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Periodic Solutions to a Hysteresis Model in Micromagnetics

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We show the existence of periodic solutions to a model describing a rate-independent hysteresis response in bulk ferromagnets. The magnetic microstructure is treated in terms of Young measures and the whole formulation is based on energy functionals for the Gibbs and dissipative energies. Our proof of the periodicity of solutions is mainly based on uniqueness of a solution to time incremental problems and on the Tychonoff fixed point theorem.

Keywords: Ferromagnetism, fixed point theorem, hysteresis, periodic solution

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1. Introduction

In this paper we show the existence of a periodic solution to a hysteresis model of bulk ferromagnets established in [27, 28]. The model is based on the Brown's theory of micromagnetics [3, 10, 14, 15] which is here enriched by a suitable rate-independent dissipation mechanism. The basic assumption is that the transformation of the magnetization from one pole to another one requires a certain amount of energy. This energy is related to the coercive force H_c. The rate-independence allows for the description of pure hysteresis losses [10] and is well accepted for a fairly wide range of frequencies of external magnetic fields. There are also other attempts in the literature to build phenomenological rate-independent dissipation mechanisms into the models. On the microscopical level, let us mention the dry-friction-type models by Bergqvist [2], Jiles [12], or Visintin [29, 30]. On the macroscopic level, we refer e.g. to Visintin [31]. We assume sufficiently slow processes so that the released heat can be absorbed by the environment (hence the process is isothermal). The model fully relies on energy principles and is based on the two main requirements, namely, *stability* and the *energy inequality.* Roughly speaking, we say that $q = q(t)$ (magnetic "configuration" of the body) is a solution process if

$$\forall\ \tilde q:\ I(t,q(t)) \le I(t,\tilde q) + \mathcal{D}(q(t),\tilde q) \quad \text{and} \tag{1}$$

$$I(t,q(t)) + \mathrm{Var}(\mathcal{D},q;s,t) \le I(s,q(s)) + \int_s^t \partial_t I(\theta,q(\theta))\,\mathrm{d}\theta, \tag{2}$$

where I is Gibbs' stored energy of the system, "Var" stands for the total variation, $\mathcal{D}$ is a dissipation functional ensuring a rate-independent response and $0 \le s \le t \le T$, where $[0,T]$ is the process time interval. The formulation of rate-independent evolutionary

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processes in continuum mechanics by means of (1) and (2) first appeared in [18], see also [20]. Its application to problems of rate-independent hysteresis in micromagnetism appeared in [28]. In particular, in [28] the existence of a solution is proved for the model including a virgin magnetization process.

Not much is known about properties of solutions to (1) and (2). Mielke and Theil [19, Th. 7.1] showed the uniqueness of the solution if I is smooth and uniformly convex in q. However, these assumptions do not hold in our application. As our functional I comes from the convexification it is not strictly convex and it is affine along the easy axis. Nevertheless, analyzing time-discrete problems corresponding to (1) and (2) we can show the uniqueness of a time-discrete solution for a suitable form of $\mathcal{D}$. Using the Tychonoff fixed point theorem we prove the existence of discrete periodic solutions. Passing to the limit for a time step $\tau \to 0$ we show the existence of periodic solutions even in the continuous case. Our method can be applied to other problems, as e.g. inelastic response of shape memory alloys [17], if one can show uniqueness of solutions in time discrete problems.

2. Model

2.1. Stored energy, its relaxation

The theory of rigid ferromagnetic bodies [3, 14, 15] assumes that a *magnetization* $m : \Omega \to \mathbb{R}^n$, describing the state of a body $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is subjected to the *Heisenberg-Weiss constraint*, i.e., has a given (in general, temperature dependent) magnitude

$$| \, m(x) \, | = M_{\rm s} \ \text{ for almost all } x \in \Omega,$$

where $M_{\rm s} > 0$ is the *saturation magnetization*, considered here as a constant (since temperature is considered constant, too).

In the no-exchange formulation, which is valid for large bodies [8], the Helmholtz free energy of a rigid ferromagnetic body $\Omega \subset \mathbb{R}^n$ consists of two parts. The first part is the *anisotropy energy* $\int_\Omega \varphi(m(x))\,{\rm d}x$ related crystallographic properties of the ferromagnet. A typical $\varphi : S := \{s \in \mathbb{R}^n;\ |s| = M_{\rm s}\} \to \mathbb{R}$ is a nonnegative function vanishing only at a few isolated points on S determining *directions of easy magnetization*, e.g. at two points for uniaxial materials or at six (or eight) for cubic ones. Throughout the paper we will assume that φ is a restriction of some smooth function $\tilde\varphi$, i.e.,

$$\varphi = \tilde\varphi|_S;\ \tilde\varphi \in C^\infty(\mathbb{R}^n)\ ,\ \tilde\varphi \geq 0 \text{ and even.} \tag{3}$$

The second part of the Helmholtz energy, $\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m(x)|^2\,{\rm d}x$, is the *energy of the demagnetizing field* ∇u_m self-induced by the magnetization m; its potential u_m is governed by

$$\operatorname{div}\big(- \nabla u_m + m\chi_\Omega \big) = 0 \quad \text{in } \mathbb{R}^n\,, \tag{4}$$

where $\chi_\Omega : \mathbb{R}^n \to \{0, 1\}$ is the characteristic function of Ω. The demagnetizing-field energy thus penalizes non-divergence-free magnetization vectors. Standardly, we will understand (4) in the weak sense, i.e. $u_m \in H^1(\mathbb{R}^n)$ will be called a weak solution to (4) if the integral identity $\int_{\mathbb{R}^n} \big(m\chi_\Omega - \nabla u_m(x)\big) \cdot \nabla v(x)\,{\rm d}x = 0$ holds for all $v \in H^1(\mathbb{R}^n)$, where

$H^1(\mathbb{R}^n) \equiv W^{1,2}(\mathbb{R}^n)$ denotes the Sobolev space of functions from $L^2(\mathbb{R}^n)$ with all first derivatives (in the distributional sense) also in $L^2(\mathbb{R}^n)$. Altogether, the Helmholtz energy $E(m)$, has the form

$$E(m) = \int_\Omega \varphi(m(x))\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m(x)|^2\,\mathrm{d}x. \tag{5}$$

If the ferromagnetic specimen is exposed to some external magnetic field $h = h(x)$, the so-called *Zeeman's energy* of interactions between this field and magnetization vectors equals to $H(m) := -\int_\Omega h(x)\cdot m(x)\,\mathrm{d}x$. Finally, the following variational principle governs equilibrium configurations:

$$\begin{cases} \text{minimize} \quad G(m) := E(m) - H(m) \\ \qquad\qquad\qquad = \displaystyle\int_\Omega (\varphi(m(x)) - h(x)\cdot m(x))\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m(x)|^2\,\mathrm{d}x, \\ \text{subject to} \quad (4),\ (m, u_m) \in \mathcal{A} \times H^1(\mathbb{R}^n), \end{cases} \tag{6}$$

where the introduced notation G stands for *Gibbs' energy* and $\mathcal{A}$ is the set of admissible magnetizations

$$\mathcal{A} := \{m \in L^\infty(\Omega;\mathbb{R}^n);\ |\,m(x)\,| = M_\mathrm{s} \text{ for almost all } x \in \Omega\}.$$

As $\mathcal{A}$ is not convex we cannot rely on direct methods in proving the existence of a solution. In fact, the solution to (6) need not exist in $\mathcal{A} \times H^1(\mathbb{R}^n)$; cf. [11] for the uniaxial case. Due to nonconvexity of $\mathcal{A}$ weak limits of minimizing sequences of (6) do not necessarily live in $\mathcal{A} \times H^1(\mathbb{R}^n)$.

It is, therefore, natural to look for an extension (=relaxation) of our problem in which we would properly describe behavior of (6) along minimizing sequences. It is well-known [8, 21, 22] that such relaxation can be achieved by extending the Helmholtz energy by continuity on the convex set of the so-called *Young measures* [33] (see also [22, 26]):

$$\bar{E}(\nu) = \int_\Omega \varphi\bullet\nu\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_{(\mathrm{id}\,\bullet\,\nu)}(x)|^2\,\mathrm{d}x, \tag{7}$$

where $[v\bullet\nu](x) := \int_{\mathbb{R}^n} v(s)\nu_x(\mathrm{d}s)$ and $\mathrm{id}: \mathbb{R}^n \to \mathbb{R}^n$ is the identity. The set of Young measures $\mathcal{Y}(\Omega;S) \subset L^\infty_\mathrm{w}(\Omega;\mathrm{rca}(S)) \cong L^1(\Omega;C(S))^*$ is the set of all weakly measurable essentially bounded mappings $x \mapsto \nu_x : \Omega \to \mathrm{rca}(S) \cong C(S)^*$ such that ν_x is a probability Radon measure supported on the sphere S for a.a. $x\in\Omega$; the adjective "weakly measurable" means that $v\bullet\nu$ is Lebesgue measurable for any $v \in C(S)$. Above $C(S)$ denotes the vector space of continuous functions on S endowed with the maximum norm and $\mathrm{rca}(S)$ is the vector space of regular countably additive set functions; cf. [32] for details. A natural embedding of a magnetization $m \in L^\infty(\Omega;\mathbb{R}^n)$, $|m(x)| = M_\mathrm{s}$, to $\mathcal{Y}(\Omega;S)$ is $\nu = i(m)$ defined by $\nu_x = \delta_{m(x)}$ with δ_s denoting the Dirac measure at $s \in S$. We say that a sequence $\{\nu^k\}_{k\in\mathbb{N}} \subset \mathcal{Y}(\Omega;S)$ converges weakly* to ν if $\lim_{k\to\infty}\langle\nu^k, f\rangle = \langle\nu, f\rangle$ for any $f \in L^1(\Omega;C(S))$ or, equally, for any $f = g\otimes v$ with $g \in L^1(\Omega)$ and $v \in C(S)$, where the tensorial notation means naturally $[g\otimes v](x,s) = g(x)v(s)$. From the last fact, we can also say that $\nu^k \to \nu$ weakly* if and only if w*-$\lim_{k\to\infty} v\bullet\nu^k = v\bullet\nu$ for all $v\in C(S)$. where the weak*-limit is understood in $L^\infty(\Omega)$. Considering the weak* topology on $L^\infty_\mathrm{w}(\Omega;\mathrm{rca}(S))$

makes $\mathcal{Y}(\Omega;S)$ convex, metrizable compact set containing densely the set of admissible magnetizations $\mathcal{A}$ if embedded via i.

As shown in [8, 21, 22], a correct relaxation (=natural extension) of (6) is

$$\begin{cases} \text{minimize} & \bar{G}(\nu) := \bar{E}(\nu) - H(\mathrm{id}\bullet\nu), \\ \text{subject to} & (4) \text{ with } m = \mathrm{id}\bullet\nu, \quad (\nu, u_m) \in \mathcal{Y}(\Omega;S) \times H^1(\mathbb{R}^n), \end{cases} \tag{8}$$

The model (8) represents a so-called *mesoscopic level* model because, a minimizing Young measure ν records some, but not full information about spatial oscillations of a minimizing sequence of (6) around each "macroscopic" point x through volume fractions described as the probability distribution ν_x. This information makes possible to describe the effective magnetic properties by means of the first moment, the "macroscopic" magnetization $m = \mathrm{id}\bullet\nu$, and moreover seems sufficient for designing a dissipative mechanism in a good agreement with experiments, which will be just exploited further.

2.2. Rate-independent dissipation

For usual loading regimes and magnetically hard materials, one must consider a certain dissipation. Moreover, the dissipation mechanism in ferromagnets can be influenced by impurities in the material without affecting substantially the stored energy. Hence, both mechanisms (energy storage and dissipation) are, to some extent, independent of each other and, as the dissipation mechanisms are determined on the atomistic level, it seems that the only efficient way how to incorporate them in a higher-level model is phenomenology.

Our, to some extent simplified, standpoint is that the amount of dissipated energy within the phase transformation from one pole to the other can be described by a single, phenomenologically given number (of the dimension $\mathrm{J/m^3{=}Pa}$) depending on the coercive force H_c [5]. Hence, we need to identify the particular poles according to the magnetization vector. Inspired by [18, 20] and considering L poles ($L = 2$ for uniaxial magnets or 6 or 8 for cubic magnets), we define a continuous mapping $\mathfrak{L} : S \to \triangle_L$ where $\triangle_L := \{\xi \in \mathbb{R}^L;\ \xi_i \geq 0,\ i = 1, ..., L,\ \sum_{i=1}^L \xi_i = 1\}$. In other words, $\{\mathfrak{L}_1, ..., \mathfrak{L}_L\}$ forms a *partition of unity* on S such that $\mathfrak{L}_i(s)$ is equal 1 if s is in i-th pole, i.e. $s \in S$ is in a neighborhood of i-th easy-magnetization direction. Of course, $\mathfrak{L}(m)$ in the (relative) interior of $\triangle_L$ indicates m in the region where no definite pole is specified. Hence $\mathfrak{L}$ plays the role of what is often called an *order parameter*.

In terms of the mesoscopic microstructure described by the Young measure ν, the "mesoscopic" order parameter is naturally defined as

$$\lambda = \Lambda\nu := \mathfrak{L}\bullet\nu \tag{9}$$

where $[\mathfrak{L}\bullet\nu](x) := \int_S \mathfrak{L}(s)\nu_x(\mathrm{d}s)$. Thus Λ is just a continuous extension of the mapping $m \mapsto \mathfrak{L}(m)$, i.e. if $\{m_k\}$ converges to ν weakly* in $L^\infty(\Omega;\mathbb{R}^n)$, then $\mathfrak{L}(m_k) \rightharpoonup \Lambda\nu$ weakly* in $L^\infty(\Omega;\mathbb{R}^L)$.

To described phenomenologically the dissipative energetics, one must prescribe a (pseudo)*potential of dissipative forces* as a function of the rate of λ. For rate-independent processes, this potential must be convex and homogeneous of degree one. Considering a norm $|\cdot|_L$ on $\mathbb{R}^L$, one can postulate $\varrho(\dot{\lambda}) = H_c|\dot{\lambda}|_L$. The energy needed to transform i-th pole to j-pole is then $H_c|e_i - e_j|_L$ with e_i the unit vector with 1 at the i-th position.

(At a given time t) will be described by the couple $q = q(t) \equiv (\nu, \lambda) = (\{\nu_{x,t}\}_{x\in\Omega}, \lambda(\cdot, t))$. Let us denote by $\mathcal{Q}$ the convex set of admissible configurations:

$$\mathcal{Q} := \Big\{ q = (\nu, \lambda) \in \mathcal{Y}(\Omega; S) \times L^\infty(\Omega; \mathbb{R}^L) \\ \lambda(x) \in \triangle_L, \quad \Lambda\nu = \lambda \text{ for a.a. } x \in \Omega \Big\}. \tag{10}$$

Seemingly, the volume fraction λ is redundant due to the link $\lambda = \Lambda\nu$ but it is advantageous to include it separately as a component of q because ν and λ will have a different quality, cf. Definition 2.5 below. Here and in what follows, we use the convention to denote by calligraphical letters objects related with this q-configuration.

We will see that the desired effect will be obtained by choosing the specific dissipation potential $\varrho : \mathbb{R}^L \to \mathbb{R}$ in the form:

$$\varrho(\dot{\lambda}) = H_c|\dot{\lambda}|_L\,. \tag{11}$$

For the analysis below, we will need to consider rather a certain *regularization* of the stored energy $\mathcal{E}$ which would control spatial smoothness of λ. For this, we will augment $\mathcal{E}$ by a higher-order term

$$\mathcal{E}_\rho(\nu, \lambda) := \bar{E}(\nu) + \begin{cases} \rho||\lambda||^2_{H^\alpha(\Omega;\mathbb{R}^L)} & \text{if } \lambda \in H^\alpha(\Omega; \mathbb{R}^L), \\ +\infty & \text{otherwise,} \end{cases} \tag{12}$$

where $H^\alpha(\Omega) \equiv W^{\alpha,2}(\Omega)$ denotes the usual Sobolev-Slobodetskiĭ space and where we assume

$$\alpha, \rho > 0, \text{ fixed.} \tag{13}$$

From now on, we will work with this regularized relaxed stored energy $\mathcal{E}_\rho$ rather than $\mathcal{E}$.

Remark 2.1. The regularizing term in (12) represents higher-order energies associated to spatial changes of λ. We admit that its physical meaning is not entirely clear. In any way, analogous terms were obtained in mechanics as a limit of the so-called Ericksen-Timoshenko beam. We refer to [23, 24] for details.

2.3. Formulation of the problem

Following [16] we define the "dissipation distance" by ("co" denotes the convex hull):

$$d(\lambda_1, \lambda_2) := \inf \Big\{ \int_0^1 \varrho(\frac{\mathrm{d}\lambda}{\mathrm{d}t})\,\mathrm{d}t; \quad \lambda \in C^1\big([0,1]; \mathbb{R}^L\big), \\ \lambda(t) \in \mathrm{co}\mathfrak{L}(S),\ \lambda|_{t=0} = \lambda_1,\ \lambda|_{t=1} = \lambda_2 \Big\}. \tag{14}$$

Let us still introduce the total "dissipation distance"

$$\mathcal{D}(q_1, q_2) := \int_\Omega d(\lambda_1, \lambda_2)\,\mathrm{d}x, \qquad q_i = (\nu_i, \lambda_i). \tag{15}$$

The following properties of d and $\mathcal{D}$ will be used:

Lemma 2.2. *The "distance" d satisfies the triangle inequality, i.e.*

$$d(\lambda_1,\lambda_2) \le d(\lambda_1,\lambda_3) + d(\lambda_3,\lambda_2) \tag{16}$$

for all $\lambda_i \in \mathrm{co}\mathfrak{L}(S)$, $i = 1,2,3$, and the following holds:

$$d(\lambda_1,\lambda_2) = H_\mathrm{c}|\lambda_1 - \lambda_2|_L\,. \tag{17}$$

Hence, the distance $\mathcal{D}$ satisfies the triangle inequality, i.e.

$$\forall q_1,q_2,q_3 \in \mathcal{Q}: \qquad \mathcal{D}(q_1,q_2) \le \mathcal{D}(q_1,q_3) + \mathcal{D}(q_3,q_2). \tag{18}$$

Proof. This is easy. □

Let us abbreviate the Gibbs energy by

$$\mathcal{G}(t,q) := \mathcal{E}_\rho(q) - \langle \mathcal{H}(t), q\rangle, \tag{19}$$

where

$$\langle \mathcal{H}(t), q\rangle = \big[H(t)\big](\mathrm{id}\bullet\nu) = \langle \nu, h(\cdot,t)\otimes \mathrm{id}\rangle. \tag{20}$$

Let us agree to identify quite naturally the mapping $t \mapsto \nu(t) = \{[\nu(t)]_x\}_{x\in\Omega}$ with a Young measure $(x,t) \mapsto \nu_{x,t}$.

Definition 2.3. We say that a process $q = q(t)$ is stable if

$$\forall \tilde q \in \mathcal{Q}: \qquad \mathcal{G}(t,q) \le \mathcal{G}(t,\tilde q) + \mathcal{D}(q(t),\tilde q) \tag{21}$$

for all $t \in [0,T]$.

An important notion is the so-called set of stable states, $S(t)$, at a time instant t

$$S(t) = \{q \in \mathcal{Q};\ \forall \tilde q \in \mathcal{Q} : \mathcal{G}(t,q) \le \mathcal{G}(t,\tilde q) + \mathcal{D}(q,\tilde q)\}\,. \tag{22}$$

Definition 2.4. We say that the process $q = q(t)$ satisfies the energy inequality if for a.a. $s \in [0,T]$ and a.a. $t \in [0,T]$, $s \le t$,

$$\underbrace{\mathcal{G}(t,q(t))}_{\text{effective Gibbs' energy at time } t} + \underbrace{\mathrm{Var}(\mathcal{D},q;s,t)}_{\text{dissipated energy}} \le \underbrace{\mathcal{G}(s,q(s)))}_{\text{Gibbs' energy at time 0}} - \underbrace{\int_s^t \left\langle \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}, q(\theta)\right\rangle \mathrm{d}\theta}_{\text{reduced work of external field}} \tag{23}$$

where the total variation over the time interval $[s,t]$ is defined standardly, without using explicitly any time derivative, as

$$\begin{aligned}\mathrm{Var}(\mathcal{D},q;s,t) \ &:= \ \sup\sum_{i=1}^{J} \mathcal{D}(q(t_{i-1}),q(t_i)) \\ &\equiv \ \sup\sum_{i=1}^{J}\int_\Omega d(\lambda(t_{i-1}),\lambda(t_i))\mathrm{d}x,\end{aligned} \tag{24}$$

where the supremum is taken over all $J \in \mathbb{N}$ and over all partitions of $[s,t]$ in the form $s = t_0 < t_1 < ... < t_{J-1} < t_J = t$.

Definition 2.5. The process $q = q(t)$, $q \equiv (\nu, \lambda)$, will be considered as a solution if $\nu \in \mathcal{Y}(\Omega \times [0,T]; S)$, $\lambda \in \mathrm{BV}([0,T]; L^1(\Omega; \mathbb{R}^L))$ and $q(t) \in \mathcal{Q}$ for all $t \in [0,T]$, and it is stable in the sense (21) for all $t \in [0,T]$ and satisfies the energy inequality (23) for a.a. $s, t \in [0,T]$, $s \le t$.

Remark 2.6. By the "reduced work" in (23) we mean (up to a sign) the usual work, i.e. $\int_s^t \left\langle \mathcal{H}(\theta), \frac{\mathrm{d}q}{\mathrm{d}t}(\theta) \right\rangle \mathrm{d}\theta$, but reduced by $\langle \mathcal{H}(s), q(s) \rangle - \langle \mathcal{H}(t), q(t) \rangle$ which is just the gap between Gibbs' and Helmholtz' energies at time instances s and t.

3. Incremental problems

The existence of a response q with the above mentioned properties was shown even in a more general case in [28] by a semi-discretization in time, using the implicit Euler scheme. For simplicity, let us consider an equi-distant partition of the time interval $[0,T]$ with a time step $\tau > 0$, assuming T/τ an integer. Even more, we consider a sequence of τ's converging to zero and such that, τ_i/τ_{i+1} is an integer, i.e. each next partition is a refinement of the preceding one.

Then we put $q_\tau^0 = q_0$, a given initial condition, and, for $k = 1, ..., T/\tau$ we define q_τ^k recursively as a solution of the minimization problem

$$\left\{ \begin{array}{ll} \text{Minimize} & I(q) := \mathcal{G}(k\tau, q) + \mathcal{D}(q_\tau^{k-1}, q) \\ \text{subject to} & q \equiv (\nu, \lambda) \in \mathcal{Q}, \end{array} \right. \tag{25}$$

where $\mathcal{Q}$ is from (10), $\mathcal{G}$ is from (19), and $\mathcal{D}$ from (15). If a solution (i.e. a *global* minimizer) to (25) is not unique, we just take an arbitrary one for q_τ^k. Then we define the piecewise constant interpolation $q_\tau \in L^\infty(0,T; L^\infty_{\mathrm{w}}(\Omega; \mathrm{rca}(S)) \times L^\infty(\Omega; \mathbb{R}^L))$ so that $q_\tau|_{((k-1)\tau, k\tau]} = q_\tau^k$ for $k = 1, ..., T/\tau$ while for $t = 0$ we put $q_\tau(0) = q_0$. Besides, assuming that for some $i \in \mathbb{N}$

$$h \in W^{1,1}(0, iT; L^1(\Omega; \mathbb{R}^n)),\ h(\cdot, t + jT) = h(\cdot, t)\ ,\ 1 \le j \le i-1\ ,\ t \in [0,T] \tag{26}$$

we have certainly $\mathcal{H} \in C(0, iT; L^1(\Omega; C(S))$ and we can define the piece-wise constant approximation of $\mathcal{H}$, denoted by $\mathcal{H}_\tau$, by $\mathcal{H}_\tau(t) = \mathcal{H}(k\tau)$ for $t \in ((k-1)\tau, k\tau]$ and by $\mathcal{H}_\tau(t) = \mathcal{H}(0)$ for $t = 0$. Besides, we will still need the piecewise affine interpolation, denoted by $\mathcal{H}_\tau^{\mathrm{aff}}$, i.e. $\mathcal{H}_\tau^{\mathrm{aff}}$ is affine in time if restricted on the interval $[(k-1)\tau, k\tau]$ for $k = 1, ..., T/\tau$ and $\mathcal{H}_\tau^{\mathrm{aff}}(k\tau) = \mathcal{H}(k\tau)$ for $k = 0, ..., T/\tau$. Also, we will assume that the initial condition q_0 is admissible and even stable:

$$q_0 \in \mathcal{Q} \quad \text{and} \quad \mathcal{E}_\rho(q_0) \le \mathcal{E}_\rho(\tilde{q}) + \mathcal{D}(q_0, \tilde{q}) + \langle \mathcal{H}(0), q_0 - \tilde{q} \rangle \qquad \forall \tilde{q} \in \mathcal{Q}; \tag{27}$$

note that it implies, in particular, that $\mathcal{E}_\rho(q_0) < +\infty$.

Let us define a sufficiently large set $\mathcal{P}$ where the values of all the processes $q_\tau(\cdot)$ will certainly live; here it is natural to put

$$\mathcal{P} := \Big\{ (\nu, \lambda) \in \mathcal{Q};\quad ||\lambda||_{H^\alpha(\Omega; \mathbb{R}^L)} \le C_1 \Big\}; \tag{28}$$

the constant C_1 can be now considered arbitrary but sufficiently large, and will be fixed later, see (31). Note that (27) says, in particular, that $q_0 \in \mathcal{P}$. We will endow $\mathcal{P}$ by the (weak*×weak)-topology of $L^\infty_{\mathrm{w}}(\Omega; \mathrm{rca}(S)) \times H^\alpha(\Omega; \mathbb{R}^L)$.

Lemma 3.1. *The set $\mathcal{P}$ is compact.*

Proof. It follows from the weak* compactness of $\mathcal{Y}(\Omega;S)$, the weak compactness of the balls in $H^\alpha(\Omega;\mathbb{R}^L)$ and the closeness of the set $\triangle_L$ and continuity of Λ involved in (10). □

The following theorem was basically proven in [28]. We include the proof for the convenience of the reader.

Proposition 3.2. *Let* (13), (26) *and* (27) *hold. Let $q_\tau = (\nu_\tau, \lambda_\tau)$ be a solution constructed recursively from solutions to* (25) *at the prescribed time increments. Moreover, this q_τ is stable in the sense of Definition* 2.3 *with $\mathcal{H}_\tau$ taken instead of $\mathcal{H}$, i.e.*

$$\forall \tilde{q} \in \mathcal{Q}: \qquad \mathcal{E}_\rho(q_\tau(t)) \leq \mathcal{E}_\rho(\tilde{q}) + \mathcal{D}(q_\tau(t),\tilde{q}) + \langle \mathcal{H}_\tau(t), q_\tau(t) - \tilde{q}\rangle \tag{29}$$

for all $t \in [0,T]$, and satisfies the two-side approximate energy inequality

$$\begin{aligned}\int_{s_1}^{t_1} \Big\langle -\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}, q_\tau(\theta)\Big\rangle \,\mathrm{d}\theta &\leq \mathcal{G}(t_1, q_\tau(t_1)) + \mathrm{Var}(\mathcal{D}, q_\tau; s_1, t_1) - \mathcal{G}(s_1, q_\tau(s_1)) \\ &\leq \int_{s_1}^{t_1} \Big\langle -\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}, q_\tau(\theta - \tau)\Big\rangle \,\mathrm{d}\theta,\end{aligned} \tag{30}$$

where naturally $q_\tau(t) := q_0$ for $t < 0$ and $s_1 \leq t_1$ belong to the set $\{k\tau\}_{k=0}^{T/\tau}$. Also, the following a-priori estimates hold:

$$||\lambda_\tau||_{L^\infty(0,T;H^\alpha(\Omega;\mathbb{R}^L)\cap L^\infty(\Omega;\mathbb{R}^L))\cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^L)} \leq C_1, \tag{31}$$

$$||\nu_\tau||_{L^\infty(0,T;L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S))} \leq C_2, \tag{32}$$

$$||\mathfrak{G}_\tau||_{\mathrm{BV}([0,T])} \leq C_3, \tag{33}$$

where $\mathfrak{G}_\tau(t) := \mathcal{E}_\rho(q_\tau(t)) - \langle \mathcal{H}_\tau(t), q_\tau(t)\rangle$ denote Gibbs's energy of the approximate trajectory.

Proof. The existence of a solution to (25) follows from the coerciveness of I in λ, where $q = (\nu,\lambda)$ and from the sequential weak lower semicontinuity of I. Taking C_1 in (28) sufficiently large we can replace $\mathcal{Q}$ by $\mathcal{P}$ in (25). As in [20, Thm. 3.4], by using successively (25) and (18), we get

$$\mathcal{G}(k\tau, q_\tau^k) \leq \mathcal{G}(k\tau,\tilde{q}) + \mathcal{D}(q_\tau^{k-1},\tilde{q}) - \mathcal{D}(q_\tau^{k-1}, q_\tau^k) \leq \mathcal{G}(k\tau,\tilde{q}) + \mathcal{D}(q_\tau^k,\tilde{q}) \tag{34}$$

for any $k = 1, ..., T/\tau$. In view of the definition of q_τ and $\mathcal{H}_\tau$, it just means that the stability condition (29) holds for all $t \in (0,T]$. For $t = 0$, the stability follows from the stability of the initial condition.

Moreover, as in [20, Formula (2.13)], we can test the first inequality in (34) by $\tilde{q} = q_\tau^{k-1} \in \mathcal{Q}$, and sum it for all $k = J, ..., K$. After a small re-arrangement, it gives

$$\begin{aligned}&\mathcal{G}(K\tau, q_\tau^K) - \mathcal{G}(J\tau, q_\tau^J) + \sum_{k=J}^{K} \mathcal{D}(q_\tau^{k-1}, q_\tau^k) \\ \leq\ &\sum_{k=J}^{K-1} \langle \mathcal{H}_\tau^k - \mathcal{H}_\tau^{k+1}, q_\tau^k\rangle = \int_{J\tau}^{K\tau} \Big\langle -\frac{\mathrm{d}\mathcal{H}_\tau^{\mathrm{aff}}}{\mathrm{d}t}, q_\tau(t-\tau)\Big\rangle \,\mathrm{d}t,\end{aligned} \tag{35}$$

from which the second inequality in (30) follows because

$$\mathrm{Var}(\mathcal{D},q_\tau;0,K\tau)=\sum_{k=J}^{K}\mathcal{D}(q_\tau^{k-1},q_\tau^k) \tag{36}$$

and because

$$\int_{J\tau}^{K\tau}\Big\langle\frac{\mathrm{d}\mathcal{H}_\tau^{\mathrm{aff}}}{\mathrm{d}t},q_\tau(t-\tau)\Big\rangle\,\mathrm{d}t=\int_{J\tau}^{K\tau}\Big\langle\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t},q_\tau(t-\tau)\Big\rangle\,\mathrm{d}t. \tag{37}$$

As (36) concerns, realize that q_τ is piece-wise constant with $K-J-1$ jumps at the time instances $t=k\tau$, $k=J,...,K-1$, hence its total variation according the formula (24) can explicitly be evaluated as (36). As (37) concerns, the particular construction of $\mathcal{H}_\tau^{\mathrm{aff}}$ as a piece-wise affine interpolation between the values $\{\mathcal{H}(k\tau)\}_{k=0,...,K}$ yields that, for $k=J,...,K$, it holds $\int_{(k-1)\tau}^{k\tau}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\tau^{\mathrm{aff}}\,\mathrm{d}t=\mathcal{H}(k\tau)-\mathcal{H}((k-1)\tau)=\int_{(k-1)\tau}^{k\tau}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}\,\mathrm{d}t$ (in Bochner's sense), which remains equal if tested by any function constant of the subinterval $((k-1)\tau,k\tau]$, in particular by $q_\tau(\cdot-\tau)$.

Using (34) to express the stability of q_τ^{k-1} with respect to q_τ^k, we get

$$\mathcal{G}(k\tau,q_\tau^k)+\mathcal{D}(q_\tau^{k-1},q_\tau^k)-\mathcal{G}((k-1)\tau,q_\tau^{k-1})\geq\langle\mathcal{H}_\tau^{k-1}-\mathcal{H}_\tau^k,q_\tau^k\rangle. \tag{38}$$

Summing up (38) for $J\leq k\leq K$, we obtain

$$\begin{aligned}&\mathcal{G}(K\tau,q_\tau^K)-\mathcal{G}(J\tau,q_\tau^J)+\sum_{k=J}^{K}\mathcal{D}(q_\tau^{k-1},q_\tau^k) \qquad (39)\\ \geq\ &\sum_{k=J\tau}^{K-1}\langle\mathcal{H}_\tau^k-\mathcal{H}_\tau^{k+1},q_\tau^{k+1}\rangle\ =\int_{J\tau}^{K\tau}\Big\langle-\frac{\mathrm{d}\mathcal{H}_\tau^{\mathrm{aff}}}{\mathrm{d}t},q_\tau(t)\Big\rangle\,\mathrm{d}t,\end{aligned}$$

which gives the first inequality in (30) when again (37) (now with q_τ instead of $q_\tau(\cdot-\tau)$) taken into account.

The estimate (32) is obvious because ν_τ is a Young measure.

Note that the $L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^L))$ estimate in (31) is trivial since λ_τ ranges a set $\triangle_L$ which is bounded in $\mathbb{R}^L$.

By (17), we have

$$\begin{aligned}\sum_{k=J}^{K}\mathcal{D}(q_\tau^{k-1},q_\tau^k)\ &=\ \sum_{k=J}^{K}\int_\Omega H_\mathrm{c}\big|\lambda_\tau^{k-1}-\lambda_\tau^k\big|_L\,\mathrm{d}x \qquad (40)\\ &=\ H_\mathrm{c}\Big\|\sum_{k=J}^{K}\big|\lambda_\tau^{k-1}-\lambda_\tau^k\big|_L\Big\|_{L^1(\Omega;\mathbb{R}^L)}.\end{aligned}$$

Coming back to (35), we get the left-hand side of (40) bounded and we get $\sum_{k=1}^K\big|\lambda_\tau^k-\lambda_\tau^{k-1}\big|_L$ bounded as well in the L^1-norm and hence the BV-estimate in (31) follows by considering $K=T/\tau$.

Finally, by (34) and (38), we have the estimate

$$\begin{aligned}&\big|\mathcal{G}(k\tau,q_\tau^k)-\mathcal{G}((k-1)\tau,q_\tau^{k-1})\big|\\ &\le\big|\mathcal{D}(q_\tau^{k-1},q_\tau^k)\big|+\max\Big(\big|\langle\mathcal{H}_\tau^k-\mathcal{H}_\tau^{k-1},q_\tau^{k-1}\rangle\big|,\big|\langle\mathcal{H}_\tau^k-\mathcal{H}_\tau^{k-1},q_\tau^k\rangle\big|\Big)\end{aligned}\tag{41}$$

for $k=1,...,T/\tau$. By (31) we can see that $\sum_{k=1}^{T/\tau}|\mathcal{D}(q_\tau^{k-1},q_\tau^k)|$ is bounded independently of τ. Thanks to the assumption (26) and the L^∞-estimate (32) for ν_τ, also the last term in (41) is limitedly summable; indeed, one can estimate

$$\begin{aligned}&\sum_{k=1}^{T/\tau}\max\Big(\big|\langle\mathcal{H}_\tau^k-\mathcal{H}_\tau^{k-1},q_\tau^{k-1}\rangle\big|,\big|\langle\mathcal{H}_\tau^k-\mathcal{H}_\tau^{k-1},q_\tau^k\rangle\big|\Big)\qquad(42)\\ \le\ &\max_{k=0,...,T/\tau}\big\|\mathrm{id}\bullet\nu_\tau^k\big\|_{L^\infty(\Omega;\mathbb{R}^n)}\sum_{k=1}^{T/\tau}\big\|h_\tau^k-h_\tau^{k-1}\big\|_{L^1(\Omega;\mathbb{R}^n)}\\ \le\ &M_{\rm s}\sum_{k=1}^{T/\tau}\Big\|\int_{(k-1)\tau}^{k\tau}\frac{\partial h}{\partial t}\mathrm{d}t\Big\|_{L^1(\Omega;\mathbb{R}^n)}\le M_{\rm s}\sum_{k=1}^{T/\tau}\int_{(k-1)\tau}^{k\tau}\Big\|\frac{\partial h}{\partial t}\Big\|_{L^1(\Omega;\mathbb{R}^n)}\mathrm{d}t\\ =\ &M_{\rm s}\Big\|\frac{\partial h}{\partial t}\Big\|_{L^1(\Omega\times[0,T];\mathbb{R}^n)}\end{aligned}$$

which is bounded by the assumption (26); here we denoted $h_\tau^k:=h(\cdot,k\tau)$. Altogether, (41) gives $||\mathfrak{G}_\tau||_{\mathrm{BV}([0,T])}=\sum_{k=1}^{T/\tau}|\mathcal{G}(k\tau,q_\tau^k)-\mathcal{G}((k-1)\tau,q_\tau^{k-1})\big|$ bounded independently of τ, which proves (33). □

4. Auxiliary macroscopic problems

Our aim is to show that in a special situation one can show uniqueness of the macroscopic magnetization arising in the solution to (25). We recall that the magnetization is the first moment of the appropriate Young measure, i.e.

$$m(x)=\int_S s\nu_x(\mathrm{d}s)\ \text{for a.a. } x\in\Omega.\tag{43}$$

In the sequel, we sometimes write $m=\bar\nu$ instead of (43). Our main assumption is that $\mathfrak{L}$ has an affine extension to the ball $B=\{s\in\mathbb{R}^n;|s|\le M_{\rm s}\}$. In other words, we suppose that there is an affine mapping $\ell:B\to\triangle_L$ such that

$$\mathfrak{L}=\ell|_S.\tag{44}$$

As an example we take e.g. $L=2$, i.e., an uniaxial magnet and set $\mathcal{L}=(\mathcal{L}_1,\mathcal{L}_2)$ with $\mathcal{L}_1(s)=\frac12(1+s_2/M_{\rm s})$ and $\mathcal{L}_2(s)=\frac12(1-s_2/M_{\rm s})$. This together with $|b|_L:=M_{\rm s}\max(|b_1|,|b_2|)$ for any $b\in\mathbb{R}^2$ gives the dissipation potential (11) $\varrho(\dot\lambda)=H_{\rm c}|\mathrm{d}m_2/\mathrm{d}t|$, where $m=(m_1,m_2,m_3)$, if $n=3$. Therefore the dissipation is proportional to $|\mathrm{d}m_2/\mathrm{d}t|$. Similar magnetic dissipation terms has been already used by Visintin [29]. Obviously, we can take $\ell=(\ell_1,\ell_2)$ with $\ell_1(s)=\frac12(1+s_2/M_{\rm s})$ and $\ell_2(s)=\frac12(1-s_2/M_{\rm s})$ for any $s\in B$. The idea to relate dissipation only to some components of m is supported by physical experiments, where one observes almost no hysteresis if the specimen is magnetized perpendicularly to the easy axis; cf. [25]. However, our method works in cases when dissipation depends on all components of m, too.

Then we define

$$\mathcal{Q}^{**} := \Big\{ q^{**} = (m,\lambda) \in L^2(\Omega;\mathbb{R}^n)\times L^\infty(\Omega;\mathbb{R}^L) \\ |m(x)| \le M_{\rm s},\ \lambda(x) \in \triangle_L,\ \ \ell(m) = \lambda \text{ a.e. in } \Omega \Big\} \tag{45}$$

and

$$\mathcal{E}^{**}_\rho(m,\lambda) := E^{**}(m,\lambda) + \begin{cases} \rho||\lambda||^2_{H^\alpha(\Omega;\mathbb{R}^L)} & \text{if } \lambda \in H^\alpha(\Omega;\mathbb{R}^L), \\ +\infty & \text{otherwise,} \end{cases} \tag{46}$$

where

$$E^{**}(m,\lambda) := \int_\Omega \varphi^{**}(m(x))\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m(x)|^2\,\mathrm{d}x$$

with φ^{**} being the convex envelope of $\hat\varphi$ where

$$\hat\varphi(m) = \begin{cases} \varphi(m) & \text{if } |m| = M_{\rm s} \\ +\infty & \text{otherwise.} \end{cases}$$

Let us abbreviate the appropriate Gibbs energy by

$$\mathcal{G}^{**}(t,q^{**}) := \mathcal{E}^{**}_\rho(q^{**}) - (H(t),q^{**})\,, \tag{47}$$

where $(H(t),q^{**}) := \int_\Omega h(t,x)\cdot m(x)\,\mathrm{d}x$. The total dissipation distance depends only on λ therefore we will write for $q_1^{**} = (m_1,\lambda_1), q_2^{**} = (m_2,\lambda_2) \in \mathcal{Q}^{**}$ without ambiguity

$$\mathcal{D}(q_1^{**},q_2^{**}) = \int_\Omega H_{\rm c}|\lambda_1 - \lambda_2|_L\,\mathrm{d}x,$$

Note that under the assumption (44)

$$\mathcal{D}(q_1^{**},q_2^{**}) = \int_\Omega H_{\rm c}|\ell(m_1(x)) - \ell(m_2(x))|_L\,\mathrm{d}x \tag{48}$$

and that the dissipation distance is convex in m_1, for instance.

Similarly we define an auxiliary minimization problem in terms of (m,λ).

$$\begin{cases} \text{Minimize} & I^{**}(q^{**}) := \mathcal{G}^{**}(k\tau,q^{**}) + \mathcal{D}(q_\tau^{**k-1},q^{**}) \\ \text{subject to} & q^{**} \equiv (m,\lambda) \in \mathcal{Q}^{**}, \end{cases} \tag{49}$$

If a solution (i.e. a *global* minimizer) to (49) is not unique, we just take an arbitrary one for q_τ^{**k}. Then we define the piecewise constant interpolation $q_\tau^{**} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)\times L^\infty(\Omega;\mathbb{R}^L))$ so that $q_\tau|_{((k-1)\tau,k\tau]} = q_\tau^{**k}$ for $k = 1,...,T/\tau$ while for $t = 0$ we put $q_\tau^{**}(0) = q_0^{**}$.

Again, we suppose that

$$q_0^{**} \in \mathcal{Q}^{**} \quad \text{and} \quad \mathcal{E}_\rho^{**}(q_0^{**}) \le \mathcal{E}_\rho^{**}(\tilde{q}) + D(q_0^{**}, \tilde{q}) + (H(0), q_0^{**} - \tilde{q}) \qquad \forall \tilde{q} \in \mathcal{Q}^{**}. \tag{50}$$

Similarly, as before we define the set

$$\mathcal{P}^{**} := \Big\{(m, \lambda) \in \mathcal{Q}^{**}; \quad \|\lambda\|_{H^\alpha(\Omega;\mathbb{R}^L)} \le C_1\Big\}. \tag{51}$$

We will endow $\mathcal{P}^{**}$ by the (weak×weak)-topology of $L^2(\Omega;\mathbb{R}^n) \times H^\alpha(\Omega;\mathbb{R}^L)$.

The following lemma is obvious.

Lemma 4.1. *The set $\mathcal{P}^{**}$ is compact.*

Proposition 4.2. *Let* (13)*,* (26) *and* (27) *hold. Then $q_\tau^{**} = (m_\tau, \lambda_\tau)$ "constructed" recursively by* (49) *does exist.*

Proof. The objective function I^{**} is convex and continuous in the strong topology of $L^2(\Omega;\mathbb{R}^n) \times H^\alpha(\Omega;\mathbb{R}^L)$ and therefore weakly lower semicontinuous. Moreover, I^{**} is coercive in λ. The assertion therefore follows by the direct method. □

Proposition 4.3. *If* (3) *and* (44) *hold we have* min(25)= min(49).

Proof. Let $(\nu, \lambda) \in \mathcal{Q}$ and take $(m, \lambda) \in \mathcal{Q}^{**}$ such that $\bar{\nu} = m$. Then we have by the Jensen inequality

$$\int_\Omega \int_S \varphi(A)\nu_x(\mathrm{d}A)\,\mathrm{d}x \ge \int_\Omega \varphi^{**}(m(x))\,\mathrm{d}x.$$

Note that other terms in I and I^{**} are the same. In particular, note that $\int_S \mathfrak{L}(A)\nu_x(\mathrm{d}A) = \int_S \ell(A)\nu_x(\mathrm{d}A) = \ell(m(x))$ for almost all $x \in \Omega$. Thus, $\min I^{**} \le \min I$.

Conversely, if $(m, \lambda) \in \mathcal{Q}^{**}$, by the definition of φ^{**} we can always find $(\nu, \lambda) \in \mathcal{Q}$ (see [22, p. 93]) such that

$$\varphi^{**}(m(x)) = \int_S \varphi(s)\nu_x(\mathrm{d}s) \tag{52}$$

and

$$m(x) = \int_S s\nu_x(\mathrm{d}s). \tag{53}$$

Therefore, for a given $(m, \lambda) \in \mathcal{Q}^{**}$ we always find $(\nu, \lambda) \in \mathcal{Q}$ such that $I^{**}(m, \lambda) = I(\nu, \lambda)$. The proposition is proved. □

Corollary 4.4. *Let* (3) *and* (44) *hold. If $(\nu, \lambda) \in \mathcal{Q}$ is a solution to* (25)*, $(m, \lambda) \in \mathcal{Q}^{**}$ with $m = \mathrm{id} \bullet \nu$, solves* (49) *and* (52) *holds for almost all $x \in \Omega$.*

Conversely, if $(m, \lambda) \in \mathcal{Q}^{**}$ solves (49) *and $(\nu, \lambda) \in \mathcal{Q}$ is such that* (52) *holds for almost all $x \in \Omega$ then $(\nu, \lambda) \in \mathcal{Q}$ is a solution to* (25)*.*

Proof. Suppose that $(\nu, \lambda) \in \mathcal{Q}$ is a minimizer of I with $m = \bar{\nu}$ and assume that

$$\int_\Omega \int_S \varphi(s)\nu_x(\mathrm{d}s)\,\mathrm{d}x > \int_\Omega \varphi^{**}(m(x))\,\mathrm{d}x.$$

Then $\min_{\mathcal{Q}} I > \min I^{**}_{\mathcal{Q}^{**}}$ which contradicts Proposition 4.3. Hence,

$$\int_\Omega \left(\int_S \varphi(s)\nu_x(\mathrm{d}s) - \varphi^{**}(m(x)) \right) \mathrm{d}x \leq 0 \tag{54}$$

but as $\int_S \varphi(s)\nu_x(\mathrm{d}s) - \varphi^{**}(m(x)) \geq 0$ for a.a. $x \in \Omega$ we have equality in (54), $(m, \lambda) \in \mathcal{Q}^{**}$ is solution to (49) and (52) holds for a.a. $x \in \Omega$.

If $(m, \lambda) \in \mathcal{Q}^{**}$ solves (49), $m = \mathrm{id} \bullet \nu$ and (52) holds then obviously (ν, λ) solves (49). □

Proposition 4.5. *Let $\varphi(s) = \alpha \sum_{i=1}^{n-1} s_i^2$, $\alpha > 0$, $|s| = M_{\mathrm{s}}$, and let (44) hold. Then the problem (49) has a unique solution for any $T/\tau \geq k \geq 1$.*

Proof. Under the assumptions $\varphi^{**}(s) = \alpha \sum_{i=1}^{n-1} s_i^2$ for all $s \in \mathbb{R}^n$, $|s| \leq M_{\mathrm{s}}$; cf. [8]. We will proceed by induction. Suppose that $q_\tau^{**k-1} \in \mathcal{Q}^{**}$, is given uniquely.

Let $q^{**} = (m, \lambda), \hat{q}^{**} = (\hat{m}, \hat{\lambda}) \in \mathcal{Q}^{**}$ be two different minimizers to I^{**}. Then $\nabla u_m = \nabla u_{\hat{m}}$ a.e. in $\mathbb{R}^n$. Indeed, if they were different the convexity I^{**}, the strict convexity of the demagnetizing field energy, i.e. of $\|\cdot\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^n)}$, and the linearity of the map $\mathcal{Q}^{**} \to L^2(\mathbb{R}^n) : m \mapsto \nabla u_m$ would give us a magnetization $\theta q^{**} + (1-\theta)\hat{q}^{**} \in \mathcal{Q}^{**}$, $0 < \theta < 1$ which gives a strictly lower energy than q^{**} and $\hat{q}^{**}$. Similarly, as φ^{**} is strictly convex in first $(n-1)$ variables we get that $m_i = \hat{m}_i$ a.e. in Ω for $i = 1, \ldots, n-1$. Put $\beta := m\chi_\Omega - \hat{m}\chi_\Omega$. Then $\mathrm{div}\, \beta = 0$ because $u_\beta = 0$ a.e. in Ω, where u_β is calculated from (4). Moreover, the only nonzero component of β is the nth one. Therefore $\beta = (0, \ldots, \beta_n)$ with $\beta_n = \beta_n(x_1, \ldots, x_{n-1})$. Since β has a compact support we get $\beta = 0$ identically. Consequently $\lambda - \hat{\lambda} = \ell(m) - \ell(\hat{m}) = 0$. The proposition is proved. □

Remark 4.6.

(i) Note that the proof works also if $\rho = 0$, i.e., if there is no regularization in (46). Therefore our uniqueness result does not depend on a particular form of the regularization term.
(ii) Note that if $\rho > 0$ then due to (32) the solution to (49) is unique in $\mathcal{P}^{**}$.
(iii) The proof of Proposition 4.5 for $n = 2$ and $\ell = 0$ can be found in [4].

Due to proved uniqueness one can define a mapping $Z : \mathcal{P}^{**} \to L^2(\Omega; \mathbb{R}^n) \times H^\alpha(\Omega; \mathbb{R}^L)$ assigning to an initial condition $q_\tau^{**}(0)$ the solution at the final time T, i.e., $Z(q_\tau^{**}(0)) = q_\tau^{**}(T)$.

Proposition 4.7. *The mapping $Z : \mathcal{P}^{**} \to L^2(\Omega; \mathbb{R}^n) \times H^\alpha(\Omega; \mathbb{R}^L)$ is weakly sequentially continuous.*

Proof. Notice that Z is the composition of $\{Z_k\}_{k=1}^{T/\tau}$, $Z_k : \mathcal{P}^{**} \to L^2(\Omega; \mathbb{R}^n) \times H^\alpha(\Omega; \mathbb{R}^L)$: $Z_k(q_\tau^{**}((k-1)\tau)) = q_\tau^{**}(k\tau)$, i.e. $Z = Z_{T/\tau} \circ Z_{T/\tau - 1} \circ \ldots \circ Z_1$.

Without loss of generality we can take $k = 1$. Let us take a sequence of initial conditions $\{g_{0j}\}_{j\in\mathbb{N}} \subset \mathcal{P}^{**}$ such that $\lim_{j\to\infty} g_{0j} = g_0$ (in the topology of $\mathcal{P}^{**}$). Further, we denote $g_{0j} := (m_{0j}, \lambda_{0j})$, $g_0 := (m_0, \lambda_0)$, $g = (m, \lambda) \in \mathcal{P}^{**}$ and

$$F_j(g) := \mathcal{G}^{**}(\tau, g) + \mathcal{D}(g_{0j}, g)$$

and

$$F(g) := \mathcal{G}^{**}(\tau, g) + \mathcal{D}(g_0, g).$$

Then we have for any $g \in \mathcal{P}^{**}$

$$\begin{aligned} |F_j(g) - F(g)| &= |\mathcal{D}(g_{0j}, g) - \mathcal{D}(g_0, g)| \\ &= \left| \int_\Omega H_c(|\lambda_{0j}(x) - \lambda(x)|_L - |\lambda_0(x) - \lambda(x)|_L)\,\mathrm{d}x \right| \\ &\le \int_\Omega H_c \left| |\lambda_{0j}(x) - \lambda(x)|_L - |\lambda_0(x) - \lambda(x)|_L \right| \mathrm{d}x \\ &\le \int_\Omega H_c |\lambda_{0j}(x) - \lambda_0(x)|_L\,\mathrm{d}x = H_c \|\lambda_{0j} - \lambda_0\|_{L^1(\Omega;\mathbb{R}^L)}. \end{aligned} \tag{55}$$

The last term in (55) goes to zero as $j \to +\infty$ because of the compact embedding $H^\alpha(\Omega; R^L) \hookrightarrow L^1(\Omega;\mathbb{R}^L)$. Therefore $\lim_{j\to\infty} F_j = F$, uniformly in $\mathcal{P}^{**}$. Further $\{F_j\}_{j\in\mathbb{N}}$ and F have unique minima on $\mathcal{P}^{**}$. Let $\mathcal{P}^{**} \ni \gamma_j = \mathrm{argmin} F_j$, $\mathcal{P}^{**} \ni \gamma = \mathrm{argmin} F$. We see that F_j, $j \in \mathbb{N}$, are lower semicontinuous on $\mathcal{P}^{**}$. The same holds for F. Therefore F_j Γ-converges to F (see [6, Rem. 5.3]) and, consequently as minima of F_j and F are unique, by [6, Cor. 7.24], $\lim_{j\to\infty} \gamma_j = \gamma$ in the compact topology of $\mathcal{P}^{**}$. Thus, Z_1 is sequentially continuous and consequently Z is weakly sequentially continuous, as well. □

Corollary 4.8. *Z has a fixed point in $\mathcal{P}^{**}$, i.e., there is a solution to* (49) *giving a piecewise constant interpolation q_τ^{**} such that $q_\tau^{**}(0) = q_\tau^{**}(T)$.*

Proof. $\mathcal{P}^{**}$ is a closed convex and bounded subset of the reflexive Banach space $L^2(\Omega;\mathbb{R}^2) \times H^\alpha(\Omega;\mathbb{R}^L)$ equipped with the norm $\|(m,\lambda)\|_{L^2(\Omega;\mathbb{R}^2)\times H^\alpha(\Omega;\mathbb{R}^L)} = \|m\|_{L^2(\Omega;\mathbb{R}^n)} + \|\lambda\|_{H^\alpha(\Omega;\mathbb{R}^L)}$. By Proposition 4.7 Z is a weakly sequentially continuous mapping and, moreover, it maps $\mathcal{P}^{**}$ into itself. Therefore, by the Tychonoff fixed point theorem (see e.g. [7]), it has a fixed point. □

Corollary 4.8 shows that for a T-periodic external field h there exists a periodic solution q_τ^{**} recursively constructed from (49). Moreover, this periodic solution is fully determined by $q_\tau^{**}(0)$, i.e., by the initial condition.

Corollary 4.9. *Let* (26) *hold. Let $\varphi(s) = \alpha \sum_{i=1}^{n-1} s_i^2$, $\alpha > 0$, $|s| = M_s$. There is a periodic solution to* (25)*, i.e., there is a solution q_τ recursively constructed from solutions to* (25) *such that $q_\tau(0) = q_\tau(T)$.*

Proof. Since h is periodic, Corollary 4.8 gives the existence of a periodic solution $q_\tau^{**} = (m_\tau, \lambda_\tau)$ constructed from (49).

It suffices to find a Young measure ν such that (52) and (53) are satisfied with m_τ instead of m. Then $q_\tau := (\nu_\tau, \lambda_\tau)$ gives the desired periodic solution if it is kept fixed for given m_τ.

The existence of such ν_τ follows from Proposition 4.3. In order to make our construction explicit we follow [8] and take $m_\tau(x) \in \operatorname{span}\{e_n(x), e_\perp(x)\}$, where $e_n(x) \in \mathbb{R}^n$ is the unit vector with 1 on the n-th position and $e_\perp(x) \in \mathbb{R}^n$ is a unit vector orthogonal to $e_n(x)$. Then for a.a. $x \in \Omega$ $\nu_{\tau,x} = \omega(x)\delta_{s^+(x)} + (1-\omega(x))\delta_{s^-(x)}$, where

$$s^\pm(x) = \pm[M_\mathrm{s}^2 - (m_\tau(x)\cdot e_\perp(x))^2]^{1/2} e_n(x) + (m_\tau(x)\cdot e_\perp(x)) e_\perp(x)$$

and $\omega(x) = 1$ if $m_\tau(x) = M_\mathrm{s} e_\perp(x)$ and

$$\omega(x) = 0.5 + \frac{m_\tau(x)\cdot e(x)}{2[M_\mathrm{s}^2 - (m_\tau(x)\cdot e_\perp(x))^2]^{1/2}}$$

otherwise.

The stability of $q_\tau(0)$ follows from the periodicity of h and stability of $q_\tau(T)$ proved in (34). Indeed, we have for any $\tilde{q} \in \mathcal{Q}$

$$\begin{aligned} \mathcal{G}(0, q_\tau(0)) &= \mathcal{E}_\rho(q_\tau(0)) - \langle \mathcal{H}(0), q_\tau(0)\rangle \\ &= \mathcal{E}_\rho(q_\tau(T)) - \langle \mathcal{H}(T), q_\tau(T)\rangle \\ &= \mathcal{G}(T, q_\tau(T)) \\ &\le \mathcal{G}(T, \tilde{q}) + \mathcal{D}(\tilde{q}, q_\tau(T)) \\ &= \mathcal{G}(0, \tilde{q}) + \mathcal{D}(\tilde{q}, q_\tau(0)). \end{aligned}$$

which shows the stability of $q_\tau(0)$. □

The following lemma is a version of [17, Prop. 2.7]. We also follow its proof.

Lemma 4.10. *Let $\{(t_k, q_k)\}_k$ be a sequence such that $t_k \to t$ and $q_k \to q$ weakly* in $L^\infty_\mathrm{w}(\Omega; \mathrm{rca}(S)) \times H^\alpha(\Omega; \mathbb{R}^L)$ and let $q_k \in S(t_k)$ for all $k \in \mathbb{N}$. Then*

(i) $q \in S(t)$ *and*

(ii) $\lim_{k\to\infty} \mathcal{G}(t_k, q_k) = \mathcal{G}(t, q)$.

Proof. As $\nu_k \to \nu$ in $L^\infty_\mathrm{w}(\Omega; \mathrm{rca}(S))$ we have $\liminf_{k\to\infty} \int_\Omega \varphi \bullet \nu_k \,\mathrm{d}x \ge \int_\Omega \varphi \bullet \nu \,\mathrm{d}x$. Other terms in $\mathcal{E}_\rho$ are sequentially weakly lower semicontinuous. Note, in particular, that as $\nu_k \to \nu$ in $L^\infty_\mathrm{w}(\Omega; \mathrm{rca}(S))$ then $\mathrm{id} \bullet \nu_k \to \mathrm{id} \bullet \nu$ weakly in $L^2(\Omega; \mathbb{R}^n)$ which ensures the sequential weak lower semicontinuity of the magnetostatic energy term. Therefore, we have $\liminf_{k\to\infty} \mathcal{E}_\rho(q_k) \ge \mathcal{E}_\rho(q)$. Due to (26) we have $h \in C([0,T]; L^1(\Omega; \mathbb{R}^n))$, hence

$$\lim_{k\to\infty} \mathcal{H}(t_k, q_k) = \lim_{k\to\infty} \int_\Omega h(t_k, x)\cdot [\mathrm{id} \bullet \nu_k](x)\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega h(t_k, x)[\mathrm{id} \bullet \nu](x)\,\mathrm{d}x.$$

Altogether, it yields the lower semicontinuity of $\mathcal{G}$, i.e.,

$$\mathcal{G}(t, q) \le \liminf \mathcal{G}(t_k, q_k). \tag{56}$$

Further we have $\lim_{k\to\infty} \mathcal{D}(q_k, \tilde{q}) = \mathcal{D}(q, \tilde{q})$ for any $\tilde{q} \in \mathcal{Q}$ since by the triangle inequality

$$\lim_{k\to\infty} |\mathcal{D}(q_k, \tilde{q}) - \mathcal{D}(q, \tilde{q})| \le \mathcal{D}(q_k, q) = \lim_{k\to\infty} H_c \|\lambda_k - \lambda\|_{L^1(\Omega;\mathbb{R}^L)} = 0 \tag{57}$$

because of the compact embedding $H^\alpha(\Omega; \mathbb{R}^L)$ to $L^1(\Omega; \mathbb{R}^L)$.

Using lower semicontinuity of $\mathcal{G}$ and (57) we have

$$\mathcal{G}(t,q) \leq \liminf_{k\to\infty} \mathcal{G}(t_k,q_k) \leq \lim_{k\to\infty} \mathcal{G}(t_k,\tilde{q}) + \mathcal{D}(q_k,\tilde{q}) = \mathcal{G}(t,\tilde{q}) + \mathcal{D}(q,\tilde{q}). \tag{58}$$

Applying (58) with $\tilde{q} := q$ we get (ii) because all inequalities are in fact equalities. If we show that $q \in \mathcal{Q}$ (58) then implies (i). It is sufficient to show that $\lambda = \mathcal{L}\bullet\nu$. Clearly $\nu \in \mathcal{Y}(\Omega;S)$ because $\mathcal{Y}(\Omega;S)$ is sequentially compact; cf. [32]. We already have $\lambda_k \to \lambda$ strongly in $L^1(\Omega;\mathbb{R}^L)$, $\lambda_k = \mathcal{L}\bullet\nu_k$. Thus, $\lambda = \mathcal{L}\bullet\nu$ due to linearity of "$\bullet$". □

The following proposition uses the concept of nets from the general topology; cf. [9]. We only briefly recall this notion, which generalizes the notion of sequences, here. The set $\{x_\xi\}_{\xi\in\Xi} \subset X$ is a net if the index set Ξ is directed, i.e., partially ordered and such that any two elements have a common majorant. The net $\{x_\xi\}_{\xi\in\Xi}$ is said to be convergent to x_0 or $\lim_{\xi\in\Xi} x_\xi = x_0$ if for every neighborhood U of x_0 there is ξ_0 such that for all $\xi \succeq \xi_0$ $x_\xi \in U$. Corresponding to subsequences one considers finer nets. A net $\{y_{\tilde{\xi}}\}_{\tilde{\xi}\in\tilde{\Xi}}$ is a finer net than $\{x_\xi\}_{\xi\in\Xi}$ if there is a mapping $\phi : \tilde{\Xi} \to \Xi$ such that for any $\tilde{\xi} \in \tilde{\Xi}$ it holds that $y_{\tilde{\xi}} = x_{\phi(\xi)}$ and for any $\xi \in \Xi$ there is $\tilde{\xi} \in \tilde{\Xi}$ such that $\phi(\tilde{\xi}) \succeq \xi$ if $\tilde{\xi} \succeq \tilde{\xi}$. Compact sets enjoy the property that every net possesses a finer net that converges.

Proposition 4.11. *Let* (13)*, and* (26) *be valid. Then there are a process* $q = (\nu,\lambda) \in \mathcal{Y}(\Omega\times[0,T];S) \times \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^L))$, $q(0) = q(T)$ *and a net* $\{q_{\tau_\xi}\}_{\xi\in\Xi}$, $q_{\tau_\xi}(0) = q_{\tau_\xi}(T)$ *such that:*

(*i*) $\lim_{\xi\in\Xi} \lambda_{\tau_\xi}(t) = \lambda(t)$ *weakly* in* $L^\infty(\Omega;\mathbb{R}^L)$*, strongly in* $L^2(\Omega;\mathbb{R}^L)$ *for all* $t \in [0,T]$,

(*ii*) $\lim_{\xi\in\Xi} \nu_{\tau_\xi}(t) = \nu(t)$ *weakly* in* $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S))$ *for all* $t \in [0,T]$,

(*iii*) $\lim_{\xi\in\Xi} \mathcal{G}_{\tau_\xi}(t,q_{\tau_\xi}(t)) = \mathcal{G}(t,q)$ *for all* $t \in [0,T]$.

Moreover, $\lambda = \Lambda\nu$ *a.e. on* Ω *for every* $t \in [0,T]$ *and* q *thus obtained is a solution process according to Definition* 2.5 *and* $q(0) = q(T)$.

Proof. The proof is modification of the proof of [17, Theorem 4.3]. The point (*i*) follows from the a-priori estimate (32) and from the generalized Helly's theorem [20, Cor. 2.8] and [1]. Using them, one can select a subsequence indexed for simplicity by τ such that $\lambda_\tau(t) \to \lambda(t)$ weakly in $H^\alpha(\Omega;\mathbb{R}^L)$ and $\lambda_\tau(t) \to \lambda(t)$ weakly* in $L^\infty(\Omega;\mathbb{R}^L)$ for any $t \in [0,T]$. The compact embedding $H^\alpha(\Omega;\mathbb{R}^L) \subset L^2(\Omega;\mathbb{R}^L)$ gives the claimed convergence. Moreover, we assume that the subsequence is chosen in such a way that $\mathfrak{G}_\tau$ from (33) converges pointwise to some $\mathfrak{G}$ in $\mathrm{BV}([0,T])$.

To prove (*ii*), following [17], we notice that the set $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S)) \cong L^1(\Omega;C(S))^*$ endowed by the weak* topology is compact and metrizable. Then $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S))^{[0,T]}$ endowed by the product topology is compact as well by the Tychonoff theorem. Considering $\nu_\tau = \{\nu_\tau(t)\}_{t\in[0,T]}$ as elements of $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S))^{[0,T]}$ for $\tau > 0$ there is a finer net $\{\nu_{\tau_\xi}\}_{\xi\in\Xi}$ which converges in $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S))^{[0,T]}$. Let us denote its limit by $\nu := \{\nu(t)\}_{t\in[0,T]}$. Thus $\lim_{\xi\in\Xi} q_{\tau_\xi} = q(t)$.

In order to prove (*iii*) we exploit the definition of $\mathcal{G}_\tau$ and (21) so that we can write for a fixed $t \in [0,T]$, $q_\tau(t) \in S(\theta(t,\tau))$ such that $T \geq \theta(t,\tau) \geq t$ and $\lim_{\tau\to 0} \theta(t,\tau) = t$. Note that $\theta(t,\tau) = \min_{k\in\mathbb{N}\cup\{0\}}\{k\tau; k\tau \geq t\}$. Metrizability of the weak* topology on $\mathcal{P}$ allows us to work with a sequence $\{q_{\tau_{\xi_k}}\}_{k\in\mathbb{N}}$ which converges to $q(t)$ weakly*. Using $q_{\tau_{\xi_k}} \in$

$S(\theta(t,\tau_{\xi_k}))$ and applying Lemma 4.10 we get $q(t) \in S(t)$. In particular, we have $\lambda(t) = \mathfrak{L}\bullet\nu$. Realizing $\lim_{\tau\to 0}\mathcal{G}_\tau(t,q_\tau) = \mathcal{G}(t,q(t))$ similarly as in (58) we obtain $G(t,q(t)) \leq \mathfrak{G}(t) \leq \mathcal{G}(t,\tilde{q}) + \mathcal{D}(q(t),\tilde{q})$ Taking $\tilde{q} := q(t)$ we get $\mathcal{G}(t,q(t)) = \mathfrak{G}(t)$. Hence (iii) holds.

Let us show the energy inequality (23) for $s = 0$ and almost all t in $[0,T]$. We pass to the limit in (30) taking $s_1 = 0$ and t_1 such that it belongs to grid points of some partition of $[0,T]$. In particular, it also belongs to any finer partition, i.e. with τ small enough. The set of all such t_1's is dense in $[0,T]$. Moreover,

$$\begin{aligned}\lim_{\xi\in\Xi}\int_0^{t_1}\langle\mathcal{H}_{\tau_\xi}(t),q_{\tau_\xi}(t)\rangle\,\mathrm{d}t &= \lim_{\xi\in\Xi}\int_0^{t_1}\int_\Omega h_{\tau_\xi}(x,t)m_{\tau_\xi}(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^{t_1}\int_\Omega h(x,t)m(x,t)\,\mathrm{d}x\mathrm{d}t = \int_0^{t_1}\langle\mathcal{H}(t),q(t)\rangle\,\mathrm{d}t\end{aligned}\tag{59}$$

with h_τ denoting naturally the piecewise constant approximation of h inducing the approximation $\mathcal{H}_\tau$, and $m_\tau = \mathrm{id}\bullet\nu_\tau$ and $m = \mathrm{id}\bullet\nu$. By the convergence $\nu_\tau\rightharpoonup\nu$ claimed at the point (i), $m_\tau\rightharpoonup m$ weakly* in $L^\infty(\Omega;\mathbb{R}^n)$, and by (26), $||h-h_\tau||_{L^1(\Omega\times(0,T);\mathbb{R}^n)} = \mathcal{O}(\tau)$, hence (59) indeed holds.

The work of external field $\int_0^{t_1}\langle\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H},q_\tau(\cdot-\tau)\rangle\mathrm{d}t$, cf. the right-hand side of (30), converges to $\int_0^{t_1}\langle\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H},q\rangle\mathrm{d}t$ by similar argument as already used for (59); here we need that the shifted $m_\tau(\cdot-\tau)$ has the same weak* limit as m_τ, which can quite easily be proved by testing it by functions of the type $\chi_{[k\tau,l\tau]}(t)g(s)$ which forms a dense subset in $L^1(\Omega\times[0,T];\mathbb{R}^n)$.

Then, we can pass to the limit in (30), proving thus (23) for $s = 0$ and each t of the form $k\tau \in [0,T]$, $k = 1,...,T/\tau$, τ from the considered sequence of time steps. The (only countable) set of such t's is dense in $[0,T]$ and thus (23) must hold also at each $t \in [0,T]$ at which all functions involved in (23) are continuous. Those functions have, however, bounded variations and are thus continuous with the exception of at most countable number of points. Hence (23) holds for $s = 0$ everywhere on $[0,T]$ with the only exception of at most countable number of points, i.e. (23) (with $s = 0$) holds a.e. on $[0,T]$. In other words,

$$f(t) := \mathcal{G}(t,q(t)) - \mathcal{G}(0,q(0)) + \mathrm{Var}(q;0,t) - \int_s^t\langle\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t},q(\theta)\rangle\,\mathrm{d}\theta = 0 \tag{60}$$

for almost all $t \in [0,T]$. In particular, taking $0 \leq s \leq t$ we have that $f(t) - f(s) = 0$ for almost all $s,t \in [0,T]$. This means that (23) holds for almost all $s,t \in [0,T]$ even as equality. □

Remark 4.12. Refining the argumentation one can even prove that $f(t) = 0$ for all $t \in [0,T]$. See [17] for details.

Now we can extend the Definitions 2.3 and 2.4 for any the terminal time iT, $i \in \mathbb{N}$ instead of T.

Proposition 4.13. *Let* (13) *and* (26) *hold. Then there is a process* $q = (\nu,\lambda) \in \mathcal{Y}(\Omega\times[0,iT];S)\times\mathrm{BV}([0,iT];L^1(\Omega;\mathbb{R}^L))$ *such that* $q(t+jT) = q(t)$ *for any* $t \in [0,T]$ *and any* $1 \leq j \leq i-1$.

Proof. The process q can be constructed by the T-periodic extension of the process whose existence was established in Proposition 4.11. Clearly the extended definitions of a solution, i.e., Definitions 2.3 and 2.4 are satisfied by such process. □

Remark 4.14. If $\mathfrak{L}$ does not have an affine extension to B we cannot reformulate (25) by means of (49). Nevertheless, relying on $\rho > 0$ we see that $\mathcal{G}$ is strictly convex in λ and therefore λ_τ^k is given uniquely for any minimizer in (25). Moreover, the magnetostatic potential u_τ^k is given uniquely in each time step k as well because of the strict convexity of the magnetostatic energy. This means that for any two minimizers ν_τ^k and μ_τ^k of (25) the spatial average of the macroscopic magnetization is given uniquely, i.e.,

$$|\Omega|^{-1}\int_\Omega\int_S A\nu_{\tau,x}^k(\mathrm{d}A)\,\mathrm{d}x = |\Omega|^{-1}\int_\Omega\int_S A\mu_{\tau,x}^k(\mathrm{d}A)\,\mathrm{d}x.$$

By similar arguments as above we can show that there exists a solution $q = (\nu, \lambda)$ to our problem according to Definition 2.5 such that $|\Omega|^{-1}\int_\Omega\int_S A\nu_x(0)(\mathrm{d}A)\,\mathrm{d}x = |\Omega|^{-1}\int_\Omega\int_S A\nu_x$ $(T)(\mathrm{d}A)\,\mathrm{d}x$. Hence, this solution gives a periodic hysteresis loop which only depends on the spatial average of the magnetization. See [13] for details.

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References

[1] V. Barbu, T. Precupanu: Convexity and Optimization in Banach Spaces, 2nd ed., D. Reidel, Dordrecht (1986).

[2] A. Bergqvist: Magnetic vector hysteresis model with dry friction-like pinning, Physica B 233 (1997) 342–347.

[3] W. F. Brown, Jr.: Magnetostatic Principles in Ferromagnetism, Springer, New York (1966).

[4] C. Carstensen, A. Prohl: Numerical analysis of relaxed micromagnetics by penalised finite elements, Num. Math. 90 (2001) 65–99.

[5] S. Chikazumi: Physics of Magnetism, Wiley, New York (1964).

[6] G. Dal Maso: An Introduction to Γ-Convergence, Birkhäuser, Boston (1993).

[7] K. Deimling: Nonlinear Functional Analysis, Springer, Berlin (1985).

[8] A. DeSimone: Energy minimizers for large ferromagnetic bodies, Arch. Rat. Mech. Anal. 125 (1993) 99–143.

[9] R. Engelking: General Topology, Heldermann Verlag, Lemgo (1989).

[10] A. Hubert, R. Schäffer: Magnetic Domains, Springer, Berlin (1998).

[11] R. D. James, D. Kinderlehrer: Frustration in ferromagnetic materials, Continuum Mech. Thermodyn. 2 (1990) 215–239.

[12] D. Jiles: A self consistent generalized model for the calculation of minor loop excursions in the theory of hysteresis, IEEE Trans. Mag. 28 (1992) 2603.

[13] M. Kružík: Periodicity properties of solutions to a hysteresis model in micromagnetics, in: Proceedings of ENUMATH 2003 (Prague, August 2003), M. Feistauer et al. (eds.), Springer, Berlin (2004) 605–614.

[14] L. D. Landau, E. M. Lifshitz: On the theory of the dispersion of magnetic permeability of ferromagnetic bodies, Physik Z. Sowjetunion 8 (1935) 153–169.

[15] L. D. Landau, E. M. Lifshitz: Course of Theoretical Physics 8, Pergamon Press, Oxford (1960).

[16] A. Mielke: Energetic formulation of multiplicative elastoplasticity using dissipation distances, Continuum Mech. Thermodyn. 15 (2003) 351–382.

[17] A. Mielke, T. Roubíček: A rate-independent model for inelastic behavior of shape-memory alloys, Multiscale Modeling Simul. 1 (2003) 571–597.

[18] A. Mielke, F. Theil: Mathematical model for rate-independent phase transformations, in: Models of Cont. Mechanics in Analysis and Engineering, H.-D. Alber et al (eds.), Shaker, Aachen (1999) 117–129.

[19] A. Mielke, F. Theil: On rate-independent hysteresis models, Nonlin. Diff. Eq. Appl. 11 (2004) 151–189.

[20] A. Mielke, F. Theil, V. Levitas: A variational formulation of rate-independent phase transformations using extremum principle, Arch. Rat. Mech. Anal. 162 (2002) 137–177.

[21] P. Pedregal: Relaxation in ferromagnetism: The rigid case, J. Nonlin. Sci. 4 (1994) 105–125.

[22] P. Pedregal: Parametrized Measures and Variational Principles, Birkhäuser, Basel (1997).

[23] X. Ren, L. Truskinovsky: Finite scale microstructures in nonlocal elasticity, J. Elasticity 59 (2000) 319–355.

[24] R. Rogers, L. Truskinovsky: Discretization and hysteresis, Physica B 233 (1997) 370–375.

[25] I. V. Roshchin, J. Yu, A. D. Kent, G. W. Stupian, M. S. Leung: Magnetic properties of Fe microstructures with focused ion beam-fabricated nano-constrictions, IEEE Trans. Mag. 37 (2001) 2101–2103.

[26] T. Roubíček: Relaxation in Optimization Theory and Variational Calculus, W. de Gruyter, Berlin (1997).

[27] T. Roubíček, M. Kružík: Microstructure evolution model in micromagnetics, Z. Angew. Math. Phys. 55 (2004) 159–182.

[28] T. Roubíček, M. Kružík: Mesoscopic model for ferromagnets with isotropic hardening, Z. Angew. Math. Phys. 56 (2005) 107–135.

[29] A. Visintin: Modified Landau-Lifshitz equation for ferromagnetism, Physica B 233 (1997) 365–369.

[30] A. Visintin: On some models of ferromagnetism, in: Free Boundary Problems I (Chiba, 1999), N. Kenmochi (ed.), Gakuto Int. Series in Math. Sci. Appl. 13 (2000) 411–428.

[31] A. Visintin: A Weiss-type model of ferromagnetism, Physica B 275 (2000) 87–91.

[32] J. Warga: Optimal Control of Differential and Functional Equations, Academic Press, New York (1972).

[33] L. C. Young: Generalized curves and existence of an attained absolute minimum in the calculus of variations, C. R. Soc. Sci. Lett. Varsovie, Classe III 30 (1937) 212–234.