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A Necessary Condition for the Quasiconvexity of Polynomials of Degree Four

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Using ideas from Compensated Compactness, we derive a necessary condition for any fourth degree polynomial on $\mathbb{R}^p$ to be sequentially lower semicontinuous with respect to weakly convergent fields defined on $\mathbb{R}^N$. We use that result to derive a necessary condition for the quasiconvexity of fourth degree polynomials of $m \times N$ gradient matrices of vector fields defined on $\mathbb{R}^N$. This condition is violated by the example given by Šverák for $m \geq 3$ and $N \geq 2$, of a fourth degree polynomial which is rank-one convex, but it is not quasiconvex. These classes of functions are used in the approach to Nonlinear Elasticity based on the Calculus of Variations.

Keywords: Compensated compactness, lower semicontinuity, quasiconvexity, rank-one convexity

1. Introduction

The method of Compensated Compactness was developed by Murat and Tartar in the seventies, see [12], as a way to generalize the Div-Curl Lemma which was one of the keystones of the general Theory of Homogenization based on H-convergence that they developed, see for example [9], or the other papers due to them on the same volume. The purpose of Compensated Compactness is to characterize the nonlinear functions that are sequentially weakly continuous, or even just lower semicontinuous, when attention is restricted to sequences that take their values on some fixed set and satisfy some linear partial differential equations with constant coefficients.

Particularly important is the case when the sequences are formed by gradients of vector fields. Throughout the paper N, m and p are positive integers greater than or equal to two and $M_{m\times N}$ denotes the space of the $m \times N$ real matrices. If we have a sequence of vector fields with m components, which are defined on a subset of $\mathbb{R}^N$, it is customary to speak of their gradients as a sequence of $m\times N$-matrix valued fields. A function $F : M_{m\times N} \to \mathbb{R}$ is said rank-one convex if for any matrix U in $M_{m\times N}$ and any pair $\eta \in \mathbb{R}^m$, $\xi \in \mathbb{R}^N$, the function

$$\phi(t) = F(U + t\,\eta \otimes \xi)$$

is convex when $t \in (0, 1)$. A function F that is sequentially weakly lower semicontinuous (for short s.w.l.s.c.) with respect to weakly convergent sequences of gradient matrices is necessarily rank-one convex. This has been known at least since Morrey [6] and can be proved very easily using Compensated Compactness. However the reciprocal question: whether all rank-one convex functions will necessarily be s.w.l.s.c. for sequences of gradient matrices, has not been answered for $m = 2$. For $N = m = 2$ some of the previous works

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are [1], [7] and [10]. For $m \geq 3$ it was answered negatively by Šverák in [11] in the early nineties through a counterexample of a fourth degree polynomial.

The same problem, the comparison between rank-one convexity and s.w.l.s.c, also arises when the existence of solutions to the Nonlinear Elasticity system is studied in the context of the Calculus of Variations, approach followed by J. M. Ball in [2]. Let $\Omega \subset I\!R^N$ be a bounded, regular open set. The system of Nonlinear Elasticity reads as:

$$-\text{div}\ \sigma(x, \nabla u) = f \qquad x \in \Omega,$$

where: $u : \Omega \to I\!R^N$ represents the displacement fields, which should also satisfy some boundary conditions, $f : \Omega \to I\!R^N$ are the external forces and $\sigma : \Omega \times M_{N\times N} \to M_{N\times N}$ gives the internal stresses. Then assuming that

$$\sigma_{ij}(x, \nabla u) = \frac{\partial W}{\partial (\nabla u)_{ij}}(x, \nabla u),$$

where W is termed the strain-energy density of the body, one can use the framework of the Direct Method of the Calculus of Variations and try to minimize the total strain energy

$$J(u) = \int_\Omega W(x, \nabla u(x))\, dx.$$

In this approach one needs two ingredients: first, one has to generate a minimizing sequence that belongs to a sequentially compact set for weak convergence and secondly, J should be s.w.l.s.c. In that form the weak limit of the sequence will be a minimizer of the problem and then, under smoothness assumptions, a solution to the nonlinear elasticity system. The assumptions needed over W to make J s.w.l.s.c., led to the notion of quasiconvexity, namely W is quasiconvex if for any matrix U in $M_{N\times N}$ and $x_0 \in \Omega$, any $\omega \subset \Omega$ and any $\varphi \in C_c^\infty(\omega, I\!R^N)$,

$$\int_\omega W(x_0, U + D\varphi(x))\, dx \geq |\omega| W(x_0, U),$$

where $D\varphi$ is the gradient of φ. Quasiconvexity is equivalent to s.w.l.s.c. for the $W^{1,\infty}$ weak $\star$ topology, see [6]. The problem however, is that to check either of those conditions is extremely difficult, therefore the search for necessary and/or sufficient conditions for quasiconvexity becomes very interesting. There is an important negative result, namely that in the class of infinitely differentiable functions there is no local characterization of quasiconvexity if $m \geq 3$ and $N \geq 2$. This result is due to Kristensen, see [5].

Our purpose here is to explore further necessary conditions for quasiconvexity. First in Section 2 we get a simple characterization of rank-one convexity which will be used in the following sections, this because, as we said already, it is known that rank-one convexity is a necessary condition for quasiconvexity. In Section 3 we recall a result from [13] giving a first additional necessary condition for s.w.l.s.c., namely Proposition 3.1 below, and which is valid not only for sequences of gradients. Then we prove that if $N = m = 2$ and we restrict the result to sequences of gradients, this condition is also necessary for rank-one convexity, see Proposition 3.2, making it then not fine enough to distinguish between the two classes if $N = m = 2$. Still in Section 3 we prove Proposition 3.3 saying that for

$N, m \geq 2$ any rank-one convex fourth degree polynomial also has to satisfy the same necessary condition, meaning now that Proposition 3.1 does not explain why Šverák's counterexample works. In the case $max\{N, m\} \geq 3$ it might still be possible to use Proposition 3.1 to find a rank-one convex function that is not quasiconvex, which then cannot be a fourth degree polynomial.

We try then to go further and get in Section 4 another necessary condition for s.w.l.s.c., but only for fourth degree polynomials, this will be Proposition 4.1, which when specialized to sequences of gradients will finally give a necessary condition for the quasiconvexity of a fourth degree polynomial, this will be in Corollary 4.2. If $N = 2$ and $m = 3$ the example created by Šverák violates this condition, which is checked in Section 5. We also explain there why his construction can not be replicated if $N = m = 2$. For this case we tried, and failed, to create a rank-one convex fourth degree polynomial that violates the necessary condition from Corollary 4.2, giving us then the impression that the condition we found might also be necessary for rank-one convexity. Finally in Section 6 we present some closing remarks.

2. A Characterization of Rank-one Convexity

If $F \in C^2(M_{m\times N}, I\!R)$ then a condition equivalent to rank-one convexity is the Legendre-Hadamard condition, namely that for any $U \in M_{m\times N}$

$$\sum_{i,k=1}^{m}\sum_{j,l=1}^{N}\frac{\partial^2 F(U)}{\partial U_{ij}\,\partial U_{kl}}\eta_i\,\xi_j\,\eta_k\,\xi_l \geq 0 \qquad \forall\, \eta \in I\!R^m,\ \ \xi \in I\!R^N.$$

From now on we identify $M_{m\times N}$ with $I\!R^{Nm}$ by putting the first row of the matrix as the first N entries of the vector, the second row as the following N entries and so on. We can then write the Hessian matrix of F as

$$F''(d) = \begin{bmatrix} A_{11}(d) & ... & A_{1m}(d) \\ : & : & : \\ A_{1m}^T(d) & ... & A_{mm}(d) \end{bmatrix}, \tag{1}$$

where each matrix $A_{ij}(d)$ is an $N\times N$ block comprised of the second derivatives of F with respect to λ_k and λ_l for $k = N(i-1)+1, ..., N\,i$ and $l = N(j-1)+1, ..., N\,j$ and $i, j = 1, ..., m$.

Then the Legendre-Hadamard condition for F becomes that for any $d \in I\!R^{Nm}$ $F''(d)$ must be a positive semidefinite matrix over the following cone in $I\!R^{Nm}$

$$\Lambda = \left\{ \lambda \in I\!R^{Nm} : \quad \text{there exist } \eta \in I\!R^m,\ \xi \in I\!R^N \text{ such that } \lambda = \begin{pmatrix} \eta_1\,\xi \\ : \\ \eta_m\,\xi \end{pmatrix} \right\},$$

namely we need to have that for any $\eta \in I\!R^m$, $\xi \in I\!R^N$ and $d \in I\!R^{Nm}$

$$F''(d)(\lambda,\lambda) = \eta_1^2\,\xi^T A_{11}(d)\,\xi + ... + 2\,\eta_1\,\eta_m\,\xi^T A_{1m}(d)\,\xi + ... + \eta_m^2\,\xi^T A_{mm}(d)\,\xi \geq 0.$$

Which is simply that for any $\xi \in I\!R^N$, the following matrix be positive semidefinite on $I\!R^m$

$$\begin{bmatrix} \xi^T A_{11}(d)\xi & ... & \xi^T A_{1m}(d)\xi \\ : & : & : \\ \xi^T A_{1m}^T(d)\xi & ... & \xi^T A_{mm}(d)\xi \end{bmatrix}, \tag{2}$$

In particular if $m = 2$ this is equivalent to ask for the following two conditions to hold:

a) $A_{11}(d)$ and $A_{22}(d)$ are positive semidefinite $\forall\, d \in I\!R^{2N}$
b) $\xi^T A_{11}(d)\, \xi\, \xi^T A_{22}(d)\, \xi \geq (\xi^T A_{12}(d)\, \xi)^2 \;\; \forall\, \xi \in I\!R^N,\; d \in I\!R^{2N}$.

If we restrict F to be a fourth-degree polynomial, then the Hessian matrix of F at any point $t\, d$, with $t \in I\!R$ and $d \in I\!R^{Nm}$, is given by

$$F''(t\,d)(\cdot\,,\cdot) = F''(0)(\cdot\,,\cdot) + t\, F^{(3)}(0)(d,\cdot,\cdot) + \frac{1}{2}\, t^2\, F^{(4)}(0)(d,d,\cdot,\cdot),$$

then F is rank-one convex if and only if for all $\lambda \in \Lambda$, $d \in I\!R^{Nm}$, $t \in I\!R$ one has that

$$F''(0)(\lambda,\lambda) + t\, F^{(3)}(0)(d,\lambda,\lambda) + \frac{1}{2}\, t^2\, F^{(4)}(0)(d,d,\lambda,\lambda) \geq 0,$$

which in turn is equivalent to ask for the following three conditions to hold

c) $F''(0)(\cdot\,,\cdot)$ is positive semidefinite over Λ,
d) $F^{(4)}(0)(d,d,\lambda,\lambda) \geq 0 \quad \forall\, d \in I\!R^{Nm},\; \lambda \in \Lambda$,
e) $F^{(3)}(0)(d,\lambda,\lambda)^2 - 2\, F''(0)(\lambda,\lambda)\, F^{(4)}(0)(d,d,\lambda,\lambda) \leq 0 \quad \forall\, d \in I\!R^{Nm},\; \lambda \in \Lambda$.

3. A first additional necessary condition for quasiconvexity

We work in the framework of Compensated Compactness, which in its classical formulation, see [12], gives that for a function to be quasiconvex, it has to be convex in all directions belonging to the cone Λ just defined, giving then the necessity for a quasiconvex function to be rank-one convex. Going further the following result due to Luc Tartar, proposition VII.21 in [13], gives a first additional necessary condition for s.w.l.s.c. under more general differential constraints.

Proposition 3.1. *Let $\Omega \subset I\!R^N$ be a regular open set and F be a C^3 function $F : I\!R^p \to I\!R$, such that for any sequence $U^n \in L^\infty(\Omega, I\!R^p)$ for which the following hold:*

i) $U^n \rightharpoonup U^\infty$ *in* L^∞ *weak* $\star$
ii) $F(U^n) \rightharpoonup V^\infty$ *in* L^∞ *weak* $\star$
iii) $\sum_{j,k} A_{ijk} \frac{\partial U_j^n}{\partial x_k} = 0$ *for* $i = 1, .., q$,

one necessarily has that $V^\infty \geq F(U^\infty)$ almost everywhere in Ω. Then calling

$$V = \left\{ (\lambda,\xi) \in I\!R^p \times (I\!R^N \setminus \{0\}) : \sum_{j,k} A_{ijk} \lambda_j\, \xi_k = 0 \textit{ for } i = 1, .., q \right\},$$

we have that if $(\lambda^i, \xi^i) \in V$ for $i = 1, 2, 3$ with rank$\{\xi^1, \xi^2, \xi^3\} < 3$ and $a_0 \in I\!R^p$ are such that $F''(a_0)(\lambda^i, \lambda^i) = 0$ for $i = 1, 2, 3$, then one must have

$$F^{(3)}(a_0)(\lambda^1, \lambda^2, \lambda^3) = 0.$$

In particular if the U^ns are gradients of $I\!R^m$-valued fields, then $p = Nm$ and the set V becomes

$$V_0 = \left\{ (\lambda,\xi) \in I\!R^{Nm} \times (I\!R^N \setminus \{0\}) : \lambda = \begin{pmatrix} \eta_1 \xi \\ \vdots \\ \eta_m \xi \end{pmatrix} \quad \eta_i \in I\!R \text{ for } i = 1, ..., m \right\}.$$

So when we project V_0 onto $\mathbb{R}^{Nm}$ we get the set Λ mentioned in Section 2.

One way to prove that rank-one convexity does not imply quasiconvexity would then be to construct a rank-one convex function F for which one can choose $a_0 \in \mathbb{R}^{Nm}$ and pairs $(\lambda^1, \xi^1), (\lambda^2, \xi^2), (\lambda^3, \xi^3) \in V_0$ with $rank\{\xi^1, \xi^2, \xi^3\} < 3$, such that $F''(a_0)(\lambda^i, \lambda^i) = 0$ for $i = 1, 2, 3$, but $F^{(3)}(a_0)(\lambda^1, \lambda^2, \lambda^3) \neq 0$, so that it can not be quasiconvex. However in the particular case when $N = m = 2$, this idea does not help since the following holds.

Proposition 3.2. *Let $F : \mathbb{R}^{Nm} \to \mathbb{R}$ be a rank-one convex function of class C^3. Let also $(\lambda^i, \xi^i) \in V_0$ for $i = 1, 2, 3$ be such that $F''(0)(\lambda^i, \lambda^i) = 0$ for $i = 1, 2, 3$. Then if either two of the ξ^i are parallel or $N = m = 2$, we necessarily have that $F^{(3)}(0)(\lambda^1, \lambda^2, \lambda^3) = 0$.*

Proof. Let us say that ξ^1 and ξ^2 are parallel, then we get that $\lambda^1 \pm \lambda^2 \in \Lambda$, but this implies that $F''(0)(\lambda^1 \pm \lambda^2, \lambda^1 \pm \lambda^2) \geq 0$ and, since $F''(0)(\lambda^1, \lambda^1) = F''(0)(\lambda^2, \lambda^2) = 0$, we get that $F''(0)(\lambda^1, \lambda^2) = 0$. Now, since $F''(sv)(\lambda^1, \lambda^1)$ is non negative for all $s \in \mathbb{R}$ and any $v \in \mathbb{R}^{Nm}$, and takes the value 0 at $s = 0$, we need to have that $F^{(3)}(0)(\lambda^1, \lambda^1, v) = 0$ and analogously $F^{(3)}(\lambda^2, \lambda^2, v) = 0$. Now similarly $F''(s\lambda^3)(\lambda^1 + \lambda^2, \lambda^1 + \lambda^2) \geq 0$, which gives that $F^{(3)}(0)(\lambda^1 + \lambda^2, \lambda^1 + \lambda^2, \lambda^3) = 0$, from where we conclude that $F^{(3)}(0)(\lambda^1, \lambda^2, \lambda^3) = 0$.

Now we assume $N = m = 2$ and that no two of the ξ^i are parallel, from $F''(0)(\lambda^i, \lambda^i) = 0$, we get from Section 2 that if we call $A = A_{11}$, $B = A_{12}$ and $C = A_{22}$, then

$$P(\xi) = \left(\xi^T A\, \xi\right) \left(\xi^T C\, \xi\right) - \left(\xi^T B\, \xi\right)^2 \tag{3}$$

should have at least three different nonzero roots, the ξ^i's, and from condition b) for rank-one convexity in the previous section, P is non-negative, hence those three roots have to be double roots. If there is no root with $\xi_2 = 0$, we divide through by $(\xi_2)^4$ and call $x = \xi_1/\xi_2$, to write P as a polynomial of degree four in x, which will then need to have three different double roots, which is impossible.

Now if we have that $(\xi_1^1, 0)$ is a zero of P, with $\xi_1^1 \neq 0$, then

$$P(\xi) = (a_{11}c_{11} - b_{11}^2)(\xi_1^1)^4,$$

and then $a_{11}c_{11} - b_{11}^2 = 0$. For the other roots we have that $\xi_2 \neq 0$, then we divide through by $(\xi_2)^4$ in the definition of P, to write now P as a polynomial of degree three in x, but the positivity of P, will force the coefficient of x^3 to be zero, so P will be quadratic and positive, and then it will have at most one more double root, giving then also a contradiction. $\square$

The following result shows for $m, N \geq 2$ that the necessary condition of Proposition 3.1 is also satisfied by all rank-one convex fourth degree polynomials, showing then that one needs a finer condition to explain why it is enough to look into the class of the fourth degree polynomials to encounter Šverák's famous counterexample.

Proposition 3.3. *Let $m, N \geq 2$ and $F : \mathbb{R}^{Nm} \to \mathbb{R}$ be a rank-one convex fourth degree polynomial. Let also $(\lambda^i, \xi^i) \in V_0$ for $i = 1, 2, 3$ be such that $F''(0)(\lambda^i, \lambda^i) = 0$ for $i = 1, 2, 3$. Then we also have that $F^{(3)}(0)(\lambda^1, \lambda^2, \lambda^3) = 0$.*

Proof. From condition e) in Section 2 and because

$$F''(0)(\lambda^1, \lambda^1) = F''(0)(\lambda^2, \lambda^2) = 0,$$

we get that

$$F^{(3)}(0)(\lambda^1,\lambda^1,\lambda^2) = F^{(3)}(0)(\lambda^1,\lambda^1,\lambda^3) = F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^2) = 0.$$

Then we write the Taylor expansions:

$$F^{(3)}(0)(\lambda^1,\lambda^1,\lambda^2) = F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3) + F^{(4)}(0)(\lambda^1,\lambda^2,\lambda^3,\lambda^1-\lambda^3),$$
$$F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^2) = F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3) + F^{(4)}(0)(\lambda^1,\lambda^2,\lambda^3,\lambda^2-\lambda^3),$$

which when subtracted give that

$$F^{(4)}(0)(\lambda^1,\lambda^2,\lambda^3,\lambda^1-\lambda^2) = 0.$$

Finally we write the expansion

$$F^{(3)}(0)(\lambda^1,\lambda^1,\lambda^3) = F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3) + F^{(4)}(0)(\lambda^1,\lambda^2,\lambda^3,\lambda^1-\lambda^2),$$

which now gives that $F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3) = 0$. □

4. Another necessary condition for quasiconvexity

One can then turn to derive a different necessary condition for quasiconvexity, valid for a more restrictive class of functions. We obtain first the following result, dealing with sequences which are not necessarily gradients.

Proposition 4.1. *Let $\Omega \subset I\!R^N$ be a bounded regular open set and F be a fourth degree polynomial in $I\!R^p$, satisfying $F(0) = 0$ and such that for any sequence $U^n \in L^\infty(\Omega, I\!R^p)$ for which $U^n \rightharpoonup 0$ and $F(U^n) \rightharpoonup V^\infty$ both in L^∞ weak $\star$, one has that $V^\infty \geq 0$. Then $F''(0)$ must be positive semidefinite and for any $d^1, d^2, d^3 \in I\!R^p$ it should be true that*

$$\begin{aligned} &4\,F^{(3)}(0)(d^1,d^2,d^3)^2 \\ \leq\ &\sum_{i=1}^{3} F''(0)(d^i,d^i)\left(\sum_{i=1}^{3} F^{(4)}(0)(d^i,d^i,d^i,d^i) + 4\sum_{i,j=1,i<j}^{3} F^{(4)}(0)(d^i,d^i,d^j,d^j)\right). \end{aligned} \tag{4}$$

Proof. Let ξ^1 and ξ^2 be two linearly independent vectors in $I\!R^N$, set $\xi^3 = \xi^1 + \xi^2$, $t \in I\!R$ and take the following sequence defined over Ω

$$U^n(x) = t\sum_{i=1}^{3} d^i \cos(n\,\xi^i\cdot x),$$

then $U^n \rightharpoonup 0$ in L^∞ weak $\star$ and since $F(0) = 0$, if we make a Taylor expansion of F at zero, we have that $F(U^n)$ is equal to

$$\begin{aligned} &t\sum_{i=1}^{3} F'(0)(d^i)\cos(n\,\xi^i\cdot x) + \frac{1}{2}t^2\sum_{i,j=1}^{3} F''(0)(d^i,d^j)\cos(n\,\xi^i\cdot x)\,\cos(n\,\xi^j\cdot x) \\ &+\frac{1}{6}t^3\sum_{i,j,k=1}^{3} F^{(3)}(0)(d^i,d^j,d^k)\cos(n\,\xi^i\cdot x)\,\cos(n\,\xi^j\cdot x)\,\cos(n\,\xi^k\cdot x) \\ &+\frac{1}{24}t^4\sum_{i,j,k,l=1}^{3} F^{(4)}(0)(d^i,d^j,d^k,d^l)\cos(n\,\xi^i\cdot x)\,\cos(n\,\xi^j\cdot x)\,\cos(n\,\xi^k\cdot x)\,\cos(n\,\xi^l\cdot x). \end{aligned}$$

Now when we take the weak $\star$ limit in L^∞ as n goes to infinity, the linear term vanishes because

$$\cos(n\,\xi^i\cdot x) \rightharpoonup 0.$$

For the quadratic part, and due to the linear independence of ξ^1 and ξ^2, will only survive the terms with $i=j$ and

$$\cos^2(n\,\xi^i\cdot x) \rightharpoonup \frac{1}{2}.$$

For the cubic part

$$\cos^2(n\,\xi^i\cdot x)\cos(n\,\xi^j\cdot x) \rightharpoonup 0 \qquad \text{for all } i \text{ and } j$$

and

$$\cos(n\,\xi^1\cdot x)\cos(n\,\xi^2\cdot x)\cos(n\,\xi^3\cdot x) \rightharpoonup \frac{1}{4}.$$

For the fourth order term

$$\begin{aligned}\cos^4(n\,\xi^i\cdot x) &\rightharpoonup 3/8,\\ \cos^3(n\,\xi^i\cdot x)\cos(n\,\xi^j\cdot x) &\rightharpoonup 0 \qquad \text{if } i\neq j,\\ \cos^2(n\,\xi^i\cdot x)\cos^2(n\,\xi^j\cdot x) &\rightharpoonup \frac{1}{4} \qquad \text{if } i\neq j\end{aligned}$$

and

$$\cos^2(n\,\xi^i\cdot x)\cos(n\,\xi^j\cdot x)\cos(n\,\xi^k\cdot x) \rightharpoonup 0 \qquad \text{if } i\neq j, i\neq k \text{ and } j\neq k.$$

Therefore

$$\begin{aligned}V^\infty &= \lim_{n\to\infty} F(U^n)\\ &= \frac{1}{4}t^2\sum_{i=1}^{3}F''(0)(d^i,d^i)+\frac{1}{4}t^3F^{(3)}(0)(d^1,d^2,d^3)\\ &\quad +\frac{1}{24}t^4\left(\frac{3}{8}\sum_{i=1}^{3}F^{(4)}(0)(d^i,d^i,d^i,d^i)+\frac{3}{2}\sum_{i<j}^{3}F^{(4)}(0)(d^i,d^i,d^j,d^j)\right)\\ &= \frac{1}{64}t^2\left(16\sum_{i=1}^{3}F''(0)(d^i,d^i)+16\,t\,F^{(3)}(0)(d^1,d^2,d^3)\right.\\ &\quad\left.+t^2\sum_{i=1}^{3}F^{(4)}(0)(d^i,d^i,d^i,d^i)+4\,t^2\sum_{i<j}^{3}F^{(4)}(0)(d^i,d^i,d^j,d^j)\right).\end{aligned}$$

In this expression the coefficients of t^2 and t^4 must be non negative and we also need that (4) holds. $\square$

This proof is inspired on the one of Proposition 3.1 given in [13].

The following corollary, which provides with a necessary condition for a fourth degree polynomial to be quasiconvex, is now immediate.

Corollary 4.2. *Let $\Omega\subset I\!R^N$ be a bounded regular open set and F be a fourth degree polynomial in $I\!R^{Nm}$, satisfying $F(0)=0$ and such that for any sequence $U^n\in L^\infty(\Omega, I\!R^{Nm})$ for which the following hold:*

i) $U^n \rightharpoonup 0$ *in* L^∞ *weak* $\star$

ii) $F(U^n) \rightharpoonup V^\infty$ *in* L^∞ *weak* $\star$

iii) *there exist functions* $g_i^n \in W^{1,\infty}(\Omega)$ *with* $i = 1, ..., m$, *such that* $U_k^n(x) = \frac{\partial g_i^n(x)}{\partial x_j}$ *in* Ω, *with* $k = N(i-1) + j$, *for* $i = 1, ..., m$ *and* $j = 1, ..., N$,

one has that $V^\infty \geq 0$. *Then if* ξ^1 *and* ξ^2 *are linearly independent vectors in* $\mathbb{R}^N$ *and* $\lambda^1, \lambda^2 \in \mathbb{R}^{Nm}$ *are such that* $(\lambda^1, \xi^1), (\lambda^2, \xi^2) \in V_0$, *we have that for any* $\lambda^3 \in \mathbb{R}^{Nm}$ *such that* $(\lambda^3, \xi^1 + \xi^2) \in V_0$, *condition* (4) *necessarily holds with* $d^i = \lambda^i$ *for* $i = 1, 2, 3$.

Proof. We just take the sequence

$$U^n(x) = t \sum_{i=1}^{3} \lambda^i \cos(n\, \xi^i \cdot x),$$

then $U^n \rightharpoonup 0$ in L^∞ weak $\star$ and it satisfies condition *iii)* because $(\lambda^i, \xi^i) \in V_0$ for $i = 1, 2, 3$. We then just apply Proposition 4.1. $\square$

5. Šverák's Example

One can now check that the example given by Šverák in [11] violates condition (4). Let us recall his example, one defines L as the linear subspace of $\mathbb{R}^6$ spanned by

$$\lambda^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

with the directions of oscillation being the following

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \xi^3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that $(\lambda^i, \xi^i) \in V$ for $i = 1, 2, 3$. Additionally one has the plus that in L the only rank-one directions are those along the lines spanned by the λ^i's themselves, fact used by Šverák to create first a convex function on L and then extend it to the whole $\mathbb{R}^6$ as a rank-one convex function. Let $f : L \to \mathbb{R}$ be defined as

$$f \begin{pmatrix} r \\ 0 \\ 0 \\ s \\ t \\ t \end{pmatrix} = -r\, s\, t$$

and call P the orthogonal projection from $\mathbb{R}^6$ over L. Define for any pair ϵ, k of positive real constants, the following fourth degree polynomial on $\mathbb{R}^6$

$$F(X) = f(P\,X) + \epsilon \|X\|^2 + \epsilon \|X\|^4 + k \|X - P\,X\|^2.$$

Now for any $\epsilon > 0$ it is proved in [11] that one can choose a positive value for k, so that this function becomes rank-one convex.

Then

$$F''(0)(\lambda^1,\lambda^1) = F''(0)(\lambda^2,\lambda^2) = 2\,\epsilon, \qquad F''(0)(\lambda^3,\lambda^3) = 4\,\epsilon,$$
$$F^{(3)}(0)(\lambda^1,\lambda^2,\lambda^3) = -1,$$
$$F^{(4)}(0)(\lambda^1,\lambda^1,\lambda^1,\lambda^1) = F^{(4)}(0)(\lambda^2,\lambda^2,\lambda^2,\lambda^2) = 24\,\epsilon,$$
$$F^{(4)}(0)(\lambda^1,\lambda^1,\lambda^2,\lambda^2) = 8\,\epsilon,$$
$$F^{(4)}(0)(\lambda^1,\lambda^1,\lambda^3,\lambda^3) = F^{(4)}(0)(\lambda^2,\lambda^2,\lambda^3,\lambda^3) = 16\,\epsilon$$

and

$$F^{(4)}(0)(\lambda^3,\lambda^3,\lambda^3,\lambda^3) = 96\,\epsilon.$$

Condition (4) becomes $1 \leq 608\,\epsilon^2$, which is then easily violated.

It is not difficult to show that if $m = 2$, for any pair of linearly independent directions $\xi^1, \xi^2 \in I\!R^2$, taking again $\xi^3 = \xi^1 + \xi^2$, and for any three vectors $\lambda^1, \lambda^2, \lambda^3 \in I\!R^4$ such that $(\lambda^i, \xi^i) \in V_0$ for $i = 1, 2, 3$, there always exists a nonzero vector $\lambda \in span\{\lambda^1, \lambda^2, \lambda^3\}$, with $\lambda \neq \lambda^i$ for $i = 1, 2, 3$, and $\xi \in I\!R^2 \setminus \{0, \xi^1, \xi^2, \xi^3\}$ with $(\lambda, \xi) \in V_0$.

6. Some Remarks

Condition (4) alone is not sufficient for quasiconvexity because it is not even sufficient for rank-one convexity, since for example for $N = m = 2$, $F(d) = -d_1^4$ satisfies (4), but it is not rank-one convex.

Proposition 3.1, which gave us the idea for Proposition 4.1, and its proof are closely inspired on Theorem 18 in [12], which gives a necessary condition for sequential weak continuity. This condition was later on proved in [8], to be also sufficient under the extra hypothesis of constant rank, namely that for any $\xi \in I\!R^N \setminus \{0\}$ if we call

$$\Lambda_\xi = \{\lambda \in I\!R^p \text{ such that } (\lambda, \xi) \in V\},$$

the dimension of Λ_ξ is independent of ξ. This condition is certainly satisfied by the problem studied here, since the dimension of Λ_ξ is always equal to m. Then one could try to use the idea from the proof in [8] to prove that condition (4) together with conditions c), d) and e) from Section 2, are also sufficient for quasiconvexity.

Corollary 4.2 gives an additional necessary condition for quasiconvexity, besides rank-one convexity, only for those fourth degree polynomials with nonzero cubic part. Therefore it does not help when applied to the example of [1], which has only fourth order terms. The same remark applies for the example of a fourth degree polynomial which is rank-one convex, but not polyconvex, presented in [4] which is also homogeneous.

We posses very strong numerical evidence that there is no rank-one convex fourth degree polynomial that violates condition (4). Since it does not seem to be easy to get a better necessary condition in the whole class of fourth degree polynomials, one possible future step could be to go to polynomials of degree six. Alternatively, thinking in terms of the examples mentioned in the previous paragraph, it would be useful to have extra necessary conditions for quasiconvexity specialized to the case of homogeneous fourth degree polynomials.

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