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Perimeter Estimates for Reachable Sets of Control Systems*

Piermarco Cannarsa
Dipartimento di Matematica, Università di Roma Tor Vergata,
Via della Ricerca Scientifica 1, 00133 Roma, Italy
cannarsa@axp.mat.uniroma2.it

Pierre Cardaliaguet
Université de Bretagne Occidentale, UMR 6205,
6 Av. Le Gorgeu, BP 809, 29285 Brest, France
pierre.cardaliaguet@univ-brest.fr

Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

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The reachable set in time $T > 0$, $\mathcal{R}(T)$, is here investigated for the symmetric control system $\dot{x}(t) = f(x(t))u(t)$, $u(t) \in \overline{B}$. It turns out that, for $f(x)$ smooth and nondegenerate, $\mathcal{R}(T)$ has finite perimeter, and a sharp estimate for the time-dependence of the perimeter and volume of such a set can be obtained.

Keywords: Control theory, attainable sets, interior ball condition, sets of finite perimeter

1. Introduction

In this paper we investigate regularity properties of the reachable set of a control system of the form

$$x'(t) = f(x(t))u(t) \qquad u(t) \in \overline{B}, \tag{1}$$

where $\overline{B}$ denotes the closed unit ball of $\mathbb{R}^N$ centered at 0. We shall assume that $f : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is sufficiently smooth, and $f(x)$ is invertible for any $x \in \mathbb{R}^N$.

Given a nonempty compact subset $\mathcal{I}$ of $\mathbb{R}^N$ and a time $t \geq 0$, we will study the reachable set from $\mathcal{I}$ at time t, which is defined as

$$\mathcal{R}(t) = \left\{ x \in \mathbb{R}^N \mid \exists x(\cdot) \text{ solution to (1), with } x(0) \in \mathcal{I} \text{ and } x(t) = x \right\}. \tag{2}$$

Our main objective is to show that, for any $t > 0$, $\mathcal{R}(t)$ is a set of finite perimeter (in the sense of De Giorgi), and its perimeter can only increase with time in a controlled way. More precisely, we will show that, for any fixed horizon $T > 0$, there are positive constants c_1 and c_2 such that, for any $0 < t_1 \leq t_2 \leq T$,

$$\begin{aligned} &\int_{\partial^*\mathcal{R}(t_2)} |f^*(x)\nu_{\mathcal{R}(t_2)}(x)| \, d\mathcal{H}^{N-1}(x) \\ \leq\ & \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)} \int_{\partial^*\mathcal{R}(t_1)} |f^*(x)\nu_{\mathcal{R}(t_1)}(x)| \, d\mathcal{H}^{N-1}(x), \end{aligned} \tag{3}$$

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where $\partial^*\mathcal{R}(t)$ denotes the reduced boundary of $\mathcal{R}(t)$, $\mathcal{H}^{N-1}$ is the standard $(N-1)-$dimensional Hausdorff measure, and $\nu_{\mathcal{R}(t)}(x)$ is the measure theoretic outward unit normal of $\mathcal{R}(t)$ at $x \in \partial^*\mathcal{R}(t)$. Consequently, we will obtain that the volume of $\mathcal{R}(t)$, denoted by $|\mathcal{R}(t)|$, is a locally Lipschitz continuous function of t on $(0, +\infty)$ and

$$\frac{d}{dt}|\mathcal{R}(t)| = \int_{\partial^*\mathcal{R}(t)} \left| f^*(x)\nu_{\mathcal{R}(t)}(x) \right| d\mathcal{H}^{N-1}(x) \qquad \text{a.e. } t > 0. \tag{4}$$

This work is strongly motivated by [1], where volume and perimeter estimates for reachable sets of control systems of special form ($f(x) = c(x)I_n$, where $c : \mathbb{R}^N \to \mathbb{R}$) turn out to be crucial for the study of first order front propagation problems. The estimates given here are much sharper than those of [1], and we think these improvement should be of some help for other problems.

Let us briefly explain our method of proof. The starting point is a joint paper [5] by Frankowska and the first author where—under more general structural assumptions on the system—$\mathcal{R}(t)$ is shown to satisfy a uniform interior ball condition with a radius proportional to t. This property entails that reachable sets are of finite perimeter in the sense of De Giorgi (see [1] or [8]). Moreover, the reduced boundary of $\mathcal{R}(t)$ can be characterized as follows: $\partial^*\mathcal{R}(t)$ coincides ($\mathcal{H}^{N-1}$ a.e.) with the set of points $x \in \partial\mathcal{R}(t)$ such that the contingent cone $T_{\mathcal{R}(t)}(x)$ of $\mathcal{R}(t)$ at x is exactly a half-space (see Lemma 3.4).

Another crucial tool for the proof is the notion of extremal solution (or boundary trajectory). Let us recall that an extremal solution $x(\cdot)$ of (1) on the time interval $[0, T]$ is a solution which remains on the boundary of $\mathcal{R}(t)$ for every $t \in [0, T]$. Following Theorem 3.8 of [7] (see also Theorem the 8.4.6 of [6] and Lemma 5.1), we note that, if $x(\cdot)$ is an extremal solution on $[0, T]$, then the contingent cone of $\mathcal{R}(t)$ at $x(t)$ is a half-space for all $t \in (0, T)$. Thus, loosely speaking, extremal solutions remain in the reduced boundary of $\mathcal{R}(t)$ (see Lemma 5.1).

The key remark of the paper is an estimate that bounds the distance between points reached by two extremal solutions at the same time: if $x_1(\cdot)$ and $x_2(\cdot)$ are extremal solutions, then we show that, for suitable contants $C_1, C_2 \geq 0$,

$$|f^{-1}(x_1(t_2))(x_1(t_2) - x_2(t_2))| \leq \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2-t_1)} |f^{-1}(x_1(t_1))(x_1(t_1) - x_2(t_1))|$$

for any t_1, t_2 with $0 < t_1 < t_2 \leq T$. Note that the above estimate measures the difference $x_1(t_2) - x_2(t_2)$ in terms of $x_1(t_1) - x_2(t_1)$, with respect to a position-depending norm. The reason why such an inequality is true is the fact that reachable sets have the interior ball property, and so—loosely speaking—outward normals at different points cannot point at "too opposite directions". Then, the proof of (3) follows introducing a Hausdorff-like measure, $\mathcal{H}_f^{N-1}$, that reduces to the above expression for sets of finite perimeter.

The paper is organized in the following way. In Section 1, we recall basic facts concerning the sets of finite perimeter and extremal solutions of system (1). Section 2 is devoted to the study of sets satisfying a uniform interior ball condition. In Section 3 we define the measure $\mathcal{H}_f^{N-1}$, and give an equivalent expression for such a measure. In Section 4 we state and prove our perimeter estimate. Finally, Section 5 is devoted to the proof of our volume estimate (4).

We complete this introduction by stating the assumption we need throughout this paper:

$$\begin{array}{ll} (a) & f:\mathbb{R}^N \to \mathbb{R}^{N\times N} \text{ is of class } \mathcal{C}^2; \\ (b) & f(x) \text{ is invertible for any } x\in\mathbb{R}^N; \\ (c) & f \text{ and } f^{-1} \text{ are globally bounded.} \end{array} \tag{5}$$

2. Some notation

We start this section by collecting standard notations used throughout the paper. We denote by $\langle\cdot,\cdot\rangle$ and by $|\cdot|$ the scalar product and the euclidean norm of $\mathbb{R}^N$, by $B(x,r)$ the closed ball of radius r centered at the point x. Recall that $\overline{B} = B(0,1)$. If K is a subset of $\mathbb{R}^N$, $d_K(x)$ denotes the distance of the point x to the set K: $d_K(x) = \inf_{y\in K}|y-x|$. For $r>0$, we denote by $K+rB$ the set of points $x\in\mathbb{R}^N$ such that $d_K(x)\le r$.

For a vector $v\in\mathbb{R}^N$ we denote by $(v)^-$ the negative polar cone of v, i.e., the set of vectors $w\in\mathbb{R}^N$ such that $\langle v,w\rangle\le 0$.

Finally, if K is a subset of $\mathbb{R}^N$, $\mathbf{1}_K$ is the function equal to 1 in K and to 0 outside.

We now recall some well-known results on functions with bounded variation. For a general presentation and proofs, see for instance [2, 9]. A function $u\in L^1(\mathbb{R}^N,\mathbb{R})$ whose gradient Du in the sense of distribution is a vector valued Radon measure with finite total variation is called a function of bounded variation. The total variation of Du in an open set U is denoted by $\|Du\|(U)$ and given by

$$\sup\left\{\int u\,\mathrm{div}\,\phi dx \mid \phi\in\mathcal{C}^1_c(U),\ |\phi|\le 1\right\}.$$

If B is a Borel set, then the total variation of u on B is defined as:

$$\inf\{\|Du\|(U)\mid U \text{ open and } B\subset U\}.$$

A measurable set $E\subset\mathbb{R}^N$ is said to be of finite perimeter if the function $\mathbf{1}_E$ has bounded variation. The perimeter of E in a Borel set B is then given by $P(E,B):=\|D\mathbf{1}_E\|(B)$. For sets of finite perimeter, one can define the essential boundary ∂^*E of E, which is countably $(N-1)$ rectifiable with finite $\mathcal{H}^{N-1}$ measure. The outer unit normal $\nu_E(x)$ is then defined for all points x of ∂^*E. If we set $E_t=\{y\in\mathbb{R}^N\mid x+t(y-x)\in E\}$, then the function $\mathbf{1}_{E_t}$ converges to $\mathbf{1}_{H^-(x)}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t\to 0^+$, where $H^-(x)=\{y\in\mathbb{R}^N\mid\langle\nu_E(x),y-x\rangle\le 0\}$ (see Theorem 1, p. 199 in [9]). Moreover, the measure $P(E,\cdot)$ coincides with the restriction of $\mathcal{H}^{N-1}$ to ∂^*E. The measure theoretic boundary of E, denoted by ∂_*E, is the set of points $x\in\mathbb{R}^N$ such that

$$\limsup_{t\to 0^+}\frac{|B(x,t)\cap E|}{t^N}>0 \quad\text{and}\quad \limsup_{t\to 0^+}\frac{|B(x,t)\backslash E|}{t^N}>0,$$

where $|A|$ denotes the Lebesgue measure of a set A. It is known that

$$\partial^*E\subset\partial_*E \quad\text{and}\quad \mathcal{H}^{N-1}(\partial_*E\backslash\partial^*E)=0.$$

Next we go back to the controlled system (1) and to the reachable set defined in (2). If $T>0$ and $x\in\partial\mathcal{R}(T)$, then it is well-known that there is a solution $x(\cdot)$ to (1) such that

$$x(0)\in\mathcal{I},\ x(T)=x \quad\text{and}\quad x(s)\in\partial\mathcal{R}(s),\ \forall s\in[0,T].$$

Such a solution is called an extremal (or a boundary) solution on the time interval $[0, T]$. Let us now introduce the Hamiltonian of the problem

$$H(x,p) = \sup_{u \in \overline{B}} \langle f(x)u, p \rangle = |f^*(x)p|,$$

and the polar of H given by

$$H^0(x,q) = |f^{-1}(x)q|. \tag{6}$$

We note for later use that

$$\frac{\partial H}{\partial p}(x,p) = \frac{f(x)f^*(x)p}{|f^*(x)p|} \qquad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N_*.$$

Pontryagin maximum principle states that, if $x(\cdot)$ is an extremal solution, then it is solution to an Hamiltonian system:

Theorem 2.1 (Pontryagin Maximum Principle). *If $x(\cdot)$ is an extremal solution on the time interval $[0, T]$, then there is an absolutely continuous map $p : [0, T] \to \mathbb{R}^N \backslash \{0\}$ such that*

$$\begin{cases} x' = \dfrac{\partial H}{\partial p}(x,p) \\ p' = -\dfrac{\partial H}{\partial x}(x,p). \end{cases} \tag{7}$$

The map $p(\cdot)$ is called the adjoint state of the extremal solution $x(\cdot)$.

3. Interior ball property

We say that a closed set $K \subset \mathbb{R}^N$ has the interior ball property of radius $r > 0$ if

$$\forall x \in \partial K,\ \exists p \in \mathbb{R}^N,\ |p| = 1, \text{ such that } B(x - rp, r) \subset K.$$

Reachable sets have the interior ball property. More precisely, Frankowska and the first author proved in [5] the following:

Theorem 3.1 ([5]). *Under assumption (5), for any $T > 0$, there is a constant c_T such that $\mathcal{R}(t)$ has the interior ball property of radius $c_T t$ for any $t \in (0, T]$.*

More precisely, if $x(\cdot)$ is an extremal solution on some time interval $[0, T]$ (with $T > 0$), and if we denote by $p(\cdot)$ its adjoint, then

$$B\left(x(t) - c_T t \frac{p(t)}{|p(t)|}, c_T t\right) \subset \mathcal{R}(t) \qquad \forall t \in (0, T].$$

If K is a closed subset of $\mathbb{R}^N$, we denote by $T_K(x)$ the contingent cone to K at $x \in \partial K$, i.e., the set of vectors $v \in \mathbb{R}^N$ for which there exist $h_n \to 0^+$, $v_n \to v$ with $x + h_n v_n \in K$ (see [3]). We need for later use the following remark which is directly inspired by a result due to Quincampoix [11].

Lemma 3.2. *If K is a closed subset of $\mathbb{R}^N$, then $\partial T_K(x) \subset T_{\partial K}(x)$.*

Proof. Let $v \in \partial T_K(x)$. There are sequences $h_n \to 0^+$, $v_n \to v$ such that $x + h_n v_n \in K$ because $v \in T_K(x)$. Moreover, since $v \in \partial T_K(x)$, there is a sequence $w_k \to v$ such that $w_k \notin T_K(x)$. This implies that there is some $\tau_k > 0$ such that the segment $]x, x+\tau_k w_k]$ has an empty intersection with K. Let us choose a subsequence (h_{n_k}) of (h_n) such that $h_{n_k} \le \tau_k$ for any k. Then, since $x + h_{n_k} v_{n_k} \in K$ and $x + h_{n_k} w_k \notin K$, the segment $[x + h_{n_k} v_{n_k}, x + h_{n_k} w_k]$ contains a point x_k of ∂K. This point can be written as $x_k = x + h_{n_k} z_k$ where $z_k \in [v_{n_k}, w_k]$. Since $v_{n_k} \to v$ and $w_k \to v$, we have $z_k \to v$, which proves that $v \in T_{\partial K}(x)$. □

Lemma 3.3. *If a set $K \subset \mathbb{R}^N$ has the interior ball property of radius $r > 0$, there is some closed subset K_0 of K such that $K_0 + rB \subset K$ and $\partial K \subset \partial(K_0 + rB)$.*

We note that the equality does not hold in general. In particular sets with interior ball property are not of "positive reach" in the sense of Federer [10].

Proof. Let us set $K_0 = \{x \in K \mid d_{\partial K}(x) \ge r\}$. Then $K_0 + rB \subset K$, from the definition of K_0. Furthermore, for any $x \in \partial K$, there is some $p \in \mathbb{R}^N$, with $|p| = 1$ and $B(x - rp, r) \subset K$. Then $x - rp \in K_0$ and therefore $x \in K_0 + rB$. Since $x \in \partial K$ and $x \in K_0 + rB \subset K$, we have $x \in \partial(K_0 + rB)$. This proves that $\partial K \subset \partial(K_0 + rB)$. □

Lemma 3.4. *If the closed set K has the interior ball property, then K is a set of finite perimeter and the following inclusions hold*

$$\partial^* K \subset \{x \in \partial K \mid T_K(x) \text{ is a half-space}\} \subset \partial_* K, \tag{8}$$

where $\partial^* K$ is the reduced boundary of K and $\partial_* K$ its measure theoretic boundary. Furthermore, if $x \in \partial^* K$, then $T_K(x) = (\nu_K(x))^-$.

Remark. Since $\mathcal{H}^{N-1}(\partial_* K \backslash \partial^* K) = 0$, the sets in (8) are equal up to sets of zero $\mathcal{H}^{N-1}$ Hausdorff measure.

Proof of Lemma 3.4. Let us first prove that K is a set of finite perimeter. For this let us consider K_0 as in Lemma 3.3 and let us set $K_1 = K_0 + rB$. From Lemma 2.4 of [1], we know that $\mathcal{H}^{N-1}(\partial K_1)$ is finite. Hence so is $\mathcal{H}^{N-1}(\partial K)$. Next we note that $\partial_* K \subset \partial K$ because K is closed. So $\mathcal{H}^{N-1}(\partial^* K)$ is finite, which proves that K is a set of finite perimeter (see Theorem 1 p. 222 of [9]).

Next we show the inclusions (8). Let

$$x \in S := \{x \in \partial K \mid T_K(x) \text{ is a half-space}\}.$$

Let us show that x belongs to the measure theoretic boundary $\partial_* K$ of K, i.e.,

$$\limsup_{t \to 0^+} \frac{|B(x,t) \cap K|}{t^N} > 0 \quad \text{and} \quad \limsup_{t \to 0^+} \frac{|B(x,t) \backslash K|}{t^N} > 0.$$

The first assertion is obvious, because, from the interior ball condition, there is some $p \in \mathbb{R}^N$, $|p| = 1$, with $B(x - rp, r) \subset K$ and so

$$\limsup_{t \to 0^+} \frac{|B(x,t) \cap K|}{t^N} \ge \limsup_{t \to 0^+} \frac{|B(x,t) \cap B(x - rp, r)|}{t^N} = \frac{1}{2}.$$

As for the second, since $T_K(x)$ is a half-space, there is some $v \in \mathbb{R}^N \backslash T_K(x)$. From the definition of $T_K(x)$, there is some $\epsilon > 0$ such that the troncated cone $x + (0, \epsilon)(v + \epsilon B)$ does not intersect K. Then

$$\limsup_{t\to 0^+} \frac{|B(x,t)\backslash K|}{t^N} \geq \limsup_{t\to 0^+} \frac{|B(x,t) \cap [x + (0,\epsilon)(v+\epsilon B)]\,|}{t^N} > 0.$$

So x belongs to the measure theoretic boundary of K.

Let now $x \in \partial^* K$. Let us set $K_t = \{y \in \mathbb{R}^N \mid x + t(y-x) \in K\}$. As recalled in the previous section, the function $\mathbf{1}_{K_t}$ converges to $\mathbf{1}_{H^-(x)}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, where $H^-(x) = \{y \in \mathbb{R}^N \mid \langle \nu_K(x), (y-x)\rangle \leq 0\}$. We claim that

$$T_K(x) = \{v \in \mathbb{R}^N \mid \langle \nu_K(x), v\rangle \leq 0\}, \tag{9}$$

which completes the proof of the first equality in (8). For proving (9), we first show that the projection $\Pi(x)$ of x onto K_1 is reduced to the singleton $\{x - r\nu_K(x)\}$ and that $(\nu_K(x))^- \subset T_K(x)$. Indeed, let $y \in \Pi(x)$. Then, $x \in \partial B(y,r)$ and $B(y,r) \subset K$, so that, after scaling, $B(x + (y-x)/t, r/t) \subset K_t$. Therefore

$$\lim_{t\to 0+} \mathbf{1}_{K_t}(z) = 1 \quad \text{for any } z \text{ with } \langle z - x, x - y\rangle < 0.$$

Since $\lim_{t\to 0+} \mathbf{1}_{K_t}(z) = \mathbf{1}_{H^-(x)}$, this proves that $(x-y)/r = \nu_K(x)$, i.e., $y = x - r\nu_K(x)$. Thus $\Pi(x) = \{x - r\nu_K(x)\}$. Moreover, since $B(x - r\nu_K(x), r) \subset K$, we have $(\nu_K(x))^- \subset T_K(x)$.

We now show (9). Let us first consider $v \in \partial T_K(x)$. From Lemma 3.2, $\partial T_K(x) \subset T_{\partial K}(x)$. So there are $h_n \to 0^+$, $v_n \to v$ such that $x_n := x + h_n v_n \in \partial K$. Since K has the interior ball property of radius r, there is some $p_n \in \mathbb{R}^N$ with $|p_n| = 1$ and $B(x_n - rp_n, r) \subset K$. Let us note that $p_n \to \nu_K(x)$. Indeed, if q is an accumulation point of the sequence (p_n), then $B(x - rp, r) \subset K$. Hence $x - rq \in \Pi(x) = \{x - r\nu_K(x)\}$, which proves that $p_n \to \nu_K(x)$. Since x belongs to ∂K, x does not belong to the interior of the ball $B(x_n - rp_n, r)$. So $|x_n - rp_n - x| \geq r$. Using the definition of x_n, we get

$$-2h_n\langle v_n, p_n\rangle + h_n^2|v_n|^2 \leq 0\,.$$

Dividing by h_n and letting $n \to +\infty$ gives $\langle v, \nu_K(x)\rangle \leq 0$. Since $v \in \partial T_K(x)$ with $(\nu_K(x))^- \subset T_K(x)$, we also have $\langle v, \nu_K(x)\rangle \geq 0$, so that $\langle v, \nu_K(x)\rangle = 0$. To summarize, we have proved that $(\nu_K(x))^- \subset T_K(x)$ and that $\partial T_K(x) \subset \{v \in \mathbb{R}^N \mid \langle v, \nu_K(x)\rangle = 0\}$. This shows (9). □

Next we need a technical inequality, linking two points x_1 and x_2 of the boundary of a set K, at which this set has an interior ball, and the two normals to this set at these points.

Lemma 3.5. *Let K be a closed subset of $\mathbb{R}^N$, $r > 0$, $x_1, x_2 \in \partial K$ for which there are $p_1, p_2 \in \mathbb{R}^N$ with $|p_1| = |p_2| = 1$ and $B(x_i - rp_i, r) \subset K$ for $i = 1, 2$. Then, for any invertible matrix A, we have*

$$\langle x_1 - x_2, \frac{p_1}{|A^{-*}p_1|} - \frac{p_2}{|A^{-*}p_2|}\rangle \leq \frac{\Lambda^{\frac{1}{2}}}{r\lambda}|A(x_1 - x_2)|^2\,,$$

where $A^{-*} = (A^{-1})^*$ and where λ (resp. Λ) is the smallest (resp. largest) eigenvalue of AA^*.

Proof. Let $K' = AK$, $x_i' = Ax_i$, $q_i = A^{-*}p_i/|A^{-*}p_i|$ for $i = 1, 2$. We claim that

$$A^{-1}B(x_i' - r'q_i, r') \subset B(x_i - rp_i, r)\,, \tag{10}$$

where $r' = \lambda\Lambda^{-\frac{1}{2}}r$. Indeed, let $z \in B(x_i' - r'q_i, r')$. Then $|z - x_i' + r'q_i|^2 \leq (r')^2$, which is equivalent to saying that $|z - x_i'|^2 + 2r'\langle z - x_i', q_i\rangle \leq 0$. Therefore

$$|A(A^{-1}z - x_i)|^2 + 2r'\langle A^{-1}z - x_i, \frac{p_i}{|A^{-*}p_i|}\rangle \leq 0. \tag{11}$$

Let us now check that this inequality implies that $A^{-1}z$ belongs to the ball $B(x_i - rp_i, r)$, i.e., $|A^{-1}z - x_i|^2 + 2r\langle A^{-1}z - x_i, p_i\rangle \leq 0$. Since λ is the smallest eigenvalue of AA^*, and Λ the largest, we have

$$\begin{aligned} & |A^{-1}z - x_i|^2 + 2r\langle A^{-1}z - x_i, p_i\rangle \\ \leq\ & \frac{1}{\lambda}|A(A^{-1}z - x_i)|^2 + 2r\langle A^{-1}z - x_i, p_i\rangle \\ \leq\ & \frac{1}{\lambda}\left(|A(A^{-1}z - x_i)|^2 + 2r'\langle A^{-1}z - x_i, \frac{p_i}{|A^{-*}p_i|}\rangle\right) \ \leq\ 0, \end{aligned}$$

thanks to (11) So our claim (10) is proved.

Since $B(x_i - rp_i, r) \subset K$ for $i = 1, 2$, (10) implies that $B(x_i' - r'q_i, r') \subset K'$. We now mimic the proof of Lemma 2.1 of [1] to show that

$$\langle x_1' - x_2', q_1 - q_2\rangle \leq \frac{\Lambda^{\frac{1}{2}}}{r\lambda}|x_1' - x_2'|^2. \tag{12}$$

Since x_2' does not belong to the open ball centered at $x_1' - r'q_1$ and of radius r', we have

$$|x_2' - (x_1' - r'q_1)|^2 \geq (r')^2, \quad \text{whence} \quad |x_2' - x_1'|^2 + 2r'\langle q_1, x_2' - x_1'\rangle \geq 0.$$

In the same way, since x_1' does not belong to the open ball centered at $x_2' - r'q_2'$ and of radius r', we have $|x_2' - x_1'|^2 + 2r'\langle q_2, x_1' - x_2'\rangle \geq 0$. Putting the two inequalities together gives (12).

Using inequality (12) and writing explicitely what are x_i' and q_i gives the desired result. □

4. On some Hausdorff measure

We now introduce the Hausdorff measure $\mathcal{H}_f^{N-1}$ adapted to our framework. For any set $E \subset \mathbb{R}^N$ and $\delta > 0$, we set

$$\mathcal{H}_{f,\delta}^{N-1}(E) = \inf\left\{\alpha_{N-1}\sum_{i=1}^{\infty}\left(\frac{\operatorname{diam}_f(K_i)}{2}\right)^{N-1}\right\},$$

where α_{N-1} is the volume of the unit ball of $\mathbb{R}^{N-1}$, and where the infimum is taken over the families (K_i) of compact subsets of $\mathbb{R}^N$ such that

$$E \subset \bigcup_{i=1}^{\infty} K_i \quad \text{and} \quad \operatorname{diam}_f(K_i) \leq \delta,$$

with, for any set K,

$$\mathrm{diam}_f(K) = \sup_{x,y\in K} |\det(f(x))|^{\frac{1}{N-1}} \left|f^{-1}(x)(x-y)\right|.$$

Then we set

$$\mathcal{H}_f^{N-1}(E) = \lim_{\delta\to 0^+} \mathcal{H}_{f,\delta}^{N-1}(E).$$

We note that $\mathcal{H}_f^{N-1}$ can easily be estimated by the usual Hausdorff measure:

Lemma 4.1. *Under assumption* (5) *on f, there is a constant $C > 1$ such that*

$$\frac{1}{C}\mathcal{H}^{N-1}(E) \le \mathcal{H}_f^{N-1}(E) \le C\,\mathcal{H}^{N-1}(E) \qquad \text{for any set } E \subset \mathbb{R}^N.$$

Proof. This is just due to the fact that there is a constant $C' > 0$ such that $|f(y)| \le C'$ and $|f^{-1}(y)| \le C'$ for any $y \in \mathbb{R}^N$, which implies that

$$\frac{1}{C}|p| \le |\det(f(x))|^{\frac{1}{N-1}} \left|f^{-1}(x)p\right| \le C\,|p| \qquad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N,$$

for some constant C. □

For sets of finite perimeter, $\mathcal{H}_f^{N-1}$ can also be defined in the following way:

Lemma 4.2. *Let K be a set of finite perimeter. Then*

$$\mathcal{H}_f^{N-1}(\partial^* K) = \int_{\partial^* K} |f^*(x)\nu_K(x)|\, d\mathcal{H}^{N-1}(x).$$

where $\partial^* K$ is the reduced boundary of K.

Proof of Lemma 4.2. Let us define, for any Borel set E and any $x_0 \in \partial^* K$

$$\mu(E) = \int_{\partial^* K\cap E} |f^*(x)\nu_K(x)|\, d\mathcal{H}^{N-1}(x),$$

$$\mu_{x_0}(E) = \int_{\partial^* K\cap E} |f^*(x_0)\nu_K(x)|\, d\mathcal{H}^{N-1}(x),$$

and

$$m = \mathcal{H}_f^{N-1}\lfloor\partial^* K \quad \text{and} \quad m_{x_0} = \mathcal{H}_{f(x_0)}^{N-1}\lfloor\partial^* K,$$

where $\mathcal{H}_{f(x_0)}^{N-1}$ is defined as $\mathcal{H}_f^{N-1}$ with the constant matrix $f(x_0)$ instead of f. We note that μ, μ_{x_0}, m and m_{x_0} are Radon measures and, from Lemma 4.1, that m is absolutely continuous with respect to μ. The derivative h of m with respect to μ is given by

$$h(x) = \lim_{t\to 0^+} \frac{m(B(x,t))}{\mu(B(x,t))} \qquad \text{for } \mu\text{-almost every } x. \tag{13}$$

Our aim is to show that $h(x) = 1$ for $\mu-$almost every x. For this, we fix a point $x_0 \in \partial^* K$ for which (13) holds. From the continuity of f and f^{-1} and the definition of $\mathcal{H}_f^{N-1}$ and μ, we have

$$\lim_{t\to 0^+} \frac{m(B(x_0,t))}{m_{x_0}(B(x_0,t))} = 1 \quad \text{and} \quad \lim_{t\to 0^+} \frac{\mu(B(x_0,t))}{\mu_{x_0}(B(x_0,t))} = 1.$$

So it remains to show that

$$h(x_0) = \lim_{t\to 0^+} \frac{m_{x_0}(B(x_0,t))}{\mu_{x_0}(B(x_0,t))} = 1.$$

We actually show the following stronger result: for any open set U, we have

$$m_{x_0}(U) = \mu_{x_0}(U). \tag{14}$$

Proof of (14). Let us first note that, for any set $E \subset \mathbb{R}^N$, we have

$$m_{x_0}(E) = |\det(f(x_0))|\, \mathcal{H}^{N-1}(f^{-1}(x_0)E). \tag{15}$$

Indeed, for any set S, we have

$$\mathrm{diam}_{f(x_0)}(S) = |\det(f(x_0))|^{\frac{1}{N-1}}\, \mathrm{diam}(f^{-1}(x_0)S)$$

where $\mathrm{diam}(S) = \sup_{y,z\in S} |y-z|$ is the standard diameter. Then using the definition of the usual Hausdorff measure and of $\mathcal{H}^{N-1}_{f(x_0)}$ gives (15).

Let now set $K' = f^{-1}(x_0)K$, $U' = f^{-1}(x_0)U$ and let $\phi \in \mathcal{C}^1_c(U')$. Let us note that the function $\psi(z) = f(x_0)\phi(f^{-1}(x_0)z)$ belongs to $\mathcal{C}^1_c(U)$ and that

$$\mathrm{div}\,\psi(z) = \mathrm{div}\,\phi(f^{-1}(x_0)z) \qquad \forall z \in \mathbb{R}^N.$$

After a change of variables, we get

$$\begin{aligned} \int_{K'\cap U'} \mathrm{div}\,\phi(y)dy &= |\det(f^{-1}(x_0))| \int_{K\cap U} (\mathrm{div}\,\phi)(f^{-1}(x_0)z)dz \\ &= |\det(f^{-1}(x_0))| \int_{K\cap U} \mathrm{div}\,\psi(z)dz. \end{aligned}$$

Since K has a finite perimeter and $\psi \in \mathcal{C}^1_c(U)$, the Gauss-Green formula gives

$$\begin{aligned} -\int_{K\cap U} \mathrm{div}\,\psi(z)dz &= \int_{\partial^* K\cap U} \langle \psi(z), \nu_K(z)\rangle d\mathcal{H}^{N-1}(z) \\ &= \int_{\partial^* K\cap U} \langle \phi(f^{-1}(x_0)z), f^*(x_0)\nu_K(z)\rangle d\mathcal{H}^{N-1}(z). \end{aligned}$$

From its definition, K' has a finite perimeter and $\partial^* K' = f^{-1}(x_0)\partial^* K$. Therefore

$$\begin{aligned} m_{x_0}(U) &= |\det(f(x_0))|\, \mathcal{H}^{N-1}(\partial^* K' \cap U') \\ &= |\det(f(x_0))| \sup\left\{ \int_{K'\cap U'} \mathrm{div}\,\phi(y)dy \mid \phi \in \mathcal{C}^1_c(U'),\ |\phi| \le 1 \right\} \\ &= \sup\left\{ \int_{\partial^* K\cap U} \langle \phi(f^{-1}(x_0)z), f^*(x_0)\nu_K(z)\rangle d\mathcal{H}^{N-1}(z) \mid \phi \in \mathcal{C}^1_c(U'),\ |\phi| \le 1 \right\} \\ &= \int_{\partial^* K\cap U} |f^*(x_0)\nu_K(z)| d\mathcal{H}^{N-1}(z) \\ &= \mu_{x_0}(U). \end{aligned}$$

Therefore (14) holds, which completes the proof of the Lemma. □

5. The perimeter estimate

Let us start with a Lemma which states that the reachable set is rather regular along extremal solutions.

Lemma 5.1. *Let x be an extremal trajectory on the time interval $[0,T]$. Then $T_{\mathcal{R}(t)}(x(t))$ is a half-space for any $t \in (0,T)$ and*

$$x(t) \in \partial_* \mathcal{R}(t) \qquad \forall t \in (0,T),$$

where $\partial_* \mathcal{R}(t)$ is the measure theoretic boundary of $\mathcal{R}(t)$.

Proof of Lemma 5.1. Let us introduce the minimal time function given by:

$$\tau(x) = \min\{t \geq 0 \mid x \in \mathcal{R}(t)\} \qquad \forall x \in \mathbb{R}^N.$$

We note for later use that

$$\mathcal{R}(t) = \{x \in \mathbb{R}^N \mid \tau(x) \leq t\} \qquad \forall t > 0. \tag{16}$$

Following Theorem 3.1 of [12] the function τ is locally semi-concave in $\mathbb{R}^N \backslash K$. Then, arguing as in the proof of Theorem 8.4.6 of [6], one can easily check that τ is differentiable at $x(t)$ for any $t \in (0,T)$. We also note that $H(x(t), D\tau(x(t))) = 1$ for $t \in (0,T)$, because τ is a viscosity solution of this Hamilton-Jacobi equation. Thus $D\tau(x(t)) \neq 0$ for $t \in (0,T)$. Using (16), this shows that $T_{\mathcal{R}(t)}(x(t)) = (D\tau(x(t))^-$ is a half-space. We complete the proof thanks to Lemma 3.4 . □

Next we investigate how two extremal trajectories depart from each other. Recall that $H^0(x,p) = |f^{-1}(x)p|$ for any $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$.

Lemma 5.2. *Let $T > 0$ be fixed. There are constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that for any extremal solution x_1 and x_2 on the time interval $[0,T]$, we have*

$$H^0(x_1(t_2), x_1(t_2) - x_2(t_2)) \leq \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2 - t_1)} H^0(x_1(t_1), x_1(t_1) - x_2(t_1)),$$

for any t_1, t_2 with $0 < t_1 < t_2 \leq T$.

Proof. Let us first introduce some notations. Let p_1 and p_2 be the adjoints of the extremal solutions x_1 and x_2, with $|p_1(0)| = |p_2(0)| = 1$. Let $R > 0$ be a constant such that any solution starting from $\mathcal{I}$ remains in $B(0,R)$ on the time interval $[0,T]$. We define a constant C such that:

$$\left|\frac{\partial H^0}{\partial x}(x,q)\right| \leq C|q|, \ \left|\frac{\partial H^0}{\partial q}(x,q)\right| \leq C,$$

and

$$\left|\frac{\partial H}{\partial x}(x,p)\right| \leq C|p|, \ \left|\frac{\partial H}{\partial p}(x,p)\right| \leq C,$$

for all $(x,p,q) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, with $|x| \leq R$, $p \neq 0$, $q \neq 0$. We also choose C in such a way that

$$|q| \leq C\, H^0(x,q) \qquad \forall (x,q) \in \mathbb{R}^N \times (\mathbb{R}^N \backslash \{0\}) \text{ with } |x| \leq R. \tag{17}$$

Since, for $i = 1, 2$, and $t \in (0, T)$,

$$\frac{d}{dt}\frac{1}{2}|p_i(t)|^2 = -\langle \frac{\partial H}{\partial x}(x_i(t), p_i(t)), p_i(t)\rangle \geq -C|p_i(t)|^2$$

we have

$$|p_i(t)| \geq e^{-CT} \qquad \forall t \in [0, T].$$

Finally, we denote by C' a constant such that

$$\left|\frac{\partial^2 H}{\partial x \partial p}(x, p)\right| \leq C'$$

for all $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$, with $|x| \leq R$ and $|p| \geq e^{-CT}$.

From the definition of C and the equation (7) satisfied by (x, p) we have

$$\begin{aligned}
&\frac{d}{dt}H^0(x_1(t), x_1(t) - x_2(t)) \\
= \; & \langle \frac{\partial H^0}{\partial x}(x_1, x_1 - x_2), \frac{\partial H}{\partial p}(x_1, p_1)\rangle + \langle \frac{\partial H^0}{\partial q}(x_1, x_1 - x_2), \frac{\partial H}{\partial p}(x_1, p_1) - \frac{\partial H}{\partial p}(x_2, p_2)\rangle \\
\leq \; & C^2|x_1 - x_2| + \langle \frac{\partial H^0}{\partial q}(x_1, x_1 - x_2), \frac{\partial H}{\partial p}(x_1, p_1) - \frac{\partial H}{\partial p}(x_1, p_2)\rangle \\
& \quad + \langle \frac{\partial H^0}{\partial q}(x_1, x_1 - x_2), \frac{\partial H}{\partial p}(x_1, p_2) - \frac{\partial H}{\partial p}(x_2, p_2)\rangle
\end{aligned}$$

We note that (omitting the x_1 argument in f and f^{-1})

$$\begin{aligned}
& \langle \frac{\partial H^0}{\partial q}(x_1, x_1 - x_2), \frac{\partial H}{\partial p}(x_1, p_1) - \frac{\partial H}{\partial p}(x_1, p_2)\rangle \\
= \; & \langle \frac{f^{-*}f^{-1}(x_1 - x_2)}{|f^{-1}(x_1 - x_2)|}, \frac{ff^*p_1}{|f^*p_1|} - \frac{ff^*p_2}{|f^*p_2|}\rangle \\
= \; & \frac{1}{H^0(x_1, x_1 - x_2)}\langle x_1 - x_2, \frac{p_1}{H(x_1, p_1)} - \frac{p_2}{H(x_1, p_2)}\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{d}{dt}\, H^0(x_1(t), x_1(t) - x_2(t)) \\
\leq \; & \frac{1}{H^0(x_1, x_1 - x_2)}\langle x_1 - x_2, \frac{p_1}{H(x_1, p_1)} - \frac{p_2}{H(x_1, p_2)}\rangle + C(C + C')|x_1 - x_2|.
\end{aligned}$$

Combining Theorem 3.1 with Lemma 3.5 applied to the matrix $A = f^{-1}(x_1(t))$ gives that

$$\langle x_1 - x_2, \frac{p_1}{H(x_1, p_1)} - \frac{p_2}{H(x_1, p_2)}\rangle \; \leq \; \frac{C_2}{t}\left(H^0(x_1, x_1 - x_2)\right)^2,$$

where $C_2 = \max_{|x| \leq R} \Lambda(x)/(c_T\lambda(x))$, $\lambda(x)$ (resp. $\Lambda(x)$) being the minimal (resp. maximal) eigenvalue value of $f^{-1}(x)f^{-*}(x)$, and c_T is the constant given by Theorem 3.1.

Using (17), there is a constant $C_1 = C^2(C+C')$ such that

$$\frac{d}{dt} H^0(x_1(t), x_1(t)-x_2(t)) \leq \left(C_1 + \frac{C_2}{t}\right) H^0(x_1(t), x_1(t)-x_2(t)).$$

Then Gronwall Lemma gives

$$H^0(x_1(t_2), x_1(t_2)-x_2(t_2)) \leq \left(\frac{t_2}{t_1}\right)^{C_2} e^{C_1(t_2-t_1)} H^0(x_1(t_1), x_1(t_1)-x_2(t_1)).$$

□

We are now ready to state the main result of this paper.

Theorem 5.3. *Under assumption* (5), *the set* $\mathcal{R}(t)$ *is a set of finite perimeter for any* $t>0$. *Moreover, for any* $T>0$, *there are constant* c_1 *and* c_2 *such that for any* t_1, t_2 *with* $0<t_1<t_2\leq T$ *we have*

$$\begin{aligned} &\int_{\partial^*\mathcal{R}(t_2)} |f^*(x)\nu_{\mathcal{R}(t_2)}(x)|\, d\mathcal{H}^{N-1}(x) \\ \leq\ & \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)} \int_{\partial^*\mathcal{R}(t_1)} |f^*(x)\nu_{\mathcal{R}(t_1)}(x)|\, d\mathcal{H}^{N-1}(x). \end{aligned}$$

Theorem 5.3 is an application of the following Proposition:

Proposition 5.4. *With the notation and assumption of Theorem* 5.3, *we have*

$$\mathcal{H}_f^{N-1}(\partial\mathcal{R}(t_2)) \ \leq\ \left(\frac{t_2}{t_1}\right)^{c_2} e^{c_1(t_2-t_1)}\, \mathcal{H}_f^{N-1}(\partial^*\mathcal{R}(t_1)).$$

Remark. Note that we can estimate the measure of the topological boundary of $\mathcal{R}(t_2)$ in terms of the measure of the reduced boundary of $\mathcal{R}(t_1)$.

Proof of Theorem 5.3. The set $\mathcal{R}(t)$ has the interior ball property from Theorem 3.1, and therefore it is a set of finite perimeter thanks to Lemma 3.4. To get the estimate on the perimeter, it is enough to note that $\partial^*\mathcal{R}(t_2) \subset \partial\mathcal{R}(t_2)$ and to combine Proposition 5.4 with Lemma 4.2. □

Proof of Proposition 5.4. Since

$$\mathcal{H}^{N-1}(\partial_*\mathcal{R}(t_1)\backslash\partial^*\mathcal{R}(t_1)) = 0.$$

and since, from Lemma 4.1, $\mathcal{H}_f^{N-1}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, we have

$$\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1)\backslash\partial^*\mathcal{R}(t_1)) = 0.$$

Since $\mathcal{R}(t_1)$ has the interior ball property, we already know that $\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1)) < +\infty$. Let C_1 and C_2 be the constants given in Lemma 5.2. Let $R>0$ be a constant such that any solution starting from $\mathcal{I}$ remains in $B(0,R)$ on the time interval $[0,T]$. We also denote by $k>0$ a constant such that

$$\frac{|\det(f(y))|}{|\det(f(x))|} \le e^{k|y-x|} \qquad \forall (x,y)\in B(0,R)\times B(0,R). \tag{18}$$

Note that such a constant exists thanks to assumption (5). We also set $M=\|f\|_\infty$.

Let us fix $\epsilon>0$, $\delta>0$, K_i compact subsets of $\mathbb{R}^N$ such that

$$0<\mathrm{diam}_f(K_i) \ \le\ \delta(t_1/t_2)^{C_2}e^{-(C_1+kM/(N-1))(t_2-t_1)} \qquad \forall i\ge 1,$$

$$\partial_*\mathcal{R}(t_1)\subset\bigcup_{i=1}^{\infty}K_i$$

and

$$\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1))\ge \alpha_{N-1}\sum_{i=1}^{\infty}\left(\frac{\mathrm{diam}_f(K_i)}{2}\right)^{N-1}-\epsilon,$$

where α_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$.

We denote by K_i' the subset of points z of $\partial\mathcal{R}(t_2)$ for which there is an extremal solution x on $[0,t_2]$ with $x(t_2)=z$ and $x(t_1)\in K_i$. Then, from Lemma 5.1, we know that

$$\partial\mathcal{R}(t_2)\subset\bigcup_{i=1}^{\infty}K_i'.$$

We now estimate the diameter $\mathrm{diam}_f(K_i')$ of K_i'. Let z_1, z_2 belong to K_i', x_1, x_2 be extremal trajectories such that $x_j(t_2)=z_j$ and $x_j(t_1)\in K_i$ for $j=1,2$. Then from Lemma 5.2 and the definition of k in (18), we have

$$\begin{aligned}
&|\det(f(z_1))|^{\frac{1}{N-1}}H^0(z_1,z_1-z_2)\\
=\ &|\det(f(z_1))|^{\frac{1}{N-1}}H^0(x_1(t_2),x_1(t_2)-x_2(t_2))\\
\le\ &|\det(f(x_1(t_1)))|^{\frac{1}{N-1}}e^{k|x_1(t_1)-z_1|/(N-1)}(t_2/t_1)^{C_2}e^{C_1(t_2-t_1)}H^0(x_1(t_1),x_1(t_1)-x_2(t_1))\\
\le\ &(t_2/t_1)^{C_2}e^{(C_1+kM/(N-1))(t_2-t_1)}\mathrm{diam}_f(K_i)
\end{aligned}$$

because $|x_1(t_1)-z_1|\le M(t_2-t_1)$ since $\|f\|_\infty\le M$. Hence

$$\mathrm{diam}_f(K_i')\le (t_2/t_1)^{C_2}e^{(C_1+kM/(N-1))(t_2-t_1)}\mathrm{diam}_f(K_i)\le\delta.$$

Therefore, setting $c_1=(N-1)C_1+kM$ and $c_2=(N-1)C_2$, we get

$$\begin{aligned}
\mathcal{H}_{f,\delta}^{N-1}(\partial\mathcal{R}(t_2)) \ &\le\ \alpha_{N-1}\sum_{i=0}^{\infty}\left(\frac{\mathrm{diam}_f(K_i')}{2}\right)^{N-1}\\
&\le\ (t_2/t_1)^{c_2}e^{c_1(t_2-t_1)}\alpha_{N-1}\sum_{i=0}^{\infty}\left(\frac{\mathrm{diam}_f(K_i)}{2}\right)^{N-1}\\
&\le\ (t_2/t_1)^{c_2}e^{c_1(t_2-t_1)}(\mathcal{H}_f^{N-1}(\partial_*\mathcal{R}(t_1))+\epsilon).
\end{aligned}$$

Letting first $\delta\to 0^+$, then $\epsilon\to 0^+$, gives the result. □

6. Application to a volume estimate of the reachable set

Let $\mathcal{R}(t)$ be the reachable set at time t for the controlled system (1) with initial set $\mathcal{I}$, which is compact. We denote by $|\mathcal{R}(t)|$ the volume of $\mathcal{R}(t)$.

Corollary 6.1. *Under assumption* (5)*, the map* $t \to |\mathcal{R}(t)|$ *is locally Lipschitz continuous in* $(0, +\infty)$ *and*

$$\frac{d}{dt}|\mathcal{R}(t)| = \int_{\partial^*\mathcal{R}(t)} |f^*(x)\nu_{\mathcal{R}(t)}(x)| d\mathcal{H}^{N-1}(x) \text{ a.e. } t > 0.$$

Proof. As in the proof of Lemma 5.1 we introduce the minimal time function given by:

$$\tau(x) = \min\{t \geq 0 \mid x \in \mathcal{R}(t)\} \qquad \forall x \in \mathbb{R}^N,$$

and we recall that τ is locally Lipschitz continuous in $\mathbb{R}^N$ and satisfies the equation $H(x, D\tau(x)) = 1$ a.e. $x \in \mathbb{R}^N \backslash \mathcal{I}$. In particular, there is a constant $C > 0$ such that $|D\tau(x(t))| \geq 1/C$ a.e. $x \in \mathbb{R}^N \backslash \mathcal{I}$. Then the coarea formula states that

$$|\mathcal{R}(t)\backslash\mathcal{I}| = \int_0^t \int_{\{\tau=s\}} \frac{1}{|D\tau(y)|} d\mathcal{H}^{N-1}(y)ds = \int_0^t \int_{\{\tau=s\}} H\left(y, \frac{D\tau(y)}{|D\tau(y)|}\right) d\mathcal{H}^{N-1}(y)ds,$$

since $H(x, D\tau(x)) = 1$ a.e. $x \in \mathbb{R}^N \backslash \mathcal{I}$. Comparing the coarea formula for Lipschitz continuous functions with that same formula for BV functions (see for instance Proposition 2, p. 118 and Theorem 1, p. 185 of [9]) shows that

$$\mathcal{H}^{N-1}(\{\tau = s\}\backslash\partial^*\mathcal{R}(s)) = 0 \qquad \text{for almost all } s > 0. \tag{19}$$

Let us choose a level $s > 0$ such that equality (19) holds and such that $D\tau(y)$ exists for $\mathcal{H}^{N-1}-$almost every $y \in \partial^*\mathcal{R}(s)$. Let $y \in \partial^*\mathcal{R}(s)$ be a point of differentiability of τ. Then, since $D\tau(y) \neq 0$ and $\mathcal{R}(s) = \{z \in \mathbb{R}^N \mid \tau(z) \leq s\}$, we have $T_{\mathcal{R}(s)}(y) = (D\tau(y))^-$ and therefore, from Lemma 3.4, $\nu_{\mathcal{R}(s)}(y) = D\tau(y)/|D\tau(y)|$. This holds for $\mathcal{H}^{N-1}-$almost every $y \in \partial^*\mathcal{R}(s)$. So we have proved that

$$|\mathcal{R}(t)| = |\mathcal{I}| + \int_0^t \int_{\partial^*\mathcal{R}(s)} \left|f^*(y)\nu_{\mathcal{R}(s)}(y)\right| d\mathcal{H}^{N-1}(y)ds.$$

The map $s \to \int_{\partial^*\mathcal{R}(s)} \left|f^*(y)\nu_{\mathcal{R}(s)}(y)\right| d\mathcal{H}^{N-1}(y)$ being locally bounded from Theorem 5.3, the result is proved. □

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