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Strongly Nonlinear Elliptic Unilateral Problems without Sign Condition and L^1 Data

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In this paper, we prove the existence of solutions to unilateral problems involving nonlinear operators of the form

$$Au + H(x, u, \nabla u) = f$$

where A is a Leray Lions operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ and $H(x, u, \nabla u)$ is a nonlinearity which satisfies the following growth condition $|H(x, s, \xi)| \leq \gamma(x)+g(s)|\xi|^p$ with $\gamma \in L^1(\Omega)$ and $g \in L^1(I\!R)$, and without assuming any sign condition on $H(x, s, \xi)$. The right hand side f belongs to $L^1(\Omega)$.

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1. Introduction

The objective of this paper is to study the obstacle problem with L^1 data associated to the nonlinear operator of the form

$$Au + H(x, u, \nabla u) = f \tag{1}$$

in a bounded subset Ω of $I\!R^N$, $N \geq 2$. The principal part A is a differential elliptic operator of the second order in divergence form, acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$

$$Au = -\text{div}a(x, u, \nabla u),$$

and H is a nonlinear lower order term having a growth condition of the form $|H(x, s, \xi)| \leq \gamma(x) + g(s)|\xi|^p$ with $\gamma \in L^1(\Omega), g \in L^1(I\!R)$ and $g \geq 0$. More precisely, this paper deals with the existence of solutions to the following problem

$$(\mathcal{P}) \begin{cases} u \geq \psi \ a.e. \ \text{ in } \ \Omega. \\ T_k(u) \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega H(x, u, \nabla u)T_k(u - v)\, dx \\ \qquad \leq \displaystyle\int_\Omega fT_k(u - v)\, dx, \\ v \in K_\psi \cap L^\infty(\Omega), \ \forall k > 0. \end{cases}$$

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where $f \in L^1(\Omega)$ and $K_\psi = \{u \in W_0^{1,p}(\Omega),\ u \geq \psi \ \ a.e. \text{ in } \Omega)\}$.
Our principal goal in this paper is to prove the existence result for the unilateral problem $(\mathcal{P})$ without assuming any sign condition on H. For that, we prove the strong convergence of truncations $T_k(u_n)$ in $W_0^{1,p}(\Omega)$, where u_n is a solution of the approximate problem.
Recently Porreta has proved in [16] the existence result for the problem (1) in the case of an equation with a measure right hand side. Another result in this direction can be found in [9] where the problem (1) is studied with $f \in L^m(\Omega)$. In this last work, the authors proved that there exists a bounded weak solution for $m > \frac{N}{2}$, and unbounded entropy solution for $\frac{N}{2} > m > \frac{2N}{N+2}$. A different approach (without sign condition) was used in [7], under the assumption $b(x,s,\xi) = \lambda s - |\xi|^2$ with $\lambda > 0$. We recall also that the authors used in [8] the methods of lower and upper-solutions. In this direction, we can refer to [11, 12, 13, 14, 18].
For the case of sign condition, many important works have appeared during these last decades. Namely, [3, 5, 6, 17] for equations and [2, 3] for inequality.
This paper is organized as follows: Section 2 contains the basic assumptions and the statement of result, in Section 3 we prove our main result.

2. Preliminaries and statements of the result

Through this paper Ω will be a bounded subset of $\mathbb{R}^N, N \geq 2$ and $p \in \mathbb{R}$ such that $1 < p < \infty$.
For $k > 0$ and for $s \in \mathbb{R}$, we denote by $T_k(s)$ usual truncation defined by

$$T_k(s) = \begin{cases} k & \text{if } s > k \\ s & \text{if } |s| \leq k \\ -k & \text{if } s < -k \end{cases}$$

and by $\mathcal{T}_0^{1,p}(\Omega)$ the space of the measurable function u is defined on Ω almost everywhere, and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. We recall also that for $0 < q < \infty$ the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ can be defined as the set of measurable function $f : \Omega \longrightarrow \mathbb{R}$ such that the corresponding distribution functions $\Phi_f(k) = meas\{x \in \Omega,\ |f(x)| > k\}$ satisfy an estimate of the form $\Phi_f(k) \leq ck^{-q}$, where c is a positive constant. (For more details we refer to [1]).
Let us consider the nonlinear operator A from $W_0^{1,p}(\Omega)$ into its dual of the form

$$Au = -\text{div} a(x, u, \nabla u) \tag{2}$$

where $a(x,s,\xi)$ is a Carathéodory vector valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ satisfying the following assumptions, for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N, (\xi \neq \eta)$ and for all $s \in \mathbb{R}$:

$$a(x,s,\xi)\xi \geq \alpha|\xi|^p, \tag{3}$$

$$|a(x,s,\xi)| \leq \beta(k(x) + |s|^{p-1} + |\xi|^{p-1}), \tag{4}$$

$$(a(x,s,\xi) - a(x,s,\eta), \xi - \eta) > 0, \tag{5}$$

with α, β are some positive constants and $k(x)$ is a positive function in $L^{p'}(\Omega)$ (p' is the conjugate exponent of p).
Furthermore, we will consider a Carathéodory function $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that, for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)|\xi|^p, \tag{6}$$

where $g : I\!R \to I\!R_+$ is continuous, positive and belongs to $L^1(I\!R)$, while $\gamma(x) \in L^1(\Omega)$. Moreover, assume that

$$f \in L^1(\Omega). \tag{7}$$

Finally let ψ be a measurable function with values in $\overline{I\!R}$ such that

$$\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{8}$$

and let us define

$$K_\psi = \{u \in W_0^{1,p}(\Omega),\ u \geq \psi \ \ a.e. \ \text{ in } \ \Omega)\}. \tag{9}$$

The aim of this paper is to prove the following

Theorem 2.1. *Assume that the assumptions* $(3)-(8)$ *hold. Then, the following problem:*

$$(\mathcal{P}) \begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega) \quad u \geq \psi \ a.e. \quad in \ \ \Omega \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, \\ \forall v \in K_\psi \cap L^\infty(\Omega), \quad and \ \ \forall k > 0, \end{cases}$$

has at least one solution.

Remark 2.2. Let us remark that in the case of $\psi = -\infty$ Theorem 2.1 states the existence of solution in the case of equation i.e. the following problem

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega). \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \text{and} \ \ \forall k > 0, \end{cases}$$

has at least one solution.

3. Proof of Theorem 2.1

3.1. Approximate problem

In order to prove the Theorem 2.1, let us consider the sequence of approximate problem

$$(\mathcal{P}_n) \begin{cases} u_n \in K_\psi \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n-v)\,dx + \int_\Omega H_n(x,u_n,\nabla u_n)(u_n-v)\,dx \\ \qquad \leq \displaystyle\int_\Omega f_n(u_n-v)\,dx, \\ \forall v \in K_\psi, \end{cases}$$

where f_n are regular functions such that $f_n \in L^\infty(\Omega)$ and strongly converge to f in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ and where

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1 + \frac{1}{n}|H(x,s,\xi)|}$$

Note that $|H_n(x,s,\xi)| \leq |H(x,s,\xi)|$ and $|H_n(x,s,\xi)| \leq n$, then for fixed $n \in I\!N$ the approximate problem $(\mathcal{P}_n)$ has at least one solution ([15]).

3.2. A priori estimate

Let $v = u_n - \eta \exp(G(u_n))T_k(u_n^+ - \psi^+)$, where $G(s) = \int_0^s \frac{g(t)}{\alpha}\, dt$ (the function g appears in (6)) and $\eta \geq 0$. Since $v \in W_0^{1,p}(\Omega)$ and for η small enough, we have $v \geq \psi$, thus v is admissible test function in $(\mathcal{P}_n)$, then

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla(\eta\exp(G(u_n))T_k(u_n^+ - \psi^+))\, dx \\ &\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)(\eta\exp(G(u_n))T_k(u_n^+ - \psi^+))\, dx \\ \leq\ & \int_\Omega f_n(\eta\exp(G(u_n))T_k(u_n^+ - \psi^+))\, dx \end{aligned}$$

which implies that

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla(\exp(G(u_n))T_k(u_n^+ - \psi^+))\, dx \\ &\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx \\ \leq\ & \int_\Omega f_n\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx. \end{aligned}$$

Then

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))T_k(u_n^+ - \psi^+))\, dx \\ &\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^+ - \psi^+)\exp(G(u_n))\, dx \\ \leq\ & -\int_\Omega H_n(x,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx + \int_\Omega f_n\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx \\ \leq\ & \int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx + \int_\Omega g(u_n)|\nabla u_n|^p\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx \\ &\qquad + \int_\Omega f_n\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx, \end{aligned}$$

in view of (3), we obtain

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^+ - \psi^+)\exp(G(u_n))\, dx \\ \leq\ & \int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx + \int_\Omega f_n\exp(G(u_n))T_k(u_n^+ - \psi^+)\, dx \leq c_1 k \end{aligned}$$

where c_1 is a positive constant not depending on n.
Consequently, we have.

$$\begin{aligned}&\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n^+ \exp(G(u_n))\,dx\\ \le\ &\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla \psi^+ \exp(G(u_n))\,dx + c_1 k.\end{aligned}$$

Thanks to (3) and Young's inequality, we deduce

$$\int_{\{|u_n^+-\psi^+|\le k\}} |\nabla u_n^+|^p\,dx \le c_2 k. \tag{10}$$

Since $\{x\in\Omega,\ |u_n^+|\le k\}\subset\{x\in\Omega,\ |u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$, hence

$$\int_\Omega |\nabla T_k(u_n^+)|^p\,dx = \int_{\{|u_n^+|\le k\}} |\nabla u_n^+|^p\,dx \le \int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} |\nabla u_n^+|^p\,dx.$$

Moreover, (10) implies that,

$$\int_\Omega |\nabla T_k(u_n^+)|^p\,dx \le c_3 k \quad \forall k>0 \tag{11}$$

where c_3 is a positive constant.
On the other hand, taking $v = u_n + \exp(-G(u_n))T_k(u_n^-)$ as test function in $(\mathcal{P}_n)$, we obtain

$$\begin{aligned}&-\int_\Omega a(x,u_n,\nabla u_n)\nabla(\eta \exp(-G(u_n))T_k u_n^-))\,dx\\ &\qquad -\int_\Omega H_n(x,u_n,\nabla u_n)(\eta\exp(-G(u_n))T_k(u_n^-))\,dx\\ \le\ & -\int_\Omega f_n(\eta\exp(-G(u_n))T_k(u_n^-))\,dx.\end{aligned}$$

Using (6), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(-G(u_n))T_k(u_n^-)\,dx\\ &\qquad -\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))\,dx\\ \le\ &\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n^-)\,dx + \int_\Omega g(u_n)|\nabla u_n|^p\exp(-G(u_n))T_k(u_n^-)\,dx\\ &\qquad -\int_\Omega f_n\exp(-G(u_n))T_k(u_n^-)\,dx.\end{aligned} \tag{12}$$

In virtue of (3) and since $\gamma, k\in L^1(\Omega)$, and $\|f_n\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)}$, we have:

$$\begin{aligned}&-\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))\,dx\\ =\ &\int_{\{u_n\le 0\}} a(x,u_n,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))\,dx \le c_3 k\end{aligned}$$

by using again (3), we deduce that

$$\int_{\{u_n\le 0\}} |\nabla T_k(u_n)|^p \, dx \le c_3 k$$

i.e.,

$$\int_\Omega |\nabla T_k(u_n^-)|^p \, dx \le c_4 k \tag{13}$$

where c_4 is a positive constant.
Combining (11) and (13), we conclude

$$\int_\Omega |\nabla T_k(u_n)|^p \, dx \le ck \tag{14}$$

with a positive constant c.

3.3. Strong convergence of truncation

In view of (14), we can apply Lemmas 4.1 and 4.2, in [1], which imply that (u_n) is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N(p-1)}{N-p}}$ and $(|\nabla u_n|)_n$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N(p-1)}{N-1}}$. Moreover, reasoning as in the proof of Theorem 6.1 in [1], we conclude that, there exists a function u and a subsequence, still denoted by $(u_n)_n$, such that:

$$\begin{array}{l} u_n \longrightarrow u \quad \text{a.e. in} \quad \Omega \\ T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \quad \text{and a.e. in} \quad \Omega \quad \text{for every} \quad k>0. \end{array} \tag{15}$$

We will use the following function of one real variable, which is defined as follow:

$$\begin{cases} h_j(s) = 1 & \text{if } |s| \le j \\ h_j(s) = 0 & \text{if } |s| \ge j+1 \\ h_j(s) = j+1-s & \text{if } j \le s \le j+1 \\ h_j(s) = s+j+1 & \text{if } -j-1 \le s \le -j \end{cases} \tag{16}$$

where j is a nonnegative real parameter.
In order to prove the strong convergence of truncation $T_k(u_n)$, we first proove the following assertions:

Assertion (i)

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_{\{j\le |u_n| \le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \, dx = 0. \tag{17}$$

Assertion (ii)

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n)) \, dx = 0. \tag{18}$$

Assertion (iii)

$$\begin{aligned} \lim_{j\to\infty} \lim_{n\to\infty} \int_\Omega &(a(x,T_k(u_n),\nabla T_k(u_n)) \\ &-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n) \, dx = 0. \end{aligned} \tag{19}$$

Assertion (iv)

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } \ W_0^{1,p}(\Omega) \text{ as } n \to +\infty. \tag{20}$$

Proof of assertion (i). Consider the following function $v = u_n - \eta \exp(G(u_n))T_1(u_n - T_j(u_n))^+$.
For j large enough and η small enough, we can deduce that $v \geq \psi$, and since $v \in W_0^{1,p}(\Omega)$, v is a test function in $(\mathcal{P}_n)$. Then, we obtain,

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla(\exp(G(u_n)T_1(u_n - T_j(u_n))^+)\,dx \\ &\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)\exp(G(u_n)T_1(u_n - T_j(u_n))^+\,dx \\ \leq\ &\int_\Omega f_n \exp(G(u_n)T_1(u_n - T_j(u_n))^+\,dx. \end{aligned}$$

From the growth condition (6), we have,

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))T_1(u_n - T_j(u_n))^+\,dx \\ &\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n - T_j(u_n))^+ \exp(G(u_n))\,dx \\ \leq\ &\int_\Omega \gamma(x)\exp(G(u_n))T_1(u_n - T_j(u_n))^+\,dx \\ &\qquad + \int_\Omega g(u_n)|\nabla u_n|^p \exp(G(u_n))T_1(u_n - T_j(u_n))^+\,dx \\ &\qquad + \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_j(u_n))^+\,dx \end{aligned}$$

which, thanks to (3), gives:

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n - T_j(u_n))^+ \exp(G(u_n))\,dx \\ \leq\ &\int_\Omega \gamma(x)\exp(G(u_n))T_1(u_n - T_j(u_n))^+\,dx + \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_j(u_n))^+\,dx, \end{aligned} \tag{21}$$

by Lebesgue's theorem the right hand side goes to zero as n and j tend to infinity. Therefore, passing to the limit first in n, then in j, we obtain from (21)

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\leq u_n \leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{22}$$

On the other hand, consider the test function $v = u_n + \exp(-G(u_n))T_1(u_n - T_j(u_n))^-$ in $(\mathcal{P}_n)$ is clearly admissible, then

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla(-\exp(-G(u_n))T_1(u_n - T_j(u_n))^-)\,dx \\ &\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)(-\exp(-G(u_n))T_1(u_n - T_j(u_n))^-)\,dx \\ \leq\ &\int_\Omega f_n(-\exp(-G(u_n))T_1(u_n - T_j(u_n))^-)\,dx \end{aligned}$$

which implies that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx\\
&\qquad -\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^-\exp(-G(u_n))\,dx\\
&\qquad +\int_\Omega H_n(x,u_n,\nabla u_n)\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx\\
\leq\ &\int_\Omega f_n\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx.
\end{aligned}$$

From (3) and (6), it is possible to conclude that

$$\begin{aligned}
&-\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^-\exp(-G(u_n))\,dx\\
\leq\ &\int_\Omega \gamma(x)\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx \qquad (23)\\
&\qquad -\int_\Omega f_n\exp(-G(u_n))T_1(u_n-T_j(u_n))^-\,dx
\end{aligned}$$

the second term in the right hand side can be neglected since it is nonnegative, and by Lebesgue's theorem the first term goes to zero as n and j tend to infinity. Then (23) becomes

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{-j-1\leq u_n\leq -j\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx=0. \qquad (24)$$

Finally, (17) follows from (22) and (24).

Proof of assertion (ii). Let $v=u_n+\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))$ (h_j is defined in (16)), v is a test function in $(\mathcal{P})$. Then we have,

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n)))\,dx\\
&\qquad +\int_\Omega H_n(x,u_n,\nabla u_n)(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n)))\,dx\\
\leq\ &\int_\Omega f_n(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n)))\,dx
\end{aligned}$$

by using (6), we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx\\
&\qquad -\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)^-\exp(-G(u_n))(1-h_j(u_n))\,dx\\
&\qquad +\int_\Omega a(x,u_n,\nabla u_n)\nabla h_j(u_n)\exp(-G(u_n))T_k(u_n)^-\,dx\\
&\qquad -\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx\\
&\qquad -\int_\Omega \exp(-G(u_n))g(u_n)|\nabla u_n|^pT_k(u_n)^-(1-h_j(u_n))\,dx\\
\leq\ &-\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx
\end{aligned}$$

Thanks to (3), we can deduce that

$$\begin{aligned} & -\int_{\{u_n\le 0\}} a(x,u_n,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))(1-h_j(u_n))\,dx \\ & \qquad -\int_{\{-j-1\le u_n\le -j\}} a(x,u_n,\nabla u_n)\nabla u_n\exp(-G(u_n))T_k(u_n)^-\,dx \\ & \qquad +\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx \\ \ge & \int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx. \end{aligned} \tag{25}$$

In view of (13), the second integral tends to zero as n and j go to infinity. And by Lebesgue's theorem, it is possible to conclude that the third and the fourth integral converge to zero as n and j go to infinity. Then (15) implies that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{u_n\le 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx = 0. \tag{26}$$

On the other hand, take $v = u_n - \eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))$. This is a test function admissible in $(\mathcal{P}_n)$. Then, we have

$$\begin{aligned} & \int_\Omega a(x,u_n,\nabla u_n)\nabla(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n)))\,dx \\ & \qquad +\int_\Omega H_n(x,u_n,\nabla u_n)(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n)))\,dx \\ \le & \int_\Omega f_n(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n)))\,dx \end{aligned}$$

using (6) this implies

$$\begin{aligned} & \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\,dx \\ & \qquad +\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^+-\psi^+)\exp(G(u_n))(1-h_j(u_n))\,dx \\ & \qquad -\int_{\{j\le u_n\le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\exp(G(u_n))T_k(u_n^+-\psi^+)\,dx \\ \le & \int_\Omega g(u_n)|\nabla u_n|^p\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\,dx \\ & \qquad +\int_\Omega f_n\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\,dx \\ & \qquad +\int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\,dx, \end{aligned}$$

and by using (3), we get

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^+-\psi^+)\exp(G(u_n))((1-h_j(u_n))\,dx \\
\leq &\int_{\{j\leq u_n\leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))T_k(u_n^+-\psi^+)\,dx \\
&\quad+\int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\,dx \\
&\quad+\int_\Omega f_n\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\,dx \\
= &\ \varepsilon_1(j,n).
\end{aligned} \tag{27}$$

In virtue of (17) and Lebesgue's theorem, we can conclude that $\varepsilon_1(j,n)$ converges to zero as n and j go to infinity.
From (27), we have

$$\begin{aligned}
&\int_{\{|u_n^+-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n^+\exp(G(u_n))(1-h_j(u_n))\,dx \\
\leq &\int_{\{|u_n^+-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla \psi^+\exp(G(u_n))(1-h_j(u_n))\,dx+\varepsilon_1(j,n).
\end{aligned} \tag{28}$$

Thanks to the growth condition (4) and Young's inequality, it is possible to conclude that

$$\int_{\{|u_n^+-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n^+\exp(G(u_n))(1-h_j(u_n))\,dx\leq \varepsilon_2(j,n)$$

where $\varepsilon_2(j,n)$ tends to 0 as n and j go to infinity.
Since $\exp(G(u_n))$ is bounded, we obtain

$$\int_{\{|u_n^+-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n^+\exp(G(u_n))(1-h_j(u_n))\,dx\leq \varepsilon_3(j,n). \tag{29}$$

Since $\{x\in\Omega,\ |u_n^+|\leq h\}\subset\{x\in\Omega,\ |u_n^+-\psi^+|\leq h+\|\psi^+\|_\infty\}$, hence,

$$\begin{aligned}
&\int_{\{|u_n^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n(1-h_j(u_n))\,dx \\
\leq &\int_{\{|u_n^+-\psi^+|\leq k+\|\psi^+\|_\infty\}} a(x,u_n,\nabla u_n)\nabla u_n(1-h_j(u_n))\,dx\leq\varepsilon_3(j,n)
\end{aligned}$$

which yields for all $k>0$

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{u_n\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx=0. \tag{30}$$

Using (26) and (30), we conclude (18).

Proof of assertion (iii). On one hand, let $v=u_n-\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)$ with h_j is defined in (16) and η small enough such that $v\in K_\psi$, then, we take v as test function in $(\mathcal{P}_n)$, we obtain

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla(\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n))\,dx \\ &\qquad +\int_\Omega H_n(x,u_n,\nabla u_n)(\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n))\,dx \\ \leq\ &\int_\Omega f_n(\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n))\,dx, \end{aligned}$$

similarly, using (3) and (6), we obtain

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+\exp(G(u_n))h_j(u_n)\,dx \\ &\qquad +\int_{\{j\leq|u_n|\leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\exp(G(u_n))(T_k(u_n)-T_k(u))^+\,dx \\ \leq\ &\int_\Omega \gamma(x)\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n))\,dx \\ &\qquad +\int_\Omega f_n\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n))\,dx \end{aligned}$$

i.e.,

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\exp(G(u_n))h_j(u_n)\,dx \leq \varepsilon_4(j,n) \qquad (31)$$

applying again (17) and Lebesgue's theorem, we deduce that $\varepsilon_4(j,n)$ goes to zero as n and j tend to infinity. Moreover, (31) becomes

$$\begin{aligned} &\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))\exp(G(u_n))h_j(u_n)\,dx \\ &\qquad +\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\exp(G(u_n))h_j(u_n)\,dx \\ \leq\ &\varepsilon_4(j,n). \end{aligned}$$

Since $h_j(u_n)=0$ if $|u_n|>j+1$, we obtain

$$\begin{aligned} &\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\exp(G(u_n))h_j(u_n)\,dx \\ =\ &\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))\nabla T_k(u)\exp(G(u_n))h_j(u_n)\,dx \\ \leq\ &\varepsilon_5(j,n). \end{aligned}$$

Which gives,

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))\exp(G(u_n))h_j(u_n)\,dx \leq \varepsilon_6(j,n)$$

where

$$\varepsilon_6(j,n)=c(\int_{\{|u_n|>k\}} |a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))||\nabla T_k(u)|\exp(G(u_n))h_j(u_n)\,dx+\varepsilon_5(j,n))$$

which goes to zero as n and j tend to infinity.
Consequently

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}(a(x,T_k(u_n),\nabla T_k(u_n)) \\ -a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n)\,dx=0. \tag{32}$$

On the other hand, taking $v=u_n+\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n)$ as test function in $(\mathcal{P}_n)$ and reasoning as in (32) it is possible to conclude that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\leq 0\}}(a(x,T_k(u_n),\nabla T_k(u_n)) \\ -a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n)\,dx=0. \tag{33}$$

Combining (32) and (33), we deduce (19).

Proof of assertion (iv). First we have

$$\begin{aligned} &\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\,dx \\ =\ &\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n)\,dx \\ &\quad+\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n)) \\ &\qquad-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))(1-h_j(u_n))\,dx. \end{aligned}$$

Thanks to (19) the first integral of the right hand side converges to zero as n and j tend to infinity. For the second term, we have

$$\begin{aligned} &\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n)) \\ &\qquad-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))(1-h_j(u_n))\,dx \\ =\ &\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx \\ &\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)(1-h_j(u_n))\,dx \\ &\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))(1-h_j(u_n))\,dx. \end{aligned}$$

By (18) the first integral of the right hand side goes to zero as $n,j\longrightarrow+\infty$, and since $(a(x,T_k(u_n),\nabla T_k(u_n)))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega)$ uniformly on n while $\nabla T_k(u)(1-h_j(u_n))$ converges to zero. Hence, the second integral converges to zero. For the third integral, it converges to zero because $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $\prod_{i=1}^N L^p(\Omega)$.
Finally, we conclude that

$$\lim_{n\to+\infty}\int_\Omega(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\,dx=0.$$

Then Lemma 3.1, of [8], implies that

$$T_k(u_n)\longrightarrow T_k(u)\quad\text{strongly in}\quad W_0^{1,p}(\Omega). \tag{34}$$

3.4. Passing to the limit

Thanks to (34), we obtain for a subsequence

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in} \quad \Omega.$$

Now, we show that:

$$H_n(x, u_n, \nabla u_n) \longrightarrow H(x, u, \nabla u) \ \text{ strongly in } \ L^1(\Omega). \tag{35}$$

On the one hand, let $v = u_n + \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds$. Since $v \in W_0^{1,p}(\Omega)$ and $v \geq \psi$, v is an admissible test function in $(\mathcal{P}_n)$. Then,

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n)\nabla(-\exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds)\, dx \\
&\qquad + \int_\Omega H_n(x, u_n, \nabla u_n)(-\exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds)\, dx \\
\leq\ & \int_\Omega f_n(-\exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds)\, dx.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx \\
&\qquad + \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \exp(-G(u_n))g(u_n)\chi_{\{u_n<-h\}}\, dx \\
\leq\ & \int_\Omega \gamma(x)\exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx \\
&\qquad + \int_\Omega g(u_n)|\nabla u_n|^p \exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx \\
&\qquad - \int_\Omega f_n \exp(-G(u_n))\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx
\end{aligned}$$

using (3) and since $\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds \leq \int_{-\infty}^{-h} g(s)\, ds$, we get

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \exp(-G(u_n))g(u_n)\chi_{\{u_n<-h\}}\, dx \\
\leq\ & \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})\int_{-\infty}^{-h} g(s)\, ds(\|\gamma\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}) \\
\leq\ & \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})\int_{-\infty}^{-h} g(s)\, ds(\|\gamma\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)})
\end{aligned}$$

using again (3), we obtain

$$\int_{\{u_n<-h\}} g(u_n)|\nabla u_n|^p\, dx \leq c\int_{-\infty}^{-h} g(s)\, ds$$

and since $g \in L^1(I\!R)$, we deduce that

$$\lim_{h\to+\infty} \sup_{n\in I\!N} \int_{\{u_n<-h\}} g(u_n)|\nabla u_n|^p \, dx = 0. \tag{36}$$

On the other hand, let $M = \exp(-G(u_n) \int_0^{+\infty} g(s) \, ds$ and $h \geq M + \|\psi^+\|_{L^\infty(\Omega)}$. Consider $v = u_n - \exp(G(u_n)) \int_0^{u_n} g(s)\chi_{\{s>h\}} \, ds$. Since $v \in W_0^{1,p}(\Omega)$ and $v \geq \psi$, v is an admissible test function in $(\mathcal{P}_n)$. Then, similarly to (36), we deduce that

$$\lim_{h\to+\infty} \sup_{n\in I\!N} \int_{\{u_n>h\}} g(u_n)|\nabla u_n|^p \, dx = 0. \tag{37}$$

Combining (34), (36), (35) and Vitali's Theorem, we conclude (28).
On the other hand, let $\varphi \in K_\psi \cap L^\infty(\Omega)$ and take $v = u_n - T_k(u_n - \varphi)$ as a test function in (P_n). We get,

$$\begin{cases} u_n \in K_\psi \quad \forall k > 0. \\ \displaystyle\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi) \, dx + \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - \varphi) \, dx \\ \qquad \displaystyle\leq \int_\Omega f_n T_k(u_n - \varphi) \, dx, \\ \forall \varphi \in K_\psi \cap L^\infty(\Omega). \end{cases} \tag{38}$$

Finally, from (34) and (35), we can pass to the limit in (38). This completes the proof of Theorem 2.1.

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