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Convex Bodies with Sheafs of Elliptic Sections*

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A convex body in E^d is an ellipsoid if all the sections given by hyperplanes are elliptic. We study whether we can restrict the hyperplanes to those that contain one of two fixed linear varieties.

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1. Introduction

Let D be a convex body (i.e., a compact convex set with non-empty interior) in the d-dimensional Euclidean space E^d ($d \geq 3$) and let S be the boundary of D. It is well known that S is an ellipsoid if and only if the section of S given by any hyperplane is ellipsoidal. The question of whether it is actually necessary to consider "any" hyperplane to characterize S as ellipsoid or it is enough to consider "some" hyperplanes is at the origin of an important family of characterizations of ellipsoids. To recall some of them and to state the new characterization to be given in Theorem 2.1, we shall consider the following definition.

Let l be a linear variety with $0 \leq \dim l \leq d-2$. We say that D (indistinctly, S) is *elliptic through* l if for every hyperplane P such that $l \subset P$ and $P \cap \operatorname{Int} D \neq \emptyset$ the section $P \cap S$ is ellipsoidal.

If D is elliptic through some point p, then S is an ellipsoid. This result was proved by Busemann [7] for $p \in \operatorname{Int} D$ and extended by Burton [5] to $p \in E^d$. Petty [11] pointed out that it is also true for p in the projective space P^d. Recently, the authors [3] proved that if S is centrally symmetric and there exist three hyperplanes P_i, $i = 1, 2, 3$, such that the sections of S by hyperplanes parallel to any P_i are elliptic then S is an ellipsoid. We can complete E^d to P^d by adding the hyperplane at infinity, and then express the last result by saying that if S is centrally symmetric and there exist in the hyperplane at infinity three $(d-2)$-dimensional linear varieties, l_i, $i = 1, 2, 3$, such that S is elliptic through any l_i, then S is an ellipsoid. If either S is not centrally symmetric or only two linear varieties

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are considered, the result is false. Theorem 2.1 will show that only two linear varieties are necessary if they are in E^d and some of them cuts the interior of D.

Results similar to the above, but related to the central symmetry or homotheticity of the sections can be found in [1], [2], [4], [6], [8]–[11].

2. Results

Theorem 2.1. *Let S be the boundary of a convex body $D \subset E^d$ $(d \geq 3)$ and let l_1 and l_2 be two linear varieties such that $1 \leq \dim l_i \leq d-2$, $i = 1, 2$, $l_1 \not\subset l_2$, $l_2 \not\subset l_1$ and $l_1 \cap \operatorname{Int} D \neq \emptyset$. If S is elliptic through l_1 and l_2, then S is an ellipsoid.*

The convex bodies in Example 2.2 show that in Theorem 2.1 it is necessary that one of the varieties cuts the interior of D. Moreover, it is interesting to note that B and C are strong counterexamples in the sense that no point in their boundaries has a neighborhood contained in a quadric. On the other hand, Example 2.3 shows that neither is one variety alone sufficient to characterize ellipsoids, even if it cuts the interior of D. In the convex body E not only the sections through l are elliptic but also those that cut the spherical region.

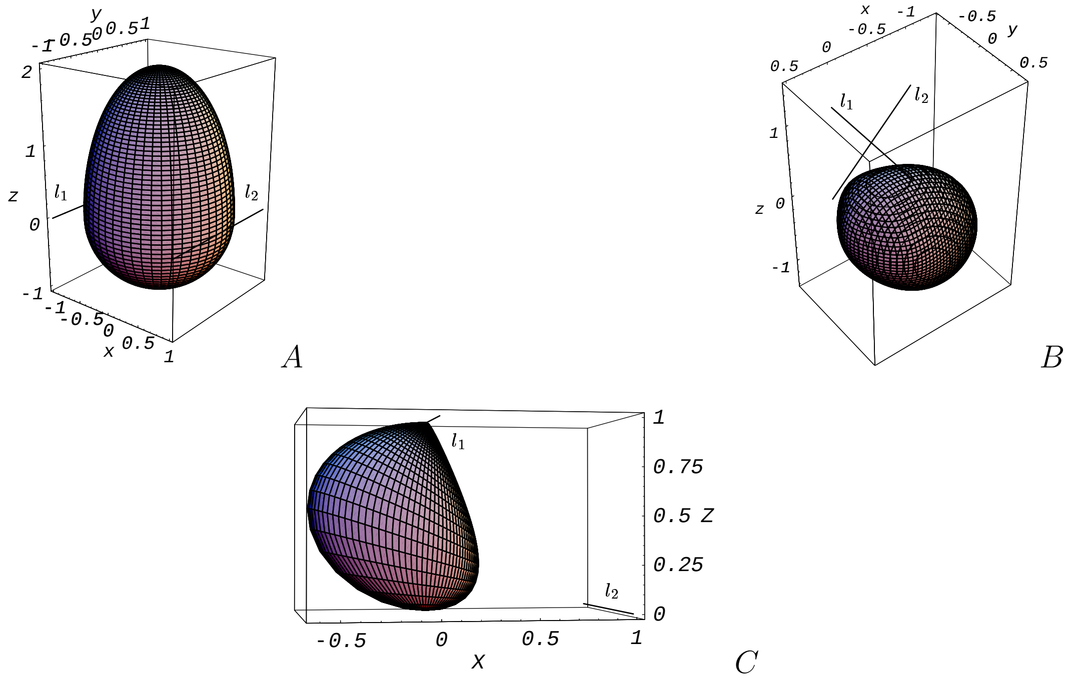


Figure 2.1: Convex bodies elliptic through l_1 and l_2.

Example 2.2. The sets

$$\begin{aligned}
A &= \left\{(x,y,z) \in \mathbb{R}^3 : \; x^2+y^2+z^2 \leq 1 \text{ if } z \leq 0; \; x^2+y^2+\frac{z^2}{4} \text{ if } z \geq 0\right\}, \\
B &= \big\{(x,y,z) \in \mathbb{R}^3 : \; 4(x^2+y^2+z^2)(x+z-1)^2 - (x+z-1)^4 \\
&\qquad + 16x^2y^2 \leq 0, \; x+z-1 \neq 0\big\}, \\
C &= \big\{(x,y,z) \in \mathbb{R}^3 : \; x^2(x-1) + y^2(x+z-1) + z(x+z-1)^2 \geq 0, \\
&\qquad z < 1, \; 2x+z < 1\big\} \cup \{(0,0,1)\}
\end{aligned}$$

are convex bodies elliptic through two lines. None of them is an ellipsoid (see Figure 2.1).

Example 2.3. The sets

$$D = \{(x, y, z) \in \mathbb{R}^3 : (x + z)^2 + y^2 \leq 1,\ (x - z)^2 + y^2 \leq 1\},$$
$$E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1,\ (x + z)^2 + y^2 \leq 1\}$$

are convex bodies elliptic through the line $l \equiv \{x = 0, z = 0\}$. None of them is an ellipsoid (see Figure 2.2).

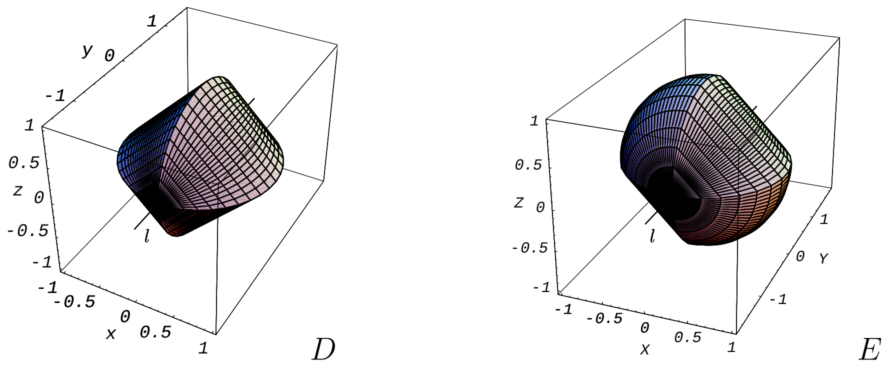


Figure 2.2: Convex bodies elliptic through l.

In the proof of Theorem 2.1 we shall use the technical Lemma 2.4. We shall also use Proposition 2.5, stated in Nakagawa [10] without proof.

Lemma 2.4. *Let P_1, P_2, P_3 be three planes in E^3 that cut each other in parallel lines*

$$r_{12} = P_1 \cap P_2, \quad r_{13} = P_1 \cap P_3, \quad r_{23} = P_2 \cap P_3.$$

Let $E_i \subset P_i$, $i = 1, 2, 3$, be three ellipses such that

(i) *E_1 and E_2 are centred at r_{12} and $E_1 \cap r_{12} = E_2 \cap r_{12}$,*
(ii) *E_1 and E_3 meet in two points of r_{13},*
(iii) *r_{23} is tangent to E_2 and E_3 at the same point.*

Then there exists a unique real quadric C containing the three ellipses.

Proposition 2.5. *Let S be the boundary of a convex body D in E^3. Let l and P be, respectively, a line and a plane that meet in an interior point of D and $l \not\subset P$. If $P \cap S$ is an ellipse and S is elliptic through l, then S is an ellipsoid.*

3. Proofs and remarks

Proof of Lemma 2.4. Let us consider a coordinate system $\{o, e_1, e_2, e_3\}$, where the origin o is the centre of E_1 and E_2, $e_1 \in E_1 \cap r_{12}$, $e_2 \in E_2 \cap r_{23} = E_3 \cap r_{23}$, and $e_3 \in E_1$ has the conjugate direction of e_1 with respect to E_1. Let $p \in E_3$ be the symmetric point of e_2 with respect to the centre of E_3. Then p has the coordinates $(0, y_0, z_0)$ with $y_0 < 0$ and $z_0 \neq 0$. It is easily checked that the surface defined by

$$x^2 + y^2 + z^2 + \Big(\frac{1 - y_0^2 - z_0^2}{y_0 z_0}\Big) yz = 1$$

is the only quadric that contains the three ellipses. $\square$

Proof of Proposition 2.5. Let u and u' be the points where l cuts S. Since S is elliptic through l, it has a unique supporting plane at the points u and u', that we denote by H_u and $H_{u'}$, respectively. Let $p \in l \cap P$ and let p' be the fourth harmonic of p with respect to u and u'. Considering (if necessary) a projectivity that transforms the plane defined by the point p' and the line $H_u \cap H_{u'}$ (at infinity, if H_u and $H_{u'}$ are parallel) into the plane at infinity, we can assume that H_u and $H_{u'}$ are parallel and that S is symmetric about p.

Let E denote the ellipse $P \cap S$. Let H_p be the plane through p parallel to H_u and let $w \in H_p \cap E$. Let E_{uw} be the ellipse where the plane defined by l and w cuts S. Then, the lines pu and pw have conjugate directions with respect to E_{uw}. Let $v \in E$ be such that pv and pw have conjugate directions with respect to E. Let E_{uv} be the ellipse defined in S by the plane generated by l and v. On account of how the ellipses E, E_{uv} and E_{uw} were defined it follows that there exists a unique quadric C that contains them. Let x be any point of S that is not in $E \cup E_{uv} \cup E_{uw}$ and let Q be the plane defined by l and x. The ellipse $Q \cap S$ and the conic $C \cap S$ have four points in common (i.e., u, u', and $Q \cap E$) and the same tangent at u and u'. Therefore, $Q \cap S = C \cap S$ and then $x \in C$. We thus get $S \subset C$, which clearly forces $S = C$, and consequently S is an ellipsoid. □

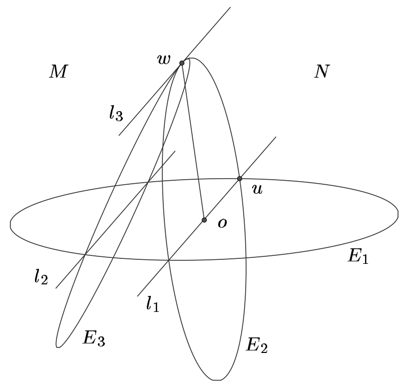
Proof of Theorem 2.1. ($d = 3$) The proof follows from Proposition 2.5 if either the lines l_1 and l_2 meet in an interior point of D or they are not coplanar. On the contrary, assume that l_1 and l_2 are coplanar and meet in an exterior point of D. Applying a projective transformation if necessary, we can take l_1 and l_2 to be parallel. If the tangent planes at the points where l_1 cuts S are parallel then S is centrally symmetric. If those planes cut each other in a line m then we can apply a similar argument to that used in Proposition 2.5 by sending m to infinity and retaining there the point where the parallel lines l_1 and l_2 meet. In sum, we can consider that l_1 and l_2 are parallel and that S is symmetric about a point $o \in l_1$. Nevertheless, we must observe that if l_2 cuts m (which is only possible if l_2 does not touch D) then we send all the line l_2 to infinity. In that case the planes of the sheaf defined by l_2 become parallel planes, although this will not affect the rest of the proof.

We shall denote the sheaf of planes of axis the line l by the abbreviation *l-sheaf*.

Let us now set o as the origin of the space. We proceed by considering two cases.

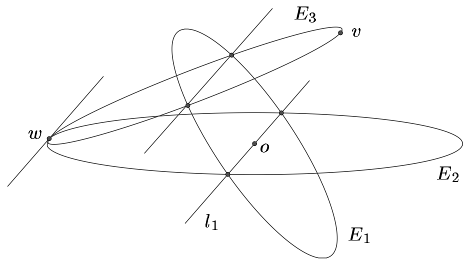
Case 1. Assume that $l_2 \cap \operatorname{Int} D \neq \emptyset$. Let E_1 be the ellipse that defines in S the plane that contains l_1 and l_2, and let E_2 be the ellipse defined by another plane of the l_1-sheaf. Let $u \in l_1 \cap E_1$ and let $w \in E_2$ with the conjugate direction of u with respect to E_2. Let l_3 be the line parallel to l_1 that goes through w. Hence l_3 is tangent to E_2 at w. Let E_3 be the ellipse determined by the plane of the l_2-sheaf that contains l_3. Clearly l_3 is also tangent to E_3 at w. From Lemma 2.4 it follows that there exists a unique quadric C that contains the three ellipses.

The plane of E_2 divides the space into two semispaces. Let M be the semispace that contains l_2 and let N be the other. Let $x \in S \cap N$. The plane defined by l_2 and x cuts S in an ellipse E and cuts C in a conic E'. The two sections share two points in E_2 and two points in E_3. I.e., $E \cap E_2 = E' \cap E_2$ and $E \cap E_3 = E' \cap E_3 = l_2 \cap E_3 = E_1 \cap E_3$. At each of the two points of $E_1 \cap E_3$ the planes tangent to S and C coincide. Hence the tangents to E and E' at these points also coincide. Therefore $E = E'$ and then $x \in C$. Let now $x \in S \cap M$. The plane defined by l_1 and x cuts S in an ellipse. But half of this



Case 1

ellipse is in N and hence the entire ellipse is in C. We thus get $S \subset C$, and necessarily S is an ellipsoid.



Case 2

Case 2. Assume that $l_2 \cap \operatorname{Int} D = \emptyset$. Recall that in this case the planes of the l_2-sheaf can be parallel. Let P_1 be a plane that contains l_1 but does not contain l_2, and let E_1 be the ellipse $P_1 \cap S$. Let P_3 be a plane of the l_2-sheaf that does not contain l_1 and that cuts E_1 in two points, and let $E_3 = P_3 \cap S$. Let v and w be the points of the ellipse E_3 where the tangent is parallel to l_1. Let P_2 be the plane of the l_1-sheaf that goes through w, and let $E_2 = P_2 \cap S$. Again from Lemma 2.4 it follows that there exists a unique quadric C that contains the three ellipses.

Now let P_4 be the plane of the l_1-sheaf that goes through v. The planes P_2 and P_4 divide the space into four quadrants that we number so that E_3 lies in the first quadrant and the third quadrant is opposite the first.

Let P be a plane of the l_1-sheaf and let $E = P \cap S$ and $E' = P \cap C$. If $P = P_2$, then $E = E' = E_2$. If P intersects the first and third quadrants, then it meets E_3 in two points and hence the ellipse E and the conic E' share these points and also those on $E_1 \cap E_2$ where they also share the tangents. Therefore, $E = E'$. We get the same result if $P = P_4$ because E and E' have the same tangent at v. Assume now that P intersects the second and fourth quadrants and is not in the l_2-sheaf. Let $l \subset P$ be a line parallel to l_1 that cuts S, and let P' be the plane of the l_2-sheaf that contains l. We can take l close enough to l_1 so that P' cuts S at points of either the first or the third quadrants. Since S and C coincide in those quadrants it follows that the entire ellipse $P' \cap S$ coincides with the conic $P' \cap C$. Therefore, E and E' share the two points at $l \cap S$ and the two points (with same tangents) at $E_1 \cap E_2$. Hence $E = E'$.

We have thus proved that $P \cap S = P \cap C$ for every plane P of the l_1-sheaf except at most for the one that also belongs to the l_2-sheaf. By continuity we get that $S = C$ and consequently S is an ellipsoid.

($d \geq 4$) We can assume that $\dim l_1 = \dim l_2 = d-2$, as otherwise we can consider $(d-2)$-dimensional linear varieties that contain l_1 and l_2, respectively. Having proved that the result is true for $d = 3$, we can proceed by induction on d assuming that it is true for dimension $d-1$. Throughout the proof the formula that relates the dimensions of the sum and intersection of two linear varieties must be borne in mind.

Let $p \in l_1 \cap \operatorname{Int} D$ be such that $p \notin l_2$. Our aim is to see that $E \cap S$ is elliptic for "almost every" hyperplane E that contains p. To this end, let $q \in l_2 \setminus l_1$ and let l be the line defined by p and q. We begin by showing that S is elliptic through l. Thus, let P be a hyperplane in E^d such that $l \subset P$. If $l_i \subset P$ for $i = 1$ or 2, then $P \cap S$ is ellipsoidal by hypothesis. On the other hand, assume that $l_i \not\subset P$ for $i = 1$, 2. Then $\dim P \cap l_i = d-3$, because $P \cap l_i \neq \emptyset$. Since $p \in P \cap l_1$ and $p \notin l_2 \cap P$, we have $P \cap l_1 \neq P \cap l_2$. Let $i = 1$ or 2, and let $\tilde{P} \subset P$ be a $(d-2)$-dimensional linear variety such that $P \cap l_i \subset \tilde{P}$ and $\tilde{P} \cap \operatorname{Int}_P(P \cap D) \neq \emptyset$. Since $\tilde{P} \cap l_i = P \cap l_i$ we have that $\dim \tilde{P} + l_i = d-1$. Thus $\tilde{P} + l_i$ is a hyperplane that contains l_i and $(\tilde{P} + l_i) \cap \operatorname{Int} D \neq \emptyset$. By hypothesis $(\tilde{P} + l_i) \cap S$ is ellipsoidal and, consequently $\tilde{P} \cap S$ is ellipsoidal. I.e., $P \cap D$ is elliptic through $P \cap l_i$. Applying the hypothesis of induction we conclude that $P \cap S$ is ellipsoidal.

Now, let l_2' be a $(d-2)$-linear variety that contains l. Since S is elliptic through l, it is also elliptic through l_2'. Let E be an hyperplane that contains p. If $E \cap l_1 \neq E \cap l_2'$, then arguments similar to that in the previous paragraph show that $E \cap S$ is ellipsoidal. Otherwise, we have $E \cap l_1 = E \cap l_2' \subset l_1 \cap l_2' \subset E$, from which it follows that $d-3 \leq \dim l_1 \cap l_2' \leq d-2$. Since $l_1 \neq l_2'$, we get $\dim l_1 \cap l_2' = d-3$. Therefore, $E \cap S$ is ellipsoidal for every hyperplane E that contains p except at most for those that contain the $(d-3)$-dimensional linear variety $l_1 \cap l_2'$. By continuity it follows that S is elliptic through p so that S is an ellipsoid. $\square$

Remarks on Example 2.2. The set A is a convex body elliptic through any line l of the plane $z = 0$ that does not intersect its interior. Figure 2.1 shows the lines $l_1 \equiv \{y = -1,\ z = 0\}$ and $l_2 \equiv \{y = 1,\ z = 0\}$.

The set B was obtained by applying to the set

$$\widehat{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + x^2 y^2 \leq 1\}$$

the projectivity that fixes the plane $x + z + 2 = 0$ (which does not intersect $\widehat{B}$) as the plane at infinity. The planes either parallel to plane $x = 0$ or to plane $y = 0$ define elliptical sections in $\widehat{B}$. Moreover, in [3] it is proved that $\widehat{B}$ is convex. From the above, one has that B is a convex body elliptic through the lines $l_1 \equiv \{x = 0,\ z = 1\}$ and $l_2 \equiv \{y = 0,\ x + z = 1\}$. These lines meet at the point $(0, 0, 1)$ and do not touch B.

Now consider the function $f(x, y, z) = x^2 + y^2(1-z) - z(1-z)^2$ and the set

$$\widehat{C} = \left\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) \leq 0,\ 0 \leq z < 1\right\} \cup \{(0, 0, 1)\}$$

The set C follows from $\widehat{C}$ by applying the projectivity that sends the plane $x + 1 = 0$ (which does not intersect $\widehat{C}$) to infinity. The set $\widehat{C}$ is bounded because it is inside the parallelepiped defined by $-1 < x < 1$, $-1 < y < 1$ and $0 \leq z \leq 1$. To see that $\widehat{C}$ is closed, let $(x_n, y_n, z_n) \in \widehat{C}$ be such that $x_n \to x$, $y_n \to y$ and $z_n \to z$. Then $f(x, y, z) \leq 0$

and $0 \leq z \leq 1$. It only remains to see that if $z = 1$ then $x = y = 0$. From $f(x, y, 1) \leq 0$ it follows that $x = 0$. From $f(x_n, y_n, z_n) \leq 0$ we get that $y_n^2 \leq z_n(1 - z_n)$, hence that $y^2 \leq z(1 - z) = 0$, and finally that $y = 0$.

The task is now to see that $\widehat{C}$ is convex. Consider the convex set $D = \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 - z \leq 0\}$ and the function

$$(y, z) \in D \rightsquigarrow F(y, z) = -\sqrt{(z - 1)(y^2 + z^2 - z)} \in \mathbb{R}.$$

That function is well defined and it is straightforward to see that it is convex. We conclude that $\widehat{C}$ is convex because it is the boundary of the set $\{(\pm F(y, z), y, z) : (y, z) \in D\}$.

The sections of $\widehat{C}$ by planes parallel to the plane $z = 0$ are ellipses. The line $\{x = 0,\ z = 1\}$ is tangent to $\widehat{C}$ at $(0, 0, 1)$ and the sections of $\widehat{C}$ by planes that contain that line are also ellipses. The projectivity defined above transforms the convex body $\widehat{C}$ into the convex body C, the line $l_1 \equiv \{x = 0,\ z = 1\}$ remains fixed, and the line at infinity where the planes parallel to $z = 0$ meet into the planes that meet at the line $l_2 \equiv \{x = 1,\ z = 0\}$. □

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