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Inner Estimation of the Eigenvalue Set and Exponential Series Solutions to Differential Inclusions

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We obtain inner estimations, around special eigenvalues, for the eigenvalue set of a properly nonlinear closed convex process. We also consider a differential inclusion associated with a general closed convex process and we construct smooth power series solutions of exponential type for some initial states.

Keywords: Convex process, eigenvalues, differential inclusion

1. Introduction

Set-valued analysis is a flexible framework which permits to treat in a unified manner a wide variety of applications, ranging from equilibrium problems in theoretical economics to the control of dynamical systems. Although multivalued maps share some properties with their singlevalued analogues, the set-valued structure gives rise to important differences in many aspects of the theory. A particularly interesting multivalued concept is that of convex process on a vector space, that is, a set-valued map whose graph is a convex cone containing the origin. This natural generalization of a linear transformation was first introduced by Rockafellar [8, 9], and since his pioneering work many authors have investigated the properties of this notion.

This paper is concerned with eigenvalue as well as differential inclusion problems associated with some closed convex process $F : H \rightrightarrows H$, with H being either a finite dimensional Euclidean space or a Hilbert space. Our goal is twofold. On the one hand, we expect to contribute to the understanding of this important class of set-valued maps when some regularity and boundedness conditions hold. On the other hand, we intend to stress similarities and differences between linear and properly nonlinear convex processes.

This paper is organized as follows. In Section 2 we recall some definitions and basic properties of set-valued maps. Section 3 is devoted to an inner estimation of the eigenvalue set $\sigma(F) = \{\lambda \in I\!R \mid \lambda x \in F(x) \ \text{ for some } \ x \neq 0\}$ of the type

$$\left[\lambda_0 - \hat{\lambda}, \lambda_0 + \hat{\lambda}\right] \subset \sigma(F),$$

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when $\dim H < \infty$, the convex process F is properly nonlinear and λ_0 is a particular element of $\sigma(F)$. To this end, we begin with a characterization of linear convex processes, that is, set-valued maps whose graph is a vector subspace, and we establish a useful property for properly nonlinear convex processes. Then, using Brower's fixed point theorem, we establish an inner estimation of the set $(F - \lambda I)(B_H)$ for each λ near λ_0, where B_H is the closed unit ball in H with $\dim H < \infty$. As a consequence of this result, we finish the section with the proof of the inner estimation of $\sigma(F)$. In Section 5 we turn our attention to the differential inclusion problem

$$(P;\xi) \begin{cases} \dot{\varphi}(t) \in F(\varphi(t)), & t \in [0,T], \\ \varphi(0) = \xi. \end{cases}$$

Under a special condition on the initial state ξ we give an elementary construction of power series solutions to $(P;\xi)$ of the exponential type:

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{t^k x_k}{k!},\ t \in [0,T],\ x_0 = \xi,\ x_{k+1} \in F(x_k).$$

We give sufficient conditions on F in order to ensure that such solutions are well defined, and we discuss some connections with exponential solutions associated with eigenvalues when the initial state is a (generalized) eigenvector of F. Finally, when the domain of F is the whole space we give a continuity property for the constructed solutions with respect to the initial state.

Let us mention that spectral theory of set-valued maps in Hilbert spaces has been studied by several authors in the last years. For a clear introduction and a brief historical account on the eigenvalue analysis of set-valued maps, the reader can consult Seeger [11, 12] and Lavilledieu and Seeger [5]. The important case when the map is a convex process is studied by Leizarowitz [6] and, by Aubin, Frankowska and Olech [3] where the eigenvalue problem is related to the controllability of a differential inclusion. The book of Aubin and Frankowska [2] also deals with this subject. Concerning estimations of the eigenvalue set, an outer estimation of the type $\sigma(F) \subset [\lambda_F, \lambda^F]$ has been given by Correa and Seeger in [4]. On the other hand, it is well known that the existence of eigenvalues allows us to obtain solutions of differential inclusions where the right hand side is a convex process, see for instance Lavilledieu and Seeger [5] and Smirnov [13]. Concerning exponential behavior of solutions, we can cite the work of Wolenski [14].

2. Preliminaries

Let $(H, \langle\cdot,\cdot\rangle)$ be a real *Hilbert space* with associated norm $|\cdot|$. Let us recall some definitions of set-valued analysis. The *graph* of the set-valued map $F : H \rightrightarrows H$ is defined by $\operatorname{Graph} F := \{(x,y) \in H \times H \mid y \in F(x)\}$, the *domain* of F is given by $\operatorname{Dom} F := \{x \in H \mid \exists y \in H, (x,y) \in \operatorname{Graph} F\} = \{x \in H \mid F(x) \neq \emptyset\}$, and the *image* of F is defined to be $\operatorname{Im} F := \{y \in H \mid \exists x \in H, (x,y) \in \operatorname{Graph} F\} = \{y \in H \mid F^{-1}(y) \neq \emptyset\}$, where $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

The set-valued map F is said to be *closed* if $\operatorname{Graph} F$ is a closed subset of $H \times H$, and it is said to be a *process* if its graph is a cone, i.e., $\forall \alpha > 0,\ \forall x \in H,\ F(\alpha x) = \alpha F(x)$. We say that F is *linear* (see [2]) if $\operatorname{Graph} F$ is a linear subspace of $H \times H$; otherwise, we say that F is *properly nonlinear*.

A set-valued map F is called *convex process* [8, 9] if $\operatorname{Graph} F$ is a convex cone containing the origin. Equivalently, F is a convex process if and only if the three following properties hold:

(a) Normalization: $0 \in F(0)$.
(b) Positive homogeneity: $\forall \alpha > 0$, $\forall x \in H$, $F(\alpha x) = \alpha F(x)$.
(c) Super-additivity: $\forall x, y \in H$, $F(x + y) \supset F(x) + F(y)$.

Throughout this paper, $F : H \rightrightarrows H$ stands for a convex process. A convex process F is said to be *fully defined* or *strict* when $\operatorname{Dom} F = H$. Note that the domain and image of a closed convex process are convex cones which are not necessarily closed.

We say that F is *fully bounded* if $F(B_H)$ is bounded, where $B_H = \{x \in H \mid |x| \leq 1\}$ is the closed unit ball in H. Thus, if F is fully bounded then it maps bounded sets to bounded sets. Defining

$$\|F\|_{sup} := \sup_{x \in B_H \cap \operatorname{Dom} F} \left(\sup_{y \in F(x)} |y| \right), \tag{1}$$

it is clear that F is fully bounded if and only if $\|F\|_{sup} < +\infty$. Note that if F is fully bounded then $F(0) = \{0\}$; in the finite dimensional context, we get the opposite implication. It is easy to see that if F is a strict convex process satisfying $F(0) = \{0\}$, then F is linear, indeed $F(x) = \{Ax\}$ with $A : H \to H$ being a linear transformation. Hence, a strict closed convex process F is fully bounded if and only if F is a continuous linear operator.

As in [7], we define now

$$\|F\| := \sup_{x \in B_H \cap \operatorname{Dom} F} \left(\inf_{y \in F(x)} |y| \right) \tag{2}$$

and we say that F is *bounded*, or *normed*, if $\|F\| < +\infty$. By [7, Theorem 1], the following three properties are equivalent for any convex process F:

(a) F is bounded;
(b) F is lower semicontinuous (lsc) at 0, that is, for each open set U in H with $F(x) \cap U \neq \emptyset$ there exists a neighborhood V of x such $F(x') \cap U \neq \emptyset$ for all $x' \in V \cap \operatorname{Dom} F$;
(c) F^{-1} is open at 0, that is, for each open neighborhood U of 0 in H, there exists a neighborhood V of 0 in $\operatorname{Im} F$ such that $V \subset F^{-1}(U)$.

Recall the generalized closed graph theorem given in [7]: if F is a strict closed convex process then F is bounded and $\|F\|$-Lipschitz, that is,

$$\forall x, y \in H, \; F(x) \subset F(y) + \|F\| |x - y| B_H. \tag{3}$$

Thus, F is lsc and upper semicontinuous (usc) at every point of H. Recall that F is usc at x if for each open set U in H with $F(x) \subset U$ there exists a neighborhood V of x such that $F(x') \subset U$, $\forall x' \in V$.

The *kernel* of F is defined by $\operatorname{Ker} F := \{x \in H \mid 0 \in F(x)\}$. A real number λ is an *eigenvalue* of F if $\lambda x \in F(x)$ for some $x \neq 0$. The element $x \neq 0$ such that $\lambda x \in F(x)$ is called *eigenvector* associated with λ. We define $E_\lambda(F) := \{x \in H \mid \lambda x \in F(x)\}$ the set consisting of all eigenvectors associated with λ together with the origin, and $\sigma(F)$ the set of all the eigenvalues of F.

3. Inner estimation of the eigenvalue set

3.1. Linear and properly nonlinear convex processes

In this section we give a basic property of properly nonlinear convex processes that we will use in the next section for obtaining an inner estimation of the eigenvalue set $\sigma(F)$. We begin with a characterization of linear convex processes.

Proposition 3.1. *If a closed convex process F is linear, then $\operatorname{Dom} F$ is a linear subspace of H and there exists a linear operator $A : \operatorname{Dom} F \to H$ and a linear subspace L of H such that $F(x) = Ax + L$ for all $x \in \operatorname{Dom} F$. In fact, Ax may be chosen to be the minimal norm element in $F(x)$ and $L = F(0)$. Moreover, the linear operator A is continuous if and only if $\operatorname{Dom} F$ is closed.*

Proof. If the convex process F is linear, then for every $x \in \operatorname{Dom} F$ and $y \in F(x)$ we have that $-y \in F(-x)$. This implies that $\operatorname{Dom} F$ and $F(0)$ are linear subspaces of H. Furthermore, we obtain that

(i) $F(\alpha x) = \alpha F(x)$ for all $x \in \operatorname{Dom} F$ and $\alpha \in I\!R \setminus \{0\}$;
(ii) $F(x + y) = F(x) + F(y)$ for all $x, y \in \operatorname{Dom} F$.

Then, $F(x) - F(x) = F(0)$ for all $x \in \operatorname{Dom} F$. From this we deduce that $F(x) = \Pi_{F(x)}(0) + F(0)$ for all $x \in \operatorname{Dom} F$, where $\Pi_C(0)$ is the minimal norm element in the closed convex set C. Let us check now that $x \mapsto \Pi_{F(x)}(0)$ is a linear operator in $\operatorname{Dom} F$. Let $x \in \operatorname{Dom} F$. Then, by the characterization of the unique minimal norm element of $F(x)$ we have $\langle \Pi_{F(x)}(0), z - \Pi_{F(x)}(0)\rangle \geq 0$ for all $z \in F(x)$. On the other hand, we can write $F(0) = F(x) - \Pi_{F(x)}(0)$ and then, since $F(0)$ is a linear subspace, the above characterization of $\Pi_{F(x)}(0)$ can be written by the equality $\langle \Pi_{F(x)}(0), p\rangle = 0$ for all $p \in F(0)$. By (i) and (ii) we conclude that $\alpha \Pi_{F(x)}(0) = \Pi_{F(\alpha x)}(0)$ for all $\alpha \in I\!R$ and $\Pi_{F(x)}(0) + \Pi_{F(y)}(0) = \Pi_{F(x+y)}(0)$ for all $x, y \in \operatorname{Dom} F$. Assume now that $\operatorname{Dom} F$ is a closed linear subspace, from the generalized closed graph theorem [7], it follows that F is bounded and then $|Ax| = |\Pi_{F(x)}(0)| \leq \|F\|\ |x|$ for all $x \in \operatorname{Dom} F$, which implies the continuity of A. Finally, it is clear that if the linear operator A is continuous, the linear subspace $\operatorname{Dom} F$ is closed. □

Recalling that $F(0) = \{0\}$ when F is fully bounded, Proposition 3.1 yields directly the following result.

Corollary 3.2. *If a linear convex process F is fully bounded, then it is a linear operator defined over a linear subspace of H. It is continuous if and only if $\operatorname{Dom} F$ is closed.*

There is an important difference between linear and properly nonlinear convex processes. For instance, it is well known that if $A : H \to H$ is a linear transformation such that $I - A$ is compact then $\operatorname{Ker} A = \{0\} \Leftrightarrow \operatorname{Im} A = H$. Such a property does not hold for properly nonlinear convex processes as the next result shows.

Proposition 3.3. *If $F : H \rightrightarrows H$ is a properly nonlinear convex process with $\operatorname{Ker} F = \{0\}$ then $\operatorname{Im} F \neq H$.*

Proof. Since F is a properly nonlinear convex process, there exists $x \in \operatorname{Dom} F$ and $y \in F(x)$ such that $-y \notin F(-x)$. We claim that $-y \notin \operatorname{Im} F$. In fact, if $-y$ belongs to

$\operatorname{Im} F$ then there exists $z \in \operatorname{Dom} F$ such that $-y \in F(z)$, which implies $0 \in F(x+z)$. Since $\operatorname{Ker} F = \{0\}$ then $z = -x$ which is a contradiction. □

In the next section, we use this property of properly nonlinear convex processes to provide an inner estimation of the spectrum under appropriate conditions.

3.2. Inner estimation of $\sigma(F)$ for a properly nonlinear convex process F

The main result of this section is an inner estimation of the spectrum $\sigma(F)$ of a properly nonlinear convex process, around an eigenvalue λ_0 verifying $\operatorname{Im}(F - \lambda_0 I) = H$. Remark that the existence of λ_0 such that $\operatorname{Im}(F-\lambda_0 I) = H$ is not equivalent to $\operatorname{Im} F = H$ except in the simple case when $\|(F - \lambda_0 I)^{-1}\| = 0$ as in the next proposition.

Proposition 3.4. *Let F be a closed convex process such that $\operatorname{Im} F = H$, $\operatorname{Dom} F \neq \{0\}$, and $\|F^{-1}\| = 0$. Then $\sigma(F) = I\!R$ and the equality $E_\lambda(F) = \operatorname{Dom} F$ holds for all $\lambda \in I\!R$.*

Proof. Since $\|F^{-1}\| = 0$, from (3) we obtain that there exists a closed convex cone K such that $F^{-1}(y) = K$ for all $y \in H$. Therefore $0 \in F^{-1}(y)$ for all $y \in H$ and then $F(0) = H$. Fix any x in $\operatorname{Dom} F$. Since F is a convex process, we obtain $F(x) + F(0) \subset F(x)$, that is, $F(x) = H$ for all $x \in \operatorname{Dom} F$. Now, it is clear that for any $\lambda \in I\!R$ and for each $x \in \operatorname{Dom} F \setminus \{0\}$ we have $\lambda x \in F(x)$. □

Now, we assume that $\dim H < \infty$ and we focus on the case $\|(F - \lambda_0 I)^{-1}\| > 0$.

Lemma 3.5. *Let $F : H \rightrightarrows H$ be a closed convex process and let us suppose that $\dim H < \infty$. If there exists λ_0 such that $\operatorname{Im}(F - \lambda_0 I) = H$ and $\|(F - \lambda_0 I)^{-1}\| > 0$, then for all real number α such that $0 < \alpha \leq 1/\|(F - \lambda_0 I)^{-1}\|$ and for all $\lambda \in I\!R$ such that $|\lambda - \lambda_0| \leq \frac{1 - \alpha\|(F-\lambda_0 I)^{-1}\|}{\|(F-\lambda_0 I)^{-1}\|}$ one has*

$$\alpha B_H \subset (F - \lambda I)(B_H).$$

Proof. With no loss of generality we may assume that $\lambda_0 = 0$. Note that $\|F^{-1}\|$ is finite because the closed convex process F^{-1} is strict. Let $\alpha \in]0, 1/\|F^{-1}\|]$ and $\lambda \in I\!R$ be such that $|\lambda| \leq \frac{1-\alpha\|F^{-1}\|}{\|F^{-1}\|}$. Fix $y \in \alpha B_H$. We define the mapping $\phi : H \rightrightarrows H$ given by

$$\phi(x) = \{z \in H \mid y + \lambda x \in F(z)\} = F^{-1}(y + \lambda x).$$

Since F^{-1} is a strict closed convex process, then $\phi(x)$ is a nonempty convex closed set and the mapping ϕ is lsc and usc. From [1, Theorem 1, pp. 70] we have that the function $m : H \longrightarrow H$ defined by $m(x) = \Pi_{\phi(x)}(0)$ is continuous, where $\Pi_C(0)$ is the minimal norm element in the closed convex set C. From definition of $\|F^{-1}\|$ we can write

$$|m(x)| = |\Pi_{F^{-1}(y\,+\,\lambda x)}(0)| \leq \|F^{-1}\|\ |y + \lambda x| \leq \|F^{-1}\|(\alpha + |\lambda|) \leq 1,$$

for all $x \in B_H$, that is, $m(B_H) \subset B_H$ and from the Brower fixed point theorem (see [15]) there exists $\bar{x} \in B_H$ such that $m(\bar{x}) = \bar{x}$. Hence $y \in (F - \lambda I)(\bar{x})$. □

Theorem 3.6. *Let $F : H \rightrightarrows H$ be a closed convex process and let us suppose that $\dim H < \infty$. If F is properly nonlinear and if there exists λ_0 such that $\mathrm{Im}\,(F - \lambda_0 I) = H$ with $\|(F - \lambda_0 I)^{-1}\| > 0$, then we have the inner estimation*

$$\left[\lambda_0 - \hat{\lambda}, \lambda_0 + \hat{\lambda}\right] \subset \sigma(F), \tag{4}$$

where $\hat{\lambda} = 1/\|(F - \lambda_0 I)^{-1}\|$.

Proof. As in the proof of Lemma 3.5, for simplicity we assume that $\lambda_0 = 0$; otherwise, we conclude by redefining $\lambda \mapsto \lambda + \lambda_0$. Let $\alpha \in]0, 1/\|F^{-1}\|]$ and $\lambda \in I\!R$ be such that $|\lambda| \leq \frac{1-\alpha\|F^{-1}\|}{\|F^{-1}\|}$. By Lemma 3.5 one has $\alpha B_H \subset (F - \lambda I)(B_H)$ and then, $\mathrm{Im}\,(F - \lambda I) = H$. Since the closed convex process F is properly nonlinear, then it will be the same for the closed convex process $F - \lambda I$. By Proposition 3.3 we obtain that $\mathrm{Ker}\,(F - \lambda I) \neq \{0\}$, that is, λ is an eigenvalue of F. Therefore, for all $\alpha \in]0, 1/\|F^{-1}\|]$ and for all $\lambda \in I\!R$ such that $|\lambda| \leq \frac{1-\alpha\|F^{-1}\|}{\|F^{-1}\|}$ we have that $\lambda \in \sigma(F)$ which proves the result with the open interval in the left hand side. We finish by noting that in the finite dimensional setting, if F is a closed convex process, then $\sigma(F)$ is closed. □

4. Exponential series solutions of differential inclusions

In this section we are concerned with the construction of a smooth solution for the differential inclusion

$$(P;\xi) \qquad \begin{cases} \dot{\varphi}(t) \in F(\varphi(t)), & t \in [0, T], \\ \varphi(0) = \xi. \end{cases}$$

when F is a closed convex process.

One may find in [5] and [13] some existence results of solutions for $(P;\xi)$ when the initial state ξ is either an eigenvector or a generalized eigenvector of F, that is, $\xi \in (F - \lambda I)^{-m}(0)$ for some $\lambda \in I\!R$ and $m \geq 1$. Notice that in virtue of the inner estimation (4) of $\sigma(F)$, that is, $(F - \lambda I)^{-1}(0) \neq \{0\}$ for any $\lambda \in [\lambda_0 - \hat{\lambda}, \lambda_0 + \hat{\lambda}]$ and since $\{(F - \lambda I)^{-m}(0)\}_{m \in \mathbb{N}}$ is a nondecreasing family of convex cones, applying the result given by Smirnov in [13] which we recall in Proposition 4.1, we can see that solutions of the type $\varphi(t) = e^{\lambda t} \sum_{j=0}^{m-1} \frac{t^j y_j}{j!}$ with $y_0 = \xi$ and $y_j \in (F - \lambda I)(y_{j-1})$ for $j = 1, \ldots, m-1$ exist for all $\lambda \in [\lambda_0 - \hat{\lambda}, \lambda_0 + \hat{\lambda}]$.

Proposition 4.1 ([13]). *Let $F : H \rightrightarrows H$ be a convex process. If $\lambda \in I\!R$, $\xi \in H$ and $m \geq 1$ are such that $0 \in (F - \lambda I)^m(\xi)$, then $x(t) = e^{\lambda t} \sum_{j=0}^{m-1} \frac{t^j y_j}{j!}$ with $y_0 = \xi$ and $y_j \in (F - \lambda I)(y_{j-1})$ for $j = 1, \ldots, m-1$ is a solution of $(P;\xi)$.*

The main result of this section is established in Theorem 4.2, where we give another sufficient condition on the initial state ξ for the existence of an exponential series solution to $(P;\xi)$. This existence result does not assume that ξ is an eigenvector or a generalized eigenvector. Indeed, we will show that for some bounded convex processes F, this sufficient condition is verified for all $\xi \in \mathrm{Dom}\,F$, and in particular for all $\xi \in H$ when F is a strict closed convex process.

From now on, let $H^{\mathbb{N}}$ stand for the vector space of sequences in H and for any $T \in [0, \infty)$, we define the vector subspaces of $H^{\mathbb{N}}$

$$\ell_T^1(H) = \left\{ \vec{x} \in H^{\mathbb{N}} \mid \sum_{k \geq 0} \tfrac{T^k |x_k|}{k!} < \infty \right\} \quad \text{and} \quad \ell_\infty^1(H) = \bigcap_{T \geq 0} \ell_T^1(H).$$

It is clear that if $T_2 \geq T_1 \geq 0$ then $\ell_\infty^1(H) \subset \ell_{T_2}^1(H) \subset \ell_{T_1}^1(H) \subset \ell_0^1(H) = H^{\mathbb{N}}$.

For any $T \in [0, \infty)$ and $\vec{x} \in \ell_T^1(H)$, we define the exponential series function $\varphi_{\vec{x}} : [0, T] \to H$ given by

$$\varphi_{\vec{x}}(t) = \sum_{k=0}^{\infty} \frac{t^k x_k}{k!}, \quad t \in [0, T]. \tag{5}$$

Of course, for each $t \in [0, T]$, $\varphi_{\vec{x}}(t)$ is well defined when $\vec{x} \in \ell_T^1(H)$ and $\varphi_{\vec{x}} \in C^\infty(0, T; H)$ with $\dot{\varphi}_{\vec{x}}(t) = \sum_{k=0}^{\infty} \frac{t^k x_{k+1}}{k!}$. Finally, we define the set of sequences generated by the iteration $x_{k+1} \in F(x_k)$ with initial state $\xi \in H$ by

$$S(F; \xi) := \{ \vec{x} \in H^{\mathbb{N}} \mid x_0 = \xi, \ x_{k+1} \in F(x_k), \ \forall k \geq 0 \}.$$

Theorem 4.2. *Given a closed convex process $F : H \rightrightarrows H$, for any $T \in [0, \infty)$ and $\xi \in H$ such that $\ell_T^1(H) \cap S(F; \xi) \neq \emptyset$, we have that for all $\vec{x} \in \ell_T^1(H) \cap S(F; \xi)$, the exponential series function $\varphi_{\vec{x}}$ defined in (5) is a solution of the differential inclusion $(P; \xi)$.*

Proof. Given $\vec{x} \in \ell_T^1(H) \cap S(F; \xi)$, we obtain $\varphi_{\vec{x}}(0) = \xi$ and since F is a convex process, we have that $x_{k+1} \in F(x_k)$ implies $\frac{t^k x_{k+1}}{k!} \in F\left(\frac{t^k x_k}{k!}\right)$, hence $\sum_{k=0}^{N} \frac{t^k x_{k+1}}{k!} \in F\left(\sum_{k=0}^{N} \frac{t^k x_k}{k!}\right)$, for all $N \in \mathbb{N}$. From the closedness of F, it follows that

$$\dot{\varphi}_{\vec{x}}(t) = \sum_{k=0}^{\infty} \frac{t^k x_{k+1}}{k!} \in F\left(\sum_{k=0}^{\infty} \frac{t^k x_k}{k!}\right) = F(\varphi_{\vec{x}}(t)).$$

□

This result has a direct extension to higher order dynamics:

Corollary 4.3. *Given a closed convex process $F : H \rightrightarrows H$, elements $\xi_0, \ \xi_1, \ldots, \ \xi_{n-1}$, and sequences $\vec{x}_j = (x_k^j)_{k \geq 0} \in \ell_T^1(H) \cap S(F; \xi_j)$ for $j = 0, \ldots, n-1$, we define the sequence $\vec{x} = (x_k)_{k \geq 0}$ by $x_{nk+j} = x_k^j$ for all $k \geq 0$ and $j = 0, \ldots, n-1$. Then, the corresponding exponential series function $\varphi_{\vec{x}}$ is a solution of the differential inclusion*

$$\begin{cases} \varphi^{(n)}(t) \in F(\varphi(t)), & t \in [0, T], \\ \varphi^{(j)}(0) = \xi_j & j = 0, \ldots, n-1. \end{cases}$$

In the following lemma, whose proof is straightforward, we give a very simple but useful sufficient condition in order to have $\vec{x} \in \ell_\infty^1(H)$.

Lemma 4.4. *Given $\vec{x} = (x_k)_{k \geq 0}$, if there exist $\alpha, \rho > 0$ such that $|x_k| \leq \alpha \rho^k$ for all k large enough, then $\vec{x} \in \ell_\infty^1(H)$.*

Proposition 4.5. *Let F be a closed convex process.*

(*i*) *If F is bounded and* $\operatorname{Im} F \subset \operatorname{Dom} F$, *then* $\forall \xi \in \operatorname{Dom} F$, $S(F;\xi) \cap \ell^1_\infty(H) \neq \emptyset$.

(*ii*) *If F is fully bounded then* $\forall \xi \in \operatorname{Dom} F$, $S(F;\xi) \subset \ell^1_\infty(H)$.

Proof. (i) Let $\xi \in \operatorname{Dom} F$. Since F is bounded, the minimal norm element $\eta = \Pi_{F(\xi)}(0) \in F(\xi)$ of the closed convex set $F(\xi)$ satisfies $|\eta| \leq \|F\||\xi|$. Setting $x_0 = \xi$, we can generate a sequence $\vec{x} = (x_k)_{k\geq 0}$ by $x_{k+1} = \Pi_{F(x_k)}(0) \in F(x_k)$ for all $k \geq 0$ and clearly, $\vec{x} \in S(F;\xi)$. Moreover, $|x_k| \leq \|F\|^k|\xi|$ and from Lemma 4.4, we have $\vec{x} \in \ell^1_\infty(H)$.

(ii) Since F is fully bounded, $\|F\|_{sup} < \infty$ (see (1)). Therefore, for all $y \in F(x)$ we have $|y| \leq \|F\|_{sup}|x|$. Hence, if $\vec{x} \in S(F;\xi)$ with $\xi \in \operatorname{Dom} F$ then $|x_k| \leq \|F\|^k_{sup}|\xi|$, and from Lemma 4.4, we have $\vec{x} \in \ell^1_\infty(H)$. □

As a direct consequence of Proposition 4.5 we obtain:

Corollary 4.6. *Let F be a closed convex process. Assume*

(*i*) *F is strict*
or

(*ii*) *F is fully bounded and for all* $x \in \operatorname{Dom} F$, $F(x) \cap \operatorname{Dom} F \neq \emptyset$.

Then $S(F;\xi) \cap \ell^1_\infty(H) \neq \emptyset$ *for all* $\xi \in \operatorname{Dom} F$.

In the following result we give a continuity property with respect to the initial state of the exponential series solutions of $(P;\xi)$ when F is a strict closed convex process.

Proposition 4.7. *If F is a strict closed convex process, then for any initial states* $\xi, \xi' \in H$, *and* $\vec{x} \in S(F;\xi) \cap \ell^1_T(H)$, *there exists* $\vec{y} \in S(F;\xi') \cap \ell^1_T(H)$ *such that* $|\varphi_{\vec{x}}(t) - \varphi_{\vec{y}}(t)| \leq |\xi - \xi'|e^{t\|F\|}$ *for all* $t \in [0,T]$.

Proof. Given $\xi, \xi' \in H$, and $\vec{x} = (x_k)_{k\geq 0} \in S(F;\xi) \cap \ell^1_T(H)$ we will construct a sequence $\vec{y} = (y_k)_{k\geq 0} \in S(F;\xi') \cap \ell^1_T(H)$ verifying the desired inequality. Set $y_0 = \xi'$. By (3), there exists $y_1 \in F(\xi')$ such that $|y_1 - x_1| \leq \|F\||\xi - \xi'|$. In this way, for each $k \geq 1$ we choose $y_{k+1} \in F(y_k)$ such that $|y_{k+1} - x_{k+1}| \leq \|F\||y_k - x_k|$. Hence we get $|y_k - x_k| \leq \|F\|^k|\xi - \xi'|$. Clearly $\vec{y} = (y_k)_{k\geq 0} \in S(F;\xi')$ and since $|y_k| \leq |x_k - y_k| + |x_k| \leq \|F\|^k|\xi - \xi'| + |x_k|$ we have $\sum_{k=0}^\infty \frac{T^k|y_k|}{k!} \leq |\xi - \xi'|e^{T\|F\|} + \sum_{k=0}^\infty \frac{T^k|x_k|}{k!} < \infty$ that is $\vec{y} \in \ell^1_T(H)$ and finally

$$|\varphi_{\vec{x}}(t) - \varphi_{\vec{y}}(t)| \leq \sum_{k=0}^\infty \frac{t^k|x_k - y_k|}{k!} \leq \sum_{k=0}^\infty \frac{t^k\|F\|^k|\xi - \xi'|}{k!} \leq |\xi - \xi'|e^{t\|F\|},$$

for all $t \in [0,T]$. □

A natural question is whether the exponential series solution $\varphi_{\vec{x}}$ of problem $(P;\xi)$ is a *slow solution*, that is, it satisfies $\dot{\varphi}_{\vec{x}}(t) = \Pi_{F(\varphi_{\vec{x}}(t))}(0)$ for every $t \in [0,T]$. As we show in the next example, this is not always the case.

Example 4.8. Let $F : H \rightrightarrows H$ be defined by $F(x) = Ax + K$ where $A : H \longrightarrow H$ is a continuous linear operator and $K \subset H$ is the closed convex cone given by $K = \{y \in H \mid \langle y, p\rangle \geq 0\}$ for some fixed $p \neq 0$. A direct calculation gives

$$\Pi_{F(x)}(0) = Ax + \Pi_K(-Ax) = \begin{cases} \langle Ax, p\rangle p & \text{if } -Ax \notin K \\ 0 & \text{if } -Ax \in K. \end{cases}$$

If ξ is such that $-A\xi \notin K$ then the slow solution $u(\cdot)$ of $(P;\xi)$ is given by

$$u(t) = \begin{cases} \xi + \frac{\langle A\xi,p\rangle}{\langle Ap,p\rangle}(e^{t\langle Ap,p\rangle} - 1)p & \text{if } \langle Ap,p\rangle \neq 0 \\ \xi + t\langle A\xi,p\rangle p & \text{if } \langle Ap,p\rangle = 0. \end{cases}$$

On the other hand, the exponential series solution $\varphi_{\vec{y}}$ for the same initial state ξ with $-A\xi \notin K$, obtained from $y_{k+1} = \Pi_{F(y_k)}(0)$ and $y_0 = \xi$ is

$$\varphi_{\vec{y}}(t) = \begin{cases} \xi + \frac{\langle A\xi,p\rangle}{\langle Ap,p\rangle}(e^{t\langle Ap,p\rangle} - 1)p & \text{if } \langle Ap,p\rangle > 0 \\ \xi + t\langle A\xi,p\rangle p & \text{if } \langle Ap,p\rangle \leq 0. \end{cases}$$

If $\langle Ap,p\rangle < 0$ we see that the slow solution and the exponential series solution are not the same. Notice that for all $t \geq 0$ one has $-Au(t) \notin K$ and $-A\varphi_{\vec{y}}(t) \in K$ for all $t \geq t^* = -1/\langle Ap,p\rangle$. If ξ is such that $-A\xi \in K$ then, both solutions are the same: $u(t) = \varphi_{\vec{y}}(t) = \xi$.

We finish this work by showing that the Smirnov solution for problem $(P;\xi)$ which we recalled in Proposition 4.1 is included in the solution that we give in Theorem 4.2 when the eigenvalue associated with the initial state ξ is nonnegative. We do not know whether this is also the case for negative eigenvalues.

Proposition 4.9. *Let $F : H \rightrightarrows H$ be a convex process. If $\lambda \geq 0$, $\xi \in H$ and $m \geq 1$ are such that $0 \in (F - \lambda I)^m(\xi)$, then there exists $\vec{x} \in S(F;\xi) \cap \ell^1_T(H)$ such that the exponential series solution $\varphi_{\vec{x}}$, defined in* (5), *coincides with the solution of $(P;\xi)$ given in Proposition* 4.1.

Proof. Denoting as usual by $\begin{pmatrix} k \\ j \end{pmatrix}$ the binomial coefficient, we will prove that if we define the sequence $\vec{x} = (x_k)_{k\geq 0}$ by

$$x_k = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \lambda^{k-j} y_j \tag{6}$$

where $y_0 = \xi$, $y_j \in (F - \lambda I)(y_{j-1})$ for $j = 1, \ldots, m-1$ and $y_j = 0$ for $j \geq m$ we have that $\vec{x} \in S(F;\xi) \cap \ell^1_T(H)$ and $\varphi_{\vec{x}}(t) = \sum_{k\geq 0} \frac{t^k x_k}{k!} = e^{\lambda t}\left(\sum_{j=0}^{m-1} \frac{t^j y_j}{j!}\right)$. Let us first verify the latter assuming that $\varphi_{\vec{x}}(t)$ is well defined. In fact,

$$\varphi_{\vec{x}}(t) = \sum_{k\geq 0} \frac{t^k x_k}{k!} = \sum_{k\geq 0}\sum_{j=0}^{k} \frac{t^k x}{k!} \begin{pmatrix} k \\ j \end{pmatrix} \lambda^{k-j} y_j = \sum_{j\geq 0}\sum_{k\geq j} \frac{t^k \lambda^{k-j} y_j}{(k-j)!\, j!}.$$

But

$$\sum_{j\geq 0}\sum_{k\geq 0} \frac{t^{k+j}\lambda^k y_j}{k!\, j!} = \sum_{k\geq 0} \frac{t^k \lambda^k}{k!} \sum_{j\geq 0} \frac{t^j y_j}{j!} = e^{\lambda t}\left(\sum_{j=0}^{m-1} \frac{t^j y_j}{j!}\right)$$

as we claimed. Let us prove now, by induction, that the sequence given in (6) is in $S(F;\xi)$. We see that $x_0 = y_0 = \xi$ and $x_1 = \lambda\xi + y_1 \in F(\xi) = F(x_0)$. Suppose now that

$x_k \in F(x_{k-1})$, we must prove that $x_{k+1} \in F(x_k)$. For this, we verify the following equality

$$x_{k+1} = \lambda x_k + \sum_{j=1}^{k} \binom{k-1}{j-1} (y_{j+1} + \lambda y_j)\lambda^{k-j}. \tag{7}$$

In fact, by direct computations,

$$\lambda x_k + \sum_{j=1}^{k} \binom{k-1}{j-1} (y_{j+1} + \lambda y_j)\lambda^{k-j} = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k+1-j} y_j$$

$$+\sum_{j=1}^{k-1} \binom{k-1}{j-1} \lambda^{k-j} y_{j+1} + y_{k+1} + \sum_{j=1}^{k} \binom{k-1}{j-1} \lambda^{k+1-j} y_j.$$

On the other hand, we have that

$$x_{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} \lambda^{k+1-j} y_j = \lambda^{k+1}\xi + y_{k+1}$$

$$+\sum_{j=1}^{k} \left[\binom{k}{j} + \binom{k-1}{j-1}\right] \lambda^{k+1-j} y_j + \sum_{j=2}^{k} \binom{k-1}{j-2} \lambda^{k+1-j} y_j,$$

from which (7) follows easily. We also have

$$x_k = \lambda x_{k-1} + \sum_{j=1}^{k} \binom{k-1}{j-1} \lambda^{k-j} y_j. \tag{8}$$

Since $x_k \in F(x_{k-1})$ and $\lambda \geq 0$, then

$$\lambda x_k \in F(\lambda x_{k-1}). \tag{9}$$

Furthermore, since $y_{j+1} + \lambda y_j \in F(y_j)$ for all $j \geq 0$, then

$$\sum_{j=1}^{k} \binom{k-1}{j-1} (y_{j+1} + \lambda y_j)\lambda^{k-j} \in F\left(\sum_{j=1}^{k} \binom{k-1}{j-1} \lambda^{k-j} y_j\right). \tag{10}$$

If we add (9) and (10), from the equalities (7) and (8), we obtain the desired inclusion $x_{k+1} \in F(x_k)$, thus $\vec{x} = (x_k)_{k\geq 0} \in S(F;\xi)$. To finish the proof, we verify that $\vec{x} \in \ell^1_\infty(H)$. Indeed, for $k \geq m$ we have that $x_k = \sum_{j=0}^{m-1} \binom{k}{j} \lambda^{k-j} y_j$, then

$$|x_k| \leq \max_i |y_i| \sum_{j=0}^{m-1} \binom{k}{j} |\lambda|^{k-j} \leq \max_i |y_i| \sum_{j=0}^{k} \binom{k}{j} |\lambda|^{k-j} = \max_i |y_i|(1+|\lambda|)^k$$

and by Lemma 4.4 we have $\vec{x} \in \ell^1_\infty(H)$. $\square$

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