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Boundary Contact Energies for a Variational Model in Phase Separation

Margherita Solci*
Dipartimento di Architettura e Pianificazione,
Università di Sassari, Piazza Duomo, 07041 Alghero, Italy
margherita@uniss.it

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The paper concerns the asymptotic behaviour of a family of energy functionals related to the Cahn–Hilliard theory for phase separation. Suitable boundary conditions are considered, modelling the presence of boundary layers; in the variational limit, together with the surface energy corresponding to the interior transition between the phases, an additional term appears, measuring the energy of the boundary layer.

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1. Introduction

According to the Van der Waals-Cahn-Hilliard theory of phase transitions for fluids, the variational problem which leads to the stable configurations involves the functional:

$$J_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \varepsilon\int_\Omega |Du|^2\,dx$$

where Ω stands for the container bounding the fluid, u is the density distribution and W is a (double-well shaped) Gibbs free energy. Here ε is a small parameter which weights the smoothing effect of the gradient term. Thus, one expects that useful information on the structure of the minima can be obtained through the asymptotic analysis, as $\varepsilon \to 0$, of J_ε (see [17, 19]).

On the line of [21] and [22], in [24] the asymptotic behaviour of an alternative sequence of energy functionals is studied:

$$(*)\qquad F_\varepsilon(u,v) = \frac{1}{\varepsilon}\int_\Omega W(u)dx + \frac{\alpha}{\varepsilon}\int_\Omega (u-v)^2 dx + \varepsilon\int_\Omega |Dv|^2 dx,$$

where the gradient term is taken with respect to a new variable v which is related to the phase variable u through the L^2-distance between u and v (weighted by a coefficient α).

In this paper we consider the behaviour of F_ε, as $\varepsilon \to 0$, in case that suitable boundary conditions are added, obtaining in the limit equilibrium configurations which satisfy a minimal interface criterion modified to account for the presence of boundary layers.

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Let us recall that both the sequence J_ε introduced above and the sequence F_ε in $(*)$ give rise, in the limit, to a minimal area problem, but with different values of the "surface tension" (see [17, 19, 24]). We notice that the surface tension arising in the limit problem for F_ε is strictly related to the one obtained in the asymptotic analysis of a family of nonlocal functionals, where the gradient term is replaced with a nonlocal term penalizing the spatial inhomogeneity, weighted with an influence kernel (see [13], [2], [3]).

The problem of imposing boundary conditions to J_ε was studied in [20] in the form of prescribed boundary values (Dirichlet problem) and in [18] by adding an integral term which accounts for the contact energy between the fluid and the container walls. Here we first tackle the problem of Dirichlet data for F_ε; more precisely, in §2-5 we shall consider the functionals

$$F^\circ_\varepsilon(u,v) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)dx + \dfrac{\alpha}{\varepsilon}\int_\Omega (u-v)^2 dx + \varepsilon\int_\Omega |Dv|^2 dx & \text{if } v \in H^1(\Omega),\ v|_{\partial\Omega} = h_\varepsilon \\ +\infty & \text{otherwise,} \end{cases}$$

where α is a real positive parameter, and h_ε is a converging sequence in $H^{\frac{1}{2}}(\partial\Omega)$. Notice that the constraint is imposed to the function v since this naturally lies in $H^1(\Omega)$, while the functional does not impose any regularity constraint on the phase variable u.

A common feature shared with the result of [17] and [20] is that the limit consists of a term measuring the energy associated with the interior interfaces between the phases, as in the unconstrained problem, and an additional term measuring the energy of the boundary layer. Then, the minimizers or quasi-minimizers converge, as $\varepsilon \to 0$, to the characteristic function of a subset E of Ω with finite perimeter in Ω, such that the perimeter is minimal under the additional condition that the angle between the interface ∂E and $\partial\Omega$ (*contact angle*) is a prescribed function of $x \in \partial\Omega$ (see §6, and [20, Proposition 5.2]). These problems are connected with the theory of capillary surfaces and other physical models (see, e.g. [12], [18] with the references therein, and also §6).

In §6 we shall consider, following the approach of L. Modica (see [18]), the case when a term of "contact energy", penalizing the difference with a given function defined on $\partial\Omega$, is added directly in the functional. In the limit, we get again a functional of the same form, with a different energy density at the boundary.

We notice that the *selection criterion* of the equilibrium configurations induced by the sequence of "two-variable" functionals with Dirichlet conditions gives, as in [24], a different surface tension with respect to the one induced by the sequence J_ε modified by boundary terms, and moreover a different constraint on the contact angle. The energies obtained in the asymptotic analysis of the functionals related to J_ε can be recovered by taking the limit of the corresponding two-variable energy when the coupling parameter α goes to $+\infty$.

Let us notice that a key rôle is played by the study of the profile problem which provides the optimal connection (in term of the unscaled energy corresponding to $\varepsilon = 1$) between the phases and the boundary data. We point out that a number of technical details in our proofs depends on the fact that, due to the presence of two variables, here we can not dispose of simple characterizations for the solutions of the profile problems.

Finally, we note that the close connection between functionals of type F_ε, depending on a phase variable and on an order parameter v, and nonlocal functionals, suggests that through this approach it could be possible to give an interpretation of boundary contact terms also for nonlocal models.

Notation and preliminaries. The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure are denoted respectively by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$; we also use the notation $|E|$ instead of $\mathcal{L}^n(E)$. The open ball with centre x and radius r is denoted by $B_r(x)$, and S^{n-1} is the boundary of $B_1(0)$.

Functions of bounded variation and sets of finite perimeter. For the general theory of this topic we refer to [10, 11, 4]; here we recall some definitions and properties we shall use in the sequel.

Given an open subset Ω of $\mathbb{R}^n$, a function $u\colon \Omega \to \mathbb{R}$ is said to be of *bounded variation* ($u \in BV(\Omega)$) if $u \in L^1(\Omega)$ and its distributional derivatives $D_i u$ are Radon measures with finite total variation in Ω.

If $u \in L^1_{loc}(\Omega)$ we say that $z \in \mathbb{R}$ is the *approximate limit* of u in $x \in \Omega$ ($z =$ap-$\lim_{y\to x} u(y)$) if

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0.$$

The set $S(u)$ of points where this property does not hold is called the *approximate discontinuity set* of u. The set $S(u)$ is a Borel set, and $|S(u)| = 0$. If $u \in BV(\Omega)$, then $S(u)$ is countably $(n-1)$-rectifiable, and there exist Borel functions $\nu_u\colon S(u) \to S^{n-1}$, and $u^+, u^-\colon S(u) \to \mathbb{R}$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$

$$\lim_{\rho\to 0} \rho^{-n} \int_{B^+_\rho(x)\cap\Omega} |u(y) - u^+(x)| dy = 0, \qquad \lim_{\rho\to 0} \rho^{-n} \int_{B^-_\rho(x)\cap\Omega} |u(y) - u^-(x)| dy = 0,$$

where $B^+_\rho(x) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle > 0\}$ and $B^-_\rho(x) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle < 0\}$.

If $u \in BV(\Omega)$ we denote by ∇u the density of the absolutely continuous part $D^a u$ of the vector measure Du with respect to the Lebesgue measure. Let $D^s u$ be the singular part of Du, and define $D^j u = Du \llcorner S(u)$ and $D^c u = Du \llcorner (\Omega \setminus S(u))$ (the *jump* and *Cantor* part of Du, respectively). We say that u is a *special function of bounded variation* ($u \in SBV(\Omega)$) if $D^c u = 0$; in that case the following decomposition of Du holds:

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S(u).$$

If E is a Borel subset of $\mathbb{R}^n$, the *essential boundary* $\partial^* E$ of E is defined as

$$\partial^* E = \{x \in \mathbb{R}^n \ :\ \limsup_{\rho\to 0} \rho^{-n} |B_\rho(x) \cap E| > 0,\ \limsup_{\rho\to 0} \rho^{-n} |B_\rho(x) \setminus E| > 0\ \}.$$

It turns out that the discontinuity set of the characteristic function χ_E coincides with $\partial^* E$, i.e. $S(\chi_E) = \partial^* E$. It can be proved (see [11]) that for any open subset Ω of $\mathbb{R}^n$

$$\int_\Omega |D\chi_E| dx = \mathcal{H}^{n-1}(\Omega \cap \partial^* E). \tag{1}$$

In particular, if E is a bounded Borel subset, then $\chi_E \in BV(\Omega)$ if and only if $\mathcal{H}^{n-1}(\Omega \cap \partial^* E) < +\infty$ (in such a case, E is said to have *finite perimeter* in Ω).

Γ-*convergence.* We recall the notion of Γ-convergence (we refer to [7], [9] for a complete analysis of the subject). Let (X,d) be a metric space, $F_\varepsilon\colon X \to \overline{\mathbb{R}}$ $(\varepsilon > 0)$ a family of functionals, and $F\colon X \to \overline{\mathbb{R}}$. We say that $\{F_\varepsilon\}$ Γ-converges to F at $x \in X$ as $\varepsilon \to 0$ if:

i) for every infinitesimal sequence $\{\varepsilon_j\}$ and for every sequence $\{x_j\}$ converging to x in X, we have $F(x) \leq \liminf_{j\to\infty} F_{\varepsilon_j}(x_j)$;

ii) for every infinitesimal sequence $\{\varepsilon_j\}$ there exists a sequence $\{x_j\}$ converging to x in X such that $F(x) = \lim_{j\to\infty} F_{\varepsilon_j}(x_j)$.

If *i)* and *ii)* hold for every $x \in X$ we say that $\{F_\varepsilon\}$ Γ-converges to F in X, and $F = \Gamma - \lim_{\varepsilon\to 0} F_\varepsilon$.

Remark 1.1. The Γ-*lower limit* and the Γ-*upper limit* of $\{F_\varepsilon\}$ are defined as follows:

$$\begin{aligned} F'(x) &= \inf\bigl\{\liminf_{j\to\infty} F_{\varepsilon_j}(x_j): \quad \varepsilon_j \to 0,\ x_j \to x\bigr\} \\ F''(x) &= \inf\bigl\{\limsup_{j\to\infty} F_{\varepsilon_j}(x_j): \quad \varepsilon_j \to 0,\ x_j \to x\bigr\}. \end{aligned}$$

The functionals F' and F'' are lower semicontinuous, and $\{F_\varepsilon\}$ Γ-converges if and only if $F' = F''$.

2. Setting of the problem. Dirichlet boundary conditions

Let F_ε be the functional introduced in §1, defined on $[L^1(\Omega)]^2$ by:

$$F_\varepsilon(u,v) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)dx + \dfrac{1}{\varepsilon}\int_\Omega (u-v)^2 dx + \varepsilon \int_\Omega |Dv|^2 dx & \text{if } v \in H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{2}$$

where

$W\colon \mathbb{R} \to [0,+\infty)$ belongs to $C^0(\mathbb{R})$ and $W(t) = 0$ if and only if $t \in \{0,1\}$; moreover W has at least linear growth at $\pm\infty$;

$\Omega \subset \mathbb{R}^n$ is bounded, open, connected, and the boundary $\partial\Omega$ is of class C^2.

Given a sequence $\{h_\varepsilon\}$ in $H^{\frac{1}{2}}(\partial\Omega)$, we consider, for every $\varepsilon > 0$, the functionals F_ε with the boundary Dirichlet condition $\mathrm{tr}\, v = h_\varepsilon$; more precisely, the functional $F^\circ_\varepsilon\colon [L^1(\Omega)]^2 \to [0,+\infty]$ is defined by

$$F^\circ_\varepsilon(u,v) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)dx + \dfrac{\alpha}{\varepsilon}\int_\Omega (u-v)^2 dx + \varepsilon \int_\Omega |Dv|^2 dx & \text{if } v \in H^1(\Omega),\ v|_{\partial\Omega} = h_\varepsilon \\ +\infty & \text{otherwise,} \end{cases} \tag{3}$$

α being a real positive parameter.

We study the asymptotic behaviour of the sequence $\{F^\circ_\varepsilon\}$, under some suitable hypoteses on the sequence $\{h_\varepsilon\}$. More precisely, we require:

(h1) $h_\varepsilon\colon \partial\Omega \to \mathbb{R}$, $h_\varepsilon \in C^0(\partial\Omega) \cap H^{\frac{1}{2}}(\partial\Omega)$, with $\|h_\varepsilon\|_{L^\infty} + \|h_\varepsilon\|_{H^{\frac{1}{2}}}$ uniformly bounded;

(h2) the distributional derivative $\frac{\partial h_\varepsilon}{\partial\sigma}$, where σ is a surface parameter in $\partial\Omega$, belongs to $L^\infty(\partial\Omega)$, and $\left|\frac{\partial h_\varepsilon}{\partial\sigma}\right| \leq c\varepsilon^{-r}$ for some $r < 1$.

(h3) $\exists\, h\colon \partial\Omega \to \mathbb{R}$, $h \in L^\infty(\partial\Omega) \cap H^{\frac{1}{2}}(\partial\Omega)$, such that

$$h_\varepsilon(x) \to h(x) \quad \text{pointwise for} \quad \mathcal{H}^{n-1}\text{-a.a.} \quad x \in \partial\Omega.$$

We start by giving some definitions. Since we shall be concerned with the problem of connecting the phases 0 or 1 with the boundary values, the following "optimal profile" problem can be considered. Fixed $\alpha > 0$, for $i \in \{0,1\}$, and $\lambda \in \mathbb{R}$, we set:

$$\Phi^\alpha(i,\lambda) = \inf\Big\{\mathcal{F}^\circ_\alpha(\varphi,\psi) \ : \ \varphi \in X_i,\ \psi \in X_i \cap Y,\ \psi(0) = \lambda\Big\} \tag{4}$$

where $\mathcal{F}^\circ_\alpha(\varphi,\psi) = \int_{-\infty}^0 (W(\varphi) + \alpha(\varphi-\psi)^2 + (\psi')^2)ds$ is the one-dimensional unscaled functional on the half-line, the set X_i is defined as

$$X_i = \{u\colon (-\infty,0) \to \mathbb{R} \ \text{ measurable s.t. } \lim_{t\to-\infty} u(t) = i\},$$

and Y denotes the space of the functions $u \in AC(-\infty,0)$ such that $u' \in L^2(-\infty,0)$. As in [24], the connection between the two phases is related to the following minimization problem:

$$c_W(\alpha) = \inf\Big\{\mathcal{F}_\alpha(\varphi,\psi) \ : \ \varphi \in X,\ \psi \in X \cap Y\Big\}, \tag{5}$$

where $\mathcal{F}_\alpha(\varphi,\psi) = \int_{\mathbb{R}} (W(\varphi) + \alpha(\varphi-\psi)^2 + (\psi')^2)ds$, the set X is defined as

$$X = \{u\colon \mathbb{R} \to [0,1] \ \text{ measurable s.t. } \lim_{t\to-\infty} u(t) = 0, \lim_{t\to+\infty} u(t) = 1\},$$

and Y, with a slight abuse of notation, denotes in this case the space of the functions $AC(\mathbb{R})$ such that $u' \in L^2(\mathbb{R})$.

Remark 2.1. The generalization of the previous definitions to the case where the minimum points of W are a and b instead of 0 and 1 is obvious. We point out the dependence on a and b only in §6, in order to make a comparison with the results obtained in [18].

Now we can state the following

Theorem 2.2. *Let the hypotheses* (*h*1) $-$ (*h*3) *hold. The sequence* $\{F^\circ_\varepsilon\}$ Γ*-converges, as* $\varepsilon \to 0$, *to* $F^\circ\colon L^1(\Omega)\times L^1(\Omega) \to [0,+\infty]$ *defined by:*

$$F^\circ(u,v) = \begin{cases} c_W(\alpha)\mathcal{H}^{n-1}(S(u)) + \displaystyle\int_{\partial\Omega} \Phi^\alpha(\tilde{u}(x),h(x))d\mathcal{H}^{n-1}(x) & \textit{if } \ u \in SBV(\Omega), \\ & u = v \in \{0,1\} \ \textit{a.e.} \\ +\infty & \textit{otherwise,} \end{cases} \tag{6}$$

where $\tilde{u}$ *stands for the trace of* u *on the boundary.*

Remark 2.3. Let us notice that, as we precise in the sequel, to prove the Γ-$\liminf$ inequality we shall use only the pointwise a.e. convergence of h_ε, and as to the set Ω it will be sufficient to assume that the boundary is of class C^1.

Clearly, it is enough to establish the result when $\alpha = 1$, the general case following with W replaced by W/α. To shorten the notation, when α is equal to 1 we omit the dependence.

It is not difficult to prove, applying rearrangement arguments, as in the proof of the corresponding result for the functionals without Dirichlet conditions (see [24, Proposition 2.2(b)]), that

$$\lim_{\alpha\to+\infty}\int_{\partial\Omega}\Phi^\alpha(\tilde u(x),h(x))d\mathcal{H}^{n-1}(x)=2\int_{\partial\Omega}\Big|\int_{\tilde u(x)}^{h(x)}\sqrt{W(s)}ds\Big|d\mathcal{H}^{n-1}(x).$$

Notice that this value is exactly the boundary energy which arises in the Γ-limit of the classical functionals of Modica and Mortola with prescribed Dirichlet conditions (see [20, Theorem 2.1]).

Remark 2.4 (Compactness). The compactness result for the functionals $\{F^\circ_\varepsilon\}$ follows *a fortiori* from compactness for $\{F_\varepsilon\}$, by the inequality $F_\varepsilon \le F^\circ_\varepsilon$. This result can be proven by an application of [24, Lemma 3.5], which establishes the precise connection between the two-variable and the nonlocal model, and of the compactness theorem for the corresponding nonlocal functionals (see [2, Theorem 3.5]).

As a consequence, the stability of minimizing sequences (see, e.g., [1, §3]) implies that a minimizing or quasi-minimizing sequence for F°_ε is relatively compact in $[L^1(\Omega)]^2$, and every cluster point (u,v) minimizes the Γ-limit F°.

3. The one dimensional case

The proof of the Γ-lim inf inequality in the general n-dimensional case relies on the *slicing method*, which considers functionals obtained by suitable restrictions of F°_ε to parallel lines. Thus, this Section is devoted to show the one dimensional version of Theorem 2.2.

The proof of the one dimensional result (and the construction of the recovery sequence in the boundary layer near $\partial\Omega$ in the n-dimensional case, as we see in §4.2) relies on the analysis of the optimal profile problem for the functional $\mathcal{F}^\circ$, introduced in (4); concerning this problem, we state here only the properties which we apply in the proofs of the Γ-convergence results.

We introduce, for $T>0$, the sets:

$$X_i^*(T)=X_i\cap\{u\ :\ u(t)=i\ \text{ for }\ t\le -T\},\quad\text{ and }\quad X_i^*=\bigcup_{T>0}X_i^*(T).$$

Now, we can state the following lemma

Lemma 3.1. *Let* $i\in\{0,1\}$.

1) *The function* $\Phi(i,\cdot)\colon\mathbb{R}\to\mathbb{R}$ *is continuous.*

2) *For every* $\lambda\in\mathbb{R}$:

$$\begin{aligned}\Phi(i,\lambda)&=\inf\{\mathcal{F}^\circ(\varphi,\psi)\ :\ \varphi\in X_i^*,\ \psi\in X_i^*\cap Y,\ \psi(0)=\lambda\}\\&=\lim_{T\to+\infty}\inf\{\mathcal{F}^\circ(\varphi,\psi)\ :\ \varphi\in X_i^*(T),\ \psi\in X_i^*(T)\cap Y,\ \psi(0)=\lambda\}.\end{aligned}\tag{7}$$

The limit is uniform if λ *varies on compact sets.*

3) *For every* $\lambda\in\mathbb{R}$ *and* $\eta>0$, *we can find* $R_\eta^{i,\lambda}>0$ *and*

(a) $\varphi^\eta_{i,\lambda}\in X_i^*(R_\eta^{i,\lambda})$ *monotone,*

(*b*) *$\psi^\eta_{i,\lambda} \in X^*_i(R^{i,\lambda}_\eta) \cap Y$ piecewise C^1 in $[-R^{i,\lambda}_\eta, 0]$, strictly monotone with $|(\psi^\eta_{i,\lambda})'| \geq c(\eta) > 0$ in $(-R^{i,\lambda}_\eta, 0)$, and $\psi^\eta_{i,\lambda}(0) = \lambda$,*

such that

$$\mathcal{F}^\circ(\varphi^\eta_{i,\lambda}, \psi^\eta_{i,\lambda}) \leq \Phi(i,\lambda) + \frac{\eta}{2}.$$

The proof of this results relies, in particular, on some rearrangement and approximation arguments which ensure that in (4) we can take the infimum on monotone functions, even when the minimum value is not attained; the uniformity of the limit in *2*) follows from the translation invariance of the one-dimensional functional $\mathcal{F}$. Notice that the construction of the recovery sequence in the boundary layer relies on the properties of the functions provided in *3*) (see §4.2).

Let $I = (a,b)$ be an open and bounded interval. We can prove a Γ-convergence result for the functionals $F^\circ_\varepsilon \colon [L^1(I)]^2 \to [0,+\infty]$ defined by:

$$F^\circ_\varepsilon(u,v;I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I W(u)dx + \dfrac{1}{\varepsilon}\int_I (u-v)^2 dx + \varepsilon\int_I (v')^2 dx & \text{if } v \in H^1(I) \\ & \text{and } \ v(a) = h_\varepsilon(a), \\ & \qquad v(b) = h_\varepsilon(b) \\ +\infty & \text{otherwise,} \end{cases} \tag{8}$$

where $\{h_\varepsilon(a)\}, \{h_\varepsilon(b)\}$ are given sequences, converging respectively to $h(a)$ and $h(b)$.

Theorem 3.2. *The sequence of functionals $\{F^\circ_\varepsilon\}$, as $\varepsilon \to 0$, Γ-converges in $[L^1(I)]^2$ to the functional $F^\circ \colon [L^1(I)]^2 \to [0,+\infty]$ defined by:*

$$F^\circ(u,v) = \begin{cases} c_W \# S(u) + \Phi(\tilde{u}(a), h(a)) + \Phi(\tilde{u}(b), h(b)) & \text{if } u \in BV(I), u = v \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

It is clearly sufficient to prove the result for $I = (-1,1)$. We denote $F^\circ_\varepsilon(\cdot,\cdot;I)$ by $F^\circ_\varepsilon(\cdot,\cdot)$.

Proof. Γ-liminf inequality.
Let $u, v \in L^1(-1,1)$; then, for every $\{u_\varepsilon\}$, $\{v_\varepsilon\}$ such that $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1(-1,1)$, we show that:

$$\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon) \geq F^\circ(u,v). \tag{9}$$

We can suppose that $u_\varepsilon \in L^2(-1,1)$ and $v_\varepsilon \in H^1(-1,1)$ with $v_\varepsilon(\pm 1) = h(\pm 1)$; moreover, that $\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon)$ is a limit and is finite. We can assume, up to subsequences, that the convergences are pointwise a.e..

Since $F^\circ_\varepsilon \geq F_\varepsilon$, where F_ε is the one-dimensional functional corresponding to (2), the result obtained in [24, Proposition 3.3] allows to deduce:

$$u = v \in \{0,1\}, \quad \text{and} \quad u \in BV(-1,1).$$

Hence, $S(u) = S(u) \cap (-a,a)$ for some $a \in (0,1)$.

Since $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) \geq c_W \# S(u)$, we obtain:

$$\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon) \geq c_W \# S(u) + \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; (-1,-a)) + \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; (a,1)).$$

Now, let us consider the subset $(a,1)$. Up to a slight modification, it is possible to assume $u_\varepsilon(a)=v_\varepsilon(a)=\tilde u(1)$. Then, setting $\overline{u}_\varepsilon(x)=u_\varepsilon(x+1)$, and $\overline{v}_\varepsilon(x)=v_\varepsilon(x+1)$:

$$\begin{aligned} F_\varepsilon(u_\varepsilon,v_\varepsilon;(a,1)) &= F_\varepsilon(\overline{u}_\varepsilon,\overline{v}_\varepsilon;(a-1,0)) \\ &= \int_{\frac{a-1}{\varepsilon}}^{0}\big(W(\overline{u}_\varepsilon(\varepsilon y))+(\overline{u}_\varepsilon(\varepsilon y)-\overline{v}_\varepsilon(\varepsilon y))^2+(\overline{v}_\varepsilon(\varepsilon y))'^2\big)dy \\ &= F_1(\varphi_\varepsilon,\psi_\varepsilon;(\frac{a-1}{\varepsilon},0)), \end{aligned}$$

where $\varphi_\varepsilon(y)=\overline{u}_\varepsilon(\varepsilon y)$, and $\psi_\varepsilon(y)=\overline{v}_\varepsilon(\varepsilon y)$. Hence

$$F^\circ_\varepsilon(u_\varepsilon,v_\varepsilon;(a,1))\geq\Phi(\tilde u(1),h_\varepsilon(1)).$$

Taking the lower limit as ε goes to 0, the continuity property of Lemma 3.1 *1)* yields:

$$\liminf_{\varepsilon\to0}F^\circ_\varepsilon(u_\varepsilon,v_\varepsilon;(a,1))\geq\Phi(\tilde u(1),h(1))=\Phi(\tilde u(b),h(b)).$$

For the term $F_\varepsilon(u_\varepsilon,v_\varepsilon;(-1,-a))$ we can argue in a completely similar way, and (9) follows.

Γ-limsup inequality.
Let $\eta>0$ and $\{\varepsilon_j\}$ be a positive infinitesimal sequence. We prove that for every u,v in $L^1(-1,1)$ there exist sequences $\{u_j\},\{v_j\}$ converging, in $L^1(-1,1)$ as $j\to\infty$, to u and v respectively, and such that:

$$\limsup_{j\to\infty}F^\circ_{\varepsilon_j}(u_j,v_j)\leq F^\circ(u,v)+\eta. \tag{10}$$

It is enough to prove the Γ-$\limsup$ inequality (10) for $u=v\in\{0,1\}$, $u\in BV(-1,1)$. Let $a\in(0,1)$ be such that $S(u)=S(u)\cap(-a,a)$. The one-dimensional Γ-convergence result obtained for the sequence $\{F_\varepsilon\}$ ensures the existence of $\{\widehat{u}_j\},\{\widehat{v}_j\}$ converging to u in $L^1(-1,1)$ such that

$$\limsup_{j\to+\infty}F_{\varepsilon_j}(\widehat{u}_j,\widehat{v}_j;(-a,a))\leq F(u,v;(-a,a)).$$

On the other hand, since the sequence $\{h_{\varepsilon_j}(1)\}$ is bounded, Lemma 3.1 *2)* gives $T_\eta>0$ and sequences $\{\varphi^+_j\},\{\psi^+_j\}$ in $X^*_{\tilde u(1)}(T_\eta)$ and $X^*_{\tilde u(1)}(T_\eta)\cap Y$ respectively, with $\psi^+_j(0)=h_{\varepsilon_j}(1)$ and

$$\mathcal{F}^\circ(\varphi^+_j,\psi^+_j)\leq\Phi(\tilde u(1),h_{\varepsilon_j}(1))+\frac{\eta}{2}.$$

In the same way we determine $\{\varphi^-_j\},\{\psi^-_j\}$ in $X^*_{\tilde u(-1)}(T_\eta)$ and $X^*_{\tilde u(-1)}(T_\eta)\cap Y$ (we can assume the same T_η as above) with $\psi^-_j(0)=h_{\varepsilon_j}(-1)$ and

$$\mathcal{F}^\circ(\varphi^-_j,\psi^-_j)\leq\Phi(\tilde u(-1),h_{\varepsilon_j}(-1))+\frac{\eta}{2}.$$

For $\varepsilon_j\leq\frac{1-a}{T_\eta}$ we set:

$$u_j(x)=\begin{cases}\varphi^-_j(-\frac{x+1}{\varepsilon_j}) & \text{if } -1<x\leq-a\\ \widehat{u}_j(x) & \text{if } -a<x<a\\ \varphi^+_j(\frac{x-1}{\varepsilon_j}) & \text{if } a\leq x<1,\end{cases}$$

and in the same way we define v_j. We notice that $v_j \in H^1(-1,1)$, and $v_j(\pm 1) = h_{\varepsilon_j}(\pm 1)$; it is easy to see that the sequences $\{u_j\}$ and $\{v_j\}$ converge in L^1 to u. Moreover:

$$F^\circ_{\varepsilon_j}(u_j, v_j) \leq c_W \# S(u) + \Phi(\tilde{u}(-1), h_{\varepsilon_j}(-1)) + \Phi(\tilde{u}(1), h_{\varepsilon_j}(1)) + \eta.$$

An application of Lemma 3.1 *1)* gives (10). □

4. The Γ-lower inequality

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and of class C^1, and let $\{h_\varepsilon\}$ be a sequence in $H^{\frac{1}{2}}(\partial\Omega)$, pointwise converging $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$ to a function $h \in L^\infty(\partial\Omega)$.

Proposition 4.1. *For every $u, v \in L^1(\Omega)$*

$$(F^\circ)'(u,v) \geq F^\circ(u,v).$$

The proof follows the steps outlined in [6]; the application of the *slicing method* in this case needs some further regularity properties of $\partial\Omega$, since the boundary data are involved.

4.1. Some preliminary remarks on regularity

For $\xi \in S^{n-1}$ we consider the projection $\pi_\xi \colon \partial\Omega \to \xi^\perp$ defined by:

$$\pi_\xi(x) = x - \langle x, \xi\rangle \xi.$$

We choose in $\mathbb{R}^n$ a basis $\{\xi, e^1_\xi, \dots, e^{n-1}_\xi\}$, where $\{e^1_\xi, \dots, e^{n-1}_\xi\}$ is an orthonormal basis of $\xi^\perp$; in this way $\xi^\perp$ is naturally identified with $\mathbb{R}^{n-1}$. Since $\partial\Omega$ is a C^1-submanifold of $\mathbb{R}^n$, and π_ξ is a C^1-map, for any Borel $A \subseteq \partial\Omega$ we can write the co-area formula as:

$$\int_A J^*(\pi_\xi)\, d\mathcal{H}^{n-1} = \int_{\xi^\perp} \mathcal{H}^0(A \cap \pi_\xi^{-1}(y))\, d\mathcal{H}^{n-1}(y), \tag{11}$$

where

$$J^*(\pi_\xi)(x) = \sqrt{\det(d\pi_\xi)_x \circ ((d\pi_\xi)_x)^*},$$

and $(d\pi_\xi)_x \colon T_x(\partial\Omega) \to \mathbb{R}^{n-1}$ denotes the differential map. Clearly, $J^*(\pi_\xi) = |\langle \xi, \nu\rangle|$.

Denoting by C the set of the x in $\partial\Omega$ such that $\operatorname{rank}(d\pi_\xi)_x < n-1$, i.e. $J^*(\pi_\xi)(x) = 0$, we have

$$C \cap (\pi_\xi)^{-1}(y) = \emptyset \qquad \mathcal{H}^{n-1} - \text{a.a. } \ y \in \xi^\perp. \tag{12}$$

Notice that, if $x \in \partial\Omega$ and $y \in \xi^\perp$, then $x \in C \cap (\pi_\xi)^{-1}(y)$ if and only if $r(\xi, y) \subset T^a_x(\partial\Omega)$, where $T^a_x(\partial\Omega)$ is the affine tangent space applied in x, and $r(\xi, y)$ is the line:

$$r(\xi, y) = \{t\xi + y \ : \ t \in \mathbb{R}\}.$$

This result is essentially the C^1 version of the Sard Theorem (see, e.g., [23, p. 56]). For further details, we refer to [23].

Moreover, we have that

$$\int_{\partial\Omega} J^*(\pi_\xi)\, d\mathcal{H}^{n-1} < +\infty,$$

and from (11) we deduce that $\partial\Omega \cap (\pi_\xi)^{-1}(y)$ is a finite set for $\mathcal{H}^{n-1}$-a.a. $y \in \mathbb{R}^{n-1}$.

This proves the following:

Lemma 4.2. *For $\xi \in S^{n-1}$:*

1. *for $\mathcal{H}^{n-1}$-a.a. $y \in \xi^\perp$ the set $\{t : t\xi + y \in \partial\Omega\}$ is finite;*
2. *for $\mathcal{H}^{n-1}$-a.a. $y \in \xi^\perp : r(\xi, y) \not\subset T^a_x(\partial\Omega)$ for every x.*

Hence, we deduce that for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.a. $y \in \xi^\perp$ the set $\Omega^{\xi,y}$ is a finite union of intervals with disjoint closure, and $\partial(\Omega^{\xi,y}) = (\partial\Omega)^{\xi,y}$.

4.2. Proof of Proposition 4.1

In order to apply the *slicing method* to obtain a lower bound for the Γ-limit, let us consider, for every $\varepsilon > 0$, $\xi \in S^{n-1}$ and $y \in \xi^\perp$, the function $h^{\xi,y}_\varepsilon : (\partial\Omega)^{\xi,y} \to \mathbb{R}$ defined by $h^{\xi,y}_\varepsilon(t) = h_\varepsilon(t\xi + y)$. The convergence hypothesis for $\{h_\varepsilon\}$ implies that, for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}-$a.a. $y \in \xi^\perp$:

$$h^{\xi,y}_\varepsilon(t) \to h^{\xi,y}(t) = h(t\xi + y) \quad \text{as } \varepsilon \to 0 \qquad \forall\, t \in (\partial\Omega)^{\xi,y}.$$

For $J \subset \mathbb{R}$ open and bounded, we set:

$$\phi^{\xi,y}_\varepsilon(\varphi, \psi; J) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_J W(\varphi)dt + \dfrac{1}{\varepsilon}\int_J (\varphi - \psi)^2 dt + \varepsilon \int_J (\psi')^2 dt & \text{if } \psi \in H^1(J), \\ & \psi = h^{\xi,y}_\varepsilon \text{ in } \partial J \cap (\partial\Omega)^{\xi,y} \\ +\infty & \text{otherwise} \end{cases}$$

If J is a finite union of intervals with disjoint closures, then an immediate generalization of Theorem 3.2, with the boundary condition imposed only on $\partial J \cap (\partial\Omega)^{\xi,y}$, allows to say that $\phi^{\xi,y}_\varepsilon$ Γ-converges to:

$$\phi^{\xi,y}(\varphi, \psi; J) = \begin{cases} c_W \# S(\varphi) + \displaystyle\sum_{t \in \partial J \cap (\partial\Omega)^{\xi,y}} \Phi(\widetilde{\varphi}(t), h^{\xi,y}(t)) & \text{if } \varphi, \psi \in BV(J), \\ & \varphi = \psi \in \{0, 1\} \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

Now define, for every $(u, v) \in [L^1(\Omega)]^2$:

$$(F^\circ_\varepsilon)^\xi(u, v) = \int_{\xi^\perp} \phi^{\xi,y}_\varepsilon(u^{\xi,y}, v^{\xi,y}; \Omega^{\xi,y}) d\mathcal{H}^{n-1}(y). \tag{13}$$

For every $u \in L^1(\Omega)$ and $v \in H^1(\Omega)$, with $v = h_\varepsilon$ on $\partial\Omega$, we can write $(F^\circ_\varepsilon)^\xi(u, v)$ as an integral over Ω by applying Fubini's Theorem and taking into account Lemma 4.2. We get

$$(F^\circ_\varepsilon)^\xi(u, v) = \frac{1}{\varepsilon}\int_\Omega W(u)dx + \frac{1}{\varepsilon}\int_\Omega (u - v)^2 dx + \varepsilon \int_\Omega |\langle Dv, \xi\rangle|^2 dx.$$

Therefore $(F^\circ_\varepsilon)^\xi \leq F^\circ_\varepsilon$; it follows, by Fatou's Lemma, that for every $\xi \in S^{n-1}$ and for every $(u, v) \in [L^1(\Omega)]^2$

$$(F^\circ)'(u, v) \geq (F^\circ)^\xi(u, v),$$

where

$$(F^\circ)^\xi(u,v) = \int_{\xi^\perp} \phi^{\xi,y}(u^{\xi,y}, v^{\xi,y}; \Omega^{\xi,y}) d\mathcal{H}^{n-1}(y).$$

Thus, if $(F^\circ)'(u,v)$ is finite, then $u = v \in \{0,1\}$, $u^{\xi,y} \in SBV(\Omega^{\xi,y})$ for a.a. $y \in \xi^\perp$, and

$$\int_{\xi^\perp} \#S(u^{\xi,y}) d\mathcal{H}^{n-1}(y) < +\infty,$$

i.e.

$$\int_{\xi^\perp} |D(u^{\xi,y})|(\Omega^{\xi,y}) d\mathcal{H}^{n-1}(y) < +\infty;$$

we deduce that if $(F^\circ)'$ is finite, then $u, v \in SBV(\Omega)$ and $u = v \in \{0,1\}$ a.e.. For such pairs (u,v) :

$$\begin{aligned}(F^\circ)^\xi(u,v) = &\int_{\xi^\perp} c_W \#S(u^{\xi,y}) d\mathcal{H}^{n-1}(y) \\ &+ \int_{\xi^\perp} \sum_{t \in (\partial\Omega)^{\xi,y}} \Phi(\tilde{u}^{\xi,y}(t), h(t\xi + y)) d\mathcal{H}^{n-1}(y).\end{aligned} \tag{14}$$

Then, applying the co-area formula we can deduce:

$$(F^\circ)^\xi(u,v) = c_W \int_{S(u)} |\langle \xi, \nu_u(x)\rangle| d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x))\ |\langle \xi, \nu(x)\rangle| d\mathcal{H}^{n-1}(x).$$

Therefore, the Γ-lower limit

$$(F^\circ)'(u,v) = \Gamma - \liminf_{\varepsilon \to 0} F^\circ_\varepsilon(u,v)$$

is finite only if $u, v \in SBV(\Omega)$ and $u = v \in \{0,1\}$ a.e.. Moreover, $F^\circ(u,v) \geq c\mathcal{H}^{n-1}(S(u))$ (thus, finite only if $\mathcal{H}^{n-1}(S(u))$ is finite).

Then, for $u = v \in SBV(\Omega; \{0,1\})$, we consider in $\mathcal{A}(\mathbb{R}^n) = \{A : \text{ open subset of } \Omega\}$ the set function

$$\mu(A) = (F^\circ)'(u, v; A);$$

if $g_i \colon \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$g_i(x) = \begin{cases} c_W |\langle \xi_i, \nu_u(x)\rangle| & \text{if } x \in S(u) \\ \Phi(\tilde{u}(x), h(x)) |\langle \xi_i, \nu(x)\rangle| & \text{if } x \in \partial\Omega \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\xi_i\}$ is a dense sequence in S^{n-1}, and $\lambda = \mathcal{H}^{n-1} \llcorner (S(u) \cup \partial\Omega)$, then

$$\mu(A) \geq \sup_i \int_A g_i d\lambda.$$

The function μ is superadditive on disjoint open sets of $\mathbb{R}^n$ when $u = v \in SBV(\Omega, \{0,1\})$, and we can conclude, applying a property of the supremum of a family of measures (see e.g. [6, Proposition 1.16]):

$$\mu(A) \geq \int_A \sup_i g_i d\lambda.$$

Since $\{\xi_i\}$ is dense in S^{n-1} :

$$g(x) = \sup_i g_i(x) = \begin{cases} c_W & \text{if } \ x \in S(u) \\ \Phi(\tilde{u}(x), h(x)) & \text{if } \ x \in \partial\Omega \\ 0 & \text{otherwise.} \end{cases}$$

In particular, choosing $A = \mathbb{R}^n$:

$$\begin{aligned} &\sup_i \Big(\int_{S(u)} c_W |\langle \xi_i, \nu_u \rangle| d\mathcal{H}^{n-1} + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) |\langle \xi_i, \nu_u \rangle| d\mathcal{H}^{n-1} \Big) = \mu(\mathbb{R}^n) \\ \geq \ &\int_{\mathbb{R}^n} g(x) d\lambda = c_W \mathcal{H}^{n-1}(S(u)) + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x) \end{aligned}$$

Then:

$$(F^\circ)'(u,v) \geq c_W \mathcal{H}^{n-1}(S(u)) + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x). \tag{15}$$

This concludes the proof of the Γ-$\liminf$ inequality.

5. The Γ-upper inequality

In this Section we conclude the proof of the Γ-convergence result by showing the inequality for the Γ-upper limit of $\{F_\varepsilon^\circ\}$.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ bounded, open, connected, with boundary of class C^2. Under conditions $(h1) - (h3)$, for every $u, v \in L^1(\Omega)$:*

$$(F^\circ)''(u,v) \leq F^\circ(u,v). \tag{16}$$

We shall prove Proposition 5.1 by exibiting, for every $\eta > 0$, a *recovery sequence* $\{(u_\varepsilon, v_\varepsilon)\}$, converging in $[L^1(\Omega)]^2$ to (u,v), such that $v_\varepsilon \in H^1(\Omega)$, tr $v_\varepsilon = h_\varepsilon$ on $\partial\Omega$, and

$$\limsup_{\varepsilon \to 0} F_\varepsilon^\circ(u_\varepsilon, v_\varepsilon) = \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq F^\circ(u,v) + \eta. \tag{17}$$

5.1. Regularity and density results

We introduce the following notation: given a subset S of $\mathbb{R}^n$ and $\varrho > 0$ define

$$S_\varrho = \{x \in \mathbb{R}^n : \ d(x, S) < \varrho\}.$$

If G is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary, then there exists $\varrho > 0$ such that the function $\Lambda \colon \partial G \times (-\varrho, \varrho) \to \mathbb{R}$ defined by:

$$\Lambda(x,t) = x + t\nu_G(x),$$

is injective, where $\nu_G \colon \partial G \to S^{n-1}$ denotes a unit normal field to ∂G. Then we can define the projection $\pi_G \colon (\partial G)_\varrho \to \partial G$ which maps x to the unique minimizer in ∂G of the function $y \mapsto d(x,y)$. Moreover (see [5, Theorem 3.1]), by the implicit function theorem, we can choose ϱ so that Λ is a local diffeomorphism; as a consequence, π_G is C^1. For convenience, we simply set $\pi = \pi_\Omega$ and $\nu = \nu_\Omega$, where ν_Ω is the inner normal field.

We prove the Γ-$\limsup$ inequality, by using a *density argument*. We recall the steps of this procedure (see, e.g., [6, p. 97]):

1. find a subset $\mathcal{T}$ of the domain of F° such that for each (u,v) in the domain of F° there exists a sequence $\{(u_j,v_j)\}$ in $\mathcal{T}$ satisfying $u_j \to u$, $v_j \to v$ in $L^1(\Omega)$, and $F^\circ(u,v) = \lim_{j\to\infty} F^\circ(u_j,v_j)$;
2. prove that $(F^\circ)''(u,v) \le F^\circ(u,v)$ for each $(u,v) \in \mathcal{T}$.

The lower semicontinuity of $(F^\circ)''$ entails, for every (u,v) in the domain of F°:

$$(F^\circ)''(u,v) \le \liminf_{j\to\infty} (F^\circ)''(u_j,v_j) \le \lim_{j\to\infty} F^\circ(u_j,v_j) = F^\circ(u,v).$$

It is clear that $F^\circ(u,v)$ is finite if and only if $u = v = \chi_E$, where E is a subset of $\mathbb{R}^n$ having finite perimeter in Ω.

In order to define the set $\mathcal{T}$, we need the following definition (see [16, §3.2, p. 74]):

Definition 5.2. Let M, N be C^1-submanifolds of $\mathbb{R}^n$. We say that M is transverse with respect to N (or that M and N are mutually transverse) if:

$$\forall\, x \in M \cap N \quad T_x M \oplus T_x N = \mathbb{R}^n.$$

Now, we define $\mathcal{T}$ as the set of the pairs (χ_E, χ_E) where E is the intersection with Ω of an open subset of $\mathbb{R}^n$ of class C^∞, whose boundary is transverse with respect to $\partial\Omega$, i.e.:

$$\mathcal{T} = \{(\chi_E, \chi_E) \ : \ E \in \mathcal{S}\},$$

where

$$\mathcal{S} = \{E = \Omega \cap A \ : \ A \subset \mathbb{R}^n \text{ open, such that } \partial A \text{ of class } C^\infty \text{ transverse to } \partial\Omega\}.$$

The required density of $\mathcal{T}$ in the domain of F° follows from Lemma 5.3 and Lemma 5.4.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ open, with compact and Lipschitz boundary. Given $E \subset \Omega$ with finite perimeter in Ω, there exists a sequence $\{E'_j\}$ of subsets of $\mathbb{R}^n$ with boundaries of class C^∞ in a neighbourhood of $\overline{\Omega}$, such that the sequence $\{E_j\}$ defined by $E_j = E'_j \cap \Omega$ approximates E in $L^1(\Omega)$, i.e. $|E \Delta E_j| \to 0$ as $j \to +\infty$. Moreover:*

1. $|D\chi_{E_j}|(\Omega) \to |D\chi_E|(\Omega)$;
2. $\tilde{\chi}_{E_j} \to \tilde{\chi}_E$ *in* $L^1(\partial\Omega; \mathcal{H}^{n-1} \llcorner \partial\Omega)$,

where $\tilde{\chi}_{E_j}, \tilde{\chi}_E$ stand for the traces of χ_{E_j} and χ_E respectively.

Proof. Except for the assertion 2, the proof is the same of the Proposition 4.7 in [6]. To prove 2, recall that

$$\chi_{E_j} \to \chi_E \qquad \text{strictly in } \ BV(\Omega)$$

(see [4, Definition 3.14, p. 125]). Since, for Ω open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary, the trace operator is continuous between $BV(\Omega)$, endowed with the topology induced by strict convergence, and $L^1(\partial\Omega; \mathcal{H}^{n-1} \llcorner \partial\Omega)$ ([4, Theorem 3.88, p. 181]), the assertion 2 follows. $\square$

Lemma 5.3 proves the density in the domain of F° of the set

$$\tilde{\mathcal{T}} = \{(\chi_E, \chi_E) \ : \ E \subset \Omega \quad \text{s.t. } \ E = \Omega \cap A \quad \text{for some } A \text{ of class } C^\infty\}.$$

The required density of the set $\mathcal{T}$ in the domain of F° follows from the application of the transversality lemma below to the smooth sets A which define the element of $\tilde{\mathcal{T}}$.

Lemma 5.4. *Let Ω be a bounded subset of $\mathbb{R}^n$ with boundary of class C^2. If $A \subset \mathbb{R}^n$ is bounded and of class C^∞, for every $\eta > 0$ there exists $f\colon \partial A \to (0, +\infty)$ of class C^∞ such that the set*

$$A_f = A \cup \{x + t\nu_A(x) \ : \ t \in [0, f(x)), \ x \in \partial A\},$$

where ν_A stands for the outer unit normal field, satisfies the following properties:

1. ∂A_f *is transverse with respect to* $\partial\Omega$;
2. $|(A \Delta A_f) \cap \Omega| \leq \eta$;
3. $\big||D\chi_{A_f}|(\Omega) - |D\chi_A|(\Omega)\big| \leq \eta$;
4. $\|\tilde{\chi}_{A_f} - \tilde{\chi}_A\|_{L^1(\partial\Omega)} \leq \eta$,

where $\tilde{\chi}_{A_f}$ and $\tilde{\chi}_A$ denote the trace of χ_{A_f} and χ_A on $\partial\Omega$ respectively.

Proof. Given A a bounded subset of $\mathbb{R}^n$ with C^∞ boundary, its boundary can be represented by the injection $i_A\colon \partial A \to \mathbb{R}^n$ of class C^∞. As observed at the beginning of this subsection, the regularity hypothesis allows us to find $\delta > 0$ such that the projection $\pi_A\colon (\partial A)_\delta \to \partial A$ is well-defined and smooth.

We set $\mathcal{G} = \{g\colon \partial A \to \mathbb{R}^n \text{ of class } C^\infty\}$, endowed with a C^1-type topology, which relies on the uniform convergence of the functions and their differentials on compact sets. More precisely, this is the *compact-open* C^1-topology, and, since ∂A is compact, it coincides with the *strong* C^1-topology (for the precise definitions and further details, we refer to [16, p. 34–35]).

We construct a neighbourhood $\mathcal{U}$ of i_A in $\mathcal{G}$ such that $g(\partial A) \subset (\partial A)_\delta$ for every $g \in \mathcal{U}$, and moreover for every $g \in \mathcal{U}$ we can find $f\colon \partial A \to (-\delta, \delta)$ of class C^∞ such that $g(\partial A) = \{x + f(x)\nu_A(x) \ : \ x \in \partial A\}$. Indeed, since ∂A is compact, the subset of $\mathcal{G}$ such that the differential of $\pi_A \circ g$ has rank $n-1$ for every point of ∂A is open in $\mathcal{G}$ and contains i_A. This defines the set $\mathcal{U}$. It follows that $\pi_A \circ g$, for $g \in \mathcal{U}$, is a topological covering map. Moreover, it has degree one: indeed, the signed distance labels the branches, and the monodromy is trivial. Since, for $g \in \mathcal{G}$, the composition $\pi_A \circ g$ is globally invertible, the implicit function theorem allows to find the required f.

Now, for an open neighbourhood $\mathcal{W} \subset \mathcal{U}$ of i_A, we consider the subset $\widetilde{\mathcal{W}}$ of the elements g in $\mathcal{W}$ such that the image of the corresponding f is in $(0, \delta)$. Let us notice that for such f we can define

$$A_f = A \cup \{x + t\nu_A(x) \ : \ x \in \partial A, \ t \in [0, f(x))\},$$

and clearly $A \subset A_f$.

The set $\widetilde{\mathcal{W}}$ is an open subset of $\mathcal{G}$, then [16, Theorem 2.1, p. 74] ensures the existence of a $\tilde{g} \in \widetilde{\mathcal{W}}$ such that $\tilde{g}(\partial A)$ is a submanifold transverse with respect to $\partial\Omega$; we denote by $\tilde{f}$ the corresponding function.

Fixed $\eta > 0$, the neighbourhood $\mathcal{W}$ can be chosen such that, for every g in $\mathcal{W}$ (then in $\widetilde{\mathcal{W}}$), the corresponding f satisfies $\|f\|_{C^1(\partial A)} \leq \eta$. Then, the set $A_{\tilde{f}}$, which is by construction transverse with respect to $\partial\Omega$, satisfies the assertions 2, 3, 4. $\square$

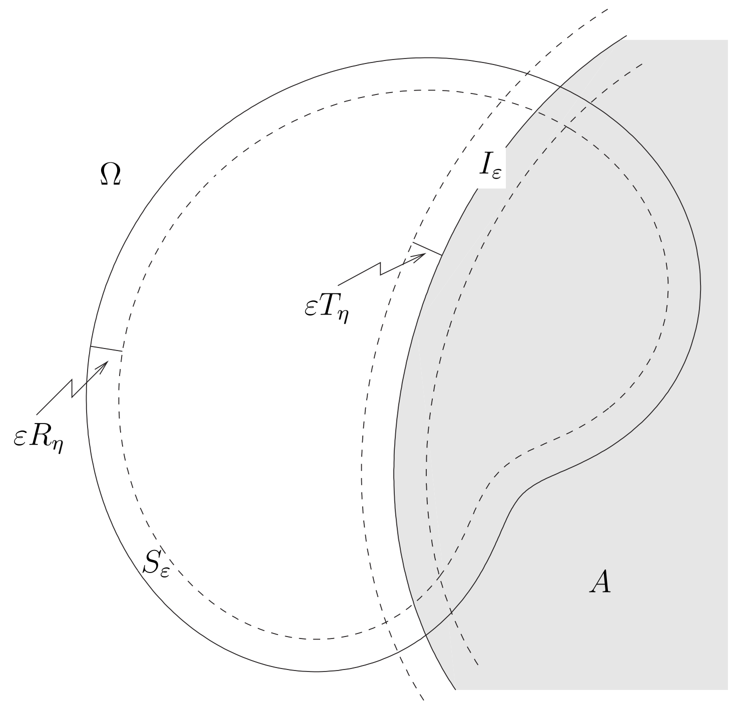

Figure 5.1: Internal and boundary layers

5.2. Recovery sequence

Now, it is sufficient to prove (16) for $(u, v) \in \mathcal{T}$: let A be an open subset of $\mathbb{R}^n$ with C^∞ boundary transverse with respect to $\partial\Omega$ and such that $u = v = \chi_E$, with $E = A \cap \Omega$.

Let $\varrho > 0$ and $\delta > 0$ be such that the projections

$$\pi\colon (\partial\Omega)_\varrho \to \partial\Omega, \quad \pi_A\colon (\partial A)_\delta \to \partial A$$

are well-defined and smooth.

Let $\eta > 0$ be fixed, $R_\eta > 0$ and T_η be positive values which will be chosen in the sequel: εR_η and εT_η will be respectively the width of the transition layer between one phase and the boundary data, and between the two phases.

Then, for $\varepsilon \leq \min\{\frac{\varrho}{R_\eta}, \frac{\delta}{T_\eta}\}$, we set (see Figure 5.1):

$$\begin{aligned} I_\varepsilon &= (\partial A)_{\varepsilon T_\eta} \cap \Omega = \{x \in \Omega \ : \ d(x, \partial E) < \varepsilon T_\eta\}, \\ S_\varepsilon &= (\partial\Omega)_{\varepsilon R_\eta} \cap \Omega = \{x \in \Omega \ : \ d(x, \partial\Omega) < \varepsilon R_\eta\}. \end{aligned}$$

Moreover, we define the set:

$$Q_\varepsilon = \{\pi(x) + t\nu(\pi(x)) \ : \ t \in (0, \varepsilon R_\eta), \ x \in I_\varepsilon \cap S_\varepsilon\}$$

(see Figure 5.2), where we recall that ν and π are respectively the inner normal field to $\partial\Omega$ and the projection on $\partial\Omega$ (see §5.1). Obviously, Q_ε includes $I_\varepsilon \cap S_\varepsilon$.

The transversality and the assumption Ω of class C^2 allow to deduce that there exists a constant $\gamma > 0$ such that, for ε sufficiently small,

$$Q_\varepsilon \subset (\partial A \cap \partial\Omega)_{\gamma\varepsilon} \cap \Omega = \{x \in \Omega \ : \ d(x, \partial A \cap \partial\Omega) < \gamma\varepsilon\}.$$

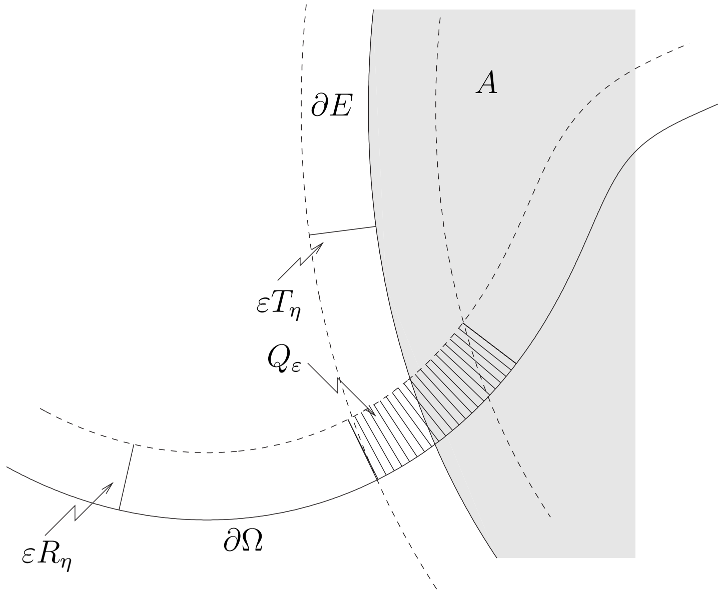

Figure 5.2: Intersection between internal and boundary layers

Remark 5.5. The property does not hold without the transversality hypothesis on A. As an example, if $\Omega = \{(x, y) \in \mathbb{R}^2 : \ y > 0\}$ and E defined by $E = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$, it is clear that $d\big((\sqrt{R_\eta \varepsilon}, R_\eta\varepsilon), (0, 0)\big) > \sqrt{R_\eta\varepsilon}$. Then, for every $\gamma > 0$ there exists ε sufficiently small that $\sqrt{R_\eta\varepsilon} > \gamma\varepsilon$, and $Q_\varepsilon \not\subset M_{\gamma\varepsilon}(\partial E \cap \partial\Omega)$.

We denote $(\partial E \cap \partial\Omega)_{\gamma\varepsilon} \cap \Omega$ by M_ε. Since $\partial E \cap \partial\Omega$ is a $n-2$ submanifold without boundary and of class C^2, for instance from [4, Theorem 2.104, p. 110] we get:

$$|M_\varepsilon| = O(\varepsilon^2)_{\varepsilon\to 0}. \tag{18}$$

Connection with the boundary data. In order to define the recovery sequence in the boundary layer near $\partial\Omega$, we give a preliminary construction, based on Lemma 3.1 *3*).

The hypothesis $(h1)$ on $\{h_\varepsilon\}$ ensures the existence of m, M finite such that

$$M > \max\{1, \ \sup_\varepsilon \max_{x\in\partial\Omega} h_\varepsilon(x)\}, \qquad \text{and} \qquad m < \min\{0, \ \inf_\varepsilon \min_{x\in\partial\Omega} h_\varepsilon(x)\}.$$

From Lemma 3.1 *3*) we get, choosing $\lambda = m$ and $\lambda = M$, functions $\varphi^\eta_{i,m}, \psi^\eta_{i,m}$ and $\varphi^\eta_{i,M}, \psi^\eta_{i,M}$ respectively, satisfying the properties enumerated in the Lemma.

For $i \in \{0, 1\}$ and $h \in [m, M]$, recalling Lemma 3.1 *3*), we define by translation (see Figure 5.3):

$$\varphi_i(h, t) = \begin{cases} \varphi^\eta_{i,M}((\psi^\eta_{i,M})^{-1}(h) + t) & \text{if } \ i < h \le M \\ i & \text{if } \ h = i \\ \varphi^\eta_{i,m}((\psi^\eta_{i,m})^{-1}(h) + t) & \text{if } \ m \le h < i, \end{cases}$$

$$\psi_i(h, t) = \begin{cases} \psi^\eta_{i,M}((\psi^\eta_{i,M})^{-1}(h) + t) & \text{if } \ i < h \le M \\ i & \text{if } \ h = i \\ \psi^\eta_{i,m}((\psi^\eta_{i,m})^{-1}(h) + t) & \text{if } \ m \le h < i, \end{cases}$$

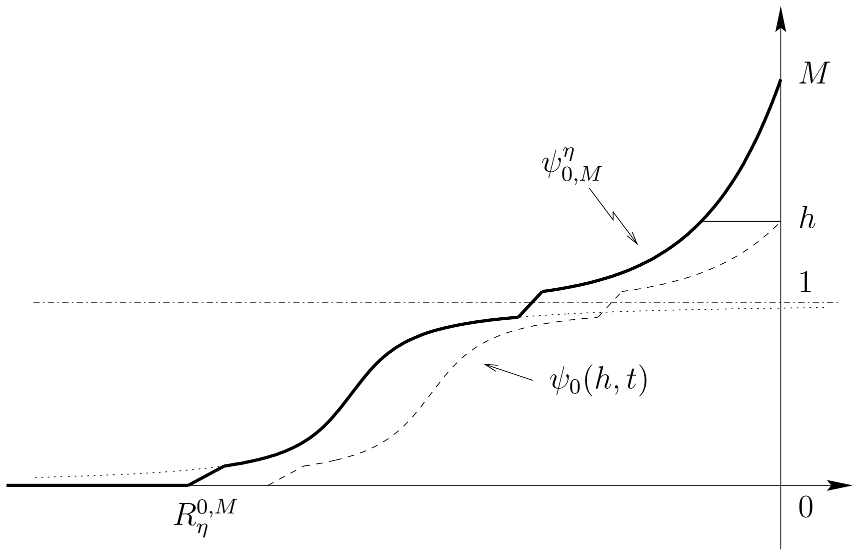

Figure 5.3: One-dimensional optimal profile

From translation invariance properties of the functionals, it follows that:

$$\mathcal{F}^{\circ}(\varphi_i(h,\cdot),\psi_i(h,\cdot)) \leq \Phi(i,h)+\eta. \tag{19}$$

By choosing R_η sufficiently large, we can construct functions $\varphi_i(h,\cdot),\psi_i(h,\cdot)$ identically equal to the constant value i also in a right neighbourhood of $-R_\eta$; in the sequel, we will use this property to glue together the partial constructions and obtain the recovery sequence.

Moreover, we note that the function ψ_i is Lipschitz in $[m,M]\times[-R_\eta,0]$.

Now, we can define the recovery sequence in the boundary layer $S_\varepsilon \setminus Q_\varepsilon$. Setting:

$$S^1_\varepsilon = (S_\varepsilon \setminus Q_\varepsilon)\cap E \quad \text{and} \quad S^0_\varepsilon = (S_\varepsilon \setminus Q_\varepsilon)\setminus E,$$

we define, for $x \in S_\varepsilon \setminus Q_\varepsilon$:

$$\hat{u}_\varepsilon(x) = \begin{cases} \varphi_1\big(\overline{h}_\varepsilon(x), d_\varepsilon(x)\big) & \text{if } x \in S^1_\varepsilon \\ \varphi_0\big(\overline{h}_\varepsilon(x), d_\varepsilon(x)\big) & \text{if } x \in S^0_\varepsilon \end{cases} \tag{20}$$

$$\hat{v}_\varepsilon(x) = \begin{cases} \psi_1\big(\overline{h}_\varepsilon(x), d_\varepsilon(x)\big) & \text{if } x \in S^1_\varepsilon \\ \psi_0\big(\overline{h}_\varepsilon(x), d_\varepsilon(x)\big) & \text{if } x \in S^0_\varepsilon \end{cases} \tag{21}$$

where, to shorten the notation:

$$\overline{h}_\varepsilon(x) = h_\varepsilon(\pi(x)) \quad \text{and} \quad d_\varepsilon(x) = -\frac{d(x,\partial\Omega)}{\varepsilon}.$$

Since π is Lipschitz and $|\nabla d(x,\partial\Omega)| = 1$ a.e., the Lipschitz property of ψ_i and the hypothesis $(h2)$ on h_ε give:

$$|D\hat{v}_\varepsilon(x)|^2 \leq \begin{cases} \dfrac{|\partial_2\psi_1(\overline{h}_\varepsilon(x), d_\varepsilon(x))|^2}{\varepsilon^2} + \dfrac{c}{\varepsilon^{2r}} & \text{if } x \in S^1_\varepsilon, \\[2ex] \dfrac{|\partial_2\psi_0(\overline{h}_\varepsilon(x), d_\varepsilon(x))|^2}{\varepsilon^2} + \dfrac{c}{\varepsilon^{2r}} & \text{if } x \in S^0_\varepsilon, \end{cases} \tag{22}$$

where $\partial_2\psi_i$ stands for the derivative of ψ_i with respect to the second variable, and c, as in the sequel, denotes a positive constant which does not depend on ε.

Since $\overline{S^1_\varepsilon} \cap \overline{S^0_\varepsilon} = \emptyset$, and $|\partial_2\psi_i|$ is bounded (recalling that ψ_i is Lipschitz), it follows that $D\hat{v}_\varepsilon \in L^\infty(S^1_\varepsilon \cup S^0_\varepsilon)$, and

$$|D\hat{v}_\varepsilon(x)| \leq \frac{c}{\varepsilon} \qquad \text{a.a. } \ x \in S^1_\varepsilon \cup S^0_\varepsilon.$$

Moreover, we observe that the sequence $\hat{v}_\varepsilon$ is uniformly bounded in L^∞.

Recalling the co-area formula, and the fact that $|\nabla d(x,\partial\Omega)| = 1$ a.e., we obtain:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{S^1_\varepsilon} W(\hat{u}_\varepsilon)dx + \frac{1}{\varepsilon}\int_{S^1_\varepsilon}(\hat{u}_\varepsilon - \hat{v}_\varepsilon)^2 dx + \varepsilon\int_{S^1_\varepsilon}|D\hat{v}_\varepsilon|^2 dx \\
\leq\ & \int_{-R_\eta}^0 \int_{\{d=s\varepsilon\}\cap S^1_\varepsilon} W\big(\varphi_1(\overline{h}_\varepsilon(x), s)\big) d\mathcal{H}^{n-1}(x)ds \\
&+ \int_{-R_\eta}^0 \int_{\{d=s\varepsilon\}\cap S^1_\varepsilon} \big(\varphi_1(\overline{h}_\varepsilon(x), s) - \psi_1(\overline{h}_\varepsilon(x), s)\big)^2 d\mathcal{H}^{n-1}(x)ds \\
&+ \int_{-R_\eta}^0 \int_{\{d=s\varepsilon\}\cap S^1_\varepsilon} |\partial_2\psi_1(\overline{h}_\varepsilon(x), s)|^2 d\mathcal{H}^{n-1}(x)ds + c\varepsilon^{1-r},
\end{aligned}$$

where d stands for the distance $d(x,\partial\Omega)$.

An application of the change of variables formula gives:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{S^1_\varepsilon} W(\hat{u}_\varepsilon)dx + \frac{1}{\varepsilon}\int_{S^1_\varepsilon}(\hat{u}_\varepsilon - \hat{v}_\varepsilon)^2 dx + \varepsilon\int_{S^1_\varepsilon}|D\hat{v}_\varepsilon|^2 dx \\
\leq\ & \int_{-R_\eta}^0 \int_{\{d=0\}\cap S^1_\varepsilon} W\big(\varphi_1(\overline{h}_\varepsilon(x), s)\big) d\mathcal{H}^{n-1}(x)ds \\
&+ \int_{-R_\eta}^0 \int_{\{d=0\}\cap S^1_\varepsilon} \big(\varphi_1(\overline{h}_\varepsilon(x), s) - \psi_1(\overline{h}_\varepsilon(x), s)\big)^2 d\mathcal{H}^{n-1}(x)ds \\
&+ \int_{-R_\eta}^0 \int_{\{d=0\}\cap S^1_\varepsilon} |\partial_2\psi_1(\overline{h}_\varepsilon(x), s)|^2 d\mathcal{H}^{n-1}(x)ds + o(1)_{\varepsilon\to 0} \\
=\ & \int_{\partial\Omega\cap\partial S^1_\varepsilon} \mathcal{F}^\circ(\varphi_1(h_\varepsilon(x),\cdot), \psi_1(h_\varepsilon(x),\cdot)) d\mathcal{H}^{n-1}(x) + o(1)_{\varepsilon\to 0}.
\end{aligned}$$

By (19), we obtain:

$$F_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon; S^1_\varepsilon) \leq \int_{\partial\Omega\cap\partial S^1_\varepsilon} \Phi(1, h_\varepsilon(x)) d\mathcal{H}^{n-1}(x) + c\eta + o(1)_{\varepsilon\to 0}.$$

The same arguments give:

$$F_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon; S^0_\varepsilon) \leq \int_{\partial\Omega\cap\partial S^0_\varepsilon} \Phi(0, h_\varepsilon(x)) d\mathcal{H}^{n-1}(x) + c\eta + o(1)_{\varepsilon\to 0}.$$

Since $\Phi(i,\cdot)$ is continuous, and $h_\varepsilon(x)$ converges to $h(x)$ $\mathcal{H}^{n-1}$- a.e. in $\partial\Omega$, the Lebesgue Theorem gives:

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon; S_\varepsilon \setminus Q_\varepsilon) \leq \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x) + c\eta. \tag{23}$$

Connection between the two phases. To construct the recovery sequence in the boundary layer between the two phases, we consider the Γ-convergence result for the functionals F_ε, in particular, Proposition 3.6 in [24] (Γ-$\limsup$ inequality) yields, for every $\eta > 0$, a family $\{(u_\varepsilon, v_\varepsilon)\}$ which we rename as $\{(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)\}$, such that $\tilde{u}_\varepsilon, \tilde{v}_\varepsilon \to u$ in $L^1(\Omega)$, satisfying the following properties:

$$0 \le \tilde{u}_\varepsilon, \tilde{v}_\varepsilon \le 1, \;\; D\tilde{v}_\varepsilon \in L^\infty(\Omega), \quad \text{and} \quad |D\tilde{v}_\varepsilon| \le c/\varepsilon \text{ a.e. in } \Omega;$$
$$\tilde{u}_\varepsilon = \tilde{v}_\varepsilon = 1 \text{ in } \{x \in E : \; d(x, \partial A) > \varepsilon T_\eta\}, \tilde{u}_\varepsilon = \tilde{v}_\varepsilon = 0 \text{ in } \{x \in \Omega \backslash E : d(x, \partial A) > \varepsilon T_\eta\},$$

and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; \Omega_\varepsilon) \le c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) + \eta. \tag{24}$$

Notice that, choosing T_η suitably, we can assume $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ taking the values 0 and 1 also in a neighbourhood of $\partial I_\varepsilon \cap \Omega$.

To connect the previous constructions, we define:

$$\overline{u}_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon(x) & \text{if } \; x \in \Omega_\varepsilon \\ \hat{u}_\varepsilon(x) & \text{if } \; x \in S_\varepsilon \setminus Q_\varepsilon \end{cases}$$

$$\overline{v}_\varepsilon(x) = \begin{cases} \tilde{v}_\varepsilon(x) & \text{if } \; x \in \Omega_\varepsilon \\ \hat{v}_\varepsilon(x) & \text{if } \; x \in S_\varepsilon \setminus Q_\varepsilon \end{cases}$$

It is easy to see that the sequence $\overline{v}_\varepsilon$ is uniformly bounded in L^∞; moreover, we recall that $|D\hat{v}_\varepsilon| \le c/\varepsilon$ a.e. in $S^1_\varepsilon \cup S^0_\varepsilon$, and $|D\tilde{v}_\varepsilon| \le c/\varepsilon$ a.e. in Ω_ε. Since, by construction, $\overline{v}_\varepsilon = 1$ in a neighbourhood of $\{x \in S^1_\varepsilon : \; d(x, \partial\Omega) = \varepsilon R_\eta\}$ and $\overline{v}_\varepsilon = 0$ in a neighbourhood of $\{x \in S^0_\varepsilon : \; d(x, \partial\Omega) = \varepsilon R_\eta\}$ we can deduce that $D\overline{v}_\varepsilon \in L^\infty(\Omega \setminus Q_\varepsilon)$, and moreover:

$$|D\overline{v}_\varepsilon(x)| \le \frac{c}{\varepsilon} \qquad \text{a.a. } \; x \in \Omega \setminus Q_\varepsilon. \tag{25}$$

Now, in order to extend the definition of the functions to the whole Ω, let us consider, for every ε, the following problem:

$$\begin{cases} \Delta H_\varepsilon = 0 & \text{in } \; \Omega \\ \text{tr } H_\varepsilon = h_\varepsilon & \text{in } \; \partial\Omega, \end{cases} \tag{26}$$

which has, since $h_\varepsilon \in C^0(\partial\Omega) \cap H^{\frac{1}{2}}(\partial\Omega)$, a unique solution $H_\varepsilon \in C^0(\overline{\Omega}) \cap H^1(\Omega)$. The maximum principle and the uniform boundedness of $\{h_\varepsilon\}$ give:

$$\|H_\varepsilon\|_{L^\infty} \le \|h_\varepsilon\|_{L^\infty} \le c \tag{27}$$

for some constant c not depending on ε. The solution of (26) minimizes the functional $\int_\Omega |DH|^2$ in the class of the functions in $H^1(\Omega)$ with fixed trace, i.e.:

$$\int_\Omega |DH_\varepsilon|^2 dx = \min\Big\{ \int_\Omega |DH|^2 dx \; : \; H \in H^1(\Omega), \;\; \text{tr } H = h_\varepsilon \Big\}.$$

Since there exists a continuous operator $\tau \colon H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ such that, for every $g \in H^{\frac{1}{2}}(\partial\Omega)$, $\text{tr } \tau(g) = g$, it follows, recalling the uniform boundedness of $\{h_\varepsilon\}$ in $H^{\frac{1}{2}}(\partial\Omega)$, that

$$\int_\Omega |DH_\varepsilon|^2 dx \quad \text{is uniformly bounded.} \tag{28}$$

We can finally give the definition of the recovery sequence. We set:

$$u_\varepsilon(x)=\begin{cases}\overline{u}_\varepsilon(x) & \text{if } x\in\Omega\setminus M_{2\varepsilon}\\ 0 & \text{otherwise in } \Omega,\end{cases}$$

$$v_\varepsilon(x)=\begin{cases}\overline{v}_\varepsilon(x) & \text{if } x\in\Omega\setminus M_{2\varepsilon}\\ \left(1-\dfrac{d(x,M_\varepsilon)}{\gamma\varepsilon}\right)H_\varepsilon(x)+\dfrac{d(x,M_\varepsilon)}{\gamma\varepsilon}\overline{v}_\varepsilon(x) & \text{if } x\in M_{2\varepsilon}\setminus M_\varepsilon\\ H_\varepsilon(x) & \text{if } x\in M_\varepsilon\end{cases}$$

Thanks to the estimates (27) and (28), we easily obtain:

$$\frac{1}{\varepsilon}\int_{M_\varepsilon}H_\varepsilon^2dx+\varepsilon\int_{M_\varepsilon}|DH_\varepsilon|^2dx\le\frac{c|M_\varepsilon|}{\varepsilon}+c\varepsilon.$$

Since $|M_\varepsilon|=O(\varepsilon^2)_{\varepsilon\to 0}$, it follows that:

$$\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,v_\varepsilon;M_\varepsilon)=0. \tag{29}$$

Recalling (27) and the uniform boundedness in L^∞ of $\overline{v}_\varepsilon$, it is easy to see that:

$$\frac{1}{\varepsilon}\int_{M_{2\varepsilon}\setminus M_\varepsilon}v_\varepsilon^2(x)dx\le\frac{c|M_{2\varepsilon}\setminus M_\varepsilon|}{\varepsilon}.$$

Moreover, we have:

$$|Dv_\varepsilon|\le|DH_\varepsilon|+\frac{1}{\gamma\varepsilon}|H_\varepsilon||\nabla d(x,M_\varepsilon)|+\frac{1}{\gamma\varepsilon}|\overline{v}_\varepsilon||\nabla d(x,M_\varepsilon)|+|D\overline{v}_\varepsilon|.$$

Since $|\nabla d(x,M_\varepsilon)|=1$ a.e., we can use again the uniform boundedness in L^∞ of $\hat{v}_\varepsilon$ and H_ε, and the estimate $|D\overline{v}_\varepsilon|\le c/\varepsilon$, to obtain $|Dv_\varepsilon|\le|DH_\varepsilon|+c/\varepsilon$; this inequality and (28) immediately imply:

$$\varepsilon\int_{M_{2\varepsilon}\setminus M_\varepsilon}|Dv_\varepsilon|^2\le c\varepsilon\int_{M_{2\varepsilon}\setminus M_\varepsilon}|DH_\varepsilon|^2+\frac{c|M_{2\varepsilon}\setminus M_\varepsilon|}{\varepsilon}\le c\varepsilon+\frac{c|M_{2\varepsilon}\setminus M_\varepsilon|}{\varepsilon},$$

and since $|M_{2\varepsilon}\setminus M_\varepsilon|=O(\varepsilon^2)_{\varepsilon\to 0}$ we conclude that

$$\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,v_\varepsilon;M_{2\varepsilon}\setminus M_\varepsilon)=0. \tag{30}$$

By construction, $u_\varepsilon,v_\varepsilon\to u$ in $L^1(\Omega)$; since the restrictions of v_ε to the subdomains are of class H^1, and the traces on the common boundaries coincide, then $v_\varepsilon\in H^1(\Omega)$. Recalling that tr $H_\varepsilon=h_\varepsilon$ on $\partial\Omega$, and tr $\overline{v}_\varepsilon=h_\varepsilon$ on $\partial\Omega\setminus M_\varepsilon$, the property tr $v_\varepsilon=h_\varepsilon$ easily follows.

Splitting the domain Ω as the previous construction suggests, estimates (23), (24), (29) and (30) easily give the result:

$$\limsup_{\varepsilon\to 0}F_\varepsilon^\circ(u_\varepsilon,v_\varepsilon)\le F^\circ(u,v)+c\eta,$$

and this concludes the proof of Proposition 5.1.

6. The boundary contact energy

In this Section we analyze the asymptotic behaviour of a sequence of two-variable functionals where a term representing the contact energy between the fluid and the container walls is taken into account, instead of the boundary Dirichlet conditions. The classical Modica-Mortola functionals modified by a boundary contact energy term have been considered by L. Modica in [18]: the result proven in [18, Theorem 2.1] can be read as a Γ-convergence theorem for the sequence of functionals $\{G_\varepsilon + I_m\}$, where I_m is the mass constraint, i.e. $I_m(u) = 0$ if $\int_\Omega u = m$, and $I_m(u) = +\infty$ otherwise in $L^1(\Omega)$, and

$$G_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)dx + \varepsilon \int_\Omega |Du|^2 dx + \int_{\partial\Omega} \sigma(\tilde u) d\mathcal{H}^{n-1} & \text{if } u \in H^1(\Omega) \text{ and } u \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

with $\tilde u$ standing for the trace of u, W being a double–well potential with the minimum points in a and b, $0 < a < b$ and σ continuous and positive. These assumptions on W and σ are coherent with the fact that the function u stands for a density distribution, hence is naturally positive, and with the equilibrium configurations of the Gibbs free-energy per unit volume W.

Fixed $m \in (a|\Omega|, b|\Omega|)$, the Γ-limit in $L^1(\Omega)$ is finite only if $u \in BV(\Omega; \{a, b\})$ with $\int_\Omega u = m$, and in this case has the form $G(u) = C_W^{a,b}\mathcal{H}^{n-1}(Su \cap \Omega) + \int_{\partial\Omega} \hat\sigma(\tilde u) d\mathcal{H}^{n-1}$, where

$$C_W^{a,b} = 2\int_a^b \sqrt{W(s)}ds \quad \text{and} \quad \hat\sigma(t) = \inf\Big\{\sigma(s) + 2\Big|\int_s^t \sqrt{W(\tau)}d\tau\Big| \ : \ s \geq 0\Big\}.$$

On the line of the problem dealt with in Theorem 2.2, here we present a convergence result for the functionals

$$\widetilde F_\varepsilon(u, v) = \begin{cases} F_\varepsilon(u, v) + \displaystyle\int_{\partial\Omega} \sigma(\tilde v) d\mathcal{H}^{n-1} & \text{if } v \in H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{31}$$

where we recall that $F_\varepsilon(u, v) = \frac{1}{\varepsilon}\int_\Omega W(u)dx + \frac{1}{\varepsilon}\int_\Omega (u - v)^2 dx + \varepsilon \int_\Omega |Dv|^2 dx$ if $v \in H^1(\Omega)$, and $\tilde v$ stands for the trace of v. The positive constraint $u, v \geq 0$ will be easily recovered (see below, Remark 6.3).

Relying on the results of the previous sections we prove the following theorem

Theorem 6.1. *The sequence of functionals $\{\widetilde F_\varepsilon\}$, as $\varepsilon \to 0$, Γ-converges in $[L^1(\Omega)]^2$ to the functional $\widetilde F\colon [L^1(\Omega)]^2 \to [0, +\infty]$ defined by:*

$$\widetilde F(u, v) = \begin{cases} c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) + \displaystyle\int_{\partial\Omega} \tilde\sigma(\tilde u) d\mathcal{H}^{n-1} & \text{if } u, v \in BV(\Omega) \\ & \text{and } u = v \in \{0, 1\} \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where, for $i \in \{0, 1\}$:

$$\tilde\sigma(i) = \inf\{\sigma(h) + \Phi(i, h) \ : \ h \in \mathbb{R}\}.$$

Outline of the Proof of Theorem 6.1. One-dimensional result.
Let $\Omega = (-1,1)$. Note that, as usual, this case is sufficient to conclude for an arbitrary finite union of bounded intervals with disjoint closure.

1. Γ-lim inf inequality.
   Since $\widetilde{F}_\varepsilon \geq F_\varepsilon$, it is not restrictive to assume $u = v \in BV((-1,1);\{0,1\})$.
   Let $\{u_\varepsilon\} \subset L^2(-1,1)$, $\{v_\varepsilon\} \subset H^1(-1,1)$, both converging in $L^1(-1,1)$ to u. Since $u \in BV(-1,1)$, we can suppose that $a > 0$ is such that $u = \tilde{u}(1)$ and $u = \tilde{u}(-1)$ in $(a,1)$ and $(-1,-a)$ respectively. Following exactly the steps of the proof of (9), and recalling the definition of $\tilde{\sigma}$, we immediately obtain:
$$\begin{aligned}\widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) &\geq F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) + \Phi(\tilde{u}(-1), v_\varepsilon(-1)) \\ &\quad + \sigma(v_\varepsilon(-1)) + \Phi(\tilde{u}(1), v_\varepsilon(1)) + \sigma(v_\varepsilon(1)) \\ &\geq F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) + \tilde{\sigma}(\tilde{u}(-1)) + \tilde{\sigma}(\tilde{u}(1)).\end{aligned}$$
   Since $\liminf F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) \geq c_W \# S(u)$, the lower inequality for the Γ-lim inf follows.
2. Γ-lim sup inequality.
   Clearly, we can assume $u = v \in BV((-1,1);\{0,1\})$. Setting $h(\pm 1) = \arg\min\{\sigma(s) + \Phi(u(\pm 1), s) \ : \ s \in \mathbb{R}\}$, we choose $\{(u_\varepsilon, v_\varepsilon)\}$ as the recovery sequence for the sequence F_ε° with $h_\varepsilon(\pm 1) = h(\pm 1)$.
   From (10), it follows that
$$\begin{aligned}\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) &= \limsup_{\varepsilon\to 0} F_\varepsilon^\circ(u_\varepsilon, v_\varepsilon) + \sigma(h(1)) + \sigma(h(-1)) + \eta \\ &\leq c_W \# S(u) + \Phi(\tilde{u}(1), h(1)) + \sigma(h(1)) \\ &\quad + \Phi(\tilde{u}(-1)) + \sigma(h(-1)) + \eta \\ &\leq \widetilde{F}(u,v) + \eta.\end{aligned}$$

The lower inequality.
Note that, if the sequence $\{(u_\varepsilon, v_\varepsilon)\}$ converging to (u,v) satisfies the further property that the traces of v_ε converge pointwise $\mathcal{H}^{n-1}$-a.e. to a bounded function h, the lower inequality
$$\liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \widetilde{F}(u,v)$$
follows from the corresponding inequality for F_ε° (with $h_\varepsilon = \mathrm{tr}\ v_\varepsilon$) and from the continuity of σ.

The proof of the general case can be obtained again applying the slicing method. We omit the details, since it is closely similar to the proof of the Proposition 4.1.

The upper inequality.
As in the proof of the Proposition 5.1, it is sufficient to prove the inequality
$$\widetilde{F}''(u,v) \leq \widetilde{F}(u,v)$$
for pairs (u,v) such that $u = v = \chi_E$, $E \in \mathcal{S}$, recalling that
$$\mathcal{S} = \{E = \Omega \cap A \ : \ A \subset \mathbb{R}^n \ \text{ with boundary of class } C^\infty \text{ transverse w.r.t. } \partial\Omega\}.$$

We set, for $i \in \{0,1\}$, $h_i = \operatorname{argmin}\{\sigma(s) + \Phi(i,s) \ : \ s \in \mathbb{R}\}$, and define:

$$h(x) = \begin{cases} h_0 & \text{if } \tilde{u}(x) = 0 \\ h_1 & \text{if } \tilde{u}(x) = 1 \end{cases}$$

Fixed $\eta > 0$, there exist:

1) a neighbourhood I_η of $\partial A \cap \partial\Omega$ in $\partial\Omega$, with $\mathcal{H}^{n-1}(I_\eta) \le \eta$;
2) $h_\eta \colon \partial\Omega \to \mathbb{R}$, of class $C^1(\partial\Omega)$, such that $\min\{h_0, h_1\} \le h_\eta \le \max\{h_0, h_1\}$, and

$$h_\eta = h \quad \text{in} \ \ \partial\Omega \setminus I_\eta.$$

Thus, for each pair (u, v), let $\{(u_\varepsilon, v_\varepsilon)\}$ be the recovery sequence converging to (u, v) for the sequence $\{F_\varepsilon^\circ\}$, with the Dirichlet data $h_\varepsilon = h_\eta$ (see Proposition 5.1). It follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \ &\le \ c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) \\ &\qquad + \int_{\partial\Omega} \Phi(\tilde{u}(x), h_\eta(x)) + \sigma(h_\eta(x)) d\mathcal{H}^{n-1}(x) + \eta.\end{aligned}$$

The choice of h_η and the continuity of Φ and σ imply that:

$$\int_{I_\eta} \Phi(\tilde{u}(x), h_\eta(x)) d\mathcal{H}^{n-1}(x) + \int_{I_\eta} \sigma(h_\eta(x)) d\mathcal{H}^{n-1}(x) \le c\eta,$$

where c is a positive constant not depending on η. Therefore,

$$\begin{aligned}&\int_{\partial\Omega} \Phi(\tilde{u}(x), h_\eta(x)) d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \sigma(h_\eta(x)) d\mathcal{H}^{n-1}(x) \\ \le \ &\int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \sigma(h(x)) d\mathcal{H}^{n-1}(x) + c\eta,\end{aligned}$$

and the definition of h allows to conclude:

$$\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \le c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) + \int_{\partial\Omega} \tilde{\sigma}(\tilde{u}(x)) d\mathcal{H}^{n-1}(x) + (c+1)\eta.$$

□

We can refine the construction of the recovery sequence, in order to take into account the *mass constraint*. For $m \in [0, |\Omega|]$, let $\{(u_\varepsilon, v_\varepsilon)\}$ as in the proof of the upper inequality in Theorem 6.1. It is easy to check that

$$\lambda_\varepsilon = m - \int_\Omega v_\varepsilon = O(\varepsilon)_{\varepsilon\to 0}. \tag{32}$$

Then, setting

$$\tilde{v}_\varepsilon = v_\varepsilon + \frac{\lambda_\varepsilon}{|\Omega|}, \qquad \tilde{u}_\varepsilon = u_\varepsilon,$$

clearly $\tilde{u}_\varepsilon, \tilde{v}_\varepsilon \to u$ in $L^1(\Omega)$, and $\int_\Omega \tilde{v}_\varepsilon = m$ for every ε. Moreover, (32) implies the convergences $\frac{1}{\varepsilon}\int_\Omega(\tilde{v}_\varepsilon - v_\varepsilon)^2 \to 0$ and $\frac{2|\lambda_\varepsilon|}{\varepsilon|\Omega|}\int_\Omega |u_\varepsilon - v_\varepsilon| \to 0$, as $\varepsilon \to 0$, recalling that $u_\varepsilon, v_\varepsilon \to u$ in $L^1(\Omega)$. Then

$$\frac{1}{\varepsilon}\int_\Omega (\tilde{u}_\varepsilon - \tilde{v}_\varepsilon)^2 = \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - v_\varepsilon)^2 + o(1)_{\varepsilon\to 0}.$$

Thus, the continuity of σ gives

$$\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) = \limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon),$$

and setting $I_m(u,v) = I_m(v)$ we have the following

Corollary 6.2. *The sequence* $\{\widetilde{F}_\varepsilon + I_m\}$ Γ*-converges in* $[L^1(\Omega)]^2$ *to the functional* $\widetilde{F} + I_m$.

Since the compactness for the sequences $\{\widetilde{F}_\varepsilon\}$ and $\{\widetilde{F}_\varepsilon + I_m\}$ follows as an immediate consequence from the corresponding result for $\{F_\varepsilon\}$, the stability of minimizing sequences implies that a minimizing or quasi-minimizing sequence satisfying the mass constraint is relatively compact in $[L^1(\Omega)]^2$, and every cluster point (u,v) minimizes $\widetilde{F}$ with the same mass constraint.

Remark 6.3. Assume that the double-well potential W vanishes in a and b, with $a, b > 0$; correspondingly, add the constraint $u, v \geq 0$ to the functionals $\tilde{F}_\varepsilon$ defined in (31). The Γ-convergence of the resulting functionals can be easily obtained by a slight modification in the proof of Theorem 6.1; the Γ-limit functional, denoted by $\widetilde{F}^+$, is finite only on pairs (u,v) such that $u = v \in BV(\Omega;\{a,b\})$, and has the form $c_W^{a,b}\mathcal{H}^{n-1}(S(u)\cap\Omega) + \int_{\partial\Omega}\overline{\sigma}(\tilde{u})d\mathcal{H}^{n-1}$, where, for $i \in \{a,b\}$, the rôle of modified contact energy is assumed by

$$\overline{\sigma}(i) = \inf\{\sigma(h) + \Phi(i,h) \ : \ h \geq 0\}. \tag{33}$$

According to Remark 2.1, $\Phi(a,h)$, $\Phi(b,h)$ and $c_W^{a,b}$ are the same as $\Phi(0,h)$, $\Phi(1,h)$ and c_W respectively, with 0 and 1 replaced by a and b.

We are now in a position to compare the result of Modica in [18] (which we recalled at the beginning of this Section) and the result just stated for the functionals $\tilde{F}_\varepsilon$ with the positivity constraint. Both of them lead to a minimization problem, in the limit, of the form

$$(P_0) \qquad \min\{P_\Omega(E) + \gamma\,\mathcal{H}^{n-1}(\partial^* E\cap\partial\Omega) \ : \ E\subset\Omega,\ |E| = \mu\},$$

with $\mu = (m - a|\Omega|)/(b-a)$, and

$$\gamma = \frac{\overline{\sigma}(b) - \overline{\sigma}(a)}{c_W^{a,b}} \quad \text{for the } \Gamma\text{-limit of } \widetilde{F}_\varepsilon,$$

$$\gamma = \frac{\hat{\sigma}(b) - \hat{\sigma}(a)}{C_W^{a,b}} \quad \text{for the } \Gamma\text{-limit of } G_\varepsilon.$$

The problem (P_0) is known as the *liquid-drop* problem, and for Ω bounded and $|\gamma| \leq 1$ it always admits at least a solution (note that a proof of this existence result can be given via Γ-convergence). For this topic, and the interpretation of the geometrical meaning of the parameters, we refer to [14], [15] and [18]. In particular, the condition $|\gamma| \leq 1$ is in correspondence with the geometrical meaning of γ, which represents the cosine of the contact angle between one phase and the container walls.

Now, let us consider the minimum problem associated to the Γ-limit functional F° obtained in the asymptotic analysis of the functionals with Dirichlet boundary conditions (with $\alpha = 1$), i.e.

$$\min\Big\{c_W\mathcal{H}^{n-1}(Su) + \int_{\partial\Omega}\Phi(\tilde{u}(x), h(x))d\mathcal{H}^{n-1}(x) \ : \ u \in BV(\Omega;\{0,1\})\Big\}.$$

The boundary term $\int_{\partial\Omega}\Phi(\tilde{u}(x), h(x))d\mathcal{H}^{n-1}(x)$ is clearly similar to $\int_{\partial\Omega}\overline{\sigma}(\tilde{u}(x))d\mathcal{H}^{n-1}(x)$, where the integrand function assumes only two values.

As in [20, §5], dealing with the case $\Omega \subset \mathbb{R}^2$, we can prove that the second term in the Γ-limit stands again for a prescribed *contact angle* between the interface $S(u_0)$ and the boundary $\partial\Omega$. An adaptation of the proof of [20, Proposition 5.2], which relies on the arguments used in [12, §1.5 and 1.6] to derive the contact angle condition for the capillary problem, gives the following property for a minimizer u_0 in $BV(\Omega;\{0,1\})$: if h is continuous in x, then the contact angle $\vartheta(x)$ between $S(u_0)$ and $\partial\Omega$ at x is given by

$$\cos\vartheta(x) = \frac{\Phi(0, h(x)) - \Phi(1, h(x))}{c_W}.$$

Notice that the translation invariance and approximation arguments imply that, when $h(x) \geq 1$ or $h(x) \leq 0$, the interface between the phases is tangent to the boundary $\partial\Omega$; this is consistent with the nature of the equlibrium configurations (for the so-called *wetting phenomenon*, we refer to [8], [18], [20]).

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