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Vector Quasi-Variational Inequalities: Approximate Solutions and Well-Posedness*

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We introduce some concepts of approximate solutions for Vector Quasi-Variational Inequalities and we investigate the associated concepts of well-posedness, in line with Tikhonov well-posedness for Optimization Problems, Non Cooperative Games and scalar Variational Inequalities.

Keywords: Vector quasi-variational inequality, set-valued mapping, well-posedness, approximate solution, monotonicity, pseudomonotonicity

1. Introduction

A vector version of Variational Inequalities was introduced by F. Giannessi in [12]. Since then, several papers have been devoted to different aspects of this topic, mainly to existence of solutions and to relationships between Vector Variational Inequalities ((VV) for short) and Vector Optimization Problems ((VO) for short): [7], [13], [14]... .

Quasi-Variational Inequalities ((QVI), for short) were introduced by A. Bensoussan and J. L. Lions in [4] and were investigated by U. Mosco [25], C. Baiocchi and A. Capelo [3], and J.-P. Aubin [1]. Vector versions of such problems or of more general problems, were considered in [6], [9] and [15], only for existence or stability of solutions.

In this paper we are interested in investigating well-posedness for Vector Quasi-Variational Inequalities ((VQ), for short), in line with Tikhonov well-posedness for Optimization Problems and Non Cooperative Games (first investigated in [26], in [5], in [22]) and Variational Inequalities (see [10] and [18]). The interest in this study is motivated by recent results on well-posedness for Multicriteria Games and Quasi-Variational Inequalities. In fact, while in scalar case a Quasi-Variational Inequality can be equivalent to a social Nash Equilibrium Problem [3], a Vector Quasi-Variational Inequality can be equivalent to a *multicriteria game*. Well-posedness for multicriteria games has been recently introduced

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and studied in [24], while various well-posedness concepts for scalar Quasi-Variational Inequalities have been considered and investigated in [20].

Our aim in this paper is to introduce well-posedness concepts for Vector Quasi-Variational Inequalities which extend those introduced in [20] for the scalar case. Relationships between well-posedness for such problems and for multicritieria games will be investigated in a separate paper, also in the case where the compactness assumptions are avoided using coercive operators.

Outline of the paper is the following. Section 2 presents the definitions and results which are used throughout the paper and Section 3 contains a Minty's type theorem for Vector Quasi-Variational Inequalities, the links among the various concepts of well-posedness and conditions implying well-posedness of Vector Quasi-Variational Inequalities.

2. Background and Preliminaries

First, we recall some concepts and notations which will be used later on.

Let E and Z be two Banach spaces, K be a nonempty, closed and convex subset of E. We consider a set-valued mapping $C : u \in K \longrightarrow C(u) \subseteq Z$, where, for every $u \in K$, $C(u)$ is a convex, closed and pointed cone of Z, with apex at the origin and nonempty interior, denoted by $intC(u)$; $\leq_{C(u)}$ will denote the partial order induced by $C(u)$, that is:

$$w \leq_{C(u)} v \quad \text{iff } v - w \in C(u).$$

Let $T \in L(E, Z)$ and $u \in E$, it will be convenient to denote $T(u) \in Z$ by $\langle T, u\rangle_{L(E,Z)}$, or $\langle T, u\rangle_L$ for short, because of the similarities of several results below with the corresponding ones of usual quasi-variational inequalities.

For results concerning cones and efficient sets, see, for example, [21].

We recall that (see [11], [2]) for a sequence of subsets A_n in Z the definitions of $\limsup A_n$ and $\liminf A_n$ in the sense of Painlevé and Kuratowski are:

$$\liminf_n A_n = \left\{ y \in Z : \ \exists\, y_n \in A_n, n \in N, \text{ with } \lim_n y_n = y \right\},$$

$$\limsup_n A_n = \left\{ y \in Z : \exists n_k \uparrow +\infty, n_k \in N, \exists y_{n_k} \in A_{n_k}, k \in N, \text{ with } \lim_k y_{n_k} = y \right\}.$$

A set-valued function F from a topological space (X, τ) to a topological space (Y, σ) is:

· *closed-valued* (resp. convex-valued) if $F(x)$ is a nonempty closed (resp. convex) subset of Y, for every $x \in X$.

· *sequentially* (τ, σ)*-closed* on a subset H of X if, for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, for every sequence $(y_n)_n$ σ-converging to y such that $y_n \in F(x_n)$ for all $n \in N$, one has $y \in F(x)$ (that is $\limsup F(x_n) \subseteq F(x)$ for all $(x_n)_n$ τ-converging to x);

· *sequentially* (τ, σ)*-lower semicontinuous* on $H \subseteq X$ if, for every $x \in H$ and every sequence $(x_n)_n$ τ-converging to x in H, for every $y \in F(x)$ there exists a sequence $(y_n)_n$ σ-converging to y such that $y_n \in F(x_n)$ for all $n \in N$ (that is $F(x) \subseteq \liminf F(x_n)$ for every $x \in H$ and for all $(x_n)_n$ τ-converging to x);

· *sequentially* (τ,σ)*-subcontinuous* on $H \subseteq X$ if, for every sequence $(x_n)_n$ τ-converging in H, every sequence $(y_n)_n$ such that $y_n \in F(x_n)$, for all $n \in N$, has a σ-convergent subsequence.

For the sake of brevity we will omit the term *sequentially.*

Let a be a positive real number and H be a subset of E, $B(H,a)$ will indicate the closed ball around H of radius a, that is $\{u \in E : d(u,H) \leq a\}$. Throughout the paper s and w will denote, respectively, the strong and the weak topology on the Banach space E.

Let S be a set-valued mapping from K to K and A be an operator from E to the set of continuous linear mappings from E to Z. For any $u \in E$ we will denote Au in place of $A(u)$ and hence $\langle Au, v\rangle_L = A(u)(v)$ for all $v \in E$ according to the notation above. Throughout the paper the following problems will be considered:

- *Weak Vector Quasi-Variational Inequality* that consists in finding $u_o \in K$ such that:

$$(WVQ) \qquad u_o \in S(u_o) \quad \text{and} \quad \langle Au_o, v-u_o\rangle_L \not\leq_{intC(u_o)} 0 \quad \forall\, v \in S(u_o)$$

- *Weak Vector Linearized Quasi-Variational Inequality* that consists in finding $u_o \in K$ such that:

$$(WVL) \qquad u_o \in S(u_o) \quad \text{and} \quad \langle Av, v-u_o\rangle_L \not\leq_{intC(u_o)} 0 \quad \forall\, v \in S(u_o)$$

- *Vector Quasi-Variational Inequality* that consists in finding $u_o \in K$ such that:

$$(VQ) \qquad u_o \in S(u_o) \quad \text{and} \quad \langle Au_o, v-u_o\rangle_L \not\leq_{C(u_o)\setminus\{0\}} 0 \quad \forall\, v \in S(u_o)$$

- *Vector Linearized Quasi-Variational Inequality* that consists in finding $u_o \in K$ such that:

$$(VL) \qquad u_o \in S(u_o) \quad \text{and} \quad \langle Av, v-u_o\rangle_L \not\leq_{C(u_o)\setminus\{0\}} 0 \quad \forall\, v \in S(u_o)$$

Observe that if u_o solves (VQ) then it solves also (WVQ), as well as if u_o solves (VL) then it solves also (WVL).

In the particular case where $Z = R$, $L(E,Z) = E^*$ and $C(u) = [0,\infty[$ for every u, the first and the third problem are a Quasi-Variational Inequality, the others are their linearized forms. When $S(u) = K$ for every $u \in K$, the last two problems become a *Vector Variational Inequality* and a *Linearized Vector Variational Inequality* (or Vector Variational Inequality of Minty's type), while the first two are their *weak* formulations. The choice of the term *weak* comes from Vector Optimization. Indeed, for a vector function $f : E \longrightarrow Z$, one considers the *Weak Vector Minimization Problem*:

$$(P) \qquad w\text{-}min_C f(u) \quad u \in K$$

that consists in finding $u_o \in K$ such that $f(u_o) \not\geq_{intC} f(v) \ \ \forall v \in K$. In a finite dimensional spaces framework, when the function f is convex and differentiable on an open set containing the set K, if $Jf(u)$ denotes the Jacobian matrix at a point $u \in K$ then u_o solves (P) if and only if it solves [13]:

$$\langle Jf(u_o), v-u_o\rangle_L \not\leq_{intC} 0 \quad \forall\, v \in K.$$

We now introduce some concepts of approximate solutions for the listed problems.

Definition 2.1. Let ε be an element of $C(K)$. An element $u \in K$ is an *ε-solution* for the Weak Vector Quasi-Variational Inequality (WVQ), if:

i) $\varepsilon \in C(u)$;

ii) $d(u, S(u)) \leq \|\varepsilon\|$ and $\langle Au, v-u\rangle_L \not\leq_{\text{int}C(u)} -\varepsilon \ \forall v \in S(u)$,

that is: $u \in B(S(u), \|\varepsilon\|)$ and there does not exist $v \in S(u)$ such that:

$$\langle Au, v-u\rangle_L \leq_{\text{int}C(u)} -\varepsilon.$$

The set of all ε-solutions to the Weak Vector Quasi-Variational Inequality (WVQ) is denoted by WQ_ε.

Definition 2.2. Let ε be an element of $C(K)$. An element $u \in K$ is an *ε-solution* for the Weak Vector Linearized Quasi-Variational Inequality (WVL), if:

i) $\varepsilon \in C(u)$;

ii) $d(u, S(u)) \leq \|\varepsilon\|$ and $\langle Av, v-u\rangle_L \not\leq_{\text{int}C(u)} -\varepsilon \ \forall v \in S(u)$,

that is: $u \in B(S(u), \|\varepsilon\|)$ and there does not exist $v \in S(u)$ such that:

$$\langle Av, v-u\rangle_L \leq_{\text{int}C(u)} -\varepsilon.$$

The set of all ε-solutions to the Weak Vector Linearized Quasi-Variational Inequality (WVL) is denoted by WL_ε.

Similarly, considering the relation $\leq_{C(\cdot)\setminus\{0\}}$ instead of $\leq_{intC(\cdot)}$, one can define sets of approximate solutions for Vector Quasi-Variational Inequalities and for Vector Linearized Quasi-Variational Inequalities, denoted respectively by Q_ε and L_ε. Obviously, for every $\varepsilon \in C(K)$:

$$Q_\varepsilon \subseteq WQ_\varepsilon \quad \text{and} \quad L_\varepsilon \subseteq WL_\varepsilon. \tag{1}$$

In line with previous papers [18], [20] concerning well-posedness for Variational and Quasi-Variational Inequalities in the scalar case, we introduce two types of approximating sequences:

Definition 2.3. A sequence $(u_n)_n$, $u_n \in K$, is an *approximating sequence* for the Weak Vector Quasi-Variational Inequality (WVQ) if there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n \in C(K)$, converging to 0 and such that $u_n \in WQ_{\varepsilon_n}$ for every $n \in N$. This amounts to:

$$\varepsilon_n \in C(u_n), \quad d(u_n, S(u_n)) \leq \|\varepsilon_n\| \quad \text{and} \quad \langle Au_n, v-u_n\rangle_L \not\leq_{\text{int}C(u_n)} -\varepsilon_n \quad \forall\, v \in S(u_n).$$

Definition 2.4. A sequence $(u_n)_n$, $u_n \in K \ \forall n \in N$, is an *approximating sequence* for the Weak Vector Linearized Quasi-Variational Inequality (WVL) if there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n \in C(K)$, converging to 0 and such that $u_n \in WL_{\varepsilon_n}$ for every $n \in N$. This amounts to:

$$\varepsilon_n \in C(u_n), \ d(u_n, S(u_n)) \leq \|\varepsilon_n\| \quad \text{and} \quad \langle Av_n, v-u_n\rangle_L \not\leq_{\text{int}C(u_n)} -\varepsilon_n \quad \forall\, v \in S(u_n).$$

Similarly, one can define approximating sequences for Vector Quasi-Variational Inequalities (VQ) (resp. for Vector Linearized Quasi-Variational Inequalities (VL)) requiring that $u_n \in Q_{\varepsilon_n}$ (resp. $u_n \in L_{\varepsilon_n}$) for every $n \in N$.

Note that the condition $d(u_n, S(u_n)) \leq \|\varepsilon_n\|$ is less restrictive than $u_n \in S(u_n)$ and the condition $\langle Av, v - u_n\rangle_L \not\leq_{\mathrm{int}C(u_n)} -\varepsilon_n\ \forall v \in S(u_n)$ is less restrictive than $\langle Av, v - u_n\rangle_L \not\leq_{\mathrm{int}C(u_n)} -\varepsilon_n\ \forall v \in B(S(u_n), \varepsilon_n)$. An example of scalar Quasi-Variational Inequality, for which the condition $u_n \in S(u_n)$ is satisfied only by the sequence whose elements are equal to the unique solution, is given in [20]. We recall it for the sake of completeness.

Example 2.5 ([20]). Let $E = R$, $K = [0, +\infty[$, $S : u \in E \longrightarrow S(u) = [0, \frac{u}{2}]$ and $A : u \in R \longrightarrow Au = u$. In this case $u \notin S(u)\ \forall u \neq 0$, while $B(S(u), \varepsilon) \cap K = [0, \frac{u}{2} + \varepsilon]$ for every real positive number ε. Then, it is easy to compute that, if $(\varepsilon_n)_n$ is a sequence of positive real numbers, an approximating sequence in the sense of Definition 2.3, can be obtained taking the elements in $[0, 2\varepsilon_n]$, if $\varepsilon_n < \frac{1}{4}$, or in $[0, \sqrt{\varepsilon_n}]$, if $\varepsilon_n \geq \frac{1}{4}$.

We conclude the section with some properties of the approximate solution sets, for which it is useful the following lemma:

Lemma 2.6 ([16]). *Let $(H_n)_n$ be a sequence of nonempty subsets of a Banach space E such that:*

i) H_n *is convex for every* $n \in N$*;*
ii) $H \subseteq \liminf_n H_n$;
iii) *there exists* $m \in N$ *such that*

$$int \bigcap_{n \geq m} H_n \neq \emptyset.$$

Then, for every $u_o \in intH$ there exists a positive real number δ such that:

$$intB(u_o, \delta) \subseteq H_n \quad \forall n \geq m.$$

If E is a finite dimensional space, then assumption iii) can be replaced by

iii′) $intH \neq \emptyset$.

Proposition 2.7. *Assume that the following assumptions hold:*

i) *the set-valued mapping $S : K \longrightarrow K$ is convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous on K;*
ii) *the operator A is continuous from (E, s) to $(L(E, Z), w)$;*
iii) *the set-valued mapping C is (s, s)-lower semicontinuous and satisfies the following condition: for every converging sequence $(u_n)_n$, there exists $m \in N$ such that*

$$int \bigcap_{n \geq m} C(u_n) \neq \emptyset.$$

Then, for every $\varepsilon \in C(K)$, the sets WQ_ε and WL_ε are closed.

Proof. Let $\varepsilon \in C(K)$ and let $(u_n)_n$ be a sequence of elements in WQ_ε converging to $u_\varepsilon \in K$, that is, for all $n \in N$, $d(u_n, S(u_n)) \leq \|\varepsilon\|$ and $\langle Au_n, v - u_n\rangle_L \not\leq_{\mathrm{int}C(u_n)} -\varepsilon$ for all $v \in S(u_n)$. If $u_\varepsilon \notin WQ_\varepsilon$, then $d(u_\varepsilon, S(u_\varepsilon)) > a > \|\varepsilon\|$ or else there exists $v_\varepsilon \in S(u_\varepsilon)$ such that $\langle Au_\varepsilon, v_\varepsilon - u_\varepsilon\rangle_L \leq_{intC(u_\varepsilon)} -\varepsilon$. In the first case, let $z_n \in S(u_n)$ such that $\|u_n - z_n\| < a$

for every $n \in N$. The set-valued mapping S being closed and subcontinuous, there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ weakly converging to some $z_\varepsilon \in S(u_\varepsilon)$. Therefore one gets: $\|u_\varepsilon - z_\varepsilon\| \leq \liminf_k \|u_{n_k} - z_{n_k}\| \leq a < d(u_\varepsilon, S(u_\varepsilon))$, which gives a contradiction. In the other case, first observe that the lower semicontinuity of C and S implies that

$$C(u_\varepsilon) \subseteq \liminf_n C(u_n) \quad \text{and} \quad S(u_\varepsilon) \subseteq \liminf_n S(u_n). \tag{2}$$

From the second inclusion in (2), there exists a sequence $(v_n)_n$ converging to v_ε such that $v_n \in S(u_n)$ for n sufficiently large. The operator A being continuous, one gets

$$\lim_n \langle Au_n, v_n - u_n \rangle_L = \langle Au_\varepsilon, v_\varepsilon - u_\varepsilon \rangle_L \leq_{intC(u_\varepsilon)} -\varepsilon.$$

Since $y_\varepsilon = (-\varepsilon - \langle Au_\varepsilon, v_\varepsilon - u_\varepsilon \rangle_L) \in intC(u_\varepsilon)$, from (2), assumption *iii*) and Lemma 2.6 there exist $m \in N$ and $\delta > 0$ such that:

$$intB(y_\varepsilon, \delta) \subseteq C(u_n) \quad \forall\, n \geq m.$$

Therefore, for n sufficiently large we have:

$$\langle Au_n, v_n - u_n \rangle_L \leq_{intC(u_n)} -\varepsilon.$$

Since $v_n \in S(u_n)$ and $u_n \in WQ_\varepsilon$, we get a contradiction.

The following example shows that, under conditions *i*)-*iii*), the sets Q_ε and L_ε may fail to be closed even in finite dimensional spaces.

Example 2.8. Let $E = R$, $K = [-1, 0]$, $Z = R^2$, $Au = (1, u)$, $S(u) = [-1, u]$ and $C(u) = C = [0, +\infty[^2$. Consider $\varepsilon = (\varepsilon_1, 0) \in C$ with $\varepsilon_1 > 0$. The inequality $\langle Au_\varepsilon, v - u_\varepsilon \rangle_L \leq_{C\setminus\{0\}} -\varepsilon$ means that $(v - u_\varepsilon, u_\varepsilon v - {u_\varepsilon}^2) \leq_{C\setminus\{0\}} (-\varepsilon_1, 0)$. Then the set Q_ε coincides with $[-1, 0[$ while $WQ_\varepsilon = [-1, 0]$.

Moreover, all the sets of approximate solutions are not always convex, even in the scalar case.

Example 2.9. Let $E = R$, $K = [-1, 1]$, $Z = R$, $Au = 0$, $C(u) = C = [0, +\infty[$ and

$$S(u) = \begin{cases} [0, u+1] & \text{if } -1 \leq u \leq 0 \\ \{1\} & \text{if } 0 < u \leq 1 \end{cases}$$

Then, for every $\varepsilon < 1$ all the sets of approximate solutions coincide with the set $[-\varepsilon, 0] \cup [1 - \varepsilon, 1]$.

For the sake of simplicity, the results in the next section will be given considering, instead of a *moving* cone, a convex, closed and pointed cone C of Z, with apex at the origin and with nonempty interior.

3. Well-posed Vector Quasi-Variational Inequalities

Using the notations introduced in Section 2, we recall that, for $\varepsilon \in C$, Q_ε, L_ε, WQ_ε and WL_ε denote, respectively, the set of ε-solutions to the Vector Quasi-Variational Inequality (VQ), to the Vector Linearized Quasi-Variational Inequality (VL), to the Weak

Vector Quasi-Variational Inequality (WVQ) and to the Weak Vector Linearized-Quasi Variational Inequality (WVL). When $\varepsilon = 0$ one obtains the sets of the exact solutions to the previous problems.

Definition 3.1. Let *(P)* be any of the listed problems; *(P)* is said to be *well-posed* if it admits at least a solution and every approximating sequence for *(P)* has a subsequence which converges to a solution of *(P)*.

In the particular case where $Z = R$, $L(E, Z) = E^*$ and $C = [0, \infty[$, the well-posedness notions for (VQ) and for (VL) coincide respectively with the notions of well-posedness and $L-$well-posedness in the generalized sense defined in [20]. Note that, as well as for Minimization Problems, many definitions of well-posedness for Variational Inequalities or Quasi-Variational Inequalities require the uniqueness of the solution; for Vector Problems this would be an utopistic assumption! Moreover, even in the scalar case, remember that a Quasi-Variational Inequality defined by a strongly monotone operator does not have necessarily a unique solution (see Example 11.2 in [3]).

F. Giannessi in [13] proved that, if E is a finite dimensional space and the operator A is monotone and continuous, a Weak Vector Variational Inequality is equivalent to the corresponding linearized problem, while this equivalence does not hold for Vector Variational Inequalities. Before proving this result for Weak Vector Quasi-Variational Inequalities in Banach spaces, we give some useful monotonicity properties for operators:

· an operator A from E to $L(E, Z)$ is W*-pseudomonotone* on a subset Y of E if, for every $u \in Y$ and $v \in Y, u \neq v$

$$\langle Av, u - v\rangle_L \geq_{intC} 0 \Longrightarrow \langle Au, u - v\rangle_L \geq_{intC} 0;$$

· an operator A from E to $L(E, Z)$ is *pseudomonotone* on a subset Y of E if, for every $u \in Y$ and $v \in Y, u \neq v$

$$\langle Av, u - v\rangle_L \geq_{C\setminus\{0\}} 0 \Longrightarrow \langle Au, u - v\rangle_L \geq_{C\setminus\{0\}} 0.$$

Proposition 3.2. *Assume that the following assumptions hold:*

i) *the set-valued mapping S is closed-valued and convex-valued on K;*

ii) *the operator A is pseudomonotone (resp. W-pseudomonotone) on K.*

Then

$$Q_o \subseteq L_o \qquad (resp.\ WQ_o \subseteq WL_o).$$

If the operator A is hemicontinuous on K (i.e. it is continuous over the segments of E to $(L(E, Z), w)$), then

$$WQ_o \supseteq WL_o.$$

Proof. We prove that if $u_o \notin L_o$ then $u_o \notin Q_o$. When $u_o \notin L_o$, then $u_o \notin S(u_o)$ or else there exists $v_o \in S(u_o)$ such that $\langle Av_o, u_o - v_o\rangle_L \geq_{C\setminus\{0\}} 0$. In both cases $u_o \notin Q_o$, since the pseudomonotonicity of the operator A on K implies that

$$\langle Au_o, u_o - v_o\rangle_L \geq_{C\setminus\{0\}} 0,$$

hence

$$\langle Au_o, v_o - u_o\rangle_L \leq_{C\setminus\{0\}} 0.$$

Assume that $u_o \notin WL_o$. Then, $u_o \notin S(u_o)$ or else there exists $v_o \in S(u_o)$ such that $\langle Av_o, u_o - v_o\rangle_L \geq_{intC} 0$. As before, one concludes that

$$\langle Au_o, v_o - u_o\rangle_L \leq_{intC} 0,$$

(since the operator A is W-pseudomonotone on K) so $u_o \notin WQ_o$.

In order to prove that $WQ_o \supseteq WL_o$ when the operator A is hemicontinuous on K, consider $u_o \in WL_o$ and assume that $u_o \notin WQ_o$. Since $u_o \in S(u_o)$, there exists $v_o \in S(u_o)$ such that $\langle Au_o, v_o - u_o\rangle_L \leq_{intC} 0$.

For every real number $t \in [0,1]$ consider $v_t = tu_o + (1-t)v_o$, which belongs to $S(u_o)$ for every $t \in [0,1]$. Since u_o is a solution to the Weak Linearized Quasi-Variational Inequality (WVL) one has

$$\langle Av_t, u_o - v_t\rangle_L = (1-t)\langle Av_t, u_o - v_o\rangle_L \not\geq_{intC} 0.$$

Then

$$\langle Av_t, u_o - v_o\rangle_L \not\geq_{intC} 0,$$

and, in light of the hemicontinuity of the operator A, one has

$$\langle Au_o, u_o - v_o\rangle_L \not\geq_{intC} 0,$$

and one gets a contradiction.

In the next result, which is concerned with relationships among well-posed problems, we use the following notions of monotonicity for operators:

- an operator A from E to $L(E,Z)$ is *W-monotone* on a subset Y of E if, for every $u \in Y$ and $v \in Y, u \neq v$
$$\langle Au - Av, u - v\rangle_L \geq_{intC} 0;$$
- an operator A from E to $L(E,Z)$ is *monotone* on a subset Y of E if, for every $u \in Y$ and $v \in Y, u \neq v$
$$\langle Au - Av, u - v\rangle_L \geq_{C\setminus\{0\}} 0.$$

Proposition 3.3. *Assume that the operator A is W-monotone on the set K. Then*

$$(WVL) \text{ well-posed} \implies (WVQ) \text{ well-posed},$$

$$(VL) \text{ well-posed} \implies (VQ) \text{ well-posed}.$$

As a consequence, when the operator A is also hemicontinuous and the Weak Vector Quasi-Variational Inequality (WVQ) and the Vector Quasi-Variational Inequality (VQ) have the same solutions, if the Weak Vector Linearized Quasi-Variational Inequality (WVL) is well-posed, then all the problems are well-posed.

Proof. Let $(u_n)_n$, $u_n \in K$, be an approximating sequence for the Weak Vector Quasi-Variational Inequality (WVQ). Then, there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n \in C$, converging to 0 in Z such that:

$$d(u_n, S(u_n)) \leq \|\varepsilon_n\| \quad \text{and} \quad \langle Au_n, v - u_n\rangle_L \not\leq_{\text{int}C} -\varepsilon_n \quad \forall\, v \in S(u_n).$$

Assume that the sequence $(u_n)_n$ is not approximating for the Weak Vector Linearized Quasi-Variational Inequality (WVL). Then, there exist $m \in N$ and $v_m \in S(u_m)$ such that $\langle Av_m, v_m - u_m \rangle_L \leq_{\mathrm{int}C} -\varepsilon_m$. Since the operator A is W-monotone on K one has:

$$\langle Au_m, v_m - u_m \rangle_L \leq_{\mathrm{int}C} -\varepsilon_m,$$

which gives a contradiction.

In a similar way, using the monotonicity of the operator A, one can prove that the Vector Quasi-Variational inequality (VQ) is well-posed whenever the Vector Linearized Quasi-Variational Inequality (VL) is well-posed.

When the set of solutions to the Weak Vector Quasi-Variational Inequality and the set of solutions to the Vector Quasi-Variational Inequality coincide, the same occurs for the linearized problems, since the operator A is monotone and hemicontinous and one has

$$Q_o \subseteq L_o \subseteq WL_o \subseteq WQ_o.$$

Then, the proof can be completed recalling that $Q_\varepsilon \subseteq WQ_\varepsilon$ and $L_\varepsilon \subseteq WL_\varepsilon$, for every $\varepsilon \in C$.

Remark 3.4. In [27] an example shows that, even if the operator A is strongly monotone (see Definition 2 in [27]), the sets of solutions to the Vector Variational Inequality and to the Weak Vector Variational Inequality defined by A do not always coincide. Nevertheless the authors gave a class of operators for which these sets coincide (see Theorem 2 in [27]).

When the space E is finite dimensional, the well-posedness is guaranteed under reasonable assumptions.

Proposition 3.5. *Assume that K is a compact and convex subset of $E = R^k$ and assume that the following assumptions hold:*

i) the set-valued mapping S is convex-valued, closed and lower semicontinuous on K;
ii) the operator A is continuous on K.

Then

- *the Weak Vector Quasi-Variational Inequality (WVQ) is well-posed if (and only if) it admits at least a solution;*
- *the Weak Vector Linearized Quasi-Variational Inequality (WVL) is well-posed if (and only if) it admits at least a solution.*

Proof. Let $(u_n)_n$ be an approximating sequence for the Weak Vector Linearized Quasi-Variational Inequality (WVL) and let $(\varepsilon_n)_n$ be a sequence converging to 0 in C such that

$$d(u_n, S(u_n)) \leq \|\varepsilon_n\| \quad \text{and} \quad \langle Av, v - u_n \rangle_L \not\leq_{intC} -\varepsilon_n \quad \forall\, v \in S(u_n). \tag{3}$$

Since the set K is compact, let $(u_{n_k})_k$ be a subsequence of $(u_n)_n$ converging to a point $u_o \in K$. We prove that u_o solves the Weak Vector Linearized Quasi-Variational Inequality (WVL).

In fact, first, we observe that $d(u_o, S(u_o)) \leq \liminf_k d(u_{n_k}, S(u_{n_k})) \leq \lim_k \|\varepsilon_{n_k}\| = 0$. Indeed, if the left inequality fails to be true, there exists a positive number η such that

$$\liminf_k d(u_{n_k}, S(u_{n_k})) < \eta < d(u_o, S(u_o)).$$

Then, there exist a subsequence $(u_{n_{k_h}})_h$ of $(u_{n_k})_k$ and a sequence $(v_h)_h$ such that:

$$v_h \in S(u_{n_{k_h}}) \quad \text{and} \quad \|u_{n_{k_h}} - v_h\| < \eta \quad \forall\, h \in N.$$

Since the set K is compact and the set-valued mapping S is closed, the sequence $(v_h)_h$ has a subsequence which converges to $w_o \in S(u_o)$ and $\|u_o - w_o\| \leq \eta$, which leads to a contradiction. Therefore one may conclude that $u_o \in S(u_o)$.

Assume now that there exists $v_o \in S(u_o)$ such that $\langle Av_o, u_o - v_o\rangle_L \geq_{intC} 0$. The lower semicontinuity of the set-valued mapping S implies that there exists a sequence $(v_k)_k$ converging to v_o such that $v_k \in S(u_{n_k})$. Since the operator A is continuous one gets

$$\langle Av_k, v_k - u_{n_k}\rangle_L \leq_{intC} -\varepsilon_k,$$

for k sufficiently large, and this contradicts (3).

The proof for a Weak Vector Quasi-Variational Inequality is similar and is omitted.

Corollary 3.6. *Assume that the assumptions of Proposition* 3.5 *are satisfied and the operator A is W-monotone. If the Vector Quasi-Variational Inequality* (VQ) *and the Weak Vector Quasi-Variational Inequality* (WVQ) *have the same solutions, then*

- *the Vector Quasi-Variational Inequality* (VQ) *is well-posed;*
- *the Vector Linearized Quasi-Variational Inequality* (VL) *is well-posed.*

Proof. From Proposition 3.5 one gets the well-posedness of the Weak Vector Quasi-Variational Inequality (WVQ). Then, since an approximating sequence $(u_n)_n$ for (VQ) is also approximating for (WVQ), there exists a subsequence of $(u_n)_n$ converging to a solution of (WVQ) which is also a solution of (VQ). The second statement can be proved similarly, observing that $Q_o \subseteq L_o \subseteq WL_o \subseteq WQ_o$.

The compactness assumption on the set K cannot be weakened, as shown by the following example.

Example 3.7. Let $E = Z = R$, $K = C = [0, +\infty[$, $Au = -e^{-u}$ and

$$S(u) = \begin{cases} [u, 1] & \text{if } u \leq 1 \\ [1, u] & \text{if } u \geq 1. \end{cases}$$

Then the sequence $u_n = n$ is approximating for the Quasi-Variational Inequality:

$$u_o \in S(u_o) \quad \text{and} \quad \langle Au_o, u_o - v\rangle_L \leq 0 \quad \forall\, v \in S(u_o)$$

and for the Linearized Quasi-Variational Inequality:

$$u_o \in S(u_o) \quad \text{and} \quad \langle Av, u_o - v\rangle_L \leq 0 \quad \forall\, v \in S(u_o),$$

but it does not have convergent subsequences.

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