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On Young Measures Controlling Discontinuous Functions

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We obtain a version of Young's Theorem, where Young-like measures can control discontinuous functions. It determines the weak limit of $\{f(u^\nu)\}$ where f is a (possibly) discontinuous scalar function, while $\{u^\nu\}$ is a sequence of measurable functions which satisfies tightness condition.

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1. Introduction and statement of results

The classical theorem of Young in one of its versions states that if $\Omega \subseteq \mathbf{R}^n$ is an open bounded domain, $f : \mathbf{R}^m \to \mathbf{R}$ is a continuous function vanishing at infinity and $u^\nu : \Omega \to \mathbf{R}^m$ is a sequence of measurable functions which satisfies tightness condition (see formulae (6) below) then there exists a subsequence $\{u^k\}$ of $\{u^\nu\}$ such that the sequence $\{f(u^k)\}$ converges $*$-weakly in $L^\infty(\Omega)$ to the function $\overline{f}$ determined by an integral formula

$$\overline{f}(x) = \int_{\mathbf{R}^m} f(\lambda)\mu_x(d\lambda), \tag{1}$$

where μ_x are probability measures on $\mathbf{R}^m$ depending measurably on $x \in \Omega$ independent of f (see e.g. [4], [16], [31], [34], [38]).

The discovery that the weak limit of $\{f(u^k)\}$ can be represented as an integral given by (1) turned out to be widely applicable in many disciplines of analysis such as calculus of variations, partial differential equations, optimal control theory, game theory and numerical analysis, see for example [3, 9, 10, 28, 31, 34] and their references.

Up to now the theorem of Young has evolved in many directions. Let me mention two of them, which are related to this work (I refer to Chapter 3 of [34] and references therein for their detailed description, as well as for description of some other generalizations):

1. Diperna and Majda [8] assume that the sequence $u^\nu : \Omega \to \mathbf{R}^m$ is bounded in $L^p(\Omega)$ (where $1 \le p < \infty$) and f is of the form $\tilde{f}(\lambda)(1+|\lambda|^p)$ where $\tilde{f}$ is continuous and bounded

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on $\mathbf{R}^m$. Then the limit of $\{f(u^\nu)\}$ does not belong to $L^1(\Omega)$ but it belongs to the space of measures on $\overline{\Omega}$. Its restriction to the subspace of measures on Ω can be identified with

$$\int_{\gamma\mathbf{R}^m} \tilde{f}(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}, \tag{2}$$

where $\tilde{m}$ is a measure on Ω generated by the sequence $\{u^\nu\}$ (it does not depend on f), $\gamma\mathbf{R}^m$ is some compactification of $\mathbf{R}^m$, $\tilde{f} : \gamma\mathbf{R}^m \to \mathbf{R}$ is the continuous function (it depends on f) and $\{\tilde{\nu}_x\}_{x\in\Omega}$ are probability measures on $\gamma\mathbf{R}^m$ which are independent of f.

2. Alibert and Bouchitté [1] present the detailed analysis of Diperna and Majda approach in the case when a compactification $\gamma\mathbf{R}^m$ of $\mathbf{R}^m$ is homeomorphic to the unit m-dimensional ball, so that $\mathcal{S}^{m-1}$ is the remainder. Their techniques allow also to study oscillation effects for sequences. It is shown (roughly speaking) that the measure obtained as a limit of $\{f(u^\nu)\}$ can be recorded in the space of measures on $\overline{\Omega}$ and describes as

$$\int_{\mathbf{R}^m} f(\lambda)\mu_x(d\lambda)dx + \int_{S^{m-1}} f^\infty(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}, \tag{3}$$

where $\tilde{m}$ is the measure on $\overline{\Omega}$ generated by the sequence (independent of f), $f^\infty(\lambda) = \lim_{t\to\infty} f(t\lambda)/t$ is assumed to be the continuous function on S^{m-1} , $\{\tilde{\nu}_x\}_{x\in\Omega}$ is a family of probability measures on S^{m-1} and $\{\mu_x\}_{x\in\Omega}$ are classical Young measures, identical with those in the classical theorem of Young. In particular the restriction of the measure given by (3) to the space of measures on Ω gives the formulae (2) for this specific compactification.

For some other interesting works handling DiPerna-Majda measures we refer e.g. to [20, 21].

In all approaches mentioned above it is assumed that f is continuous. Then it is natural to ask what can be said about the limit of $\{f(u^\nu)\}$ if f is allowed to be discontinuous. In this paper we deal with such a situation. If the sequence u^ν tends to some function u in measure then the answer is given by the Convergence Theorem, the powerful tool in Set-Valued analysis (see e.g. Chapter 7.2 in [2]). Some other related results, where the assumption that the sequence $\{u^\nu\}$ converges in the measure is violated, are also known. The general technique by Fattorini, see e.g. Sect. 12.5 in [10] admits the function $f : \mathbf{R}^m \to \mathbf{R}$ to be continuous with respect to an arbitrary normal topology in $\mathbf{R}^m$. In particular even the discrete topology is possible and the function f under consideration can be discontinuous with respect to the conventional Euclidean topology. Some other approaches can be found in the works of Balder, see e.g. [3], and also in the works by Chentsov, see e.g. [6], where essentially more abstract constructions are discussed. In all the works mentioned above the concentration effects have not been studied.

Our approach in its simplest variant can be explained in the following way (the exact formulation is given in Theorem 3.1). Function f may be discontinuous but it must be "piecewise continuous" i.e. continuous on Borel subsets A_i (where $i \in \{1,\ldots,m\}$) whose union is the whole of $\mathbf{R}^m$. We set $f_i := f\chi_{A_i}$ and deal with sequences which take their values in a compactification γA_i of the set A_i. Then we apply techniques known from Diperna and Majda and Alibert and Bouchitté approaches to represent the limit of

$\{f_i(u^\nu)\}$ in the space of measures on $\overline{\Omega}$ in the similar way as in in (2) by

$$\int_{\gamma A_i} \tilde{f}_i(\lambda)\tilde{\nu}^i_x(d\lambda)\tilde{m}^i, \tag{4}$$

where $\tilde{m}^i$ is the measure on $\overline{\Omega}$ generated by the sequence $\{u^\nu\}$ (it does not depend on f), $\tilde{f}_i : \gamma A_i \to \mathbf{R}$ is the continuous function (determined by f_i) and $\tilde{\nu}^i_x$ are probability measures on γA_i. Next, applying techniques of Alibert and Bouchitté we recognize that (4) reads as

$$\int_{\mathrm{int} A_i} f(\lambda)\mu_x(d\lambda)(dx) + \int_{\partial A_i \cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda)\overline{m}^i(dx) + \int_{\gamma A_i \setminus A_i} \tilde{f}_i(\lambda)\nu^i_x(d\lambda)m^i(dx) \tag{5}$$

with measures $\overline{m}^i$, m^i defined on $\overline{\Omega}$ and probability measures $\overline{\nu}^i_x$, ν^i_x, μ_x defined on $\partial A_i \cap A_i$, $\gamma A_i \setminus A_i$ and $\mathbf{R}^m$ respectively (where μ_x are the same as in the theorem of Young), all: $\overline{m}^i$, m^i and $\overline{\nu}^i_x$, ν^i_x, μ_x are independent of f, the set $\gamma A_i \setminus A_i$ is the remainder (see Section 2 for the detailed explanation).

Finally we sum up the limits of $\{f_i(u^\nu)\}$ together with respect to i and obtain the representation of the limit of $\{f(u^\nu)\}$.

Note that we do not compactify the whole $\mathbf{R}^m$ as it was done in the previously discussed approaches, but each set A_i separately. In particular our embedding of $\mathbf{R}^m$ equipped with the conventional Euclidean topology into its compactification need not to be continuous. However, it is continuous if one equips $\mathbf{R}^m$ with some finer topology than the Euclidean one.

Like in [1] we study also the oscillation effects for $\{f(u^\nu)\}$ and similarly as in [1] the classical Young measures $\{\mu_x\}_{x\in\Omega}$ appear in the representation formulae (5) for new Young measures.

This main result is presented in Section 3. In Section 4 we discuss the cases when f has its discontinuities in a finite number of points and along a submanifold and also the relations of our theorem with Convergence Theorem.

The presented approach is dictated by a range of nonlinear PDE's with discontinuous constraints. Such equations appear naturally in many physical models such as the Savage-Hutter model of the granular flow, see e.g. [12, 15, 17, 19], the phase flow in porus medias with discontinuous flux function, see e.g. [24, 32]), hysteresis problems, see e.g. [14, 26], traffic flow analysis, see e.g. [23], debonding of adhesive joints, the delimination of multilayered plates, the ultimate strength of fiber reinforced structures or the nonstationary heat conduction equation, see e.g. [30, 27], or dislocations of cracks in geophysics, see e.g. [29]. It seems natural to apply the non-classical Young measures in hyperelasticity theory, where the typical stored energy function W defined on 3×3 matrices satisfies $W(F) \to +\infty$ as $\det F \to 0^+$ (see e.g. Chapter 4, pages 137–138 in [5]), or one could study non-Newtonian fluids with discontinuous constrains (we refer to [25], [36] for the related theory with continuous constrains). I think it is possible now to construct measure-valued solutions of PDE's with discontinuous constraints by using the non-classical Young measures (see e.g. [7], [25] for the related theory with the classical Young measures involved). I refer also to [11], [13] for some other related works.

Thus, to my opinion, it would be natural to study PDE's with discontinuous constrains by using the non-classical Young measures. I hope that this work will contribute to such an approach.

2. Preliminaries and notation

Let A be a subset of the Euclidean space. We use the standard notation: $C(A)$, $C_0(A)$ to denote continuous and continuous vanishing at infinity (if A is unbounded) functions on A respectively. The open ball with center a and radius R is denoted by $B(a,R)$. Similar notation is reserved for the $m-1$ dimensional spheres $S^{m-1}(a,R)$ and rings $P(a,r,R)=\{x : r < |x-a| < R\}$. If $a=0$ then we omit a in our notation. The closure of the set $S \subseteq \mathbf{R}^K$ is denoted by $\overline{S}$. If $A \subseteq \mathbf{R}^m$ is an arbitrary subset and the scalar function f is defined on arbitrary set containing A by $f\chi_A$ we mean the function equal to f on A and extended by 0 outside A.

Let $S \subseteq \mathbf{R}^K$ be the Borel subset of the Euclidean space. By $\mathcal{M}(S)$ we denote the space of Radon measures on S, while $\mathcal{P}(S)$ is its subset consisting of probability measures. If $\mu \in \mathcal{M}(S)$ and f is μ-measurable, we denote $(f,\mu) := \int_S f(\lambda)\mu(d\lambda)$.

If $\mu \in \mathcal{M}(S)$ and $K \subseteq S$ is the measurable subset of S, by $\mu\angle K$ we mean the restriction of μ to K, i. e. $(\mu\angle K)(A) = \mu(A\cap K)$.

If $C \subseteq \mathbf{R}^M$ is a Borel subset, $\phi : S \to C$ is a Borel-measurable mapping and $\mu \in \mathcal{M}(S)$, by $\phi^*(\mu)$ we denote the pushforward of the measure μ to $\mathcal{M}(C)$, that is $(\phi^*\mu)(K) = \mu(\phi^{-1}(K))$ if K is the Borel subset of C.

Arrows $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ are used to denote the strong, weak, and weak $*$ convergence respectively in the given topology.

Recall that the compact topological space γA is the compactification of the topological space A if there is the dense homeomorphical embedding $\Phi : A \to \Phi(A) \subseteq \gamma A$ (see e.g. [22]). If not causing a misunderstanding we will also write A instead of $\Phi(A)$ and $\gamma A \setminus A$ instead of $\gamma A \setminus \Phi(A)$.

If S is a Borel subset of $\mathbf{R}^k$, by $L^\infty_{w*}(\Omega, \mathcal{M}(S), \mu)$ we denote the set of families $\{\mu_x\}_{x\in\Omega}$ of Radon measures on S which are weakly $*$ μ-measurable in the sense of Pettis i.e. for every $f \in C(S)$ the mapping $x \mapsto \int_S f(\lambda)\mu_x(d\lambda)$ is μ-measurable (see e.g. Definition 1 of Section V.4 in [37]). The symbol $\mathcal{P}(\Omega, S, \mu)$ stands for such families of measures $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w*}(\Omega, \mathcal{M}(S), \mu)$ which satisfy $\|\mu_x\|_{\mathcal{M}(S)} = 1$ μ-almost everywhere.

At the end of this section we state one version of the classical theorem of Young (see e.g. [4], [16], [3, Lemma 4.11 and Corollary 5.4] and [1] for its various formulations).

Theorem 2.1. *Let Ω be an open bounded subset of $\mathbf{R}^n$ and $\{u^\nu\}_{\nu\in\mathbf{N}}$ be a sequence of μ-measurable functions, $u^\nu : \Omega \to \mathbf{R}^m$. Then there exists a subsequence of $\{u^\nu\}_{\nu\in\mathbf{N}}$ still denoted by the same expression and a family of measures $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbf{R}^m), \mu)$ such that $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} \le 1$ for μ almost all x and for every function $f \in C_0(\mathbf{R}^m)$ we have*

$$f(u^\nu) \overset{*}{\rightharpoonup} (x \mapsto (f,\mu_x)) \text{ in } L^\infty(\Omega,\mu), \text{ as } \nu \to \infty.$$

If additionally the sequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ satisfies the tightness condition:

$$\limsup_{\nu\in\mathbf{N}} \mu(\{x \in \Omega : |u^\nu(x)| \ge r\}) \overset{r\to\infty}{\to} 0, \tag{6}$$

then $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} = 1$ *for* μ *almost all* x.

We say that the sequence $\{u^\nu\}$ where $u^\nu : \Omega \to \mathbf{R}^m$ generates the Young measure $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbf{R}^m), \mu)$ if for every $f \in C_0(\mathbf{R}^m)$ the sequence $\{f(u^\nu)\}$ converges weakly $*$ in $L^\infty(\Omega, \mu)$ to the function $\overline{f} : x \mapsto \int_{\mathbf{R}^m} f(\lambda)\mu_x(d\lambda)$.

3. The main result

We start by introducing general assumption we will use in the sequel.

Condition A.

1. There exist disjoint Borel subsets: $A_1, \ldots, A_k$ called *bricks* such that $\mathbf{R}^m = \cup_{i=1}^k A_i$.

2. Each A_i is compactified by some $\gamma A_i \subseteq \mathbf{R}^{N_i}$ where $N_i \in \mathbf{N}$, with the help of the dense homeomorphic embedding $\Phi_i : A_i \to \Phi_i(A_i) \subseteq \gamma A_i$.

3. We equip $\mathbf{R}^m$ with *density function* $g : \mathbf{R}^m \to [0, \infty)$ such that $g_i := g|_{A_i} \in C(A_i)$ and $g_i(\lambda) \geq \alpha$ for every $\lambda \in A_i \cap \partial A_i$ and some $\alpha > 0$.

We will deal with the following Banach space of *admissible functions*:

$$\mathcal{F} := \{f : \mathbf{R}^m \to \mathbf{R} : \tilde{f}_i := (f/g_i) \circ \Phi_i^{-1} \in C(\gamma A_i) \text{ for } i = 1, \ldots, k\}, \tag{7}$$

equipped with the supremum norm of the $\tilde{f}_i$. The notation $\tilde{f}_i \in C(\gamma A_i)$ means that the function $(f/g_i) \circ \Phi_i^{-1}$ is the restriction of some continuous function defined on γA_i to $\Phi_i(A_i)$ (if $g_i(\lambda_0) = 0$ for some λ_0 by $(f/g_i)(\lambda_0)$ we understand the limit of f/g_i at λ_0 if it exists). As this function is uniquely defined we will denote it by the same expression: $\tilde{f}_i$.

Let us remark that even if all the N_i's are equal to the same number N and sets $\gamma A_i \subseteq \mathbf{R}^N$ are disjoint, the natural "embedding" $\Phi : \mathbf{R}^m \to \cup_{i=1}^k \gamma A_i$ defined by $\Phi(\lambda) = \Phi_i(\lambda)$ for $\lambda \in A_i$ in most cases will not be continuous. Hence the set $\cup_{i=1}^k \gamma A_i$ in general does not compactify $\mathbf{R}^m$ with the conventional Euclidean topology. On the other hand, it is the compactification of $\mathbf{R}^m$ if we equip $\mathbf{R}^m$ with some finer topology, which allows discontinuities between fixed Borel subsets $\{A_i\}$.

Our main result reads as follows.

Theorem 3.1 (Representation Theorem). *Suppose that* $\Omega \subseteq \mathbf{R}^n$ *is the compact set equipped with the Radon measure* μ *and Condition A with bricks* $\{A_i\}$ *and density function* g *on* $\mathbf{R}^m$ *is satisfied. Assume further that there is given the sequence* $\{u^\nu\}$ *of* μ*-measurable functions,* $u^\nu : \Omega \to \mathbf{R}^m$ *which satisfies the condition*

$$\sup_\nu \int_{\mathbf{R}^m} g(u^\nu)\mu(dx) < \infty, \tag{8}$$

and additionally the tightness condition (6). *Then there exist*

- *i)* *a subsequence of* $\{u^\nu\}$ *denoted by the same expression,*
- *ii)* *measures* $\overline{m}^i, m^i \in \mathcal{M}(\Omega)$, *such that* $\overline{m}^i$ *is absolutely continuous with respect to* μ *and* $\text{supp} m^i \subseteq \text{supp}\mu$ *for* $i \in \{1, \ldots, k\}$,
- *iii)* *families of probability measures* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$, $\{\overline{\nu}^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \partial A_i \cap A_i, \mu)$ *and* $\{\nu^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \gamma A_i \setminus A_i, m^i)$ *where* $i \in \{1, \ldots, k\}$

such that for an arbitrary $f \in \mathcal{F}$ the subsequence $\{f(u^\nu(x))\mu(dx)\}$ converges weakly $*$ in the space of measures to the measure represented by

$$\sum_{i=1}^{k}\left(\int_{\mathrm{int}A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\partial A_i \cap A_i} f(\lambda)\overline{\nu}_x^i(d\lambda)\overline{m}^i(dx) + \int_{\gamma A_i \setminus A_i} \tilde{f}_i(\lambda)\nu_x^i(d\lambda)m^i(dx)\right), \tag{9}$$

where $\tilde{f}_i$ is defined by (7). Moreover, measures $\{\mu_x\}_{x\in\Omega}$ are the classical Young measures generated by the sequence $\{u^\nu\}$.

Remark 3.2. As $f = (f/g)g$ and f/g is the bounded function we observe that the sequence $f(u^\nu)\mu$ is bounded in the space of measures.

In the sequel we will use the following lemma. Its proof is presented in the Appendix.

Lemma 3.3. *Let $\Omega \subseteq \mathbf{R}^n$ be the compact set equipped with the Radon measure μ. Assume that A is the Borel subset of $\mathbf{R}^n$, $\Phi : A \to \Phi(A) \subseteq \gamma A$ is the homeomorphical embedding of A into its compactification γA, $g \in C(A)$ is the nonnegative function.*

Let $\{u^\nu\} : \Omega \to \mathbf{R}^m$ be the given sequence of μ-measurable functions which satisfies the condition

$$\sup_\nu \int_{x: u^\nu(x)\in A} g(u^\nu)\mu(dx) < \infty,$$

and generates the classical Young measure $\{\mu_x\}_{x\in\Omega}$. Define the sequence of measures $\{L^\nu\}_{\nu\in\mathbf{N}}$ on $\Omega \times \gamma A$ by the expression

$$(F, L^\nu) := \int_{\{x: u^\nu(x)\in A\}} F(x, \Phi(u^\nu(x)))g(u^\nu(x))\mu(dx), \text{ where } F \in C(\Omega\times\gamma A). \tag{10}$$

Then we have:

i) There exists a subsequence of $\{L^\nu\}$ still denoted by the same expression, measures $L \in \mathcal{M}(\Omega\times\gamma A)$, $\tilde{m} \in \mathcal{M}(\Omega)$ and a family of probability measures $\{\tilde{\nu}_x\} \in \mathcal{P}(\Omega, \gamma A, \tilde{m})$ such that

$$L^\nu \stackrel{*}{\rightharpoonup} L \text{ in } \mathcal{M}(\Omega\times\gamma A), \tag{11}$$

$$(F, L) = \int_\Omega\int_{\gamma A} F(x,\lambda)\tilde{\nu}_x(d\lambda))\tilde{m}(dx) \text{ where } F \in C(\Omega\times\gamma A), \tag{12}$$

$$\operatorname{supp}\tilde{m} \subseteq \operatorname{supp}\mu. \tag{13}$$

ii) Let $\tilde{m} = p(x)\mu + \tilde{m}_s$ be the Lebesgue-Nikodym decomposition of $\tilde{m}$ with respect to μ. Then if $\gamma A\setminus A \neq \emptyset$ we have $\tilde{\nu}_x(\gamma A\setminus A) = 1$ for $\tilde{m}_s$ almost all $x\in\Omega$, while if $\gamma A \setminus A = \emptyset$ we have $\tilde{m}_s = 0$.

iii) Assume that the set $U = \mathrm{int}A$ is not empty and let $U^0 = \Phi(U)$. If $f \in C(\gamma A)$ is such that the function $F(\lambda) = f(\Phi(\lambda))g(\lambda)\chi_{\lambda\in U}$ belongs to $C_0(\mathbf{R}^m)$, then we have

$$\int_U f(\Phi(\lambda))g(\lambda)\mu_x(d\lambda) = p(x)\int_{U^0} f(\lambda)\tilde{\nu}_x(d\lambda), \tag{14}$$

for μ-almost all $x\in\Omega$. In particular if we denote $\tilde{\nu}_x^0 = \tilde{\nu}_x \angle U^0 \in \mathcal{M}(U^0)$, $\mu_x^0 = g(\lambda)(\mu_x\angle U)(d\lambda) \in \mathcal{M}(U)$, then for μ almost all $x\in\Omega$, we have

$$\Phi^*(\mu_x^0) = p(x)\tilde{\nu}_x^0 \text{ in } \mathcal{M}(U^0). \tag{15}$$

Proof of Theorem 3.1. We may assume that f does not vanish on one of the bricks: $A = A_i$ only (substitute $f = f\chi_{\{\lambda\in A\}}$). To abbreviate we will omit the index "i" in the proof.

Let $u^\nu(x) \in A$. We have

$$f(u^\nu(x)) = \frac{f}{g}(u^\nu(x))g(u^\nu(x)) = F(\Phi(u^\nu(x))g(u^\nu(x)),$$

where $F = (f/g)\circ\Phi^{-1}$. According to Lemma 3.3, we have

$$f(u^\nu(x))\mu(dx) \overset{*}{\rightharpoonup} \int_{\gamma A} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx) = \mathcal{A}, \tag{16}$$

where $\tilde{m}$ and $\{\tilde{\nu}_x\}_{x\in\Omega}$ are the same as in Lemma 3.3. Using the Lebesgue-Nikodym's decomposition of $\tilde{m}$ with respect to μ as in Lemma 3.3 we verify that

$$\mathcal{A} = \int_{\gamma A} F(\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{\gamma A\setminus A} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}_s(dx).$$

Since the first integral is the sum of two: the one over $U^0 = \Phi(\mathrm{int}A)$ and the second over $\gamma A \setminus U^0$ denoted by $\gamma A \setminus \mathrm{int}A$, we derive from Lemma 3.3 that

$$\begin{aligned}\mathcal{A} \;=\; & \int_{\mathrm{int}A} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\gamma A\setminus \mathrm{int}A} F(\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) \\ & + \int_{\gamma A\setminus A} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}_s(dx) = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.\end{aligned} \tag{17}$$

Decomposing further: $\gamma A\setminus\mathrm{int}A = (\gamma A\setminus A)\cup((\gamma A\setminus\mathrm{int}A)\setminus(\gamma A\setminus A))$, noting that $(\gamma A\setminus \mathrm{int}A)\setminus(\gamma A\setminus A) = \Phi(A)\setminus\Phi(\mathrm{int}A) = \Phi(\partial A\cap A)$, and recalling that $\tilde{m} = p(x)\mu + \tilde{m}_s$, we see that

$$\mathcal{A}_2 + \mathcal{A}_3 = \int_{\Phi(\partial A\cap A)} F(\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{\gamma A\setminus A} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx) = \mathcal{B}_1 + \mathcal{B}_2. \tag{18}$$

Let $h(x) = \tilde{\nu}_x(\gamma A\setminus A)$, $\Omega_1 = \{x\in\Omega : h(x)\neq 0\}$, and choose $y \in \gamma A\setminus A$. Then the second term in (18) can be modified to

$$\mathcal{B}_2 = \int_{\gamma A\setminus A} F(\lambda)\nu_x(d\lambda)m(dx), \tag{19}$$

where ν_x is such that for every $G \in C(\gamma A\setminus A)$ we have

$$(G,\nu_x) = \begin{cases} \frac{1}{\tilde{\nu}_x(\gamma A\setminus A)}\int_{\gamma A\setminus A} G(\lambda)\tilde{\nu}_x(d\lambda) & \text{if } x\in\Omega_1 \\ G(y) & \text{if } x\notin\Omega_1, \end{cases} \tag{20}$$

and $m(dx) = h(x)\tilde{m}(dx)$. Now we deal with first term in (18). Let

$$w(x) = \int_{\Phi(\partial A\cap A)} \frac{1}{g(\Phi^{-1}(\lambda))}\tilde{\nu}_x(d\lambda).$$

Since we assume that $g(\lambda) > \alpha$ if $\lambda \in \partial A \cap A$, it follows that $w(x)$ is well defined for μ-almost all $x \in \Omega$. Choose an arbitrary $a \in \partial A \cap A$, set $\Omega_2 = \{x \in \Omega : \tilde{\nu}_x(\Phi(\partial A \cap A)) > 0\}$, and define the measure $\overline{\nu}_x$ by

$$(G, \overline{\nu}_x) = \begin{cases} \frac{1}{w(x)} \int_{\Phi(\partial A \cap A)} (G/g)(\Phi^{-1}(\lambda)) \tilde{\nu}_x(d\lambda) & \text{if } x \in \Omega_2 \\ G(a) & \text{if } x \notin \Omega_2, \end{cases} \tag{21}$$

where $G \in C(\partial A \cap A)$. Then

$$\mathcal{B}_1 = \int_{\partial A \cap A} f(\lambda) \overline{\nu}_x(d\lambda) q(x) \mu(dx), \tag{22}$$

where $q(x) = w(x)p(x)$. Now the result follows from (17), (18), (19) and (22). □

We end this section with the following remark.

Remark 3.4. 1. If we do not assume that tightness condition (6) is satisfied then representation formulae (9) still holds true but Young measures $\{\mu_x\}_{x\in\Omega}$ are not necessarily probability measures. They belong to $L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbf{R}^m), \mu)$ and satisfy the condition $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} \leq 1$.
2. If $g(\lambda) \to \infty$ as $\lambda \to \infty$ then the condition (8) implies tightness condition (6). Essentially, let $C = \sup_\nu \int_{\mathbf{R}^m} g(u^\nu)\mu(dx)$, take an arbitrary $\epsilon > 0$, and choose $L > 0$ such that $g(\lambda) > C\epsilon^{-1}$ if $|\lambda| > L$. Then applying Chebyshev's inequality we have

$$\mu(\{x : |u^\nu(x)| > L\}) \leq \mu(\{x : g(u^\nu(x)) > C\epsilon^{-1}\}) \leq \epsilon/C \int_{\mathbf{R}^m} g(u^\nu)\mu(dx) \leq \epsilon,$$

which implies the tightness condition (6).

4. The special cases

In this section we are going to illustrate Theorem 3.1 on three examples: when f is continuous, when it is allowed to have finitely many discontinuity points and when f can be discontinuous along a submanifold.

The following result is an abbreviated version of Theorem 2.5 in [1] obtained by Alibert and Bouchitte. We present its proof as the consequence of of Theorem 3.1, but let us mention that our proof of Theorem 3.1 was inspired by techniques presented in the proof of Theorem 2.5 in [1].

Theorem 4.1. *Let $\Omega \subseteq \mathbf{R}^n$ be the compact subset equipped with the Borel measure μ and $\{u^\nu\}_{\nu\in\mathbf{N}}$ be the sequence bounded in $L^1(\Omega, \mathbf{R}^m, \mu)$. Define*

$$\mathcal{F} := \{f \in C(\mathbf{R}^m) : f^\infty(\lambda) := \lim_{t\to\infty} \frac{f(t\lambda)}{t} \in C(S^{m-1})\}.$$

Then there exists

- *a subsequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ denoted by the same expression,*
- *a positive measure $m \in \mathcal{M}(\Omega)$ such that* supp$m \subseteq$ suppμ,
- *families of probability measures $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$ and $\{\nu^\infty_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, S^{m-1}, m)$*

such that for an arbitrary $f \in \mathcal{F}$ the sequence $\{f(u^\nu)\mu(dx)\}$ converges weakly $*$ in the space of measures to the measure represented by:

$$\int_{\mathbf{R}^m} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{S^{m-1}} f^\infty(\lambda)\nu_x^\infty(d\lambda)m(dx).$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ are Young measures generated by $\{u^\nu\}_{\nu\in\mathbf{N}}$.

Proof. We apply Theorem 3.1. In this case Condition A is satisfied with one brick: $A = \mathbf{R}^m$ and dense homeomorphic embedding $\Phi : \mathbf{R}^m \to \Phi(\mathbf{R}^m) = B(1) \subseteq \overline{B(1)} = \gamma\mathbf{R}^m \subseteq \mathbf{R}^m$, given by $\Phi(\lambda) = \frac{\lambda}{1+|\lambda|}$, and $\mathbf{R}^m$ is equipped with density function $g(\lambda) = 1 + |\lambda|$. Then we have $\gamma\mathbf{R}^m \setminus \mathbf{R}^m = S^{m-1}$ and $\Phi^{-1}(\lambda) = \frac{\lambda}{1-|\lambda|}$ for $\lambda \in B(1)$. It is easy to verify that for $\lambda \in S^{m-1}$ and $0 < s < 1$ the function $f/g \circ \Phi^{-1}(s\lambda)$ tends as $s \to 1$ to $\tilde{f}(\lambda) = f^\infty(\lambda)$, so it suffices to apply formulae (9). ☐

Now we will study the weak limit of $\{f(u^\nu)\mu\}_{\nu\in\mathbf{N}}$ in the case when f is allowed to have a finite number of discontinuity points. In such case we have the following result.

Theorem 4.2. *Let $\Omega \subseteq \mathbf{R}^n$ be the compact subset equipped with the Borel measure μ and $\{u^\nu\}_{\nu\in\mathbf{N}}$ be the sequence bounded in $L^1(\Omega, \mathbf{R}^m, \mu)$. Assume that $B_1, \dots, B_k \in \mathbf{R}^m$, define radial limits of the given function f at ∞ and at B_i by expressions*

$$f^\infty(\theta) := \lim_{t\to+\infty} \frac{f(t\theta)}{t} \text{ and } f_i(\theta) := \lim_{t\to 0^+} f(t\theta + B_i),\ \theta \in S^{m-1}$$

and set

$$\begin{aligned}\mathcal{F} :=&\{f \in C(\mathbf{R}^m \setminus \{B_1, \dots, B_k\}) : f^\infty \in C(S^{m-1}) \\ &\text{and } f_i \in C(S^{m-1}) \text{ for } i \in \{1, \dots, k\}\}.\end{aligned}$$

Then there exist:

- *a subsequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ denoted by the same expression,*
- *a positive measure $m \in \mathcal{M}(\Omega)$ such that* $\operatorname{supp} m \subseteq \operatorname{supp}\mu$,
- *μ-measurable functions $p_i, q_i : \Omega \to [0, 1]$ where $i = 1, \dots, k$,*
- *families of probability measures $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$ and $\{\nu_x^\infty\}_{x\in\Omega}, \{\nu_x^i\}_{x\in\Omega} \in \mathcal{P}(\Omega, S^{m-1}, \mu)$ where $i \in \{1, \dots, k\}$*

such that for an arbitrary $f \in \mathcal{F}$ the sequence $\{f(u^\nu)\mu\}$ converges weakly $*$ in the space of measures to the measure represented by

$$\begin{aligned}&\int_{\mathbf{R}^m\setminus\{B_1,\dots,B_k\}} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{S^{m-1}} f^\infty(\lambda)\nu_x^\infty(d\lambda)m(dx) \\ &\quad + \sum_{i=1}^k \int_{S^{m-1}} f_i(\lambda)\nu_x^i(d\lambda)p_i(x)\mu(dx) + \sum_{i=1}^k f(B_i)q_i(x)\mu(dx). \qquad (23)\end{aligned}$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ is the Young measure generated by the sequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ and $p_i(x) + q_i(x) = \mu_x(\{B_i\})$ for μ-almost all x. In particular we have

$$\sum_{i=1}^k p_i(x) + \sum_{i=1}^k q_i(x) = \sum_{i=1}^k \mu_x(B_i) \le 1$$

for μ-almost all x.

Proof. Let $r, R \in \mathbf{R}$, set $\theta_i(\lambda) := \frac{\lambda - B_i}{|\lambda - B_i|}$ for $i = 1, \ldots, k$ and define the following sets

$$\begin{aligned} U_i &= B(B_i, r), \text{ where } i = 1, \ldots, k, \quad U_{k+1} = \mathbf{R}^m \setminus \overline{B(R)}, \\ U_0 &= \{\lambda \in \mathbf{R}^m : |\lambda| < 2R, |\lambda - B_i| > r/2\}. \end{aligned}$$

Take r and R such that sets: $U_1, \ldots, U_{k+1}$ are disjoint. Since $U_0, U_1, \ldots, U_{k+1}$ is the open covering of $\mathbf{R}^m$ we may find the continuous partition of unity subordinate to this covering denoted by $\{\phi_r\}_{r=0,\ldots,k+1}$.

After decomposing $f = \sum_i \phi_i f$, it suffices to prove that the result is true if either 1): $f \in C(\mathbf{R}^m)$ and $B_1, \ldots, B_k \notin \operatorname{supp} f$ or 2): $f \in C(\mathbf{R}^m \setminus \{B_i\})$ and $\operatorname{supp} f \subseteq B(B_i, r)$.

In the first case it suffices to apply Theorem 4.1 (then all the f_i's are 0 and $f(B_i) = 0$ in (23)).

In the second one we decompose $\mathbf{R}^m$ by three bricks: $A_1 = P = B(B_i, r) \setminus \{B_i\}$, $A_2 = \{B_i\}$, $A_3 = \mathbf{R}^m \setminus B(B_i, r)$ and construct dense homeomorphic embeddings $\Phi_j : A_j \to \Phi_j(A_j) \subseteq \gamma A_j$ $(j = 1, 2, 3)$ where

$$\Phi_1(\lambda) = \phi(r_i(\lambda))\theta_i(\lambda) : P \to P(0, r) \subseteq \overline{B(r)} \quad \text{with} \quad \phi(s) = -s + r,$$

in particular, $\Phi_1(A_1) = P(0, r)$ and $\gamma A_1 = \overline{B(r)}$, $\gamma A_1 \setminus A_1 = \{0\} \cup S^{m-1}$ (we add the sphere S^{m-1} at $\{B_i\}$ and shrink $S^{m-1}(B_i, r)$ to the point); an embedding Φ_2 is just the *identity* function, so that $\Phi_2(A_2) = \{B_i\} = \gamma A_2$ and $\gamma A_2 \setminus A_2 = \emptyset$; an embedding $\Phi_3 : A_3 \to \Phi_3(A_3) \subseteq \gamma A_3$ is defined in an arbitrary way.

We equip $\mathbf{R}^m$ with density function $g(\lambda) \equiv 1$ on $A_1 \cup A_2$ and $g(\lambda) = 1 + |\lambda|$ on A_3. Then we recognize that the function

$$\left(\frac{f \chi_{\lambda \in A_1}}{g} \right) \circ \Phi_1^{-1} : P(0, r) \to \mathbf{R}$$

extends to the function $\tilde{f}_1$ on the ball $\overline{B(r)}$ such that $\tilde{f}_1(\lambda)(\{0\}) = 0$, $\tilde{f}_1(\lambda) = f_i(\lambda)$ for $\lambda \in S^{m-1}$. Moreover, we have $\tilde{f}_3(\lambda) = 0$ as $f \equiv 0$ on A_3. Applying Theorem 3.1 we see that (after passing to the subsequence) the sequence $\{f(u^\nu)\mu\}$ converges weakly $*$ in the space of measures to the measure $M = M_1 + M_2 + M_3$ where

$$\begin{aligned} M_1 &= \int_{P(B_i,r)\setminus\{B_i\}} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\{B_i\}\cup S^{m-1}(B_i,r)} f(\lambda)\overline{\nu}_x^1(d\lambda)\overline{m}^1(dx) \\ &\quad + \int_{\{0\}\cup S^{m-1}} \tilde{f}_1(\lambda)\nu_x^1(d\lambda)m^1(dx) = \int_{\mathbf{R}^m\setminus\{B_i\}} f(\lambda)\mu_x(d\lambda)\mu(dx) \\ &\quad + f(B_i)\overline{\nu}_x^1(B_i)\overline{m}^1(dx) + \int_{S^{m-1}} f_i(\lambda)\nu_x^1(d\lambda)m^1(dx). \\ M_2 &= f(B_i)\overline{m}_2(dx) \quad \text{and} \quad M_3 \equiv 0. \end{aligned}$$

Moreover, as the sequence $\{f(u^\nu)\}$ is bounded, we see that the measure $m^1(dx)$ must be absolutely continuous with respect to the measure μ. This gives:

$$M = \int_{\mathbf{R}^m\setminus\{B_i\}} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{S^{m-1}} f_i(\lambda)\nu_x^1(d\lambda)p_i(x)\mu(dx) + f(B_i)q_i(x)\mu(dx),$$

where p_i and q_i are such that $m^1(dx) = p_i(x)\mu(dx)$ and $q_i(x)\mu(dx) = \overline{\nu}^1_x(\{B_i\})\overline{m}^1(dx) + \overline{m}^2(dx)$. This implies (23).

Let me prove the last assertion. Take $f \in C_0(\mathbf{R}^m)$ such that supp$f \subseteq B(B_i, r)$ and $f \equiv 1$ in some neighborhood of B_i. According to (23) we have:

$$f(u^\nu)\mu(dx) \overset{*}{\rightharpoonup} \left(\int_{B(B_i,r)\setminus\{B_i\}} f(\lambda)\mu_x(d\lambda) + p_i(x) + q_i(x) \right) \mu(dx).$$

On the other hand, using the classical Young's theorem we observe that

$$f(u^\nu) \rightharpoonup \int_{B(B_i,r)} f(\lambda)\mu_x(d\lambda) \text{ in } L^1(\Omega, \mu).$$

Thus $p_i(x) + q_i(x) = \int_{\{B_i\}} f(\lambda)\mu_x(d\lambda) = \mu_x(\{B_i\})$ and the proof of theorem is finished. □

As the special case of the presented above situation let us deal with the sequence of scalar functions. Then we have the following conclusion.

Corollary 4.3. *Let $\Omega \subseteq \mathbf{R}^n$ be the compact subset equipped with the Borel measure μ and $\{u^\nu\}_{\nu\in\mathbf{N}}$ be the sequence bounded in $L^1(\Omega, \mu)$. Assume that $B_1, \dots, B_k \in \mathbf{R}$ and let the left and right hand side limits of the given function f at B be denoted by $f_-(B)$ and $f_+(B)$. Set*

$$\mathcal{F} := \{ f \in C(\mathbf{R} \setminus \{B_1, \dots, B_k\}) : \lim_{t\to\infty} |f(t)/t| = 0 \text{ and } f_-(B_i) \text{ and } f_+(B_i) \text{ are well defined for } i = 1, \dots, k \}.$$

Then there exist:

- *a subsequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ denoted by the same expression,*
- *families of probability measures $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}, \mu)$*
- *μ-measurable functions $p_i^-, p_i^+, q_i : \Omega \to [0, 1]$*

such that for an arbitrary $f \in \mathcal{F}$ the sequence $\{f(u^\nu)\}$ converges weakly in $L^1(\Omega)$ to the function

$$\overline{f}(x) = \int_{\mathbf{R}\setminus\{B_1,\dots,B_k\}} f(\lambda)\mu_x(d\lambda) + \sum_{i=1}^{k} \left(f_-(B_i)p_i^-(x) + f_+(B_i)p_i^+(x) + f(B_i)q_i(x) \right).$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ is the Young measure generated by the sequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ and $p_i^-(x) + p_i^+(x) + q_i(x) = \mu_x(\{B_i\})$ for μ-almost all x.

Proof. This follows from Theorem 4.2 if we note that under our assumptions we have $f^\infty \equiv 0$, $S^0 = \{-1, 1\}$ and $\int_{S^0} f^i(\lambda)\nu_x^i(d\lambda) = f^-(B_i)\nu_x^i(\{-1\}) + f^+(B_i)\nu_x^i(\{+1\})$. Thus it suffices to apply (23) and put $p_i^-(x) = \nu_x^i(\{-1\})p_i(x)$, $p_i^+(x) = \nu_x^i(\{+1\})p_i(x)$. □

Remark 4.4. If we additionally assume in Corollary 4.3 that $u^\nu \to u(x)$ in the measure we obtain that $\overline{f}(x)$ belongs to the convex hull of the set of accumulation points of f at $u(x)$. Essentially, it is known that in such case we have: $\mu_x = \delta_{u(x)}$ for μ almost all x (see e.g. [16]). Thus, if $u(x) \neq B_i$ we have $p_i^-(x) = p_i^+(x) = q(x) = 0$, while if $u(x) = B_i$ we have $p_i^-(x) + p_i^+(x) + q_i(x) = 1$. Hence if $u(x) \notin \{B_1, \dots, B_k\}$ we have: $\overline{f}(x) = f(u(x))$, while if $u(x) = B_i$ for some $i \in \{1, \ldots, k\}$ then $\overline{f}(x)$ is the convex combination of $f_-(B_i)$, $f_+(B_i)$ and $f(B_i)$. This gives the version of the Convergence Theorem from Set-Valued Analysis (see e.g. Chapter 7.2 in [2]) restricted to the considered by us class of functions. Note that quantities: $p_i^-(x)$, $p_i^+(x)$ and $q_i(x)$ depend on the sequence $\{u^\nu\}$ only and are the same for all $f \in \mathcal{F}$. Thus our statement is a little more precise than that of the classical Convergence Theorem. Further analysis in this direction will be presented in the forthcoming paper ([18]).

We end this section with the following theorem. Its simple but rather technical proof is left to the reader. Obviously, it is possible to construct many examples of similar nature and generalize the presented ones in various directions.

Theorem 4.5. *Suppose that* $\Omega \subseteq \mathbf{R}^n$ *is the compact subset equipped with the Borel measure* μ *and* $\{u^\nu\}_{\nu\in\mathbf{N}}$ *is the sequence such that* $\sup_\nu \|u^\nu\|_{L^\infty(\Omega,\mathbf{R}^m,\mu)} < \infty$. *Assume that* $M \subseteq \mathbf{R}^m$ *is a smooth, bounded and closed* k*-dimensional submanifold. Let* $(Y_1)_\lambda, \ldots, (Y_{m-k})_\lambda$ *be an orthonormal basis in the normal space to* M *at* $\lambda \in M$ *such that the mapping* $M \ni \lambda \mapsto ((Y_1)_\lambda, \ldots, (Y_{m-k})_\lambda)$ *is continuous. Define the following expression for* $f : \mathbf{R}^m \to \mathbf{R}$

$$\hat{f}(\lambda, \theta) := \lim_{t\to 0^+} f(\lambda + t \sum_{i=1}^{m-k} \theta_i (Y_i)_\lambda) \text{ where } \theta = (\theta_1, \ldots, \theta_{m-k}) \in S^{m-k-1}.$$

and let us consider the following class of functions

$$\mathcal{F} := \{f : \mathbf{R}^m \to \mathbf{R} : f \in C(\mathbf{R}^m \setminus M) \cap C(M) \text{ and for every } (\lambda, \theta) \in M \times S^{m-k-1} \text{ the mapping } \hat{f}(\lambda,\theta) \text{ is well defined, } \hat{f}(\lambda,\theta) \in C(M \times S^{m-k-1}) \text{ and } \lim_{\lambda\to\infty} f(\lambda) = 0\}.$$

Then the following statements hold.

i) *There exists a subsequence* $\{u^\nu\}_{\nu\in\mathbf{N}}$ *still denoted by the same expression, the family of probability measures* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$, $\{\overline{\nu}_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M, \mu)$, $\{\nu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M \times S^{m-k-1}, \mu)$ *and* μ*-measurable functions:* $p, q : \Omega \to [0,1]$ *such that for an arbitrary* $f \in \mathcal{F}$ *the sequence* $\{f(u^\nu)\}$ *converges weakly* $*$ *in* $L^\infty(\Omega)$ *to the function*

$$\overline{f}(x) = \int_{\mathbf{R}^m\setminus M} f(\lambda)\mu_x(d\lambda) + \int_M f(\lambda)\overline{\nu}_x(d\lambda)p(x) + \int_{M\times S^{m-k-1}} \hat{f}(\lambda,\theta)\nu_x(d\lambda, d\theta)q(x).$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ *is the Young's measure generated by the sequence* $\{u^\nu\}_{\nu\in\mathbf{N}}$, *and* $p(x) + q(x) = \mu_x(\{M\})$ *for* μ*-almost all* $x \in \mathbf{R}^m$.

ii) *If* $k = m - 1$ *there exist families of measures:* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$, $\{\nu_x^1\}_{x\in\Omega}$, $\{\nu_x^2\}_{x\in\Omega}, \{\overline{\nu}_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M, \mu)$, *and* μ*-measurable functions:* $q_1, q_2, p : \Omega \to [0,1]$ *such that for an arbitrary* $f \in \mathcal{F}$ *the sequence* $\{f(u^\nu)\}$ *converges weakly* $*$ *in* $L^\infty(\Omega)$ *to the function*

$$\overline{f}(x) = \int_{\mathbf{R}^m\setminus M} f(\lambda)\mu_x(d\lambda) + \int_M f(\lambda)\overline{\nu}_x(d\lambda)p(x) + \int_M f^+(\lambda)\nu_x^1(d\lambda)q_1(x) + \int_M f^-(\lambda)\nu_x^2(d\lambda)q_2(x),$$

where $f^+(\lambda) = \lim_{t\to 0^+} f(\lambda + tY_\lambda)$, $f^-(\lambda) = \lim_{t\to 0^+} f(\lambda - tY_\lambda)$ *and* Y_λ *is the orthonormal vector to* M *at* λ.
Moreover, $\{\mu_x\}_{x\in\Omega}$ *is the Young's measure generated by the sequence* $\{u^\nu\}_{\nu\in\mathbf{N}}$ *and* $p(x) + q_1(x) + q_2(x) = \mu_x(M) \le 1$ *for* μ *almost all* x.

5. Appendix

Proof of Lemma 3.3. The proof of parts $i)$ and $ii)$ is a little modification of the proof of Proposition 4.1 given in [1], but for reader's convenience we include it.

"i):"The existence of the measure L satisfying (11) follows from Banach-Alaoglu's theorem (see e.g. [35], page 131), as the space of measures on $\Omega\times\gamma A$ is dual to the separable space $C(\Omega\times\gamma A)$ and the sequence of measures $\{L^\nu\}$ is bounded.

Let $\tilde m$ be the projection of L onto $\mathcal{M}(\Omega)$, that is $(h,\tilde m) = \int_{\Omega\times\gamma A} h(x)L(dx,d\lambda)$ for every $h\in C(\Omega)$. By the slicing measure argument ([33]) there exists the family of positive measures $\{\tilde\nu_x\}_{x\in\Omega} \in L^\infty_{w*}(\Omega,\mathcal{M}(\gamma A),\tilde m)$ such that the representation formula (12) holds. We will show that $\tilde\nu_x$ are probability measures $\tilde m$ almost everywhere. Take $F(x,\lambda) = h(x)$ where $h\in C(\Omega)$ and substitute it to (10). We get from (12):

$$(h,\tilde m) = (F,L) = \int_\Omega h(x)(\int_{\gamma A} 1\tilde\nu_x(d\lambda))\tilde m(dx).$$

Since h was taken arbitrary, we deduce that $\tilde\nu_x(\gamma A)\tilde m(dx) = \tilde m(dx)$ in $\mathcal{M}(\Omega)$. This implies that $\tilde\nu_x(\gamma A) = 1$, for $\tilde m$ almost all $x\in\Omega$.

"ii):" Assume that $\gamma A\backslash A \neq \emptyset$ and let $F(\lambda) := \mathrm{dist}(\lambda, \gamma A\backslash A) = \mathrm{dist}(\lambda, \overline{\gamma A\setminus A}) \in C(\gamma A)$.

At first we note that the sequence $\{h^\nu\}$ defined by $h^\nu(x) = F(\Phi(u^\nu(x)))g(u^\nu(x))$ if $u^\nu(x)\in A$ and $h^\nu(x) = 0$ otherwise is uniformly integrable in $L^1(\Omega,\mu)$. Essentially, let $A^*_\epsilon = \{\lambda\in A : \mathrm{dist}(\Phi(\lambda), \gamma A\setminus A) < \epsilon\}$. Then for every $K\in R_+$ we have $\int_{\{|h^\nu(x)|>K\}} |h^\nu(x)|\mu(dx) =$

$$\int_{\{|h^\nu(x)|>K\}\cap\{u^\nu(x)\in A^*_\epsilon\}} |h^\nu(x)|\mu(dx) + \int_{\{|h^\nu(x)|>K\}\cap\{u^\nu(x)\in A\backslash A^*_\epsilon\}} |h^\nu(x)|\mu(dx).$$

The first term is not larger than $\epsilon\sup_\nu \int_{u^\nu\in A} g(u^\nu(x))\mu(dx)$. The second one is zero if we take K sufficiently large. This is because the set $M_\epsilon := \Phi(A\setminus A^*_\epsilon) \subseteq \mathbf{R}^N$ is compact and $g\circ\Phi^{-1}$ is continuous on M_ϵ. This implies that the second term is zero if we take $K > \mathrm{diam}\gamma A \sup_{\lambda\in M_\epsilon}|g\circ\Phi^{-1}(\lambda)|$.

Hence there exists $h\in L^1(\Omega,\mu)$ such that the subsequence of $\{h^\nu\}$ denoted by the same expression satisfies

$$h^\nu(x)\mu(dx) \overset{*}{\rightharpoonup} h\mu(dx) \text{ in } \mathcal{M}(\Omega). \tag{24}$$

On the other hand, by the just proved part i), we have for arbitrary $\phi\in C(\Omega)$,

$$\begin{aligned}\int_\Omega \phi(x)h^\nu(x)\mu(dx) &= (\phi F, L^\nu) \to (\phi F, L) = \int_\Omega \phi(x)(\int_{\gamma A} F(\lambda)\tilde\nu_x(d\lambda))\tilde m(dx)\\ &= \int_\Omega \phi(x)\overline{F}(x)p(x)\mu(dx) + \int_\Omega \phi(x)\overline{F}(x)\tilde m_s(dx),\end{aligned} \tag{25}$$

where $\overline{F}(x) = \int_{\gamma A} F(\lambda)\tilde{\nu}_x(d\lambda)$. Combining this with (24) we observe that the second term above vanishes, so $\overline{F}(x) = 0$ for $\tilde{m}_s$ almost all x. Since $F > 0$ on $\phi(A)$, we get $\tilde{\nu}_x(\phi(A)) = 0$ for $\tilde{m}_s$ almost all x, which is what we have claimed.

If $\gamma A \setminus A = \emptyset$ then A is compact and for an arbitrary $F \in C(\gamma A)$ the sequence defined by $h^\nu(x) = F(\Phi(u^\nu(x)))g(u^\nu(x))$ if $u^\nu(x) \in A$ and $h^\nu(x) = 0$ if $u^\nu(x) \notin A$ is uniformly bounded, so it is also uniformly integrable. Repeating the same computations as in (25) with $\phi \equiv 1$ we observe that $\int_{\gamma A} F(\lambda)\tilde{\nu}_x(d\lambda) = 0$ for an arbitrary $F \in C(\gamma A)$ and $\tilde{m}_s$ almost all $x \in \Omega$. This implies that $\tilde{m}_s = 0$.

"*iii*):" Take $F(\lambda) = f(\Phi(\lambda))g(\lambda)\chi_{\lambda\in U}$ and assume that $F \in C_0(\mathbf{R}^m)$. Applying the classical theorem of Young we get

$$F(u^\nu(x)) \rightharpoonup \overline{F}(x) = \int_{\mathbf{R}^m} F(\lambda)\mu_x(d\lambda) = \int_{\mathrm{int}A} f(\Phi(\lambda))g(\lambda)\mu_x(d\lambda) \text{ in } L^1(\Omega,\mu).$$

Hence $F(u^\nu(x))\mu(dx) \overset{*}{\rightharpoonup} \overline{F}(x)\mu(dx)$ in $\mathcal{M}(\Omega)$. According to the just proved parts *i*) and *ii*) we have

$$\overline{F}(x)\mu = (f,\tilde{\nu}_x)\tilde{m} = (f,\tilde{\nu}_x)p(x)\mu + (f,\tilde{\nu}_x)\tilde{m}_s.$$

Since f vanishes on $\gamma A \setminus A$, which is the support of $\tilde{\nu}_x$ for $\tilde{m}_s$ almost all x, it follows that the second term of the above decomposition vanishes. Note also that $f \equiv 0$ on $\gamma A \setminus U^0$. Hence $(f,\tilde{\nu}_x) = \int_{U^0} f(\lambda)\tilde{\nu}_x(d\lambda)$, which gives (14).

To complete the proof of part *iii*) it suffices to note that the left hand side of (14) reads as

$$\int_U f \circ \Phi(\lambda)\mu_x^0(d\lambda) = (f\circ\Phi, \mu_x^0) = (f, \Phi^*(\mu_x^0)),$$

while the right hand side of (14) reads as $(f, p(x)\tilde{\nu}_x^0)$. Now the assertion follows from an easy observation that if $V \subseteq \mathbf{R}^N$ is an open set, $\nu_1,\nu_2 \in \mathcal{M}(V)$ and $(f,\nu_1) = (f,\nu_2)$ for every $f \in C(\overline{V})$ such that $f \equiv 0$ on ∂V then $\nu_1 \equiv \nu_2$. In our case $V = U^0$. □

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