+\alpha},$$

which ends the proof. □

Let us note that the property that f is locally Lipschitz is only used for the implication $(ii) \Rightarrow (i)$, in which the Lebourg mean value theorem for locally Lipschitz functions was needed. All other implications can be adapted to the case that f is lower semicontinuous and ∂f is its Clarke-Rockafellar subdifferential (we refer to [3] or [4] for the corresponding definition).

3.2. Coincidence of α-weakly convex and $LC^{1,\alpha}$ functions

Let us now show the coincidence of the classes of locally Lipschitz α-weakly convex functions (Definition 2.2) and of $LC^{1,\alpha}$ functions (Definition 2.1). This result comes to complete statements of similar nature, previously established in [5, Corollary 3] (for approximately convex functions) and in [19], [23] (for weakly convex functions).

Theorem 3.2 (coincidence result). *Let U be a nonempty open subset of $\mathbb{R}^n$ and let $0 < \alpha \le 1$. Then a locally Lipschitz function $f : U \to \mathbb{R}$ is lower-$C^{1,\alpha}$ if and only if f is α-weakly convex.*

Proof. **($\Rightarrow$)**. Let us assume that f is lower-$C^{1,\alpha}$ and let us fix any $x_0 \in U$. Then let us consider $\delta, \sigma > 0$, a nonempty compact set S and a continuous function $F(x,s)$ according to the Definition 2.1 so that

$$f(x) \;=\; \max_{s\in S} F(x,s), \quad \text{for all } x \in B(x_0,\delta),$$

and

$$||\nabla F(y,s) - \nabla F(x,s)|| \;\le\; \sigma ||y-x||^{\alpha}, \tag{12}$$

for all $x, y \in B(x_0,\delta)$ and $s \in I(x) \cup I(y)$. Let $x \in B(x_0,\delta)$ and $u \in \mathbb{R}^n$ be such that $x+u \in B(x_0,\delta)$ and set $y = x+u$. Since S and $I(x)$ are compact, it follows (see [20, Theorem 10.31]) that

$$\partial f(x) = \operatorname{co}\{\nabla F(x,s),\ s \in I(x)\},$$

where $\operatorname{co}(A)$ denotes the convex hull of a set A. For any $x^* \in \partial f(x)$, by the Caratheodory theorem, there exist $\lambda_1, \ldots, \lambda_{n+1}$ in $\mathbb{R}_+$ with $\sum_i \lambda_i = 1$ and $s_1, \ldots, s_{n+1}$ in $I(x)$ such that

$$x^* \;=\; \sum_{i=1}^{n+1} \lambda_i \nabla F(x,s_i).$$

Applying for every $i \in \{1, \ldots, n+1\}$ the classical mean-value theorem to the differentiable function $x \longmapsto F(x,s_i)$ we obtain $z_i \in [x,y[$ such that

$$F(y,s_i) - F(x,s_i) = \langle \nabla F(z_i,s_i), y-x \rangle.$$

Since $s_i \in I(x)$, we have successively

$$\begin{aligned} f(y) &\ge F(y,s_i) \\ &= F(x,s_i) - \langle \nabla F(z_i,s_i), y-x \rangle \\ &= f(x) + \langle \nabla F(x,s_i), y-x \rangle + \langle \nabla F(z_i,s_i) - \nabla F(x,s_i), y-x \rangle. \end{aligned}$$

Multiplying by $\lambda_i \ge 0$ and adding the resulting inequalities for $i \in \{1, \ldots, n+1\}$ we obtain (recalling $y = x+u$) that

$$f(x+u) \ge f(x) + \langle x^*, u \rangle + \sum_{i=1}^{n+1} \lambda_i \, \langle \nabla F(z_i,s_i) - \nabla F(x,s_i),\, u \rangle . \tag{13}$$

Since $s_i \in I(x)$ for $i \in \{1, \ldots, n+1\}$, relation (12) yields

$$\langle \nabla F(x,s_i) - \nabla F(z_i,s_i), u \rangle \;\le\; \sigma ||u||\, ||z_i - x||^{\alpha}.$$

Since $z_i \in [x,y[$ this yields

$$\langle \nabla F(x,s_i) - \nabla F(z_i,s_i), u \rangle \;\le \sigma ||u||\, ||y-x||^{\alpha} = \sigma ||u||^{1+\alpha}.$$

Replacing into (13) we get

$$f(x+u) \;\geq\; f(x) + \langle x^*, u\rangle - \sigma||u||^{1+\alpha},$$

so the assertion follows from Theorem 3.1 (iii)⇒(i).

(⇐). Conversely, let us assume f is α-weakly convex and let us consider $x_0 \in U$. Then for some $\sigma, \delta > 0$ and all $y, z \in B(x_0, \delta)$, $z^* \in \partial f(z)$ we have

$$f(y) \;\geq\; f(z) + \langle z^*, y-z\rangle - \sigma||y-z||^{1+\alpha}. \tag{14}$$

Taking eventually $\tilde{\sigma} > \sigma$, we may assume that the above inequality is strict for all $y \neq z \in B(x_0, \delta)$ and all $z^* \in \partial f(z)$. Set

$$S = \left\{(z, z^*) \in \mathbb{R}^n \times \mathbb{R}^n, \quad ||z - x_0|| \leq \frac{\delta}{2},\; z^* \in \partial f(z)\right\}$$

Since ∂f is locally bounded and has a closed graph (see [3, Proposition 2.1.5], for example) it follows that S is compact. Moreover, S is nonempty since it contains the set $\{x_0\} \times \partial f(x_0)$. Let us now define

$$\begin{aligned} F \;:\; & B(x_0, \delta/2) \times S \longrightarrow \mathbb{R} \\ & \big(x, (z, z^*)\big) \longmapsto \; F\big(x, (z, z^*)\big) := f(z) + \langle z^*, x - z\rangle - \sigma||x-z||^{1+\alpha}. \end{aligned}$$

Then for every $x \in B(x_0, \delta/2)$ and every $s = (z, z^*) \in S$ we have in view of (14) (and the choice of $\sigma > 0$) that

$$f(x) \geq F(x, (z, z^*))$$

with strict inequality whenever $x \neq z$. Thus for every $x \in B(x_0, \delta/2)$

$$f(x) = \max_{(z,z^*)\in S} F(x, (z, z^*)),$$

and

$$I(x) = \{x\} \times \partial f(x).$$

Note also that

$$\nabla_x F(x, (z, z^*)) = \begin{cases} z^* - \sigma(1+\alpha)\,||x-z||^{\alpha-1}\,(x-z) & \text{if } \; x \neq z \\ z^* & \text{if } \; x = z \end{cases}$$

Let now any $x, y \in B(x_0, \delta)$ and $s = (z, z^*) \in I(x) \cup I(y)$. It follows that $z \in \{x, y\}$. Let us suppose (with no loss of generality) that $z = y$. Then

$$||\nabla_x F(y, s) - \nabla_x F(x, s)|| \;=\; \sigma(1+\alpha)\,||y-x||^{\alpha}.$$

Thus (4) of Definition 2.1 holds. To complete the proof, it suffices to check the continuity of $\nabla_x F(x, (z, z^*))$ on $B(x_0, \delta) \times S$. This is clear at every point $(x, (z, z^*))$ with $x \neq z$, so let us suppose that $x = z$, that is, $(x, (z, z^*)) = (x, (x, z^*))$ and let $(x_n, (z_n, z_n^*))_{n\geq 1}$ be a sequence of $B(x_0, \delta) \times S$ converging to $(x, (x, z^*))$. For all $n \in \mathbb{N}$ such that $x_n \neq z_n$ we have

$$\begin{aligned} & ||\nabla_x F(x, (x, z^*)) - \nabla_x F(x_n, (z_n, z_n^*))|| \\ = \;& \big\|\, z^* - {z_n}^* + \sigma(1+\alpha)||x_n - z_n||^{\alpha-1}(x_n - z_n)\,\big\| \\ \leq \;& ||z^* - z_n^*|| + \sigma(1+\alpha)||x_n - z_n||^{\alpha}. \end{aligned}$$

On the other hand, for all $n \in \mathbb{N}$ such that $x_n = z_n$ we have

$$||\nabla_x F(x,(x^*,z^*)) - \nabla_x F(x_n,(x_n,z_n^*))|| = \|z^* - z_n{}^*\|.$$

Thus, it follows easily that

$$||\nabla_x F(x,(x,z^*)) - \nabla_x F(x_n,(z_n,z_n^*))|| \longrightarrow 0$$

as $(x_n,(z_n,z_n^*)) \longrightarrow (x,(x,z^*))$. This shows that $\nabla_x F$ is jointly continuous, so $f \in LC^{1,\alpha}$. □

Remark 3.3 ($LC^{1,1} \equiv LC^2$). Taking $\alpha = 1$ in the above proof we obtain that the class of the lower-$C^{1,1}$ functions and of the locally Lipschitz weakly convex functions coincide. In view of the classical result of Rockafellar [19] (recalled in the introduction), we conclude that the classes $LC^{1,1}$ and LC^2 coincide.

Let us now provide a characterization of the epigraphs of $LC^{1,\alpha}$ functions, in terms of the truncated normal cone operator. We first recall the definition of the latter: if C is a nonempty subset of a Euclidean space $\mathbb{R}^m$ ($m \in \mathbb{N}^*$), then the (Clarke) normal cone of C at $u \in C$ is defined by

$$N_C(u) = \{u^* \in \mathbb{R}^m : \langle u^*, v\rangle \le 0, \ \forall v \in T_C(u)\}, \tag{15}$$

where the Clarke tangent cone $T_C(u)$ is defined as follows:

$$v \in T_C(u) \Longleftrightarrow \begin{cases} \forall \varepsilon > 0, \exists\, \delta > 0 \ \text{ such that} \\ \forall u' \in B(u,\delta) \cap C, \forall t \in]0,\delta[, \ (u' + tB(v,\varepsilon)) \cap C \neq \emptyset. \end{cases} \tag{16}$$

We put $N_C(u) = \emptyset$, whenever $u \notin C$. For any $r > 0$ we denote by $N_C^r(u)$ the truncated Clarke normal cone, that is,

$$N_C^r(u) = N_C(u) \cap B[0,r],$$

where $B[0,r]$ denotes the closed ball in $\mathbb{R}^m$ of center 0 and radius r. We further denote by

$$\operatorname{epi} f := \{(x,\beta) \in \mathbb{R}^{n+1} : \beta \ge f(x)\}$$

the epigraph of the function f defined on $\mathbb{R}^n$. By [3, p. 56], for all $u_0 = (x_0, f(x_0)) \in \operatorname{epi} f$ we have

$$N_{\operatorname{epi} f}(u_0) = \mathbb{R}^+(\partial f(x_0), -1).$$

Let us finally note that, if f is κ-Lipschitz on a ball B of $\mathbb{R}^n$, then for all x_1,x_2 in B, we have

$$||x_2 - x_1|| \le ||u_2 - u_1|| \le \sqrt{1+\kappa^2}\, ||x_2 - x_1||, \tag{17}$$

where $u_i := (x_i, f(x_i))$, $i \in \{1,2\}$ and where we use the same notation to denote the Euclidean norm of the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$.

The following result is analogous to the ones established in [4, Section 5] (for LC^2 functions) and in [1, Theorem 4.1.4] (for LC^1 functions).

Corollary 3.4 (epigraphical characterization). *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function defined on an open subset U of $\mathbb{R}^n$. The following two assertions are equivalent:*

(*i*) *the function f is lower-$C^{1,\alpha}$;*
(*ii*) *the operator $N^1_{\mathrm{epi}\, f} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^{n+1}$ is α-hypomonotone.*

Proof. ***(i)* ⇒ *(ii)*** Let $u_0 \in \mathrm{epi}\, f$. We can suppose without loss of generality that $u_0 = (x_0, f(x_0))$ for $x_0 \in U$ (otherwise $N_{\mathrm{epi}\, f}(u)$ is reduced to $\{0\}$ for all u in a neighborhood of u_0, so that (7) is clearly satisfied).

Let now $\kappa, \delta_1 > 0$ such that f is κ-Lipschitz on $B(x_0, \delta_1)$. By Theorem 3.2, the function f is weakly convex, so Theorem 3.1 (i)⇒(iii) yields that there exist $\delta_2 > 0$ and $\sigma > 0$ such that for all $x_1, x_2 \in B(x_0, \delta_2)$, $x_1^* \in \partial f(x_1)$ and $x_2^* \in \partial f(x_2)$

$$f(x_2) - f(x_1) \geq \langle x_1^*, x_2 - x_1 \rangle - \sigma ||x_1 - x_2||^{1+\alpha}. \tag{18}$$

Set $\delta = \min\{\delta_1, \delta_2\}$ and take $u_1, u_2 \in B(u_0, \delta) \cap \mathrm{epi}\, f$ (we use the same notation $B(u_0, \delta)$ to denote the ball of center u_0 and radius $\delta > 0$ in the space $\mathbb{R}^{n+1}$). In particular, u_1 has the form (x_1, β_1) with $\beta_1 \geq f(x_1)$. There are two cases:

- If $\beta_1 > f(x_1)$, then $N^1_{\mathrm{epi}\, f}(u_1) = \{0\}$.
- If $\beta_1 = f(x_1)$, then

  $$N^1_{\mathrm{epi}\, f}(u_1) = \mathbb{R}^+(\partial f(x_1), -1) \cap B[0, 1].$$

  So for every $u_1^* \in N^1_{\mathrm{epi}\, f}(u_1)$, there exists $x_1^* \in \partial f(x_1)$ such that $u_1^* = \mu_1(x_1^*, -1)$. Note also that we can bound μ_1 uniformly. Since f is κ-Lipschitz on $B(x_0, \delta)$, one has $||x_1^*|| \leq \kappa$ (see [3, Proposition 2.1.2], for example). As $||u_1^*|| \leq 1$, one obtains $\mu_1 \leq (1 + \kappa^2)^{-\frac{1}{2}}$.
  Since $\beta_2 \geq f(x_2)$, (18) implies

  $$\langle (x_1^*, -1), (x_2 - x_1, \beta_2 - \beta_1) \rangle \;\leq\; \sigma ||x_1 - x_2||^{1+\alpha}.$$

  Here again we use the same notation for the scalar products in $\mathbb{R}^n$ and in $\mathbb{R}^{n+1}$. In particular, $\langle (x, \alpha), (y, \beta) \rangle := \langle x, y \rangle + \alpha\beta$, for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$.

In both cases, for every $u_1^* \in N^1_{\mathrm{epi}\, f}(u_1)$ we have

$$\langle u_1^*, u_2 - u_1 \rangle \leq (1 + \kappa^2)^{-\frac{1}{2}} \sigma ||x_1 - x_2||^{1+\alpha},$$

which in view of (17) yields

$$\langle u_1^*, u_2 - u_1 \rangle \leq (1 + \kappa^2)^{-\frac{1}{2}} \sigma ||u_1 - u_2||^{1+\alpha}.$$

Interchanging the roles of u_1 and u_2, for every $u_2^* \in N^1_{\mathrm{epi}\, f}(u_2)$ we have

$$\langle u_2^*, u_2 - u_1 \rangle \geq -(1 + \kappa^2)^{-\frac{1}{2}} \sigma ||u_1 - u_2||^{1+\alpha}.$$

Substracting the last two equations, we get

$$\langle u_2^* - u_1^*, u_2 - u_1 \rangle \geq -2(1 + \kappa^2)^{-\frac{1}{2}} \sigma ||u_1 - u_2||^{1+\alpha},$$

which means that $N^1_{\mathrm{epi}\, f}$ is α-hypomonotone.

(ii) $\Rightarrow$ ***(i)*** Fix $x_0 \in U$ and set $u_0 = (x_0, f(x_0))$. Let δ_1 and σ such that for all $u_1, u_2 \in B(x_0, \delta_1)$, $u_1^* \in N^1_{\mathrm{epi}\, f}(u_1)$ and $u_2^* \in N^1_{\mathrm{epi}\, f}(u_2)$

$$\langle u_2^* - u_1^*, u_2 - u_1 \rangle \geq -\sigma ||u_1 - u_2||^{1+\alpha}. \tag{19}$$

Let δ_2 and κ be such that f is κ-Lipschitz on $B(x_0, \delta_1)$ and set

$$\delta = \frac{\min\{\delta_1, \delta_2\}}{\sqrt{1+\kappa^2}}.$$

Let $x_1, x_2 \in B(x_0, \delta)$, $x_1^* \in \partial f(x_1)$ and $x_2^* \in \partial f(x_2)$. For $i \in \{1, 2\}$, set $u_i = (x_i, f(x_i))$ and $u_i^* := (1+\kappa^2)^{-\frac{1}{2}}(x_i^*, -1)$. Observe that $u_i \in B(u_0, \delta_1)$ and $u_i^* \in N^1_{\mathrm{epi}\, f}(u_i)$. Thus (19) can be rephrased as

$$\langle (x_2^* - x_1^*, 0), (x_2 - x_1, f(x_2) - f(x_1)) \rangle \geq -\sigma (1+\kappa^2)^{\frac{1}{2}} ||u_1 - u_2||^{1+\alpha}.$$

Using (17) we get

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq -\sigma (1+\kappa^2) ||x_1 - x_2||^{1+\alpha}.$$

Thus ∂f is α-hypomonotone. By Theorem 3.1 (ii)$\Rightarrow$(i) and Theorem 3.2, we conclude that f is $LC^{1,\alpha}$. $\square$

3.3. Classification

Let us fix a nonempty open subset U of $\mathbb{R}^n$ and let us consider the following two particular classes of functions.

– **(locally decomposable functions)** We say that a locally Lipschitz function $f : U \to \mathbb{R}$ is *locally decomposable* on U as a sum of a convex function and a $C^{1,\alpha}$ function if for all $x_0 \in U$ there exists $\delta > 0$, a convex continuous function $k : B(x_0, \delta) \to \mathbb{R}$ and a $C^{1,\alpha}$-function $h : B(x_0, \delta) \to \mathbb{R}$ (that is, h is differentiable with α-Hölder derivative) such that

$$f(x) = k(x) + h(x), \quad \text{for all } x \in B(x_0, \delta).$$

– **(locally composite functions)** We say that a locally Lipschitz function $f : U \to \mathbb{R}$ is *locally composite* on U, if for every $x_0 \in U$ there exists $\delta > 0$, a convex continuous function $g : \mathbb{R}^m \to \mathbb{R}$ and a $C^{1,\alpha}$-function $G : B(x_0, \delta) \to \mathbb{R}^m$ such that

$$f(x) = g(G(x)), \quad \text{for all } x \in B(x_0, \delta).$$

This implies (see [20, p. 445], for example) that

$$\partial f(x) = \nabla G(x)^* \partial g(G(x)), \quad \text{for all } x \in B(x_0, \delta).$$

Proposition 3.5. *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function and $0 < \alpha \leq 1$. Consider the following conditions:*

(i) *f is locally decomposable on U as a sum of a convex continuous and a $C^{1,\alpha}$ function;*

(ii) f *is locally composite on* U *with a convex continuous and a* $C^{1,\alpha}$ *function;*
(iii) f *is a* $LC^{1,\alpha}$ *function.*

Then (i) $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii).*

Proof. (i)$\Longrightarrow$(ii). Having a local decomposition $f = k + h$, set $g(x, r) = k(x) + r$ for $(x, r) \in \mathbb{R}^n \times \mathbb{R}$ and $G(x) = (x, h(x))$ for $x \in \mathbb{R}$. It is straightforward to see that $f(x) = g(G(x))$, that G is $C^{1,\alpha}$ and that g is convex and continuous.

(ii)$\Longrightarrow$(iii). Let $x_0 \in U$, $\delta > 0$ and $g, h : B(x_0, \delta) \to \mathbb{R}$, g being convex continuous and $G \in C^{1,\alpha}(B(x_0, \delta))$ such that $f(x) = g(G(x))$ for all $x \in B(x_0, \delta)$. For all x near x_0, one has

$$\partial f(x) = \nabla G(x)^* \partial g(G(x))$$

Since ∇G is α-Hölderian, let $\sigma > 0$ such that for all $x, y \in B(x_0, \delta)$

$$||\nabla G(y) - \nabla G(x)|| \leq \sigma ||y - x||^\alpha. \tag{20}$$

Let $x, y \in B(x_0, \delta)$. For any $x^* \in \partial f(x)$, there exists $\zeta \in \partial g(G(x))$ such that $x^* = \nabla G(x)^* \zeta$. Since g is convex, it follows that

$$f(y) - f(x) = g(G(y)) - g(G(x)) \geq \langle \zeta, G(y) - G(x) \rangle. \tag{21}$$

Applying the mean value theorem to the function G on the segment $[x, y]$ we obtain $z \in [x, y[$ such that

$$G(y) - G(x) = \nabla G(z)(y - x). \tag{22}$$

By (20), it holds

$$||\nabla G(z) - \nabla G(x)|| \leq \sigma ||z - x||^\alpha \leq \sigma ||y - x||^\alpha. \tag{23}$$

Thus by (21), (22) and (23), we can write

$$\begin{aligned} f(y) - f(x) &\geq \langle \zeta, \nabla G(z)(y - x) \rangle \\ &= \langle \zeta, \nabla G(x)(y - x) \rangle + \langle \zeta, (\nabla G(z) - \nabla G(x))(y - x) \rangle \\ &\geq \langle \zeta, \nabla G(x)(y - x) \rangle - \sigma ||\zeta|| \, ||y - x||^{1+\alpha} \end{aligned}$$

Moreover, there exists a constant $\kappa > 0$ which bounds uniformly the norm of every subgradient of the convex continuous function g near x_0. Thus it holds

$$f(y) - f(x) \geq \langle x^*, y - x \rangle - \sigma\kappa \, ||y - x||^{1+\alpha},$$

and we can conclude by Theorem 3.1(iii)$\Rightarrow$(i) and Theorem 3.2. □

Remark 3.6 (conjecture). A classical result of Rockafellar [19] (see also [23], [8]) asserts that every LC^2 function is decomposable as a sum of a convex continuous and a concave quadratic function. Moreover, in view of Remark 3.3, the classes $LC^{1,1}$ and LC^2 coincide. Thus, in case $\alpha = 1$, the three assertions of Proposition 3.5 are then equivalent. It is not known if an analogous equivalence holds for the classes of LC^1 and $LC^{1,\alpha}$ functions.

Let us now give an example of a LC^1 function f, which does not belong to any of the classes $LC^{1,\alpha}$ for $\alpha > 0$. More precisely, we have the following proposition.

Proposition 3.7.

$$\bigcup_{0<\alpha<1} LC^{1,\alpha} \subsetneq LC^1$$

Proof. The inclusion follows directly from Definition 2.1. To see that the inclusion is strict, let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = -\int_0^x g(t)\, dt\,,$$

where

$$g(t) = \begin{cases} 0 & t \le 0 \\ \frac{1}{|\ln t|} & t > 0\,. \end{cases}$$

It is easily seen that g is continuous on $\mathbb{R}$, so that f is of class C^1. In particular, $f \in LC^1$. Note also that $f(0) = 0$ and $f'(0) = 0$.

Let us prove that for any $\alpha > 0$ the function f does not belong to the class $LC^{1,\alpha}$. Indeed, suppose towards a contradiction that there exists $\alpha > 0$ such that $f \in LC^{1,\alpha}$. Then by Theorem 3.2 and Theorem 3.1 (i)$\Rightarrow$(iii) there exist $\sigma, \delta > 0$ such that for all $x \in (0, 1)$,

$$f(x) \ \ge\ -\sigma\, |\, x\,|^{1+\alpha}\,.$$

Set now $\phi(x) = f(x) + \sigma|\, x\,|^{1+\alpha}$. Then the function ϕ is C^1, non-negative and $\phi(0) = 0$. It follows easily that there exists a sequence $(x_n)_{n\ge1}$ of positive real numbers converging to 0 such that $\phi'(x_n) \ge 0$. (Indeed, if for some $\delta > 0$ we have $\phi'(x) < 0$ for all $x \in (0, \delta)$, then ϕ should necessarily take negative values.) We compute $\phi'(x) = (1 + \alpha)\sigma x^\alpha - g(x)$ for $x > 0$. Then we have for all $n > 0$

$$(1 + \alpha)\sigma \ \ge\ \frac{1}{{x_n}^\alpha\, |\ln x_n|}.$$

Since $\alpha > 0$ the right-hand side tends to $+\infty$ when n grows. We thus obtain a contradiction. It follows that

$$f \in C^1 \setminus \bigcup_{\alpha>0} LC^{1,\alpha},$$

which proves the assertion. □

Let us complete our classification with the following proposition.

Proposition 3.8.

$$LC^2 \subsetneq \bigcap_{0<\alpha<1} LC^{1,\alpha}$$

Proof. Since every LC^2 function is a fortiori $LC^{1,\alpha}$ for all $0 < \alpha < 1$, the inclusion holds. To see that the inclusion is strict, let us consider the function

$$f(x) = \int_0^x g(t)\, dt\,, \qquad \text{for all } x \in \mathbb{R},$$

where

$$g(t) = \begin{cases} 0 & t \le 0 \\ t \ln t & t > 0. \end{cases}$$

Then g is continuous on $\mathbb{R}$ and clearly not Lipschitz around $t = 0$. Let us show that, for any $0 < \alpha < 1$, g is α-Hölderian in a neighborhood of 0. To this end, take x, y sufficiently small to ensure that are inside a neighborhood of 0 in which g is decreasing. We can suppose without loss of generality that $y < x$. We may suppose $x > 0$ (else the condition of α-Hölderianity is trivially fulfilled), and we distinguish three cases.

Case 1. $y \le 0$. Then we can write

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} = \frac{x|\ln x|}{|x - y|^\alpha} \le \frac{x|\ln x|}{x^\alpha} = x^{1-\alpha}|\ln x|. \tag{24}$$

Case 2. $0 < y < x/2$. In this case $0 > g(y) > g(x)$ so that

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} \le \frac{|g(x)|}{|x/2|^\alpha} \le 2^\alpha |\ln x| x^{1-\alpha}. \tag{25}$$

Case 3. $x/2 < y < x$. Applying the mean-value theorem for the function g to the segment $[x, y]$ (where g is C^∞) we obtain $z \in [x, y]$ such that

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} \le (\,|\ln\, z| + 1\,)\,|x - y|^{1-\alpha} \le (\,|\ln\, \frac{x}{2}| + 1\,)\, x^{1-\alpha}. \tag{26}$$

In all cases (24)-(26), the quantity $|x - y|^{-\alpha}\,|g(x) - g(y)|$ is bounded when x and y are sufficiently close to 0. Thus, there exist $\delta > 0$ and $M > 0$ such that for all $x, y \in]-\delta, \delta[$ with $x \neq y$ we have

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} \le M.$$

This means that g is α-Hölderian on $]-\delta, \delta[$.

It follows that f is C^1 on $\mathbb{R}$ and locally $C^{1,\alpha}$ around 0, for any $0 < \alpha < 1$. We us prove that f is not LC^2 around 0. To this end, let us assume, towards a contradiction, that there exists $\delta > 0$ such that ∂f is hypomonotone on $B(x_0, \delta)$. Since f is C^1, we have $\partial f(x) = \{g(x)\}$ for all $x \in \mathbb{R}$, and in particular $\partial f(0) = \{0\}$. Then for all $\sigma > 0$ and $x \in B(x_0, \delta)$,

$$x\, g(x) \ge -\sigma |x|^2.$$

This implies

$$\ln x \ge -\sigma \quad \text{for all } 0 < x < \delta,$$

which is a clear contradiction. $\square$

Let us resume the results in the following diagram.

$$LC^\infty = \underset{(2<k<+\infty)}{LC^k} = LC^2 = LC^{1,1} \subsetneq \underset{(0<\alpha<1)}{LC^{1,\alpha}} \subsetneq LC^1$$

$$\bigcup_{0<\alpha<1} LC^{1,\alpha} \subsetneq LC^1$$

$$LC^2 \subsetneq \bigcap_{0<\alpha<1} LC^{1,\alpha}$$

Acknowledgements. The authors are indebted to R. McCann, N. Hadjisavvas and L. Thibault for useful discussions.

References

[1] D. Aussel, A. Daniilidis, L. Thibault: Subsmooth sets: functional characterizations and related concepts, Trans. Amer. Math. Soc., to appear.

[2] F. Bernard, L. Thibault: Uniform prox-regularity of functions and epigraphs in Hilbert spaces, preprint 25p, Montpellier (2002).

[3] F. H. Clarke: Optimization and Nonsmooth Analysis, Wiley Interscience, New York (1983).

[4] F. H. Clarke, R. Stern, P. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2 (1995) 117–144.

[5] A. Daniilidis, P. Georgiev: Approximate convexity and submonotonicity, J. Math. Anal. Appl. 291 (2004) 292–301.

[6] P. Georgiev: Submonotone mappings in Banach spaces and applications, Set-Valued Analysis 5 (1997) 1–35.

[7] J.-W. Green: Approximately convex functions, Duke Math. J. 19 (1952) 499–504.

[8] J.-B. Hiriart-Urruty: Generalized differentiability, duality and optimization for problems dealing with differences of convex functions, in: Convexity and Duality in Optimization, Lecture Notes in Econom. Math. Systems 256 (1984) 37–70.

[9] D.-H. Hyers, S.-M. Ulam: Approximately convex functions, Proc. Amer. Math. Soc. 3 (1952) 821–828.

[10] R. Janin: Sur des multiapplications qui sont des gradients généralisés, C. R. Acad. Sci. Paris 294 (1982) 117–119.

[11] A. Jourani: Subdifferentiability and subdifferential monotonicity of γ-paraconvex functions, Control Cybernet. 25 (1996) 721–737.

[12] G. Lebourg: Generic differentiability of Lipschitzian functions, Trans. Amer. Math. Soc. 256 (1979) 125–144.

[13] C. Malivert, J.-P. Penot, M. Thera: Minimisation d'une fonction régulière sur un fermé non-régulier et non-convexe d'un espace de Hilbert, C. R. Acad. Sci. Paris, Sér. A 286 (1978) 1191–1193.

[14] H. V. Ngai, D. T. Luc, M. Thera: On ε-convexity and ε-monotonicity, in: Calculus of Variations and Differential Equations, A. Ioffe et al. (eds.), Research Notes in Mathematical Series, Chapman & Hall (1999) 82–100.

[15] H. V. Ngai, D. T. Luc, M. Thera: Approximate convex functions, J. Nonlinear Convex Analysis 1 (2000) 155–176.

[16] Z. Pales: On approximately convex functions, Proc. Amer. Math. Soc. 131 (2003) 243–252.

[17] J.-P. Penot: Favorable Classes of Mappings and Multimappings in Nonlinear Analysis and Optimization, J. Convex Analysis 3 (1996) 97–116.

[18] R. Poliquin, R. T. Rockafellar: Prox-regular functions in variational analysis, Trans. Amer. Math. Soc. 348 (1996) 1805–1838.

[19] R. T. Rockafellar: Favorable classes of Lipschitz continuous functions in subgradient optimization, in: Non-Differentiable Optimization, E. Nurminski (ed.), Pergamon Press, New York (1982).

[20] R. T. Rockafellar, R. Wets: Variational Analysis, Grundlehren der Mathematischen Wissenschaften 317, Springer (1998).

[21] S. Rolewicz: On the coincidence of some subdifferentials in the class of $\alpha(\cdot)$-paraconvex functions, Optimization 50 (2001) 353–363.

[22] J. E. Spingarn: Submonotone subdifferentials of Lipschitz functions, Trans. Amer. Math. Soc. 264 (1981) 77–89.

[23] J.-P. Vial: Strong and weak convexity of sets and functions, Math. Oper. Res. 8 (1983) 231–259.