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Conjugating the Inverse of a Concave Function

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Dedicated to C. Lemaréchal on the occasion of his 60^{th} birthday.

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This note is devoted to the clarification of the relationship between the Legendre-Fenchel conjugate of $\frac{1}{f}$ and that of $-f$ when f is a positive concave function.

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1. Introduction

The Legendre-Fenchel conjugate (or transform) of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is a function defined on the topological dual space $X^\star$ of X as

$$p \in X^\star \longmapsto f^*(p) := \sup_{x \in X} \left[\langle p, x\rangle - f(x) \right]. \tag{1}$$

In convex analysis the conjugacy operation $f \rightsquigarrow f^*$ plays a central role, therefore a large body of calculus rules have been developed for it; they can be found in any book on the subject. There however are some particular calculus rules which have been considered only recently, see for example [5, 3]. The present note is devoted to clarifying the calculus rule giving the conjugate of $\frac{1}{f}$ in terms of that $-f$ when f turns out to be a positive concave function. At first glance this situation can be viewed as a particular instance of a general calculus rule concerning a convex function post-composed with an increasing convex function ([2], Section 2.5 in chapter X): $x \longmapsto (-f)(x)$ post-composed with $0 > y \longmapsto \frac{-1}{y}$. We nevertheless provide a self-contained proof, insisting on the distinctive features of the resulting formula.

2. The conjugate of $\frac{1}{f}$.

The context of our work is the following one:

• X is a (real) Banach space; by $X^\star$ we denote the topological dual space of X, and $(p, x) \in X^\star \times X \longmapsto \langle p, x\rangle$ stands for the duality pairing.

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• $f : X \to \mathbb{R} \cup \{-\infty\}$ is a *concave* upper-semicontinuous (or closed) function, strictly positive on $C := \{x \in X \mid f(x) > -\infty\}$ (assumed nonempty).

If we formulate this assumption in a way more familiar to practitioners of convex analysis or optimization, this gives: $-f : X \to \mathbb{R} \cup \{+\infty\}$ is a *convex* lower-semicontinuous (or closed) function, strictly negative on $dom\,(-f) = C$.

If C turns out to be the whole of X, then f is constant on X, so that this situation is not of much interest. In applications, C happens to be a bounded (convex) set of X, on which$f$ is strictly positive.

The assumptions listed above are in force throughout the paper.

The inverse function of f , denoted as $\frac{1}{f}$, is defined on X as follows:

$$(\frac{1}{f})(x) := \begin{cases} \dfrac{1}{f(x)} & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

A classical and easily proved result is that $\frac{1}{f}$ is now *convex* on X, with domain C.

The theorem below gives the expression of $(\frac{1}{f})^*$ in terms of that of $(-f)^*$.

Theorem 2.1. *For all $p \in X^\star$,*

$$(\frac{1}{f})^*(p) = \min\left\{ \left[\alpha(-f)^*(\frac{p}{\alpha}) - 2\sqrt{\alpha}\right]_{\alpha>0},\ \sigma_C(p) \right\}, \tag{3}$$

where σ_C denotes the support function of C. If p belongs to the cone generated by $dom\,(-f)^*$ (i.e. if $p \in \mathbb{R}_+^* dom\,(-f)^*$), then

$$(\frac{1}{f})^*(p) = \inf_{\alpha>0} \left[\alpha(-f)^*(\frac{p}{\alpha}) - 2\sqrt{\alpha}\right]. \tag{4}$$

Proof. We mimic here the proof devised in [2] (p. 69 of Volume II). By definition,

$$-(\frac{1}{f})^*(p) = \inf_{x\in X} \left[\frac{1}{f(x)} - \langle p, x\rangle\right] = \inf_{x\in C} \left[-\frac{1}{-f(x)} - \langle p, x\rangle\right]$$

$$= \inf_{x\in X,\, r<0} \left[-\frac{1}{r} - \langle p, x\rangle \mid (-f)(x) \le r\right] \tag{5}$$

(because $r \longmapsto -\frac{1}{r}$ is increasing on $(-\infty, 0)$).

Let us define

$$f_1 : (x, r) \in X \times \mathbb{R} \longmapsto f_1(x, r) := \begin{cases} -\langle p, x\rangle - \dfrac{1}{r} & \text{if } x \in X \text{ and } r < 0, \\ +\infty & \text{otherwise;} \end{cases}$$

$$f_2 := i_{epi(-f)} \text{ (indicator function of the epigraph of } -f).$$

Thus, (5) can be written as

$$-(\frac{1}{f})^*(p) = \inf_{(x,r)\in X\times\mathbb{R}} [f_1(x,r) + f_2(x,r)]\,.$$

We then have to compute the conjugate of a sum of functions, however in a favorable context since $int(dom\, f_1) = X \times (-\infty, 0)$ and $dom\, f_2 = epi(-f)$ overlap. According to the classical Fenchel duality theorem

$$(\frac{1}{f})^*(p) = \min_{(s,\alpha)\in X^\star\times\mathbb{R}} [f_1^*(-s,\alpha) + f_2^*(s,-\alpha)]\,. \tag{6}$$

The computation of the above two conjugate functions is easy and gives:

$$f_1^*(-s,\alpha) = -2\sqrt{\alpha} \text{ if } s = p \text{ and } \alpha \geq 0, +\infty \text{ otherwise;}$$

$$f_2^*(s,-\alpha) = \sigma_{epi(-f)}(s,-\alpha) = \begin{cases} \alpha(-f)^*(\frac{s}{\alpha}) & \text{if } \alpha > 0, \\ \sigma_C(s) & \text{if } \alpha = 0, \\ +\infty & \text{if } \alpha < 0. \end{cases}$$

Plugging these results into (6) yields (3).

Observe that the function

$$\alpha \in \mathbb{R} \longmapsto \Theta(\alpha) := \begin{cases} \alpha(-f)^*(\frac{p}{\alpha}) - 2\sqrt{\alpha} & \text{if } \alpha > 0, \\ \sigma_C(p) & \text{if } \alpha = 0, \\ +\infty & \text{if } \alpha < 0 \end{cases}$$

is convex and lower-semicontinuous; its value at 0, that is $\sigma_C(p)$, is the limit of $\Theta(\alpha)$ when $\alpha \in dom\,\Theta \to 0^+$.

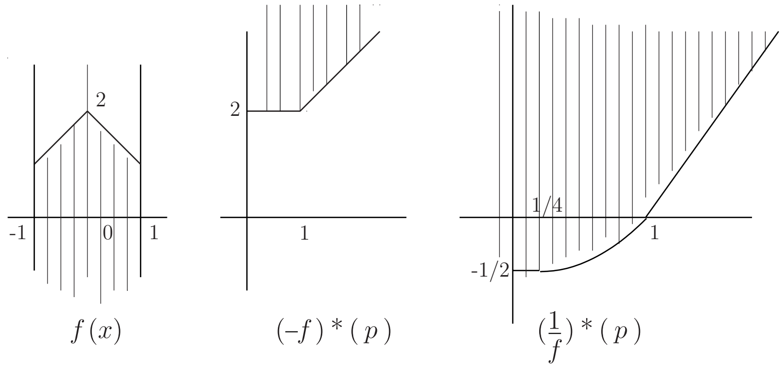
If p belongs to $\mathbb{R}_+^* dom\,(-f)^*$, the domain of Θ cannot reduce to $\{0\}$, whence

$$\min_{\alpha\geq 0} \Theta(\alpha) = \inf_{\alpha>0} \Theta(\alpha),$$

that is to say (4). □

Remarks 2.2. – It may happen that $p_0 \in dom\,(\frac{1}{f})^*$ but $p_0 \notin \mathbb{R}_+^* dom\,(-f)^*$. In that case, (4) is invalid and the minimal value in (3) is achieved "at the limit $\alpha_0 = 0$", thus $(\frac{1}{f})^*(p_0) = \sigma_C(p_0)$. See Example 2.5 below for an illustration of such a situation.

– According to formula (3), $dom\,(\frac{1}{f})^*$ contains $\mathbb{R}_+^* dom\,(-f)^*$. We will see later on that the closed (convex) cones generated by $dom\,(-f)^*$ and $dom\,(\frac{1}{f})^*$ are equal. Accordingly,



Example 2.3

$p_0 \in dom\,(\frac{1}{f})^*$ belongs to $\mathbb{R}_+^* dom\,(-f)^*$ "generically", i.e. at the exception of some "boundary-situations" such as that described above.

In view of the formulas (3) and (4) on $(\frac{1}{f})^*(p)$, on may ask the following questions:

– Could we delineate those p for which the infimum in (4) is achieved for some $\alpha > 0$?
– If so, is there any way of determining such α in terms of the given p ?

Before tackling these questions, it is worthwile to consider the next simple examples in order to grasp what can be expected and what not.

Example 2.3. Let $C = [-1, +1]$ and

$$f : x \in \mathbb{R} \longmapsto f(x) := \begin{cases} 2 - |x| & \text{if } x \in C, \\ -\infty & \text{otherwise.} \end{cases}$$

Then, $(-f)^*$ and $(\frac{1}{f})^*$ are even functions with:

$$(-f)^*(p) = \begin{cases} 2 & \text{if } 0 \leq p \leq 1, \\ p+1 & \text{if } p \geq 1; \end{cases}$$

$$(\frac{1}{f})^*(p) = \begin{cases} -1/2 & \text{if } 0 \leq p \leq 1/4, \\ 2(p - \sqrt{p}) & \text{if } 1/4 \leq p \leq 1, \\ p - 1 & \text{if } p \geq 1. \end{cases}$$

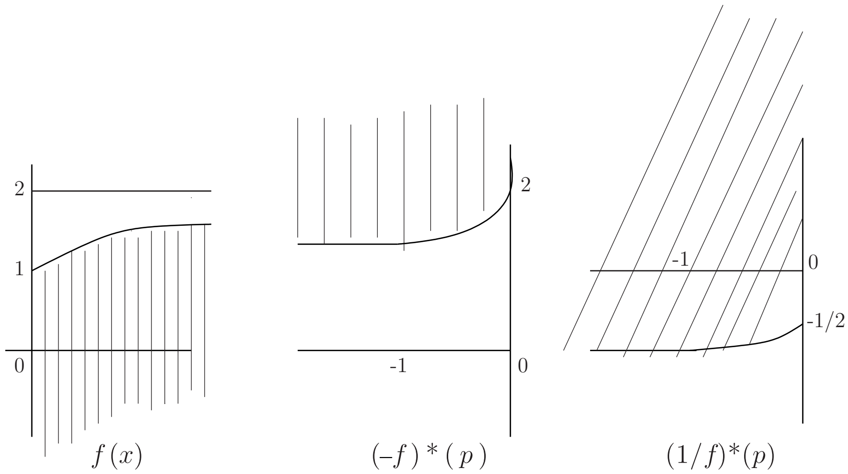
Let, for instance, $\frac{1}{4} \leq p_0 \leq 1$. The minimum value in the right-hand side of the formula

$$(\frac{1}{f})^*(p_0) = \min \left\{ [\alpha(-f)^*(\frac{p_0}{\alpha}) - 2\sqrt{\alpha}]_{\alpha > 0},\ p_0 \right\}$$

is achieved for $\alpha_0 = p_0$.

Example 2.4. Let $C = [0, +\infty)$ and

$$f : x \in \mathbb{R} \longmapsto f(x) := \begin{cases} 2 - e^{-x} & \text{if } x \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$



Example 2.4

Then, an easy calculation leads to

$$(-f)^*(p) = \begin{cases} 1 & \text{if } p \leq -1, \\ -p\,log(-p) + p + 2 & \text{if } -1 \leq p \leq 0, \\ +\infty & \text{if } p \geq 0; \end{cases}$$

while the explicit expression of $(\frac{1}{f})^*(p)$ is fairly complicated (see however the Figure above for a sketch of its graph). Let $p_0 = 0$. Then the minimum value in the right-hand side of the formula

$$(\frac{1}{f})^*(0) = \min\left\{\left[2\alpha - 2\sqrt{\alpha}\right]_{\alpha>0}, 0\right\}$$

is achieved for $\alpha_0 = \dfrac{1}{4}$.

This is a general rule. Suppose $M := \sup_{x \in C} f(x) < +\infty$ (as it is the case in Example 2.3 and the present one). Then the minimum value in the right-hand side of the formula expressing $(\frac{1}{f})^*(0)$ is achieved at $\alpha_0 = \dfrac{1}{M^2}$; the corresponding value is $-\dfrac{1}{M}$, as expected.

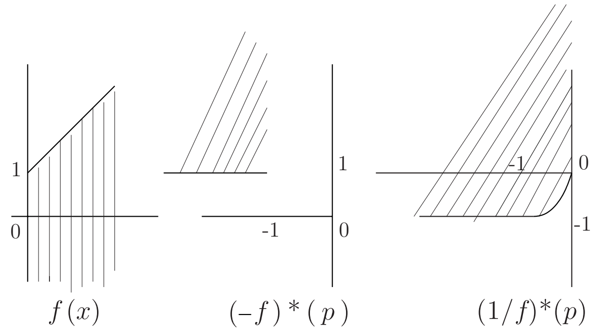
Example 2.5. Let $C = [0, +\infty)$ and

$$f : x \in \mathbb{R} \longmapsto f(x) := \begin{cases} x + 1 & \text{if } x \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Then

$$(-f)^*(p) = \begin{cases} 1 & \text{if } p \leq -1, \\ +\infty & \text{if } p > -1; \end{cases}$$

$$(\frac{1}{f})^*(p) = \begin{cases} -1 & \text{if } p \leq -1, \\ -p - 2\sqrt{-p} & \text{if } -1 \leq p \leq 0, \\ +\infty & \text{if } p > 0. \end{cases}$$



Example 2.5

Let $p_0 = 0$. Here $p_0 \in dom\,(\frac{1}{f})^*$ but $p_0 \notin \mathbb{R}_+^\star dom\,(-f)^*$. This is an example where

$$\alpha(-f)^*(\frac{p_0}{\alpha}) - 2\sqrt{\alpha} = +\infty \text{ for all } \alpha > 0,$$

whence the minimum value in the formula (3) yielding $(\frac{1}{f})^*(p_0)$ is achieved "at the limit $\alpha_0 = 0$" and has the value $\sigma_{\mathbf{C}}(p_0) = 0$.

To answer the questions posed as an introduction to the examples listed above, we need to explore furthermore the relationship between $dom\,(\frac{1}{f})^*$ and $dom\,(-f)^*$ by establishing a link between the subdifferential of $\frac{1}{f}$ that of $-f$. The connecting formula, as expected, is the one given in the statement below.

Proposition 2.6. *For all $x \in C$,*

$$\partial(\frac{1}{f})(x) = \frac{\partial(-f)(x)}{[f(x)]^2}. \tag{7}$$

Proof. We proceed to compare the directional derivates $(\frac{1}{f})'(x,d)$ and $(-f)'(x,d)$ for all $d \in X$.

First case: $x + td \notin C$ for all $t > 0$.

In that case, both $(\frac{1}{f})'(x,d)$ and $(f)'(x,d)$ equal $+\infty$.

Second case: $x + \bar{t}d \in C$ for som $\bar{t} > 0$.

Thus, the line-segment $[x, x + \bar{t}d]$ is entirely contained in C; therefore it comes from the assumption made on f (an upper-semicontinuous concave function) that the function (of the real variable) $t \longmapsto f(x+td)$ is continuous on $[0,\bar{t}]$. We infer from that,

$$\frac{\dfrac{1}{f(x+td)}-\dfrac{1}{f(x)}}{t}=\frac{\dfrac{(-f)(x+td)-(-f)(x)}{t}}{f(x+td)f(x)}$$
$$\underset{t\to 0^+}{\longrightarrow}\frac{1}{[f(x)]^2}\lim_{t\to 0^+}\frac{(-f)(x+td)-(-f)(x)}{t},$$

whence

$$(\frac{1}{f})^{'}(x,d)=\frac{(-f)^{'}(x,d)}{[f(x)]^2}. \tag{8}$$

In summary, the equality (8) holds for all $d\in X$. Since we have for any convex function φ on X

$$\partial\varphi(x)=\left\{p\in X^{\star}\mid \langle p,d\rangle\le\varphi^{'}(x,d) \text{ for all } d\in X\right\},$$

the announced relationship (7) readily follows from (8). $\square$

Not all the p in $dom\,(\frac{1}{f})^*$ are in $Im\,\partial(\frac{1}{f})$; however we have

$$Im\,\partial(\frac{1}{f})\subset dom\,(\frac{1}{f})^*\subset\overline{Im\,\partial(\frac{1}{f})} \tag{9}$$

(the second inclusion follows from the approximation theorem of Brøndsted-Rockafellar (1965)). For those p which are in $Im\,\partial(\frac{1}{f})$, we are able to provide $\alpha>0$ at which the infimum is achieved in the formula (4).

Theorem 2.7. *Let $p_0\in Im\,\partial(\frac{1}{f})$, and consider x_0 such that $p_0\in\partial(\frac{1}{f})(x_0)$. Then,*

$$(\frac{1}{f})^*(p_0)=\frac{(-f)^*\{[f(x_0)]^2p_0\}}{[f(x_0)]^2}-\frac{2}{f(x_0)}. \tag{10}$$

In other words: we are in a situation where p_0 belongs to the convex cone $\mathbf{R}^*_+ dom\,(-f)^*$, and the minimizer in the right-hand side of (4) is $\alpha_0=\dfrac{1}{[f(x_0)]^2}$.

Proof. We have $p_0\in\partial(\frac{1}{f})(x_0)$ and, according to (7), $[f(x_0)]^2p_0\in\partial(-f)(x_0)$. Using the characterization of the subdifferential of φ in terms of its conjugate ($s_0\in\partial\varphi(x_0)$ if and only if $\varphi^*(s_0)+\varphi(x_0)-\langle s_0,x_0\rangle=0$), we have:

$$(-f)^*\left\{[f(x_0)]^2p_0\right\}-f(x_0)+\langle[f(x_0)]^2p_0,x_0\rangle=0, \tag{11}$$

$$(\frac{1}{f})^*(p_0)+\frac{1}{f(x_0)}-\langle p_0,x_0\rangle=0. \tag{12}$$

Dividing (11) by $[f(x_0)]^2$ and comparing the resulting equality to (12), we derive (10).

When $p_0 \in Im\,\partial(\frac{1}{f})$, we clearly are in a situation where $p_0 \in \mathbf{R}_+^* dom\,(-f)^*$ since (according to (7)) $p_0 \in \mathbf{R}_+^* Im\,\partial(-f) \subset \mathbf{R}_+^* dom\,(-f)^*$. Then the (strictly) convex function

$$\alpha > 0 \longmapsto \alpha(-f)^*(\frac{p_0}{\alpha}) - 2\sqrt{\alpha}$$

is minimized at $\alpha_0 = \dfrac{1}{[f(x_0)]^2}$. $\square$

Remarks 2.8. – Even if $p_0 \notin Im\,\partial(\frac{1}{f})$, it may happen that the infimum in the right-hand side of (4) is achieved at some $\alpha_0 > 0$, but such α_0 is not necessarily $\dfrac{1}{[f(x_0)]^2}$ for some $x_0 \in C$. Indeed, consider again Example 2.4 and $p_0 = 0$. We note that

$$p_0 \notin Im\,\partial(\frac{1}{f}),\ p_0 \in \mathbf{R}_+^* dom\,(-f)^*,$$

$$\begin{aligned}(\frac{1}{f})^*(p_0) &= \inf_{\alpha>0}\ [\alpha(-f)^*(\frac{p_0}{\alpha}) - 2\sqrt{\alpha}] \\ &= [\alpha(-f)^*(\frac{p_0}{\alpha_0}) - 2\sqrt{\alpha_0}] \text{ for } \alpha_0 = \frac{1}{4},\end{aligned}$$

but there is no $x_0 \in C$ such that $\dfrac{1}{4} = \dfrac{1}{[f(x_0)]^2}$ (such an x_0 is "rejected at the infinity on C").

– We have :

$$\overline{dom\,(\frac{1}{f})^*} = \overline{Im\,\partial(\frac{1}{f})},\ \ \overline{dom\,(-f)^*} = \overline{Im\,\partial(-f)} \tag{13}$$

(see the comments about (9));

$$\mathbf{R}_+^* Im\,\partial(-f) = \mathbf{R}_+^* Im\,\partial(\frac{1}{f}) \tag{14}$$

(this results from (7)).

Combining (13) and (14) gives rise to the following relationship between $dom\,(\frac{1}{f})^*$ and $dom\,(-f)^*$:

$$\overline{\mathbf{R}_+^* dom\,(\frac{1}{f})^*} = \overline{\mathbf{R}_+^* dom\,(-f)^*}\,. \tag{15}$$

– There are several possible situations where $Im\,\partial(\frac{1}{f}) = dom\,(\frac{1}{f})^*$; one of them is when X is *reflexive* and C is bounded. Indeed, in that case, $(\frac{1}{f})^*$ is continuous throughout $X^\star$

and $Im\,\partial(\frac{1}{f}) = dom\,(\frac{1}{f})^* = X^\star$ ([4], Corollary 7G); thus formula (10) holds true at any $p_0 \in X^\star$.

The expression (10) for $(\frac{1}{f})^*(p)$, more comfortable and easier to handle than (3) (provided one can solve the equation $p \in \partial(\frac{1}{f})(x)$), would then allow us to pursue further the study of possible relations between the mathematical objects (from the viewpoint of convex analysis) associated with the convex functions $-f$ and $\frac{1}{f}$.

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