Journal of Convex Analysis
Volume 13 (2006), No. 2, 353–362

Existence and Relaxation Theorems for Unbounded Differential Inclusions

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Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

We are interested in the existence of solutions of the differential inclusion

$$\dot{x} \in F(t,x) \tag{1}$$

on the given time interval, say $[0,1]$. Here F is a set-valued mapping from $[0,1]\times\mathbf{R^n}$ into R^n (we shall write $F:[0,1]\times\mathbf{R^n}\rightrightarrows\mathbf{R^n}$ in what follows) with closed values which will be assumed nonempty whenever necessary.

The classical theorems of Filippov and Wazewski use, as the main assumption characterizing the dependence of F on x, the standard Lipschitz condition

$$h(F(t,x),F(t,x'))\le k(t)\|x-x'\|,$$

where $h(P,Q)$ stands for the Hausdorff distance from P to Q.

This condition, quite reasonable when F is bounded-valued, becomes unacceptably strong if the values of F can be unbounded. Meanwhile unboundedness of the values of the right-hand side set-valued mapping is a fairly natural property of differential inclusions which appear in optimal control problems, e.g. when we deal with a Mayer problem obtained as a result of reformulation of a problem with integral functional. The main purpose of this note is to provide an existence theorem with a weaker version of the Lipschitz condition which is "more acceptable" when the values of F are unbounded. This condition which could be characterized as a "global" version of Aubin's pseudo-Lipschitz property is very close to that introduced by Loewen and Rockafellar in [3].

Keywords: Differential inclusion, relaxation, global pseudo-Lipschitz condition

1. Introduction

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*The research was supported in part by the US-Israel Binational Fund under the grant 2000157.

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To briefly describe this property, we first mention that getting rid of any Lipschitz-type property is probably impossible in principle. A possible way to ease the limitation imposed by the Hausdorff-Lipschitz condition (2) is to make the estimate dependent on $\dot{x}$, e.g. as follows:

$$y \in F(t,x) \;\Rightarrow\; d(y, F(t,x')) \le \varphi(t,y)\|x - x'\|,$$

where $d(y,Q)$ is the distance from y to Q. The global version of Aubin's pseudo-Lipschitz condition would then correspond to

$$\varphi(t,y) = k(t) + \beta\|y\|,$$

where $\beta \ge 0,\ k(\cdot) \in \mathcal{L}^1$.

In geometric terms this property can be formulated as follows.

Definition 1.1. Let $Q \subset [0,1] \times \mathbf{R^n}$. We say that F satisfies the *global pseudo-Lipschitz condition on* Q if $F(t,x) \ne \emptyset$ and there are $\beta \ge 0$ and $k(\cdot) \in \mathcal{L}^1$ such that for all $N > 0$

$$F(t,x) \bigcap NB \subset F(t,x') + (k(t) + \beta N)\|x - x'\|B,$$

provided (t,x) and (t,x') belong to Q. Here B stands for the unit ball in $\mathbf{R^n}$.

The plan of the paper is the following. We first prove a "local" existence theorem which guarantees the existence of the solution of the Cauchy problem on a sufficiently small time interval. The structure of the proof is very similar to that in Filippov's original paper [2] and even the construction of iterations is exactly the same. What is different is the technique of the convergence analysis which is based on a different and no longer linear (in the absence of the possibility to use Gronwall lemma) estimating process (see Lemma 2.1 in the next section). It is actually the first theorem that should be considered an extension of Filippov's existence theorem. The second theorem about the existence of a solution defined on the entire segment, which is actually the principal result of the paper, is a much less automatic consequence of the local theorem, than the corresponding result with (2). The existence of an exact solution on the given interval is proved here under the assumption that that there is a certain sufficiently good approximate solution. The second theorem is further applied to study the relaxation problem and to get an extension of Filippov-Wazewski relaxation theorem to differential inclusions with unbounded values.

2. Existence of solutions on small intervals

Lemma 2.1. *Consider the following recursive system of inequalities:*

$$q_{n+1} - q_n \le r_{n+1}; \quad r_{n+1} \le \xi_n r_n; \quad \xi_n = \kappa + \beta q_n, \qquad q_n \ge 0,\ r_n \ge 0,$$

where κ and β being nonnegative parameters. Suppose the initial values q_1, r_1 and ξ_1 are given satisfying

$$\xi_1 \le \lambda - \beta r_1 \frac{\lambda}{1-\lambda} \tag{2}$$

for some $\lambda \in (0,1)$. Let finally (q_n, r_n, ξ_n) be a corresponding solution of the recursive system. Then

$$\sum_{n=1}^{\infty} r_n \le \frac{r_1}{1-\lambda}.$$

Proof. We claim that under the assumptions $\xi_n \le \lambda$ for all n. Indeed, $\xi_1 \le \lambda$. Assume that $\xi_i \le \lambda$ for $i = 1, ..., n$. We have

$$\xi_{n+1} = \kappa + \beta q_{n+1} \le \kappa + \beta(q_n + r_{n+1}) \le \xi_n + \beta r_{n+1}$$

Applying this estimate recursively, we get

$$\xi_{n+1} \le \xi_1 + \beta(r_2 + ... + r_{n+1}) \le \xi_1 + \beta r_1(\lambda + ... + \lambda^n) \le \xi_1 + \beta r_1 \frac{\lambda}{1-\lambda} \le \lambda$$

which proves the claim.

Consequently, $r_n \le \lambda^{n-1} r_1$, and the results follows. □

Observe that $\beta r_1 < 1$ by (2) and

$$\max_{0<\lambda<1} [\lambda - \beta r_1 \frac{\lambda}{1-\lambda}] = (1 - \sqrt{\beta r_1})^2 = \bar\lambda - \beta r_1 \frac{\bar\lambda}{1-\bar\lambda}, \quad \text{where } \bar\lambda = 1 - \sqrt{\beta r_1}. \tag{3}$$

Let us turn to the inclusion (1). We shall assume throughout that

($\mathbf{A}_1$) F is $\mathcal{L} \bigotimes \mathcal{B}$-measurable;

($\mathbf{A}_2$) Graph $F(t, \cdot)$ is a closed set for almost every t.

These assumptions are satisfied in all situations of practical interest. Let us fix furthermore a certain absolutely continuous $\bar{x}(t)$, and for $\alpha > 0$ let $Q(\alpha) = \{(t, x) : t \in [0, 1], \|x - \bar{x}(t)\| < \alpha\}$. We shall assume that

($\mathbf{A}_3$) there is an $\alpha > 0$ such that F is globally pseudo-Lipschitz on $Q(2\alpha)$.

This means in particular that there are β and $k(\cdot)$ such that

$$y \in F(t, x) \;\Rightarrow\; y \in F(t, x') + (k(t) + \beta\|y\|)\|x - x'\|B, \tag{4}$$

provided both (t, x) and (t, x') belong to $Q(2\alpha)$.

As the values of F are closed, it follows from ($\mathbf{A}_1$), ($\mathbf{A}_3$) through the standard measurable selection arguments that whenever the graphs of $x(t)$ and $x'(t)$ lie in $Q(2\alpha)$ and $y(t) \in F(t, x(t))$ almost everywhere, there is a measurable $y'(t)$ such that $y'(t) \in F(t, x'(t))$ a.e. and $\|y(t) - y'(t)\| = d(y(t), F(t, x'(t)))$ a.e.. Moreover ($\mathbf{A}_3$) implies that $y'(\cdot)$ is summable if so is $y(\cdot)$.

Theorem 2.2. *Let $x(t)$ be an absolutely continuous $\mathbf{R^n}$-valued function on a certain $[a,b] \subset [0,1]$ with its graph in $Q(\alpha)$. Assume that the function $t \to d(\dot{x}(t), F(t,x(t)))$ is summable and set*

$$r_1(t) = \int_a^t d(\dot{x}(s), F(s,x(s)))ds, \qquad q_1(t) = \int_a^t \|\dot{x}(s)\|ds$$

and

$$\kappa(t) = \int_a^t k(s)ds, \qquad \xi(t) = \kappa(t) + \beta q_1(t).$$

Let finally $\tau \in (a,b]$ be such that $r_1 = r_1(\tau)$, $q_1 = q_1(\tau)$ and $\kappa = \kappa(\tau)$ satisfy (2) *and $r_1 < (1-\lambda)\alpha$ with some $\lambda \in (0,1)$. Then there is a solution $u(t)$ of* (1) *defined on $[a,\tau]$ and such that $u(a) = x(a)$ and*

$$\int_a^\tau \|\dot{x}(t) - \dot{u}(t)\|dt \le \frac{r_1}{1-\lambda}.$$

Remark. The assumptions impose little restriction on F. The only requirement that $d(\dot{x}(t), F(t,x(t)))$ is summable holds for any $x(\cdot) \in W^{1,1}$ (with graph in Q) if, say $d(0, F(t,\bar{x}(t)))$ is summable (and, actually only under this condition - as will follow from the inequality (12) in the next section).

Proof. We shall construct inductively a sequence $(x_n(\cdot))$ of summable functions converging in $W^{1,1}$ to a desired $u(\cdot)$. Set $x_0(t) = x(t)$, $v_0(t) = \dot{x}(t)$. If $x_i \in W^{1,1}$ and $v_i = \dot{x}_i$ have been already found for $i = 0, ..., n$, and $\|x_i(t) - x(t)\| < \alpha$ for all $t \in [a,\tau]$ and i, then we set

$$r_{n+1}(t) = \int_a^t d(v_n(s), F(s,x_n(s)))ds; \quad q_n(t) = \int_a^t \|v_n(s)\|ds; \quad \xi_n(t) = \kappa(t) + \beta q_n(t)$$

and define x_{n+1} and v_{n+1} as follows: $v_{n+1}(t) \in F(t,x_n(t))$, $\|v_{n+1}(t) - v_n(t)\| = d(v_n(t), F(t,x_n(t))$ a.e. and

$$x_{n+1}(t) = x(a) + \int_a^t v_{n+1}(s)ds.$$

Clearly, this can be done. As $x_n(t)$ is within α of $x(t)$ for every t, the ($\mathbf{A}_3$) related assumption of the theorem implies that the distance from $\dot{x}_n(t)$ to $F(t,x_n(t))$ is a summable function. The existence of a $v_{n+1}(t)$ with the claimed properties now follows through the standard measurable selection argument from the fact that F is closed-valued.

By ($\mathbf{A}_3$), we have for $n \ge 1$

$$\begin{aligned} \|\dot{x}_{n+1}(t) - \dot{x}_n(t)\| &= \|v_{n+1}(t) - v_n(t)\| = d(v_n(t), F(t,x_n(t)) \\ &\le (k(t) + \beta\|v_n(t)\|)\|x_n(t) - x_{n-1}(t)\| \\ &\le (k(t) + \beta\|v_n\|(t)) \int_a^t \|v_n(s) - v_{n-1}(s)\|ds \qquad (5) \\ &= (k(t) + \beta\|v_n(t)\|) \int_a^t d(v_{n-1}(s), F(s,x_{n-1}(s))ds \\ &= (k(t) + \beta\|v_n(t)\|)r_n(t). \end{aligned}$$

Integrating the inequality and taking into account that $r_n(t)$ are non-decreasing, we get

$$r_n(t) \le (\kappa(t) + \beta q_n(t))r_{n-1}(t). \tag{6}$$

On the other hand, (6) together with the upper equality in (5) gives

$$q_{n+1}(t) - q_n(t) \le r_{n+1}(t). \tag{7}$$

In particular, setting $r_n = r_n(\tau),\ q_n = q_n(\tau)$, we get

$$r_n \le (k + \beta q_n)r_{n-1}; \quad q_{n+1} - q_n \le r_{n+1}.$$

Applying Lemma 2.1, we get in view of the choice of τ

$$\sum_{n=1}^{\infty} r_n(\tau) \le \frac{r_1}{1-\lambda} < \alpha. \tag{8}$$

By definition of $v_n(\cdot)$

$$\int_a^\tau \|\dot{x}_{n+1}(t) - \dot{x}_n(t)\|dt = r_{n+1}(\tau) \tag{9}$$

which implies along with (8) that $\|x_{n+1}(t) - x(t)\| < \alpha$ for all $t \in [a,\tau]$, so that the induction can be continued, and consequently, that $\dot{x}_n(\cdot)$ converge in $\mathcal{L}_1[0,\tau]$. Therefore (as $x_n(a) = x(a)$ for all n), $x_n(\cdot)$ converge uniformly on $[a,\tau]$ to some $u(\cdot)$ with $\dot{u}(t) = \lim \dot{x}_n(t)$. By definition (and (8))

$$\int_a^t d(\dot{x}_n(\tau), F(\tau, x_n(\tau))d\tau = r_{n+1}(t) \to 0, \quad \forall t \in [a,\tau]$$

which, according to $(\mathbf{A}_2)$ means that $\dot{u}(t) \in F(t,u(t))$, that is $u(\cdot)$ is a solution of (1) on $[a,\tau]$. It remains to refer to (8) and (9) to get the final estimate. □

3. Existence of solutions on the entire segment

In what follows we denote by $\mathcal{X}$ the collection of all solutions of (1) defined of $[0,1]$ and considered along with the $W^{1,1}$-metric. As follows from $(\mathbf{A}_2)$ this is a complete metric space.

Theorem 3.1. *Assume that there are $\alpha > 0,\ \beta \ge 0$ and a nonnegative summable $k(t)$ defined on $[0,1]$ such that $(\mathbf{A}_3)$ holds. Then for any $\gamma > 1$, and N there is a $\delta > 0$ such that the inequality*

$$d(x(\cdot),\mathcal{X}) \le \gamma e^{\int_0^1 (k(t)+\beta\|\dot{x}(t)\|)dt} \int_0^1 d(\dot{x}(t), F(t,x(t)))dt \tag{10}$$

holds for any $x(\cdot) \in W^{1,1}$ such that

$$\|x(t) - \bar{x}(t)\| \le \frac{\alpha}{2},\ \forall t \in [0,1]; \quad \int_0^1 d(\dot{x}(t), F(t,x(t)))dt < \delta; \quad \int_0^1 \|\dot{x}(t)\|dt \le N. \tag{11}$$

(The d in the left-hand side of (10) is the distance in $W^{1,1}$.)

Remark. The conclusion of Theorem 3.1 is much stronger than that of Theorem 2.2 but so is the assumption (11), basically, its second part which says that there is a good approximate solution of the differential inclusion. A possible estimate for δ as a function of α, γ and N will be given in part 3 of the proof.

Proof. 1. To begin with we mention the following simple fact: if $\eta(t)$ is a nonnegative summable function on $[0,1]$ with $\int \eta(t)dt \le p$ and $\varepsilon > 0$, then there are $k \le [p/\varepsilon]$ points (square brackets mean the integer part of the number) $0 < \tau_1 < ... < \tau_k < 1$ such that (setting $\tau_0 = 0,\ \tau_{k+1} = 1$)

$$\int_{\tau_i}^{\tau_{i+1}} \eta(t)dt < \varepsilon, \quad i = 0, ..., k.$$

(Indeed, define τ_i consecutively starting with $\tau_0 = 0$ by

$$\int_{\tau_i}^{\tau_{i+1}} \eta(t)dt = \frac{p}{k+1}.)$$

2. We can assume losing no generality that $\bar{x}(t) \equiv 0$. Observe next that $(\mathbf{A}_3)$ implies the following inequality, provided $\|x\| < \alpha$ and $\|x + w\| < \alpha$:

$$d(y, F(t, x + w)) \le (1 + \beta\|w\|)d(y, F(t, x)) + (k(t) + \beta\|y\|)\|w\|. \tag{12}$$

Indeed, take $z \in F(t, x)$ such that $\|y - z\| = d(y, F(t, x))$. Then by $(\mathbf{A}_3)$

$$z \in F(t, x + w) + (k(t) + \beta\|z\|)\|w\|B$$

which means that

$$d(z, F(t, x + w)) \le (k(t) + \beta\|z\|)\|w\| \le (k(t) + \beta(\|y\| + d(y, F(t, x))))\|w\|,$$

so that

$$\begin{aligned} d(y, F(t, x + w)) &\le \|y - z\| + d(z, F(t, x + w)) \\ &= d(y, F(t, x)) + (k(t) + \beta(\|y\| + d(y, F(t, x))))\|w\| \end{aligned}$$

and (12) follows.

3. So suppose that a $\gamma > 1$ and an N be given. We shall assume that $\gamma \le 2$ which of course is not a restrictive assumption. Take an $\varepsilon \in (0, \alpha/8)$ to make sure that

$$1 + \beta\varepsilon \le \gamma,$$

set

$$M = \int_0^1 k(t)dt + \beta N; \quad \eta = \frac{\gamma - 1}{2\gamma}$$

and finally choose δ be so small that

$$\gamma(1 + 2e^{\gamma M})\delta < \varepsilon \quad \text{and} \quad \beta\gamma\delta \le \frac{\eta}{2}. \tag{13}$$

4. Now fix an $x(\cdot) \in W^{1,1}$ satisfying (11) and a sufficiently small $\lambda > 0$. The theorem will be obtained as a result of limit calculation with $\lambda \to 0$. It will become clear in the course of the proof how small the starting λ should be but in any case not greater than 1/2.

Set

$$p = p(x(\cdot)) = \int_0^1 (k(t) + \beta\|\dot{x}(t)\|)dt; \quad n = n(\lambda) = \left[\frac{\gamma p}{\lambda}\right].$$

By (11), $p \le M$.

According to the first step of the proof, we can find $n = [\gamma p/\lambda]$ points $0 < \tau_1 < ... < \tau_n < 1$ such that

$$\xi_1^i = \int_{\tau_i}^{\tau_i+1} (k(t) + \beta\|\dot{x}(t)\|)dt \le \frac{\lambda}{\gamma}, \ i = 0, ..., k, \tag{14}$$

where we have set $\tau_0 = 0, \ \tau_{n+1} = 1$.

Since $\lambda < 1/2$, we have

$$1 - \frac{\eta}{1-\lambda} > 1 - 2\eta = 1 - \frac{\gamma - 1}{\gamma} = \frac{1}{\gamma}$$

so that

$$\xi_1^i \le \frac{\lambda}{\gamma} < \lambda - \eta\frac{\lambda}{1-\lambda} \le \lambda - \beta\gamma\delta\frac{\lambda}{1-\lambda}. \tag{15}$$

5. At the next step of the proof we apply Theorem 2.2 with $a = 0$ $\tau = \tau_1$, $x_0(t) = x(t)$, $\alpha/4$ instead of α and

$$\rho_1 = \int_0^1 d(\dot{x}(t), F(t, x(t))dt < \delta.$$

As $\rho_1 < \delta$, it follows from (15) that (2) is satisfied, provided $r_1 \le \rho_1$. On the other hand, for the same reason $\rho_1 \le \alpha/8 \le (1-\lambda)(\alpha/4)$ since $\lambda \le 1/2$. Thus Theorem 2.2 is indeed applicable and there is a solution $u(t)$ of (4) defined on $[0, \tau_1]$ and satisfying

$$u(0) = x(0); \quad \int_0^{\tau_1} \|\dot{u}(t) - \dot{x}(t)\|dt \le \frac{1}{1-\lambda}\int_0^{\tau_1} d(\dot{x}(t), F(t, x(t)))dt \le \frac{\delta}{1-\lambda}. \tag{16}$$

6. Assume now that after i-th step we have a solution $u(t)$ of (4) (an extension of the $u(\cdot)$ found at the previous step) defined on $[0, \tau_i]$ and satisfying

$$\int_0^{\tau_i} \|\dot{u}(t) - \dot{x}(t)\|dt \le \frac{\gamma}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{i-1}}\int_0^{\tau_i} d(\dot{x}(t), F(t, x(t)))dt. \tag{17}$$

We obviously get from the definition of $n(\lambda)$ that for any i

$$\left(1 - \frac{\lambda}{\gamma}\right)^i \ge \left(1 - \frac{\lambda}{\gamma}\right)^{n(\lambda)} \to e^{-p} \quad \text{as } \lambda \to 0$$

Hence for $w(t) = u(t) - x(t)$ we have if λ is sufficiently small

$$\|w(t)\| \le \frac{\gamma\delta}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{i-1}} \le \frac{\gamma\delta}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{n(\lambda)}} < \varepsilon < \frac{\alpha}{4}, \ t \in [0, \tau_i]. \tag{18}$$

We claim that (in case $i < n$) we can apply Theorem 2.2 with $a = \tau_i$, $\tau = \tau_{i+1}$, $x_0(t) = x(t) + w(\tau_i)$, $\alpha/4$ instead of α and

$$r_1 = \int_{\tau_i}^{\tau_{i+1}} d(\dot{x}_0(t), F(t, x_0(t))dt = \int_{\tau_i}^{\tau_{i+1}} d(\dot{x}(t), F(t, x_0(t))dt.$$

Indeed, as follows from the definition of ε, (12), (13), (14) and (18)

$$\begin{aligned} r_1 &\leq \gamma \int_{\tau_i}^{\tau_{i+1}} d(\dot{x}(t), F(t, x(t)))dt + (\int_{\tau_i}^{\tau_{i+1}} (k(t) + \beta\|\dot{x}(t)\|)dt)\|w(\tau_i)\| \\ &\leq \gamma\delta + \frac{\lambda}{\gamma}\frac{\gamma}{(1-\frac{\lambda}{\gamma})^{i-1}(1-\lambda)}\delta \leq \gamma\delta + \frac{\lambda}{1-\lambda}e^{\gamma p}\delta < (\gamma + 2e^{\gamma p})\delta < \varepsilon \end{aligned} \tag{19}$$

Again, for sufficiently small λ the inequality (2) will be satisfied (by (15)) along with $r_1 < (1-\lambda)(\alpha/4)$ (the latter because $r_1 < \varepsilon < \alpha/8$ and $\lambda \leq 1/2$), so Theorem 2.2 can be applied.

It follows that there exists an extension of $u(t)$ to $[\tau_i, \tau_{i+1}]$ such that (by (19), (14))

$$\begin{aligned} \int_{\tau_i}^{\tau_{i+1}} \|\dot{u}(t) - \dot{x}(t)\|dt &\leq \frac{r_1}{1-\lambda} = \frac{1}{1-\lambda}\int_{\tau_i}^{\tau_{i+1}} d(\dot{x}(t), F(t, x_0(t)))dt \\ &\leq \frac{1}{1-\lambda}\left(\gamma\int_{\tau_i}^{\tau_{i+1}} d(\dot{x}(t), F(t, x(t)))dt + \xi_1\|w(\tau_i)\|\right) \\ &\leq \frac{1}{1-\lambda}\left(\gamma\int_{\tau_i}^{\tau_{i+1}} d(\dot{x}(t), F(t, x(t)))dt + \frac{\lambda}{\gamma}\int_0^{\tau_i} \|\dot{u}(t) - \dot{x}(t)\|dt\right). \end{aligned}$$

Together with (17) this implies

$$\begin{aligned} &\int_0^{\tau_{i+1}} \|\dot{u}(t) - \dot{x}(t)\|dt \\ \leq\ &\frac{\gamma}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{i-1}}\int_0^{\tau_{i+1}} d(\dot{x}(t), F(t, x(t)))dt + \frac{\lambda}{\gamma}\int_0^{\tau_{i+1}} \|\dot{u}(t) - \dot{x}(t)\|dt, \end{aligned}$$

or

$$\int_0^{\tau_{i+1}} \|\dot{u}(t) - \dot{x}(t)\|dt \leq \frac{\gamma}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{i}}\int_0^{\tau_{i+1}} d(\dot{x}(t), F(t, x(t)))dt$$

which means that (17) holds if we replace i by $i+1$. Thus, after at most $n(\lambda)$ steps we shall have a solution $u(t)$ of (1) satisfying

$$u(0) = x(0), \quad \int_0^1 \|\dot{u}(t) - \dot{x}(t)\|dt \leq \frac{\gamma}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{n(\lambda)}}\int_0^1 d(\dot{x}(t), F(t, x(t))dt.$$

It follows that

$$d(x(\cdot), \mathcal{X}) \leq \frac{\gamma}{(1-\lambda)(1-\frac{\lambda}{\gamma})^{n(\lambda)}}\int_0^1 d(\dot{x}(t), F(t, x(t)))dt.$$

The left-hand side of the latter inequality already does not depend on λ and passing to the limit when $\lambda \to 0$ we get

$$d(x(\cdot), \mathcal{X}) \leq \gamma e^p \int_0^1 d(\dot{x}(t), F(t, x(t)))dt$$

as claimed. The proof has been completed. □

4. Relaxation

We need an additional condition to prove a relaxation theorem for unbounded differential inclusions.

$(\mathbf{A}_4)$ there are $k_1(\cdot) \in \mathcal{L}_1$ and $\beta_1 > 0$ such that

$$\|x - \bar{x}(t)\| < \alpha \;\Rightarrow\; (\overline{\text{conv}}\, F(t,x)) \bigcap NB \subset (\text{conv}\ F(t,x)) \bigcap (k_1(t) + \beta_1 N)B.$$

Theorem 4.1. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$. *Assume further that* $x(\cdot) \in W^{1,1}$ *satisfies* $\|x(\cdot) - \bar{x}(\cdot)\|_C < \alpha/4$ *and* $\dot{x}(t) \in \overline{\text{conv}}\, F(t, x(t))$ *a.e. on* $[0,1]$. *Then there is a sequence* $(x_k(\cdot))$ *of solutions of* (1) *weakly (in* $W^{1,1}$*) converging to* $x(\cdot)$.

Proof. By $(\mathbf{A}_4)$ (and standard measurable selection arguments) there are measurable $z_1(\cdot), ..., z_{n+1}(\cdot)$ and $\lambda_1(\cdot), ..., \lambda_{n+1}(\cdot)$ such that $\lambda_i \geq 0$, $\sum \lambda_i = 1$

$$z_i(t) \in F(t, x(t));\ \dot{x}(t) = \sum \lambda_i(t) z_i(t),\ \|z_i(t)\| \leq k_1(t) + \beta_1 \|\dot{x}(t)\| = \rho(t) \qquad (20)$$

almost everywhere on $[0,1]$.

From this (again using standard argument based on the Lyapunov theorem) we deduce the existence of sequences $(\lambda_{im}(\cdot))$ $(m = 1, 2, ...)$ of measurable functions assuming only values 0 and 1 and such that

$$\sum_{i=1}^{n+1} \lambda_{im}(t) = 1 \quad \text{a.e.}, \ \forall\ m = 1, 2, ... \ \text{ and } \quad \lambda_{im}(\cdot) \to \lambda_i(\cdot) \quad \text{weakly}^* \quad \text{in } \mathcal{L}_\infty.$$

Set

$$u_m(t) = x(0) + \int_0^t \sum_{i=1}^{n+1} \lambda_{im}(s) z_i(s) ds.$$

The derivatives of $u_m(\cdot)$ are bounded on $[0,1]$ by the same summable function and weakly converge in $\mathcal{L}_1$ to $\dot{x}(\cdot)$. It follows that $u_m(\cdot) \to x(\cdot)$ uniformly. We observe further that for each m the derivative of $u_m(\cdot)$ assumes one of the values $z_1(t), ..., z_{k+1}(t)$ for almost every t. Therefore

$$\|\dot{u}_m(t)\| \leq \rho(t),\ \text{a.e.}, \forall m \quad \& \quad \dot{u}_m(t) \in F(t, x(t)) \quad \text{a.e..} \qquad (21)$$

On the other hand, since $\|\dot{u}_m(t)\| \leq \rho(t)$, we have by $(\mathbf{A}_3)$

$$\begin{aligned} &|d(\dot{u}_m(t), F(t, u_m(t))) - d(\dot{u}_m(t), F(t, x(t)))| \\ \leq\ & (k(t) + \beta \|\dot{u}_m(t)\|) \|u_m(t) - x(t)\| \leq \rho(t) \|u_m(t) - x(t)\|, \end{aligned}$$

so that

$$\int_0^1 d(\dot{u}_m(t), F(t, u_m(t))dt \to 0. \tag{22}$$

Take an

$$M = \int_0^1 (k(t) + \beta\rho(t))dt,$$

and

$$\delta = \min\{\frac{\alpha}{8}, (2e^{2M}\beta)^{-1}\}.$$

Assuming without loss of generality that $2\exp(2M) \geq 16$, we can easily check that δ satisfies the requirements specified in the third part of the proof of Theorem 3.1.

Therefore, if m is sufficiently big to make sure that

$$\|u_m(\cdot) - \bar{x}(\cdot)\|_C < \frac{\alpha}{2}; \quad \int_0^1 d(\dot{u}_m(t), F(t, u_m(t)))dt < \delta,$$

then by Theorem 3.1 there is a solution $x_m(\cdot)$ of (1) such that

$$x_m(0) = u_m(0) = x(0); \quad \int_0^1 \|\dot{x}_m(t) - \dot{u}_m(t)\|dt \leq O(\int_0^1 d(\dot{u}_m(t), F(t, u_m(t)))dt) \to 0$$

as $m \to 0$. Thus $x_m(\cdot) - u_m(\cdot)$ norm converge to zero in $W^{1,1}$ and, consequently, $x_m(\cdot) \to x(\cdot)$ weakly in $W^{1,1}$. □

Acknowledgements. I wish to express my thanks to the anonymous referee for the really excellent work.

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