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On Monotone Operators and Forms

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Consider a set-valued operator mapping points of a real Banach space into convex and weak* closed subsets of the dual space. It is shown that such operators can be investigated via the notion of a form. In particular, continuity, monotonicity, maximal monotonicity, and coerciveness are considered. Moreover, a calculus of forms is derived. Having established the above connections, a probably new sum theorem in nonreflexive Banach spaces is proved, and a Browder-type theorem for forms is given.

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1. Introduction

A prominent example of a set-valued map is the subdifferential of a lower semicontinuous, convex, and proper functional $p\colon E \to \overline{\mathbb{R}} := [-\infty, +\infty]$, defined by

$$\partial p(\overline{x}) := \{x' \in E' \mid \langle x', x - \overline{x}\rangle \le p(x) - p(\overline{x}) \text{ for all } x \in E\}.$$

Here and throughout this paper, let E denote a real Banach space with dual E', and $K \subset E$ a convex and closed subset. The starting point of our analysis is the connection between support functionals, and convex and weak* closed subsets given by Hörmander's theorem, compare [22] and [1].

Theorem 1.1 (Hörmander).

- *Let M be a nonempty, convex, and weak* closed subset of E'. Then the support functional $\sigma_M(x) := \sup_{x' \in M} \langle x', x\rangle$ is proper, sublinear, and lower semicontinuous. Moreover, $M = \partial\sigma_M(0)$.*
- *Suppose $p\colon E \to \overline{\mathbb{R}}$ is proper, sublinear, and lower semicontinuous. Then the set $M_p := \partial p(0)$ is nonempty, convex, and weak* closed, moreover $\sigma_{M_p} = p$.*
- *If M_1 and M_2 are two nonempty, convex, and weak* closed subsets of E', then $M_1 = M_2$ if and only if $\sigma_{M_1} = \sigma_{M_2}$.*

If we restrict ourselves to operators $T\colon K \to \mathfrak{P}(E')$ that map into convex and weak* closed subsets of the dual, then we can associate to each set Tx a support functional. A mapping $h\colon K \times E \to \overline{\mathbb{R}}$ is called *form* if there is a nonempty set $D \subset K$ such

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that for all $x \in \complement D$ there holds $h(x,\cdot) = -\infty$; for all $x \in D$ the functional $h(x,\cdot)$ is lower semicontinuous, and sublinear; and $h(x,0) = 0$. We call the set Dom $h := D$ the *domain* of h. Using Hörmander's theorem we can show a one-to-one correspondence between forms and weak* closed-valued, and convex-valued operators. For simplicity we call these operators *formidable*. To be more precise, let $T\colon K \to \mathfrak{P}(E')$ be a formidable operator. Then its form is given by $h(x,y) = \sup_{x' \in Tx} \langle x', y \rangle$. On the other hand, suppose $h\colon K \times E \to \overline{\mathbb{R}}$ is a form. Then the associated operator is recovered by $Tx = \partial_2 h(x,0)$, where $\partial_2 h(x,0)$ denotes the subdifferential of $h(x,\cdot)$ at 0.

Phelps [14] used a form-like approach to investigate the set of points where a maximal monotone operator is single-valued, Simons [23] proved a sum theorem in nonreflexive Banach space, and in Hu and Papageorgiou [10] one finds some general observations on support functionals representing set-valued maps.

Example 1.2. Let $p\colon E \to \overline{\mathbb{R}}$ be a convex, proper, and lower semicontinuous functional. Moreover, let dom $p := \{x \in E \mid p(x) < \infty\}$ be open. Then the form of ∂p is given by $\delta_+ p(x,y)$, which is the right hand sided Gâteaux differential of p at x in direction y.

We call a form $h\colon K \times E \to \overline{\mathbb{R}}$ *monotone* if for all $x, y \in$ Dom h

$$h(x, y-x) + h(y, x-y) \leq 0.$$

The form is called *maximal monotone* if the associated operator $T := \partial_2 h(\cdot,0)$ as an operator $T\colon K \to \mathfrak{P}(E')$ is maximal monotone. One can also define different notions of monotonicity in a similar manner, compare [12].

In [16, 13] a representation of maximal monotone operators via a (modified) subdifferential of a concave-convex function is given. Representation of operators via bifunctionals can be found in [12, 29], and references therein. For a different approach compare [7, 19]. Our next step is a calculus rule.

Proposition 1.3. *Suppose $T, S\colon K \to \mathfrak{P}(E')$ are monotone and formidable operators, and h, k are their associated forms. Moreover, let the following constraint qualification be fulfilled*

$$\operatorname{int}\operatorname{dom} T \cap \operatorname{dom} S \neq \varnothing. \tag{α}$$

Then

$$[T+S](x) = \partial_2 [h+k](x,0). \tag{1}$$

Equation (1) is necessary for the maximal monotonicity of $T+S$ since the operator $\partial_2[h+k](\cdot,0)$ is a monotone extension of $T+S$. The constraint qualification (α) ensures, with the help of the monotonicity of T and S, that the sum of the convex and weak* closed sets Tx and Sx is again convex and weak* closed, hence is representable by a support functional. The proposition is proved with the sum theorem for subdifferentials.

Theorem 1.4. *Suppose $p, q\colon E \to \overline{\mathbb{R}}$ are proper, convex, and lower semicontinuous. Moreover, assume that* $\operatorname{int}\operatorname{dom} p \cap \operatorname{dom} q \neq \varnothing$. *Then we have*

$$\partial[p+q](x) = \partial p(x) + \partial q(x) \text{ for all } x \in E.$$

The theorem states that $\partial p + \partial q$ remains maximal monotone. A proof can be found in [20] and [11]. As one might expect, there is a generalization for arbitrary S and T.

Theorem 1.5 (Rockafellar [17]). *Suppose that E is a reflexive Banach space, $T, S\colon E \to \mathfrak{P}(E')$ are maximal monotone and*

$$\operatorname{int}\operatorname{dom} T \cap \operatorname{dom} S \neq \varnothing. \tag{α}$$

Then $T + S$ is maximal monotone.

Looking at our calculus rules, (α) implies that $T + S$ still maps into weak* closed and convex subsets of E', which is necessary for the maximal monotonicity of $T + S$. On the other hand, it also implies that T is $\|\cdot\|$-weak* upper semicontinuous on a subset of E. This is a strong assumption. Simons gives an overview how this condition can be relaxed, including his results, and a discussion of examples in [23] and [24]. Our sum theorem is inspired by [26]. In that paper one finds the following theorem: *Suppose $p\colon E \to \overline{\mathbb{R}}$ is proper, lower semicontinuous and convex, and $T\colon E \to E'$ is monotone. Assume further* $\operatorname{cl}\operatorname{dom} p \subset \operatorname{dom} T$. *Moreover, let T be hemicontinuous on* $\operatorname{cl}\operatorname{dom} p$, *and* $\operatorname{dom}\partial p$ *be closed. Then $\partial p + T\colon E \to \mathfrak{P}(E')$ is maximal monotone.* We can show:

Theorem 1.6. *Suppose $p\colon E \to \overline{\mathbb{R}}$ is a proper, convex, and lower semicontinuous functional, $T\colon E \to \mathfrak{P}(E')$ is a maximal monotone operator. Furthermore, suppose that $D := \operatorname{cl}\operatorname{dom} p \subset \operatorname{int}\operatorname{dom} T$. If* $\operatorname{dom}(\partial p)$ *is closed, then $\partial p + T$ is maximal monotone.*

This generalizes a result of Simons, compare [23, Theorem 41.1]. The theorem is proved in three steps. First of all we show that every pair $(x, f) \in E \times E'$ which is *monotonically related* to $\partial p + T$, i. e. for every $(y, g) \in \operatorname{gra}(\partial p + T)$ we have $\langle f, x - y\rangle + \langle g, y - x\rangle \geq 0$, must satisfy $x \in \operatorname{dom}\partial[p + T]$. Then we reduce the statement to a variational inequality, which we finally solve with the upper semicontinuity of T. Nonreflexive sum theorems, using a different sum, are investigated in [2, 18, 19, 27, 28].

In the remainder of the paper, we will see how the form gives a natural proof of a Browder-type theorem in the context of forms. First of all we reduce the solvability of an operator inclusion to a variational inequality. To solve it, we will modify [10, Lemma III.2.13 and Theorem III.2.14]. Our approach differs from [10] as the authors first solve a perturbed problem $f \in (T + \epsilon J)x_\epsilon$ and then argue on $\epsilon \to 0$. Instead, we solve $f \in Tx$ directly.

We assume that the reader is familiar with monotone operator theory and nonsmooth analysis, as in e.g. [3, 10, 14, 21, 30].

Let us finally fix some notation. We denote the *power set* of E' by $\mathfrak{P}(E')$, the *weak closure* of a set $A \subset E$ with $\operatorname{cl}^w A$, the *convex hull* of A by $\operatorname{co} A$, the *closed convex hull* of A by $\overline{\operatorname{co}}\, A$. The *domain* of $T\colon K \to \mathfrak{P}(E')$ is defined by $\operatorname{dom} T := \{x \in K \mid Tx \neq \varnothing\}$, the *graph* of T is defined as the set $\operatorname{gra} T := \{(x, f) \in K \times E' \mid f \in Tx\}$. Moreover, the *polar* of $A \subset E'$ is defined by $A^\circ := \{x \in E \mid \langle x', x\rangle \leq 1 \text{ for all } x' \in A\}$.

2. The Form of an Operator

Proposition 2.1. *Let $T\colon K \to \mathfrak{P}(E')$ be such that Tx is convex and weak* closed for all $x \in \operatorname{dom} T$. Then there is a unique form $h\colon K \times E \to \overline{\mathbb{R}}$, for which*

$$Tx = \partial_2 h(x, 0).$$

The domain D of h equals dom T*. If further Tx is weak* compact for $x \in D$, then $h(x,\cdot)$ is continuous on E and* dom $h(x,\cdot) = E$*. Moreover, if T is single-valued at $x \in$* dom T*, then $h(x,\cdot)$ is linear.*

Proof. The proposition follows from Hörmander's theorem. Let $x \in \operatorname{dom} T$, then there exists a unique support function

$$\sigma_{Tx}(y) := \sup_{x' \in Tx} \langle x', y \rangle .$$

Moreover, the set Tx is recovered by $Tx = \partial \sigma_{Tx}(0)$. Let us define

$$h(x,y) := \sup_{x' \in Tx} \langle x', y \rangle . \tag{2}$$

Then h is the unique form of T, and we have $Tx = \partial_2 h(x,0)$. Furthermore Dom $h =$ dom T, because $\sup_{\varnothing} = -\infty$. Suppose now that the set Tx is weak* compact, then the supremum in (2) is attained, hence $h(x,\cdot) = \max_{x' \in Tx} \langle x', \cdot \rangle$. This shows that dom $h(x,\cdot) = E$. Moreover, as a lower semicontinuous and convex functions is continuous on the interior of its domain we have that $h(x,\cdot)$ is continuous on E, see e. g. Chapter 2 of [4]. Finally, if Tx is only a point, then

$$h(x,y) = \sup_{x' \in Tx} \langle x', y \rangle = \langle Tx, y \rangle ,$$

which shows that $h(x,\cdot)$ is linear. □

Corollary 2.2. *Suppose that $T\colon K \to \mathfrak{P}(E')$ admits a form h. Then we have*

$$f \in Tx \Leftrightarrow f \in \partial_2 h(x,0) \Leftrightarrow \langle f, \tilde{x} \rangle \le h(x, \tilde{x}) \text{ for all } \tilde{x} \in E.$$

It is well known that a maximal monotone operator is weak* closed-valued and convex-valued. Thus, it admits a unique form. Now we show that the definition of a monotone form is a good one.

Lemma 2.3. *Suppose $T\colon K \to \mathfrak{P}(E')$ is a formidable operator with form h. The operator T is monotone if and only if h is monotone.*

Proof. Suppose that T is monotone, then we have for all $x, y \in \operatorname{dom} T$

$$\sup_{f \in Tx} \langle f, y - x \rangle \le \inf_{g \in Ty} \langle g, y - x \rangle .$$

Therefore, we deduce $h(x, y - x) \le -h(y, x - y)$. The other direction is trivial. □

Now we take a closer look at maximal monotonicity of a form. Let us call a pair $(x,f) \in K \times E'$ *monotonically related to h* if (x,f) is monotonically related to the associated operator $\partial_2 h(\cdot, 0)\colon K \to \mathfrak{P}(E')$. Our first result is the next lemma.

Lemma 2.4. *Let $h\colon K \times E \to \overline{\mathbb{R}}$ be a monotone form with domain D.*

(*i*) *The form h is maximal monotone if and only if for every pair $(x,f) \in K \times E'$ for which holds $f \notin \partial_2 h(x,0)$ there is $y \in D$ such that $\langle f, x - y \rangle < h(y, x - y)$.*

(*ii*) *If h is maximal monotone and $(x, f) \in K \times E'$ is such that for all $y \in D$ we have $\langle f, x - y\rangle \geq h(y, x - y)$, then $f \in \partial_2 h(x, 0)$.*

Proof. (*i*) Let h be maximal monotone, and let $(x, f) \in K \times E'$ be such a pair. Then (x, f) is not monotonically related to h, thus there are $y \in D$ and $g \in \partial_2 h(y, 0)$ such that $\langle f, x - y\rangle + \langle g, y - x\rangle < 0$. From Corollary 2.2 it follows

$$\langle f, x - y\rangle < \langle g, x - y\rangle \leq h(y, x - y).$$

For the converse, suppose that $(x, f) \in K \times E'$ and $f \notin \partial_2 h(x, 0)$. Then there is $y \in D$ such that $\langle f, x - y\rangle < h(y, x - y)$. In particular, there is $g \in Ty$ for which holds

$$\langle f, x - y\rangle < \langle g, x - y\rangle \leq h(y, x - y),$$

and hence $\langle f, x - y\rangle + \langle g, y - x\rangle < 0$. Thus, (x, f) is not monotonically related to h. Therefore, h is maximal monotone.

(*ii*) Let $(x, f) \in K \times E$ be such that for all $y \in D$ we have

$$\langle f, x - y\rangle \geq h(y, x - y).$$

Thus, for any $y \in D$ and any $g \in \partial_2 h(y, 0)$ it holds

$$\langle f, x - y\rangle \geq h(y, x - y) \geq \langle g, x - y\rangle.$$

This shows that (x, f) is monotonically related to h. By maximal monotonicity of h we have $f \in \partial_2 h(x, 0)$. □

Suppose that h is a form with domain $D \subset K$ and $f \in E'$. We set:

$$A_1(f) := \bigcap_{y \in E} \{x \in D \,|\, \langle f, y - x\rangle \leq h(x, y - x)\}$$

$$A_2(f) := \bigcap_{y \in D} \{x \in K \,|\, h(y, x - y) \leq \langle f, x - y\rangle\}$$

If h is a monotone form, then for any $f \in E'$ we have $A_1(f) \subset A_2(f)$. Indeed, for $x \in A_1(f)$ we obtain by monotonicity of h that for each $y \in D$

$$\langle f, y - x\rangle + h(x, y - x) \leq \langle f, y - x\rangle - h(y, x - y) \leq 0$$

and thus $x \in A_2(f)$. The set $A_1(f)$ contains all $x \in D$ for which $f \in \partial_2 h(x, 0)$. On the other hand, $A_2(f)$ collects all $x \in K$ such that (x, f) is monotonically related to h. If these sets coincide, then h is maximal monotone by definition. One might look at $A_1(f)$ and $A_2(f)$ as the primal and dual solution set of a variational inequality, which coincide if the form is maximal monotone.

Proposition 2.5. *Let $h \colon K \times E \to \overline{\mathbb{R}}$ be a monotone form. Then the following are equivalent:*

(*a*) *The associated operator $T := \partial_2 h(\cdot, 0)$ is maximal monotone.*

(*b*) *For all $f \in E'$ we have $A_1(f) = A_2(f)$.*

Proof. $(a) \Rightarrow (b)$ Since $A_1(f) \subset A_2(f)$ we have to establish only the opposite inclusion. Let $x \in A_2(f)$, then for all $y \in D$

$$h(y, x-y) \leq \langle f, x-y \rangle.$$

By Lemma 2.4 (ii) we obtain that $x \in D$ and $f \in \partial_2 h(x, 0)$. Thus, $\langle f, \tilde{x} \rangle \leq h(x, \tilde{x})$ for all $\tilde{x} \in E$. Setting $\tilde{x} = y - x$ yields $x \in A_1(f)$.

$(b) \Rightarrow (a)$ Let $(x, f) \in K \times E'$ be such that $f \notin \partial_2 h(x, 0)$. Then $x \notin A_1(f)$ and since $A_1(f) = A_2(f)$, $x \notin A_2(f)$. Therefore, there exists $y \in D$ such that

$$\langle f, y-x \rangle + h(y, x-y) > 0.$$

We obtain the assertion from Lemma 2.4 (i). □

Corollary 2.6. *Suppose $h\colon K \times E \to \overline{\mathbb{R}}$ is a monotone form, and $f \in E'$. Then*

$$\bigcap_{y \in D} \{x \in D \mid \langle f, y-x \rangle \leq h(x, y-x)\} \subset \bigcap_{y \in D} \{x \in K \mid h(y, x-y) \leq \langle f, x-y \rangle\}.$$

If the form h is maximal monotone, then we have equality.

Corollary 2.7. *Let $h\colon K \times E \to \overline{\mathbb{R}}$ be a maximal monotone form, and $f \in E'$. Then the following are equivalent.*

- $f \in \partial_2 h(x, 0)$.
- $x \in A_1(f)$.
- $x \in A_2(f)$.

Lemma 2.8. *Let $T\colon K \to \mathfrak{P}(E')$ be a formidable, monotone operator with form h. Then for all $x, y \in D :=$ dom T and all positive $\lambda \in \mathbb{R}_+$ we have $\lambda(y-x) \in$ dom $h(x, \cdot)$. Thus, $\mathbb{R}_+(D-x) \subset$ dom $h(x, \cdot)$.*

Proof. Let $x, y \in D$ and $\lambda \in \mathbb{R}_+$ be given. Using the monotonicity of T we obtain for all $f \in \partial_2 h(x, 0)$, $g \in \partial_2 h(y, 0)$ that $\langle f, y-x \rangle \leq \langle g, y-x \rangle$, and hence $h(x, y-x) \leq \langle g, y-x \rangle < \infty$. Finally, by sublinearity of $h(x, \cdot)$ we have $h(x, \lambda(y-x)) = \lambda h(x, y-x) < \infty$, which proves the lemma. □

The inclusion of the lemma can be proper. For an example, let $E = \mathbb{R}$ and let the operator $T\colon \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ be defined by

$$Tx := \begin{cases} \{0\} & \text{if } x \in [0,1], \\ \varnothing & \text{otherwise.} \end{cases}$$

Obviously, T is monotone. Let h be its form. We have dom $h(1, \cdot) = \mathbb{R}$, but $\mathbb{R}_+(D-1) = (-\infty, 0]$.

Lemma 2.9. *Suppose that $T\colon K \to \mathfrak{P}(E')$ is a formidable operator with form h. Then we have for all $x \in$ Dom h*

$$\text{dom } h(x, \cdot) = \bigcup_{\lambda > 0} \lambda (Tx)^\circ.$$

Proof. Let $y \in \bigcup_{\lambda>0} \lambda(Tx)^\circ$ be given. Then there is a $\lambda > 0$ such that for all $x' \in Tx$ we have $\langle x', y\rangle \le \lambda$. Hence we deduce $y \in \operatorname{dom} h(x,\cdot)$. On the other hand, choose $y \in \operatorname{dom} h(x,\cdot)$. Then we have $\lambda := h(x,y) < \infty$. Thus, $\langle x', y\rangle \le \lambda$ for all $x' \in Tx$, and the assertion follows. □

Let $h\colon K \times E \to \overline{\mathbb{R}}$ be a form with domain D and $A \subset E$. We call h *ω-continuous on* A if for all $y \in E$ and all $\lambda \in \mathbb{R}$ the set $\{x \in E \mid h(x, y-x) \ge \lambda\}$ is closed in A. We say that h is *ω-continuous on the segments* of A if h is ω-continuous on $[a,b]$ for all $a, b \in A$. Evidently, a form is ω-continuous if and only if the map $x \mapsto h(x, y-x)$ is upper semicontinuous for every $y \in E$.

Proposition 2.10 (Simons [23, Lemma 40.1 (d)]). *Let $T\colon K \to \mathfrak{P}(E')$ be a maximal monotone operator. Then its form is ω-continuous on* $\operatorname{int}\operatorname{dom} T$.

Proposition 2.11 (Shih and Tan [25, Lemma 1]). *Let $T\colon E \to \mathfrak{P}(E')$ be a weak* compact-valued formidable operator with nonempty and convex domain D. Moreover, suppose that for all $a, b \in D$ the map $T|_{[a,b]}\colon [a,b] \to \mathfrak{P}(E')$ is $\|\cdot\|$-weak* upper semicontinuous. Then its form is ω-continuous on the segments of D.*

The next proposition is a first application of ω-continuity. It will imply the well known characterization lemma of everywhere defined maximal monotone operators in the language of forms.

Proposition 2.12 (Shih and Tan [25, Lemma 2]). *Suppose that the form $h\colon K \times E \to \overline{\mathbb{R}}$ is monotone, ω-continuous on the segments of $D :=$ Dom h, $f \in E'$, and D is convex. Then*

$$\bigcap_{y\in D}\{x \in D \mid h(y, x-y) \le \langle f, x-y\rangle\} \subset \bigcap_{y\in D}\{x \in D \mid \langle f, y-x\rangle \le h(x, y-x)\}.$$

Lemma 2.13. *Let $h\colon E\times E \to \overline{\mathbb{R}}$ be a monotone and ω-continuous form and* Dom $h = E$. *Then h is maximal monotone.*

Proof. This follows from Proposition 2.12, Proposition 2.5, and Corollary 2.6. □

We call h *coercive with respect to* $f \in E'$ if there is an $x_0 \in D :=$ Dom h such that the set

$$\operatorname{cl}^w\{x \in D \mid \langle f, x_0 - x\rangle \le h(x, x_0 - x)\}$$

is weakly compact. Since x_0 is contained in the above set, it is nonempty. We say that a form h is *coercive* if it is coercive with respect to every $f \in E'$.

We say that $T\colon K \to \mathfrak{P}(E')$ is *coercive with respect to* $f \in E'$ if there exists $\rho > 0$ such that for all $x \in \operatorname{dom} T \cap \complement B[0,\rho]$ and $x' \in Tx$ we have $\langle x' - f, x\rangle > 0$. If we assume, for a moment, that E is a reflexive Banach space, then James theorem tells us that a bounded and weakly closed subset of E is also weakly compact. With the aid of James theorem we get the following result.

Lemma 2.14. *Let $T\colon E \to \mathfrak{P}(E')$ be a formidable, and weak* compact-valued operator with form h. If T is coercive with respect to $f \in E'$, and E is reflexive, then its form is also coercive with respect to f.*

Proof. Let us set $D := \mathrm{Dom}\ h = \mathrm{dom}\ T$. Moreover, let us define the set-valued map

$$\psi(x_0) := \{x \in D \mid \langle f, x_0 - x\rangle \leq h(x, x_0 - x)\}.$$

As T is coercive with respect to f, there exists $\rho > 0$ such that for all $x \in D \cap \complement B[0,\rho]$ and all $x' \in Tx$ we have $\langle x' - f, x\rangle > 0$. Let us fix any $x \in D \cap \complement B[0,\rho]$. Since Tx is weak* compact there is $x' \in Tx$, for which $h(x, -x) = \langle x', -x\rangle < \langle f, -x\rangle$. Thus, $h(x, 0-x) < \langle f, 0-x\rangle$ and hence $x \notin \psi(0)$. Therefore $\psi(0) \subset B[0,\rho]$. The lemma follows from James' theorem, see e. g. [15]. □

Lemma 2.15. *Suppose that $h\colon K \times E \to \overline{\mathbb{R}}$ is a monotone form which is coercive with respect to $f \in E'$, and x_0, the element which exists by coerciveness of h, equals to $x_0 = 0$. Then the associated operator $T = \partial_2 h(\cdot, 0)$ is coercive with respect to f.*

Proof. As h is coercive with respect to f the set

$$A := \mathrm{cl}^w \{x \in D \mid \langle f, 0-x\rangle \leq h(x, 0-x)\}$$

is weakly compact, and thus bounded. Hence, there is $\rho > 0$ such that $A \subset B[0,\rho]$. Thus, for all $x \in D \cap \complement B[0,\rho]$ we have

$$\langle f, -x\rangle > h(x, -x) = \sup_{x' \in Tx} \langle x', -x\rangle\,.$$

Therefore, for any $x' \in Tx$ we have $\langle f, -x\rangle > \langle x', -x\rangle$ which shows that T is coercive with respect to f. □

Usually we assume that $x_0 = 0$. If this not the case, then we consider a translated problem, for the argument see the proof of Theorem 4.2.

3. The Sum Problem

First of all we derive a calculus for forms. That is, we investigate when the sum of two formidable operators remains formidable. A first application will be Heisler's theorem. Then we restrict our attention to the monotone setting.

Lemma 3.1. *Let $T, S\colon K \to \mathfrak{P}(E')$ be formidable. Assume further that T is weak* compact-valued on* dom S*, and h, k are the associated forms. Then*

$$[T + S](x) = \partial_2[h + k](x, 0).$$

Proof. Let $x \in \mathrm{Dom}\ h \cap \mathrm{Dom}\ k$ be arbitrary. Then $\mathrm{dom}\ h(x,\cdot) = E$ and hence $h(x,\cdot)$ is continuous on E, see e. g. Chapter 2 of [4]. Thus, we deduce for all $x \in \mathrm{Dom}\ h \cap \mathrm{Dom}\ k$ that

$$0 \in \mathrm{int\ dom}\ h(x,\cdot) \cap \mathrm{dom}\ k(x,\cdot).$$

The lemma is now a consequence of the sum theorem for subdifferentials. □

Theorem 3.2 (Heisler). *Suppose that $T, S\colon E \to \mathfrak{P}(E')$ are maximal monotone, and* dom T = dom $S = E$. *Then $T + S$ is maximal monotone.*

Proof. Let h and k be the respective forms. We deduce from Proposition 2.10 that h and k are ω-continuous on E. Hence their sum $h+k$ is also monotone and ω-continuous. Furthermore, Lemma 3.1 yields

$$T + S = \partial_2[h + k](\cdot, 0).$$

The result follows by Lemma 2.13. □

A similar proof of Heisler's theorem can be found in [23]. Our next result, stated in the introduction, is a step beyond Simons results. It says that we have a calculus, provided that both forms are monotone, and Rockafellar's constraint qualification is fulfilled.

Proof of Proposition 1.3. By hypotheses (α) there are $\epsilon > 0$ and $x_0 \in \mathrm{Dom}\ h$ such that $x_0 \in \mathrm{Dom}\ k$ and $B(x_0, \epsilon) \subset \mathrm{int\ Dom}\ h$. Thus, we have for all $x \in \mathrm{Dom}\ h \cap \mathrm{Dom}\ k$ from Lemma 2.8 that

$$x_0 - x \in \mathrm{int\,dom}\ h(x, \cdot) \cap \mathrm{dom}\ k(x, \cdot).$$

Finally, the sum theorem for subdifferentials yields

$$\partial_2 h(x, 0) + \partial_2 k(x, 0) = \partial_2[h + k](x, 0),$$

which proves the proposition. □

Let $C \subset E$ be a subset. We define the *normal cone* of C by

$$N_C(\overline{x}) := \partial\, \delta_C(\overline{x}) = \{x' \in E' \mid \langle x', x - \overline{x}\rangle \leq 0 \text{ for all } x \in C\}.$$

If C is convex and closed, then $N_C \colon E \to \mathfrak{P}(E')$ is a maximal monotone operator. The next lemma is crucial for the proof of the sum theorem.

Lemma 3.3 (Simons [23, Lemma 16.1]). *Let $T \colon K \to \mathfrak{P}(E')$ be a maximal monotone operator, C be a convex and closed subset of E, and* dom $T \subset C$*. Then*

$$T + N_C(x) = T.$$

It is a well known result of Brønsted and Rockafellar [5] that for a proper, convex, and lower semicontinuous functional $p \colon E \to \overline{\mathbb{R}}$ the domain of ∂p is dense in the domain of p. See also the discussion in [23]. Thus, we have $\mathrm{dom}\,(\partial p) \subset \mathrm{dom}\ p \subset \mathrm{cl\ dom}\,(\partial p)$, and hence $\mathrm{cl\ dom}\ p = \mathrm{cl\ dom}\,(\partial p)$.

Theorem 3.4. *Let $p \colon E \to \overline{\mathbb{R}}$ be a proper, lower semicontinuous and convex functional, $T \colon E \to \mathfrak{P}(E')$ be a formidable, monotone and weak* compact-valued operator. Moreover, let $D :=$ cl dom $p \subset$ dom T. If T is $\|\cdot\|$-weak* upper semicontinuous on the segments of D and* dom (∂p) *is closed, then $\partial p + T$ is maximal monotone.*

Proof. First of all, we show that for any pair $(x, f) \in E \times E'$ that is monotonically related to $\partial p + T$, we have $x \in D = \mathrm{cl\ dom}\ p$. By the result of Brønsted and Rockafellar it holds $D = \mathrm{dom}\,(\partial p)$. Thus, D is convex. Obviously, $\partial p + T$ is monotone. From Lemma 3.3 it follows that for all $y \in D$, $x_1' \in \partial p(y)$, $x_2' \in N_D(y)$, $\lambda > 0$, and $g \in Ty$ we have

$$\langle f - (x_1' + \lambda x_2' + g), x - y\rangle \geq 0.$$

This yields

$$\langle 0 - x_2', x - y\rangle \geq 0 \text{ for all } y \in D \text{ and } x_2' \in N_D(y).$$

Since N_D is maximal monotone we get $0 \in N_D(x)$, hence $x \in D = \operatorname{dom}(\partial p)$.

Now we reduce the statement to a variational inequality. To that end, let h be the form of T. We want to show

$$f \in \partial p(x) + \partial_2 h(x, 0).$$

If we set $\tilde{h}(x,y) := p(x+y) - p(x)$, then $\partial p(x) = \partial_2 \tilde{h}(x,0)$. Thus, by the sum theorem for subdifferentials, we have to show $f \in \partial_2[\tilde{h} + h](x,0)$, and hence

$$\langle f, y - x\rangle \leq p(y) - p(x) + h(x, y - x) \quad \text{for all } y \in \operatorname{dom} p. \tag{3}$$

Finally, we show that this inequality is true. Choose $y \in D$ and define for $\lambda \in [0,1]$

$$x_\lambda := \lambda y + (1 - \lambda)x.$$

By our assumption $D = \operatorname{dom} \partial p$ is convex. Hence, $\partial p(x_\lambda)$ is nonempty for every $\lambda \in [0,1]$. Since (x,f) is monotonically related to $T + \partial p$ we deduce for all $g_\lambda \in Tx_\lambda$ and all $x_\lambda' \in \partial p(x_\lambda)$ that

$$\begin{aligned} \lambda(1-\lambda)^{-1}\big\langle x_\lambda', y - x_\lambda\big\rangle &= \big\langle x_\lambda', x_\lambda - x\big\rangle \\ &\geq \big\langle f - g_\lambda, x_\lambda - x\big\rangle \\ &= \big\langle f - g_\lambda, \lambda y - \lambda x\big\rangle. \end{aligned}$$

Since $x_\lambda' \in \partial p(x_\lambda)$ was arbitrary we have

$$p(y) - p(x_\lambda) \geq (1-\lambda)\,\langle f - g_\lambda, y - x\rangle.$$

The last inequality is valid for any $g_\lambda \in Tx_\lambda$, hence

$$p(y) - p(x_\lambda) + h(x_\lambda, y - x_\lambda) \geq (1-\lambda)\,\langle f, y - x\rangle.$$

Employing Proposition 2.11 we deduce that h is ω-continuous on the segments of D. Thus, we finally obtain that

$$\begin{aligned} p(y) - p(x) + h(x, y - x) &\geq \limsup_{\lambda\to 0^+} \big[p(y) - p(x_\lambda) + h(x_\lambda, y - x_\lambda)\big] \\ &\geq \limsup_{\lambda\to 0^+} (1-\lambda)\,\langle f, y - x\rangle \\ &= \langle f, y - x\rangle, \end{aligned}$$

from which (3), and hence the theorem follows. □

Proof of Theorem 1.6. The theorem follows from the last theorem. By Proposition 2.1 we know that T is formidable. Moreover, as a maximal monotone operator, T is $\|\cdot\|$-weak* upper semicontinuous, and weak* compact-valued on $\operatorname{int}\operatorname{dom} T$, see e.g. [10]. Thus, all prerequisites of Theorem 3.4 are fulfilled. □

4. Browder's Theorem

In the remainder of the paper we prove a variant of Browder's theorem in the context of forms. The theorem we have in mind reads: *Let E be a real reflexive Banach space. If $T\colon E \to \mathfrak{P}(E')$ is a maximal monotone, and coercive operator, then T is surjective*, see e. g. [8, Theorem V.3.5, page 163]. The next lemma is a direct consequence of the Debrunner-Flor extension lemma, which can be found in [10].

Lemma 4.1. *Let $\dim E < \infty$, $K \subset E$ be compact and convex, $q\colon K \to E'$ be continuous, and $K \subset \operatorname{dom} q$. Furthermore, suppose that $h\colon K \times E \to \overline{\mathbb{R}}$ is a monotone form. Then there exists $x \in K$ such that for all $y \in D := \operatorname{Dom} h$*

$$h(y, x-y) \le \langle q(x), x-y\rangle.$$

Theorem 4.2 (Browder). *Suppose that $h\colon K \times E \to \overline{\mathbb{R}}$ is a maximal monotone form with domain D. Furthermore, suppose that h is coercive with respect to $f \in E'$. Then there is an $x \in D$ such that*

$$f \in \partial_2 h(x, 0). \tag{4}$$

Proof. This proof is inspired by [10, Theorem III.2.14]. We may assume that x_0, the element which exists by coerciveness assumption, is equal to $x_0 = 0$. If this is not the case we consider a translated problem, i.e. we define the form $\tilde{h}(x,y) := h(x_0 + x, y)$. This implies that $\tilde{D} := \operatorname{Dom} \tilde{h} = D - x_0$. Let us note that $\tilde{h}$ is still maximal monotone. We claim that $\tilde{h}$ is coercive with respect to f. To that end consider

$$\begin{aligned}\{\tilde{x} \in \tilde{D} \,|\, \langle f, -\tilde{x}\rangle \le \tilde{h}(\tilde{x}, -\tilde{x})\} &= \{\tilde{x} + x_0 \in D \,|\, \langle f, -\tilde{x}\rangle \le h(\tilde{x} + x_0, -\tilde{x})\} \\ &= \{x \in D \,|\, \langle f, x_0 - x\rangle \le h(x, x_0 - x)\}.\end{aligned}$$

We have seen that problem (4) is equivalent to

$$\exists x \in V := \bigcap_{y \in D} \left\{ x \in K \,\middle|\, h(y, x-y) \le \langle f, x-y\rangle \right\}.$$

Let us define the following set-valued map $\psi\colon D \to \mathfrak{P}(D)$

$$\psi(y) := \left\{ x \in D \,\middle|\, \langle f, y-x\rangle \le h(x, y-x) \right\}.$$

By the coerciveness of h and Krein-Šmulian's weak compactness theorem the set $B := \overline{\operatorname{co}}\, \operatorname{cl}^w\, \psi(0)$ is weakly compact. Moreover, $V \subset B$ by Proposition 2.5.

Let $\mathfrak{L}$ be the family of all finite dimensional subspaces of E equipped with $\|\cdot\|_E$. For every $F \in \mathfrak{L}$ let $I_F\colon F \to E$ be the inclusion map and $I'_F\colon E' \to F'$ be its adjoint. Furthermore, let $h_F := h \circ I_F \times I_F$ and $f_F := I'_F \circ f$. Then h_F is monotone and $f_F \in F'$. Now we set $K_F := F \cap B \cap K$ and $D_F := D \cap F$. The set K_F is nonempty since $0 \in K_F$. By Lemma 4.1 we have with $q := f_F$

$$\exists x_F \in V_F := \bigcap_{y \in D_F} \left\{ x \in K_F \,\middle|\, h(y, x-y) \le \langle f, y-x\rangle \right\}.$$

We define for $Z \in \mathfrak{L}$

$$\Gamma_Z := \left\{ x_F \in E \,\middle|\, x_F \in V_F, Z \subset F, F \in \mathfrak{L} \right\}.$$

Since $K_F \subset B$ the set $\bigcap_{Z\in\mathfrak{L}} \mathrm{cl}^w\ \Gamma_Z$ is weakly compact. We conclude by induction that $\{\Gamma_Z\}_{Z\in\mathfrak{L}}$ has the finite intersection property. Therefore, we have

$$\exists x \in \bigcap_{Z\in\mathfrak{L}} \mathrm{cl}^w\ \Gamma_Z. \tag{5}$$

Choose $y \in D$. Then there is an $F \in \mathfrak{L}$ such that $y \in F$. By (5) it follows that there is a net $(x_\lambda^F)_{\lambda\in\Lambda} \subset \Gamma_F$ which converges weakly to x and

$$\forall \lambda \in \Lambda : h(y, x_\lambda^F - y) \leq \langle f, x_\lambda^F - y \rangle.$$

Thus,

$$h(y, x - y) \leq \langle f, x - y \rangle.$$

Since $y \in D$ was arbitrary we conclude that $x \in V$. □

Remark 4.3. If we want to show Browder's original theorem stated above, we have in addition to assume that T is also weak* compact-valued, because of Lemma 2.14. The surjectivity of T follows as a coercive operator is coercive with respect to any $f \in E'$.

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References

[1] C. D. Aliprantis, K. C. Border: Infinite Dimensional Analysis, Springer (1998).

[2] H. Attouch, J.-B. Baillon, M. Théra: Variational sum of monotone operators, J. Convex Analysis 1 (1994) 1–29.

[3] C. Baiocchi, A. Capelo: Variational and Quasivariational Inequalities. Applications to Free-Boundary Problems, Wiley (1984).

[4] V. Barbu, Th. Precupanu: Convexity and Optimization in Banach spaces, D. Reidel (1986).

[5] A. Brøndsted, R. T. Rockafellar: On the subdifferentiability of convex functions, Proc. Am. Math. Soc. 16(4) (1965) 605–611.

[6] F. E. Browder: Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces, Proceedings of Symposia in Pure Mathematics, Vol. XVIII, Part 2 (1970).

[7] R. S. Burachik, B. F. Svaiter: Maximal monotone operators, convex functions and a special family of enlargements, preprint (2000).

[8] I. Cioranescu: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer (1990).

[9] D.-Z. Du et al. (eds.): Recent Advances in Nonsmooth Optimization, World Scientific (1995).

[10] S. Hu, N. S. Papageorgiou: Handbook of Multivalued Analysis, Volume I, Kluwer (1997).

[11] A. Ioffe., V. Tichomirov: Theory of Extremal Problems, North-Holland (1979).

[12] S. Komlosi: Monotonicity and quasimonotonicity in nonsmooth analysis, in: Recent Advances in Nonsmooth Optimization, D.-Z. Du et al. (eds.), World Scientific (1995) 193–214.

[13] E. Krauss: A representation of arbitrary maximal monotone operators via subgradients of skew-symmetric saddle functions, Nonlinear Anal. 9 (1985) 1381–1399.

[14] R. R. Phelps: Convex Functions, Monotone Operators and Differentiability, 2nd Ed., Lecture Notes in Mathematics 1364, Springer (1993).

[15] R. Megginson: An Introduction to Banach Space Theory, Springer (1998).

[16] R. T. Rockafellar: Monotone operators associated with saddle functions and minimax problems, in: Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces, F. E. Browder (ed.), Proceedings of Symposia in Pure Mathematics, Vol. XVIII, Part 2 (1970) 241–250.

[17] R. T. Rockafellar: On the maximality of sums of nonlinear monotone operators, Trans. Am. Math. Soc. 149 (1970) 75–88.

[18] J. P. Revalski, M. Théra: Generalized sums of monotone operators, C. R. Acad. Sci., Paris, Sér. I, Math. 329 (1999) 979–984.

[19] J. P. Revalski, M. Théra: Enlargements and sums of monotone operators, Rapport de recherche n° 2000-11 (2000).

[20] W. Schirotzek: Differenzierbare Extremalprobleme, B. G. Teubner (1989).

[21] W. Schirotzek: Monotone Operators, Skript, Technische Universität Dresden (2003).

[22] W. Schirotzek: Nonsmooth Analysis, Skript, Technische Universität Dresden (2003).

[23] S. Simons: Minimax and Monotonicity, Lecture Notes in Mathematics 1693, Springer (1998).

[24] S. Simons: Sum theorems for monotone operators and convex functions, Trans. Am. Math. Soc. 350(7) (1998) 2953–2972.

[25] M.-H. Shih, K.-K. Tan: Browder-Hartman-Stampacchia-variational inequalities for multivalued monotone operators, J. Math. Anal. Appl. 134 (1988) 431–440.

[26] A. Taa: The maximality of the sum of monotone operators in Banach spaces and an application to hemivariational inequalities, J. Math. Anal. Appl. 204 (1996) 693–700.

[27] M. E. Verona, A. Verona: Regular maximal monotone operators, Set-Valued Anal. 6 (1998) 303–312.

[28] M. E. Verona, A. Verona: Regular maximal monotone operators and the sum theorem, J. Convex Analysis 7(1) (2000) 115–128.

[29] G. X.-Z. Yuan: KKM Theory and Applications in Nonlinear Analysis, Marcel Dekker (1999).

[30] E. Zeidler: Nonlinear Functional Analysis and its Application: Monotone Operators I & II, Springer (1989).