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Fixed Points in Contractible Spaces and Convex Subsets of Topological Vector Spaces

Youcef Askoura
Université de Bretagne Occidentale, UFR Sciences et Techniques,
Laboratoire de Mathématiques CNRS-UMR 6205,
6 avenue Victor Le Gorgeu, CS 93837, 29238 Brest Cedex 3, France
askoura@univ-brest.fr

Christiane Godet-Thobie
Université de Bretagne Occidentale, UFR Sciences et Techniques,
Laboratoire de Mathématiques CNRS-UMR 6205,
6 avenue Victor Le Gorgeu, CS 93837, 29238 Brest Cedex 3, France
christiane.godet-thobie@univ-brest.fr

Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

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In this paper, we prove new existence results of fixed points for upper semicontinuous multi-valued maps with not necessarily convex values. We study two cases, according to whether the maps are defined on contractible spaces or on convex subsets of topological vector spaces possessing the simplicial approximation property.

1. Introduction and preliminaries

In this work, we are interested to prove the existence of fixed points for upper semicontinuous multi-valued maps with not necessarily convex values. Our first result is stated in spaces without linear structures. It is a contribution situated in the reduction of assertions imposed, in the known results, to the definition domain and the values of the considered multi-valued maps. In second part, we prove an other result concerning the case in which the multi-valued maps are defined on convex subsets of topological vector spaces. In this case, we use an approximation of the domains which is known as *simplicial approximation property (s.a.p.)* [14]. This last property is already established for a class of convex sets such as admissible sets (in the sense of Klee [15]), weakly admissible sets [20] and Roberts spaces [21]. The results of this article constitute a generalization of several well-known fixed point results, such as that of Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Hukuhara, Fan, Glicksberg and others.

Throughout this paper, the spaces are assumed to be separated. If X is a topological space, 2^X denotes the set of all nonempty subsets of X. The abbreviations: t.v.s., l.c.t.v.s., u.s.c., s.a.p. mean respectively: topological vector space, locally convex topological vector space, upper semicontinuous, simplicial approximation property. A topological space is said to be acyclic if all its reduced Čech homology groups over rationals vanish. Note that contractible spaces are acyclic. The standard $n-$simplex Δ_n (denoted also Δ_N where $N = \{0, ..., n\}$) is the convex hull of the canonical basis of $\mathbb{R}^{n+1}$, where $\mathbb{R}$ is the real

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line. If A is a subset of a given t.v.s., $co(A)$ denotes the convex hull of A and $\overline{A}$ the adherence of A. A topological space E is said to have the fixed point property, provided every continuous single-valued function defined from E into E has a fixed point. A given topological space is said to be contractible if its identity map is homotopic to a constant map. By a finite dimensional subset of a t.v.s. E, we mean a subset of E contained in a finite dimensional subspace of E. A multi-valued map T from a topological space X to a topological space Y is schematized by $T : X \to 2^Y$ or $T : X \rightrightarrows Y$. $\mathbb{N}$ is the set of natural numbers.

We use the following three results:

Lemma A (Shioji [27], Eilenberg and Montgomery [7], Górniewicz [8]). *Let $\Delta_N, N = \{0, 1, ..., n\}$, be the standard n-simplex endowed with its euclidien topology, E a compact topological space, $T : \Delta_N \to 2^E$ an u.s.c. multi-valued map with closed acyclic values and $\Psi : E \to \Delta_N$ a continuous function.*

Then, $\Psi \circ T$ has a fixed point, i.e. $\exists z_0 \in \Delta_N$ such that $z_0 \in (\Psi \circ T)(z_0)$.

Theorem B (Horvath [13]). *Let E be a topological space, $N = \{0, ..., n\}$, Δ_N the standard $n-$*simplex *and $F : 2^N \to 2^E$ a multi-valued map with contractible values such that,*

$$\forall J, J' \subset N, J \neq \emptyset, J \subset J' \Longrightarrow F(J) \subset F(J').$$

Then, there exists a continuous function $f : \Delta_N \to E$ such that,

$$f(\Delta_J) \subset F(J), \forall J \subset N.$$

Theorem C (Godet-Thobie [10], [11]). *Let X be a topological space and $T : X \to 2^X$ a closed multi-valued map (with closed graph).*

Then, T has a fixed point if and only if for every open cover Ω of X, there exists $U \in \Omega$ such that $T(U) \cap U \neq \emptyset$.

2. Fixed points in contractible spaces

The following theorem is the main result of this section:

Theorem 2.1. *Let X be a contractible topological space and D a compact subset of X. Suppose that for every open cover $\mathcal{W}$ of D, there exists an open finite refinement $\mathcal{U}$, such that,*

$$\begin{array}{l}\textit{For every collection } U_i, i \in J, \textit{ of elements of } \mathcal{U}, \\ \bigcap\limits_{i\in J} U_i \neq \emptyset \Longrightarrow \bigcup\limits_{i\in J} U_i \textit{ is contractible.}\end{array} \qquad (*)$$

Then, every u.s.c. multi-valued map $T : X \to 2^D$ with closed acyclic values has a fixed point.

Proof. Let $\mathcal{W}'$ be an open covering of X, $\mathcal{W}$ a subcovering of $\mathcal{W}'$ covering D, $\mathcal{U} = \{U_i\}_{i\in N=\{0,...,n\}}$, a finite open refinement of $\mathcal{W}$ satisfying (*) and Δ_N the standard $n-$simplex. We denote $e_i, i \in N$, the vectors of the canonical basis of $\mathbb{R}^{n+1}$. Define $F : 2^N \to 2^X$

as follows:

$$F(J) = \begin{cases} \bigcup_{i \in J} U_i & \text{if } \bigcap_{i \in J} U_i \neq \emptyset, \\ X & \text{otherwise.} \end{cases}$$

We remark that F satisfies the assumptions of Theorem B. Consequently, there exists a continuous function $f : \Delta_N \to X$, such that

$$\begin{array}{lcll} f(\Delta_J) & \subset & \bigcup_{i \in J} U_i & \text{if } \bigcap_{i \in J} U_i \neq \emptyset, \\ f(\Delta_J) & \subset & X & \text{otherwise.} \end{array}$$

Consider a continuous partition of unity $\{\Psi_i, i \in N\}$ subordinated to the cover $\mathcal{U}' = \{U_i \cap D, i \in N\}$. Define the function $\Psi : D \to \Delta_N$, by $\Psi(x) = \sum_{i \in N} \Psi_i(x) e_i$.

The following diagram gathers the three maps f, T and Ψ:

$$\begin{array}{ccccc} X & & \overset{T}{\rightrightarrows} & & D \\ & \nwarrow & & \swarrow & \\ f & & \Delta_N & & \Psi \end{array}$$

The function $T \circ f : \Delta_N \to D$ is u.s.c. with closed acyclic values. From Lemma A above, there exists $x_0 \in \Delta_N$ such that $x_0 \in (\Psi \circ T \circ f)(x_0)$. Let $y_0 = f(x_0)$. We have, $x_0 \in (\Psi \circ T)(y_0)$, then, $y_0 = f(x_0) \in f((\Psi \circ T)(y_0))$. Let $a_0 \in T(y_0)$ such that $y_0 = f(\Psi(a_0))$. Put, $N(a_0) = \{i \in N \text{ such that } \Psi_i(a_0) \neq 0\}$. We have, $\Psi(a_0) \in \Delta_{N(a_0)}$, and since $f(\Delta_{N(a_0)}) \subset \bigcup_{i \in N(a_0)} U_i$ ($a_0 \in \bigcap_{i \in N(a_0)} U_i \cap D \neq \emptyset$), we have, $y_0 = f(\Psi(a_0)) \in f(\Delta_{N(a_0)}) \subset \bigcup_{i \in N(a_0)} U_i$. Since $a_0 \in \bigcap_{i \in N(a_0)} U_i$ and $y_0 \in \bigcup_{i \in N(a_0)} U_i$; there exists $i_0 \in N(a_0)$ such that $a_0, y_0 \in U_{i_0}$. Taking into account the fact that $a_0 \in T(y_0)$, we conclude $T(U_{i_0}) \cap U_{i_0} \neq \emptyset$. Take an open set W of $\mathcal{W}$ such that $W \supset U_{i_0}$. Then, $T(W) \cap W \neq \emptyset$. Since W is also an element of $\mathcal{W}'$, and the covering $\mathcal{W}'$ is arbitrarily chosen, the hypotheses of Theorem C above are satisfied. This guarantees a fixed point for T □

The previous theorem is also true if the elements of $\mathcal{U}$ are chosen among the open sets of the induced topology of D. Then we have a choice to take the elements of $\mathcal{U}$ in the topology of X or in the induced topology of D. More generally, $\mathcal{U}$ can be any finite refinement by subsets of X satisfying (*) such that their intersections with D are open in D and their union contains D.

Remark 2.2. We give some examples of spaces satisfying condition (*) of Theorem 2.1.

1) For any convex set X of a given l.c.t.v.s., there exists an open convex and finite refinement for every open covering of any compact subset of X. And it is obvious that any collection of convex sets satisfies the property (*).
2) If X is a convex subset of a t.v.s. E (over $\mathbb{R}$) whose dual E' separates points of X, then, X satisfies the conditions of the precedent theorem, because it is homeomorphic to a convex subset of the l.c.t.v.s. $\mathbb{R}^{E'}$.
3) A contractible open subset X of a given l.c.t.v.s. satisfies hypotheses of Theorem 2.1. This is due to the existence of an open convex refinement for any open cover of any subset of X. More generally, if X is a subset of a given l.c.t.v.s. and D a compact subset of X, then, it is sufficient to find a contractible open set containing D and contained in X.

Corollary 2.3. *Let X be a topological space which is homeomorphic to a retract of a convex subset of a given l.c.t.v.s. and D a compact subset of X. Then, every u.s.c. multi-valued map $T : X \to 2^D$ with closed acyclic values possesses a fixed point.*

Proof. Denote by P the homeomorphism embedding X in a set $\mathcal{H} = P(X)$ which is a retract of a convex subset K of a l.c.t.v.s. E. Let $\mathcal{D} = P(D)$. Denote by R the retraction of K to $\mathcal{H}$. Define $S : K \to 2^{\mathcal{D}}$, by $S = P \circ T \circ P^{-1} \circ R$.

$$K \overset{R}{\to} \mathcal{H} \overset{P^{-1}}{\to} X \overset{T}{\rightrightarrows} D \overset{P}{\to} \mathcal{D}.$$

S is u.s.c. with closed acyclic values. Consequently, S has a fixed point from Theorem 2.1, which is necessarily in $\mathcal{D}$, denote it by p. Then $P^{-1}(p)$ is a fixed point of T. □

The precedent corollary is demonstrated by Theorem 2.1, but it can also be deduced from the fixed point theorem of Park [23].

Remark 2.4.

1) The points 1) and 2) of Remark 2.2 are valid for the precedent corollary. In fact, a convex compact subset of a l.c.t.v.s. is a retract of itself.
2) If X is a metrizable topological space which is an AR (absolute retract) for metric spaces, then it satisfies the conditions of the precedent corollary. Indeed, since X is a metric space, it can be isometrically embedded in a Banach space B, and since it is an AR for metric spaces, it is a retract of B.
3) If X is a Tychonoff space, it can be embedded in a cube $[0,1]^F$ (where F is the set of all continuous functions from X to $[0,1]$). If, in addition, we impose to X to be an absolute retract for compact spaces, or simply a retract of a convex subset of $[0,1]^F$, it satisfies the conditions of the previous corollary.
4) If X is a metrizable compact convex admissible (in the sense of Klee [15]) subset of a given t.v.s., it is in one part an absolute extensor for compact spaces [15], then an AR for compact spaces. In another part, X is a Tychonoff space. According to the precedent point of this remark, we conclude that X satisfies the conditions of Corollary 2.3.

According to Remarks 2.2 and 2.4, results of this section are generalizations of the fixed point results that are cited in the introduction.

3. Fixed points in subsets of t.v.s.

In this section, we consider a t.v.s. E and we denote by $\mathcal{V}_0$ the set of all open balanced neighborhoods of the origin of E.

Let us begin this section by defining the s.a.p.

Definition 3.1 ([14]). A convex subset X of a t.v.s. E is said to have the *simplicial approximation property* (s.a.p.) provided: for each neighborhood V of the origin of E, there exists a finite dimensional compact convex subset K_V of X, such that for any polytope (a convex hull of finitely many vectors of X) P of X, there exists a continuous function $\rho{:}P \to K_V$, satisfying: $\rho(x) - x \in V$, for all $x \in P$.

The s.a.p. was introduced, in F-spaces (Fréchet spaces), by Kalton et al [14], who have shown that it implies the fixed point property. About convex compact admissible sets, in the sense of Klee [15], it is obvious that they have the s.a.p.. More generally, the weakly admissible sets in the sense of Nhu [20] have this property. Note that the definition of Nhu is given for metrizable t.v.s.. The generalization of the weak admissibility to general t.v.s. and the proof that it enjoys the s.a.p. is given by Okon in [22], where he deduced from this fact the fixed point property for Kakutani maps (u.s.c. maps with compact convex values) in weak admissible subsets of t.v.s.. The s.a.p. is also satisfied in Roberts spaces[1] [21].

Lemma 3.2. *Let X be a closed convex subset of E, P a compact finite dimensional subset of X and $T : P \to 2^X$ an u.s.c. multi-valued map such that, $\exists Y \subseteq X, \exists Q \in \mathcal{V}_0, \forall Q' \subset Q, Q' \in \mathcal{V}_0$, $\forall x \in P$, $[T(x) + Q'] \cap Y$ is non empty and contractible.*

Then, for every two elements $U, V \in \mathcal{V}_0$, there exists a continuous function $f : P \to X$ such that

$$\forall x \in P, f(x) \in V + T[(x + U) \cap P].$$

In other words, f is a continuous selection of $x \mapsto V + T[(x + U) \cap P]$.

Proof. Let n be the topological dimension (covering-dimension) of P (n is less than or equal to the algebraic dimension of all finite dimensional subspace of E containing P) and $V' \in \mathcal{V}_0$ such that $\sum_{i=1}^{n+1} V' = \underbrace{V' + V' + ... + V'}_{n+1 \text{ times}} \subset V \cap Q$. Let E' be the subspace of E generated by P. E' is metrizable according to the fact that it is finite dimensional. Denote by d the euclidien metric on E' and by $B(x, \lambda)$ the open ball of center x and radius λ in E'. We have, for all x in E', $B(x, \lambda) = x + B(0, \lambda)$, for all positive number λ. T is u.s.c. on P, then we have for V', $\forall x \in P, \exists \lambda(x) > 0$ such that

$$B(0, \lambda(x)) \subset U \text{ and } T([x + B(0, \lambda(x))] \cap P) \subset T(x) + V'. \tag{1}$$

The set $\mathcal{W}' = \{[x + B(0, \lambda(x)/2)] \cap P, x \in P\}$ is an open covering of P. Since P is compact, we can extract from $\mathcal{W}'$ a finite subcovering of P, denote it $\mathcal{W} = \{[x_i + B(0, \lambda(x_i)/2)] \cap P, x_i \in Z\}$, where Z is a finite subset of P. Consider an open finite refinement $\mathcal{U}$ of $\mathcal{W}$ so that the dimension of its nerve is less than or equal to n. We denote $\mathcal{U} = \{O_i, i \in I\}$. Without loss of generalities, we can put $I = \{0, 1, ..., m\}$. Let $\{\Psi_i, i \in I\}$ be a continuous partition of unity subordinated to $\mathcal{U}$. We define $\Psi : P \to \Delta_I$ (Δ_I is the standard $m-$simplex) by $\Psi(x) = \sum_{i \in I} \Psi_i(x) e_i$ ($\{e_i, i \in I\}$ is the canonical basis of $\mathbb{R}^{m+1}$). For all $J \subset I$, if $\bigcap_{i \in J} O_i \neq \emptyset$, we choose $x_{k_0} \in Z$ such that $\lambda(x_{k_0})$ is the maximum (one of them if there is many) of the $\lambda(x_k)$ satisfying the following relation

$$\bigcap_{i \in J} O_i \subset [x_k + B(0, \lambda(x_k)/2)] \cap P.$$

Denote $x_J = x_{k_0}$. Introduce $\Gamma : 2^I \to 2^X$, defined as follows:

$$\Gamma(J) = \begin{cases} \left[T(x_J) + \sum_{i \in J} V'\right] \cap Y & \text{if } \bigcap_{i \in J} O_i \neq \emptyset, \\ X & \text{otherwise.} \end{cases}$$

[1]spaces constructed by the same method as the famous one constructed by Roberts [26], see [21] for a detailed definition.

Show that Γ satisfies the assumptions of Theorem B. Since the cardinal of each $J \subset I$ such that $\underset{i\in J}{\cap} O_i \neq \emptyset$ is less than or equal to $n+1$, the values of Γ are non empty and contractible. To verify the additional requirements, let $J \subset J' \subset I$ such that $J \neq \emptyset, J \neq J'$ and $\underset{i\in J'}{\cap} O_i \neq \emptyset$, then consequently, $\underset{i\in J}{\cap} O_i \neq \emptyset$. We have, $\left(\underset{i\in J'}{\cap} O_i\right) \subset \left(\underset{i\in J}{\cap} O_i\right)$, $\left(\underset{i\in J'}{\cap} O_i\right) \subset [x_{J'} + B(0, \lambda(x_{J'})/2)] \cap P$ and $\left(\underset{i\in J}{\cap} O_i\right) \subset [x_J + B(0, \lambda(x_J)/2)] \cap P$. Then,

$$[x_{J'} + B(0, \lambda(x_{J'})/2)] \cap [x_J + B(0, \lambda(x_J)/2)] \neq \emptyset.$$

From the last relation, and the fact that $\lambda(x_J) \leq \lambda(x_{J'})$, we conclude that

$$x_J \in [x_{J'} + B(0, \lambda(x_{J'}))] \cap P.$$

Thereafter, by (1), $T(x_J) \subset T([x_{J'} + B(0, \lambda(x_{J'}))] \cap P) \subset T(x_{J'}) + V'$, which gives $\left[T(x_J) + \sum_{i\in J} V'\right] \cap Y \subset \left[T(x_{J'}) + \sum_{i\in J} V' + V'\right] \cap Y \subset \left[T(x_{J'}) + \sum_{i\in J'} V'\right] \cap Y$, and this means that $\Gamma(J) \subset \Gamma(J')$. The other situations of J and J' are obvious. We infer using Theorem B a continuous function $g : \Delta_I \to X$ satisfying $g(\Delta_J) \subset \Gamma(J), \forall J \in 2^I$. Prove that $g \circ \Psi$ satisfies the conclusion of the lemma. Let $x \in P$. Put $I(x) = \{i \in I, \Psi_i(x) \neq 0\}$. We have $x \in \underset{i\in I(x)}{\cap} O_i$ and $\Psi(x) \in \Delta_{I(x)}$. Then, $g(\Psi(x)) \in g(\Delta_{I(x)}) \subset \Gamma(I(x)) = \left[T(x_{I(x)}) + \sum_{i\in I(x)} V'\right] \cap Y$ and $x \in \underset{i\in I(x)}{\cap} O_i \subset \left[x_{I(x)} + B(0, \lambda(x_{I(x)})/2)\right] \cap P$, which gives,

$g(\Psi(x)) \in \left[T(x_{I(x)}) + \sum_{i\in I(x)} V'\right] \cap Y$ and $x_{I(x)} \in \left[x + B(0, \lambda(x_{I(x)})/2)\right] \cap P \subset [x + U] \cap P$.

Finally, $g(\Psi(x)) \in \left(T\left[(x+U) \cap P\right] + \sum_{i\in I(x)} V'\right) \subset \left(V + T\left[(x+U) \cap P\right]\right)$. □

Lemma 3.3. *Let X be a closed convex subset of E, P a compact finite dimensional subset of X and $f : P \to X$ a continuous function. Then, for every $V \in \mathcal{V}_0$, there exists a polytope D of X and a continuous function $\varphi : P \to D$ such that $\varphi(x) \in V + f(x)$, for all $x \in P$*

Proof. The beginning of this proof is similar to the one's of the previous lemma. We resume the lines of the previous proof from the beginning to the construction of the function Ψ. The role of the multi-valued map T will be taken by the function f. The open set U has not a role here, it can be replaced by the entire space E. The function Ψ will be defined as follows: for every $i \in I$, choose $x_i \in Z$ such that $O_i \subset x_i + B(0, \lambda(x_i)/2)$. $\lambda(x_i)$ is not constrained here to be the maximum of the $\lambda(x_k)$ satisfying the precedent relation. Put $\Psi(x) = \sum_{i\in I} \Psi_i(x) f(x_i)$ and we verify directly that Ψ satisfies the conclusion of the lemma. In fact, $\forall i \in I(x), x \in O_i \Rightarrow f(x) \in f(x_i) + V' \Rightarrow f(x_i) \in f(x) + V'$, where $I(x)$ is the same set of indices as in the previous proof. Then, $\Psi(x) = \sum_{i\in I(x)} \Psi_i(x) f(x_i) \in \sum_{i\in I(x)} \Psi_i(x)[f(x) + V']$. Thereby, $\Psi(x) \in \left[\sum_{i\in I(x)} \Psi_i(x)\right] f(x) + \sum_{i\in I(x)} [\Psi_i(x) V'] \subset f(x) + \sum_{i\in I(x)} V' \subset f(x) + V$. Let finally $D = co\{f(x_i), x_i \in Z\}$ and $\varphi = \Psi$ □

Lemma 3.4. *Let X be a compact subset of E, $T : X \to 2^X$ a closed (with closed graph) multi-valued map. If, for every $V \in \mathcal{V}_0$, there exists $x \in X$ such that $(x+V) \cap T[(x+V) \cap X] \neq \emptyset$, then T has a fixed point.*

Proof. Let K be a compact subset of a given t.v.s. with $\mathcal{V}$ as basis of neighborhoods of the origin. Prove that for any open set O_K containing K, there exists $V \in \mathcal{V}$ such that $K + V \subset O_K$. Indeed, for all $x \in K$, there exists $U_x \in \mathcal{V}$ such that $x + U_x \subset O_K$. Let $V_x \in \mathcal{V}$ such that $V_x + V_x \subset U_x$. K is compact, then there exist, $x_1, ..., x_n \in K$ such that $K \subset \cup_{i=1}^n (x_i + V_{x_i})$. Hence $K \subset \cup_{i=1}^n (x_i + U_{x_i}) \subset O_K$. Put $V = \cap_{i=1}^n V_{x_i}$. We have: for all $x \in K$, there exists $i \in \{1, ..., n\}$ such that $x \in x_i + V_{x_i}$. Then, $x + V \subset x_i + V_{x_i} + V_{x_i} \subset x_i + U_{x_i}$. We conclude, $K + V \subset \cup_{i=1}^n (x_i + U_{x_i}) \subset O_K$.

Let us conclude the affirmation of the present lemma.

We have for all $V \in \mathcal{V}_0$, there exists $x \in X$ such that $(x+V) \cap T[(x+V) \cap X] \neq \emptyset$ or $(x+V) \times (x+V) \cap GrT \neq \emptyset$. Then, for all $V \in \mathcal{V}_0, (D + V \times V) \cap GrT \neq \emptyset$ (D is the diagonal of $X \times X$). By the precedent argument, (concerning compacts subsets of $E \times E$), we can say that for any open set O_D containing D, O_D contains a neighborhood of D of the form $D + V \times V$, where $V \in \mathcal{V}_0$ hence: $O_D \cap GrT \neq \emptyset$. Since D and GrT are closed and $X \times X$ is a normal space the last relation is possible only if $D \cap GrT \neq \emptyset$. □

Theorem 3.5. *Let X be a compact convex subset of E with the s.a.p. and $T : X \to 2^X$ an u.s.c. multi-valued map with closed values such that:*

$$\begin{aligned} &\exists Y \subseteq X, \exists Q \in \mathcal{V}_0, \forall Q' \subset Q, Q' \in \mathcal{V}_0, \forall x \in X, [T(x) + Q'] \cap Y \\ &\textit{is non empty and contractible.} \end{aligned} \tag{2}$$

Then, T has a fixed point.

Proof. According to Lemma 3.4, it suffices to prove that for all $W \in \mathcal{V}_0$, there exists $x_0 \in X$ such that

$$(x_0 + W) \cap T[(x_0 + W) \cap X] \neq \emptyset.$$

Let $W \in \mathcal{V}_0$ and $V \in \mathcal{V}_0$ such that $V + V + V \subset W \cap Q$. Since X has the simplicial approximation property, there exists K_V a finite dimensional compact convex subset of X such that, for every polytope P of X, there exists a continuous function $\rho : P \to K_V$ such that

$$\rho(x) - x \in V, \forall x \in P. \tag{3}$$

From Lemma 3.2, there exists a continuous function $f : K_V \to X$ which is a selection of $x \longmapsto T[(x+W) \cap K_V] + V$, i.e.

$$\forall x \in K_V, f(x) \in T[(x+W) \cap K_V] + V. \tag{4}$$

The function f is a continuous single valued function, then Lemma 3.3 guarantees the existence of a polytope D of X and a continuous function $\varphi : K_V \to D$ which is a selection of $x \mapsto f(x) + V$, i.e.

$$\forall x \in K_V, \varphi(x) \in f(x) + V. \tag{5}$$

Choose in (3), P identical to D. Then ρ is defined on D. The function $\rho \circ \varphi$ ($K_V \xrightarrow{\varphi} D \xrightarrow{\rho} K_V$) possesses a fixed point, denote it by x_0. We have $x_0 \in (\rho \circ \varphi)(x_0)$. Put $y_0 = \varphi(x_0)$.

Then, $x_0 = \rho(y_0)$ which gives according to (3),

$$x_0 - y_0 \in V. \tag{6}$$

Using (4) and (5), $y_0 \in [f(x_0) + V] \subset T[(x_0+W)\cap K_V]+V+V$. Then, $y_0 \in (T[(x_0+W)\cap X]+V+V)$. Taking into account (6), $(x_0+V)\cap(T[(x_0+W)\cap X]+V+V) \neq \emptyset$ or in another form, $(x_0+V+V+V)\cap T[(x_0+W)\cap X] \neq \emptyset$. Finally, $(x_0+W)\cap T[(x_0+W)\cap X] \neq \emptyset$. □

Remark 3.6. Condition (2) of Theorem 3.5 is satisfied if the values of T are convex or simply star-shaped. In fact, it is easy to see that if A is a subset of E which is star-shaped from x_0 and $V \in \mathcal{V}_0$, then, $A + V$ is also star-shaped from x_0 ($\forall t \in [0,1], \forall a \in A, \forall v \in V, [(1-t)x_0 + t(a+v)] = [(1-t)x_0 + ta + tv] \subset A + V$). Then, for all $x \in X$, for all $V \in \mathcal{V}_0, T(x) + V$ will be a star-shaped set from a given point of $T(x)$. $[T(x) + V] \cap X$ will be also star-shaped form this same point. Consequently, for all $x \in X$, for all $V \in \mathcal{V}_0$, $[T(x) + V] \cap X$ is contractible.

Since the convex compact subsets of locally convex topological vector spaces have the s.a.p. (they are admissible), the last theorem is a generalization of the fixed point results concerning Kakutani maps (upper semicontinuous multi-valued maps with convex compact values) cited in the introduction.

Now, we give another result concerning u.s.c. multi-valued maps with ∞-proximally connected values, notion due to Dugundji. Let us begin with defining the notion of ∞-proximally connected sets.

Definition 3.7 ([6]). A compact subset A of a metric space X is said to be ∞-proximally connected provided: for every $\varepsilon > 0$, there exist $\delta \in]0, \varepsilon[$, such that for every $n \in \mathbb{N}$ and for every continuous map $g : \partial\Delta_{n+1} \to V_\delta(A)$ (where $\partial\Delta_{n+1}$ is the boundary of Δ_{n+1} and $V_\delta(A)$ is the δ-neighborhood of A), there exists a continuous map $\widetilde{g} : \Delta_{n+1} \to V_\varepsilon(A)$ such that $g(x) = \widetilde{g}(x)$, for all $x \in \partial\Delta_{n+1}$.

We can replace in the previous definition $\partial\Delta_{n+1}$ (resp. Δ_{n+1}) by the unit sphere (resp. the unit ball) of $\mathbb{R}^{n+1}$.

Note that, R_δ sets (intersection of a decreasing sequence of compact AR's), intersection of compact contractible sets, compact sets which are contractible in their $\varepsilon-$neighborhoods (for every $\varepsilon > 0$) lying in ANR spaces are ∞-proximally connected.

Given two metric spaces X and Y. Then, we call an ε-approximation (in the graph) of a multi-valued map $T : X \to 2^Y$, a continuous function $f : X \to Y$, such that the graph of f lies into the ε-neighborhood (here we refer to the metric in the product space $X \times Y$) of the graph of T. We need the following well known result.

Theorem 3.8 ([17]). *Let X be a compact metric ANR space and Y a metric space. Then, for every $\varepsilon > 0$ and any u.s.c. multi-valued map $T : X \to 2^Y$ with ∞-proximally connected values, there exists an ε-approximation $f : X \to Y$ of T.*

It is clear that the previous theorem gives the result of the Lemma 3.2 in the case of E metric.

Theorem 3.9. *Suppose that E is metrizable. Let X be a convex compact subset of E*

possessing the s.a.p.. Then, any u.s.c. multi-valued map $T : X \to 2^X$ with ∞-proximally connected values has a fixed point.

Proof. The proof is similar to that of Theorem 3.5. Just use Theorem 3.8 instead of Lemma 3.2. □

In the case of metrizable t.v.s., if we impose, in Theorem 3.5, to the set Y (in Condition (2)) to be equal to X, Theorem 3.5 will became a particular case of Theorem 3.9. Because if the values of T have there nearest neighborhoods contractible, they are ∞-proximally connected.

We end this section by an example which proves that the lower semicontinuity can not guarantees necessarily fixed points for maps with contractible values (then acyclic) even when the definition domain is the unit disc of $\mathbb{R}^2$. This fact is contrary to the well-known case of maps with convex values. A trivial argument is the existence of continuous selections in the case where the maps are lower semicontinuous with closed convex values. The following example prove, furthermore, the non existence of continuous selections for the considered map which is lower semicontinuous with closed contractible values.

Example 3.10. (This example is a slight modification of the Example 2, page 69 in [1]). Let B be the closed unit ball of $\mathbb{R}^2$, Q the rectangle $[0,1] \times [0,2\pi]$. Introduce the function $\pi : Q \to B\backslash\{0\}, (\rho,\theta) \mapsto (\rho\cos(\theta), \rho\sin(\theta))$. π is a bijection from Q to $B\backslash\{0\}$. Define the function $\Phi : Q \to 2^{\mathbb{R}}$, by $\Phi(\rho,\theta) = \{w / -\theta - 2\pi(1-\rho) \leq w \leq -\theta + 2\pi(1-\rho)\}$. We remark that $\Phi(1,\theta) = \{-\theta\}, \forall\theta \in [0, 2\pi[$. Consider the functions $r : \mathbb{R} \to [0, 2\pi[$, defined by $r(x) = x mod[2\pi]$, and $\psi : Q \to 2^{[0,2\pi[}$, defined by $\psi(\rho,\theta) = r(\Phi(\rho,\theta))$. Let in the last time the multi-valued map $F : B \to 2^B$

$$F(x,y) = \begin{cases} \pi(1, \psi(\pi^{-1}(x,y))), & \text{if } (x,y) \neq (0,0), \\ z_0 \text{ arbitrary in the unit circle} & \text{otherwise} \end{cases}$$

F is lower semicontinuous with closed contractible values, but it has no fixed point, then also no continuous selection.

4. Comments on the s.a.p.

In this section, we mean by a space possessing the s.a.p. a space in which every convex compact subset possesses it.

The s.a.p. is not largely investigated like admissibility in the sense of Klee. We know for example that there exists a σ-compact metrizable t.v.s. which is not admissible in the sense of Klee (the space constructed by Cauty [3], according to Corollary 1 in [5]). Naturally, the s.a.p. seems to be weaker than the admissibility. But, we do not know an example of a space which do not have the s.a.p. (if it exists), nor a proof that the s.a.p. is satisfied in all t.v.s. or simply in all metrizable t.v.s.. We prove below (Proposition 4.5) that the verification of the s.a.p. in metrizable t.v.s. is a sufficient condition for its satisfaction in all t.v.s.. By this result, we simplify also the resolution of the question: is the s.a.p. a property of metrizable t.v.s. or not? It suffices to find, for a negative answer, a t.v.s. (not necessarily metrizable) without the s.a.p.. We begin by a simplification of the definition of the s.a.p. which we need in the sequel.

Proposition 4.1. *Let X be a convex subset of E. Then, X has the s.a.p. if and only if for every $V \in \mathcal{V}_0$, there exists a polytope S_V of X such that, for any other polytope Δ of X, there exists a continuous function $\phi : \Delta \to S_V$ satisfying $\phi(x) - x \in V$, for all $x \in \Delta$.*

Proof. The sufficiency is obvious. Let us prove the necessity. Let $V \in \mathcal{V}_0$ and $V' \in \mathcal{V}_0$ such that $V' + V' \subset V$. Consider K the finite dimensional convex compact subset of E relative to V' in Definition 3.1. Let us prove that

$$\text{There exists a polytope } S \subset K \text{ and a continuous function } h : K \longrightarrow S \text{ such that } h(x) - x \in V', \forall x \in K. \tag{7}$$

Let E' be the subspace of E generated by K. E' is of finite dimension and his induced topology is described by the euclidien norm $\|.\|$. Denote $B(x,\lambda) = \{y \in E', \|x-y\| < \lambda\}$. There exists $\varepsilon > 0$ such that $B(0,\varepsilon) \subset V' \cap E'$ (0 is the origin of E). Since K is compact, it can be covered by a finite number of open sets of type $x_i + B(0,\varepsilon), i \in \{1,...,m\}$, where $x_i \in K$. Let $\{\Psi_i\}_{i\in\{1,...,m\}}$ be a continuous partition of unity subordinated to the cover. Put $S = co\{x_i, i = \overline{1,m}\}$ and define $h : K \longrightarrow S$ by $h(x) = \sum\limits_{i=1}^{m}\Psi_i(x)x_i$. h is continuous and: $\forall x \in K, h(x) - x = \sum\limits_{i=1}^{m}\Psi_i(x)x_i - x = \sum\limits_{i=1}^{m}\Psi_i(x)(x_i - x)$. In the last sum, if for $i \in \{1,...,m\}$, $\Psi_i(x) \neq 0$, then $x \in x_i + B(0,\varepsilon)$ or $\|x_i - x| < \varepsilon$. We conclude that $\left\|\sum\limits_{i=1}^{m}\Psi_i(x)(x_i - x)\right\| \leq \sum\limits_{i=1}^{m}\Psi_i(x)\|x_i - x\| < \varepsilon$. Hence $h(x) - x \in B(0,\varepsilon) \subset V'$. (7) is actually established. By (7), there exists a polytope S_V of K and a continuous function $h : K \longrightarrow S_V$ such that $h(x) - x \in V'$, for all $x \in K$. Let Δ be any polytope of X and $\varphi : \Delta \longrightarrow K$ the continuous function of Definition 3.1. We have $\forall x \in \Delta$, $h(\varphi(x)) - \varphi(x) \in V'$ and $\varphi(x) - x \in V'$. Then, $h(\varphi(x)) - x \in V' + V' \subset V$. Put $\phi = h \circ \varphi$ □

Lemma 4.2. *Let F be a finite dimensional euclidien space, $P = co\{u_i, i \in I\}$ a polytope (I is finite) in F and A a convex subset of P containing 0. If for every $i \in I$, there exists $\varepsilon \in]0,1[$, such that $\varepsilon u_i \in A$, then A is a neighborhood of 0 relatively to the induced topology of P.*

Proof. Denote ε_i the positive number such that $\varepsilon_i u_i \in A$, $\varepsilon = \min\limits_{i\in I}\{\varepsilon_i\}$ and, for every $i \in I$, $u'_i = \varepsilon u_i$. Then the polytope $P' = co\{u'_i, i \in I\}$ is a subset of A and $P' = \varepsilon P$. We represent P' as an intersection of half-spaces of F, $P' = \bigcap\limits_{j\in J} L_j$, where l_j is a linear form, $\alpha_j \geq 0$ and $L_j = \{x \in F, l_j(x) \leq \alpha_j\}$. We have $P = \frac{1}{\varepsilon}P' = \bigcap\limits_{j\in J} H_j$, where $H_j = \{x \in F, l_j(x) \leq \frac{\alpha_j}{\varepsilon}\}$. Write $J = J_1 \cup J_2$ where $J_1 = \{j \in J, \alpha_j > 0\}$ and $J_2 = \{j \in J, \alpha_j = 0\}$. For every $j \in J_1$, $\alpha_j > 0$, then, $L_j \subset int(H_j)$, because $l_j(x) \leq \alpha_j$ implies $l_j(x) < \frac{\alpha_j}{\varepsilon}$. For every $j \in J_1$, $l_j(0) < \alpha_j < \frac{\alpha_j}{\varepsilon}$. Then there exists $\delta_j > 0$, such that $B(0,\delta_j) \subset L_j \subset H_j$. Put $\delta = \min\{\delta_j, j \in J_1\}$. Then, $B(0,\delta) \subset \left(\bigcap\limits_{j\in J_1} L_j\right) \subset \left(\bigcap\limits_{j\in J_1} H_j\right)$. For $j \in J_2$, we have $L_j = H_j$, then $B(0,\delta) \cap P' = B(0,\delta) \cap \left(\bigcap\limits_{j\in J_1} L_j\right) \cap \left(\bigcap\limits_{j\in J_2} L_j\right) = B(0,\delta) \cap \left(\bigcap\limits_{j\in J_2} H_j\right) =$

$B(0,\delta)\cap P$. We can conclude that $B(0,\delta)\cap P'$ is a neighborhood of 0 in P. Consequently, P' and also $A\supset P'$ are neighborhoods of 0 in P . $\square$

Remark 4.3. The condition "$\forall i\in I, \exists\varepsilon\in]0,1[$ such that $\varepsilon u_i\in A$" in the previous lemma, is obviously equivalent to this "$\forall x\in P, \exists\varepsilon\in]0,1[$ such that $\varepsilon x\in A$". Then, one can think about a possible version of this lemma concerning any convex compact finite dimensional set (not necessarily a polytope). But, a very simple example can be constructed to prove the contrary: Take, in the plane, P equal to the disk centered in $(2,0)$ with radius 2 and A equal to the disk centred in $(1,0)$ with radius 1. This shows that the shape of P in the lemma is essential.

Lemma 4.4. *Let E_1, E_2 be finite euclidien spaces, $\Lambda : E_1 \to E_2$ a linear surjective application, P_1 a polytope in E_1 and P_2 a polytope in E_2. If $\Lambda(P_1) = P_2$, then $\Lambda\mid_{P_1}$ is open (relatively to the induced topology).*

Proof. Let $y_0\in P_1$ and $x_0\in P_2$ such that $\Lambda(y_0)=x_0$. Consider the euclidien metrics in E_1 and E_2 and denote by $B(z,\delta)$ the open ball centered in z with radius δ (this ball is that of E_1 or E_2 following z belongs to E_1 or E_2). We have to prove that $A=\Lambda(B(y_0,\delta)\cap P_1)$ is a neighborhood of x_0 in P_2 for any $\delta\in]0,1[$. Suppose the contrary. That is, there exists $\delta\in]0,1[$ such that the set $A=\Lambda(B(y_0,\delta)\cap P_1)$ is not a neighborhood of x_0 in P_2. We have A is convex and contains x_0. Since P_2 is a polytope, we pick, applying Lemma 4.2, an element $x\in P_2$, $x\neq x_0$ (we can take x to be a vertex of P_2) such that, for all $\varepsilon\in]0,1[$, $x_0+\varepsilon(x-x_0)\notin A$. In other words, $[x,x_0[\cap A=\emptyset$. But, $\Lambda(P_1)=P_2$, then, there is a point $y\in P_1$ such that $\Lambda(y)=x$. We have $y\neq y_0$ (because $\Lambda(y)\neq\Lambda(y_0)$). Since Λ is linear, $\Lambda([y,y_0[)=[x,x_0[$, one can see easily that $\Lambda\mid_{[y,y_0[}: [y,y_0[\to [x,x_0[$ is bijective. Let $y_1\in]y,y_0[\cap B(y_0,\delta)$. We have, $\Lambda(y_1)\in]x,x_0[\cap A$ which is a contradiction. $\square$

Proposition 4.5. *The two assertions are equivalent:*

i) *Every metrizable t.v.s. possesses the s.a.p.*

ii) *Every t.v.s. possesses the s.a.p.*

Proof. It is obvious that *ii*) implies *i*). Let us prove that *i*) implies *ii*). Suppose that every metric linear space has the s.a.p.. According to Klee [15], every t.v.s. is linearly homeomorphic to a subspace of a product of metrizable t.v.s.. One can remark easily that the s.a.p. is conserved by linear embedding. Then, it suffices to prove the s.a.p. in a product of linear metric spaces.

Let A be an arbitrary set of indices. For every $i\in A$, E_i is a linear metric space and $E_A=\prod\limits_{i\in A}E_i$. For every $B\subset A$, if for every $i\in B$, $Y_i\subset E_i$, we denote $Y_B=\prod\limits_{i\in B}Y_i$ and π_B the projection to Y_B. Let X be a convex compact subset of E_A. Put for $B\subset A$, $X_B=\pi_B(X)$. We denote, for every $i\in A$, $\mathcal{V}_i$ a basis of neighborhoods of the origin of E_i and $\mathcal{V}$ a basis of neighborhoods of the origin of E_A. Let V' be an element of $\mathcal{V}$. There exists a finite subset of indices $N\subset A$, let $N=\{\alpha_1,...,\alpha_n\}$, and elements $V_i\in\mathcal{V}_i, i\in N$, such that $V'\supset V=\prod\limits_{i\in N}V_i\times\prod\limits_{i\in A\setminus N}E_i$. X_N is a convex compact subset of E_N which is a metrizable t.v.s.. Consequently X_N possesses the s.a.p. Then, by Proposition 4.1, there exists a polytope $S_V=co\{u_1,...,u_m\}\subset X_N$ such that, for every other polytope Δ of X_N, there exists a continuous function $\varphi:\Delta\longrightarrow S_V$ satisfying $\varphi(x)-x\in V_N$, for all $x\in\Delta$.

Let $a_1, a_2, ..., a_m$ be m elements of X such that $\pi_N(a_i) = u_i, i \in \{1, ..., m\}$. Denote $K_V = co\{a_1, a_2, ..., a_m\}$. Define the multi-valued map $\Gamma : S_V \to K_V$ by: $\Gamma(x) = \pi_N^{-1}(x) \cap K_V$. It is obvious that the values of Γ are convex and compact. We remark also that Γ is lower semicontinuous. Indeed, given any open set O of K_V, we have:

$$\begin{aligned}\Gamma^{-1}(O) &= \{x \in S_V, \Gamma(x) \cap O \neq \emptyset\}\\ &= \{x \in S_V, \pi_N^{-1}(x) \cap K_V \cap O \neq \emptyset\}\\ &= \{x \in S_V, \pi_N^{-1}(x) \cap O \neq \emptyset\}\\ &= \pi_N(O).\end{aligned}$$

Apply Lemma 4.4 to affirm that $\Gamma^{-1}(O)$ is open in the induced topology of S_V. Then, Γ is lower semicontinuous. By the Michael selection theorem [18], we pick a continuous selection ξ of Γ. We have $\pi_N(\xi(x)) = x$ for all $x \in S_V$. Let Δ be a polytope of X. $\pi_N(\Delta)$ is a polytope of X_N. Then, there exists a continuous function $\varphi : \pi_N(\Delta) \longrightarrow S_V$ such that $\varphi(x) - x \in V_N$, for all $x \in \pi_N(\Delta)$. Define $\Psi : \Delta \longrightarrow K_V$ by $\Psi(x) = \xi(\varphi(\pi_N(x)))$. We have, for every x in Δ, $\pi_N[\Psi(x) - x] = \pi_N(\xi(\varphi(\pi_N(x)))) - \pi_N(x) = \varphi(\pi_N(x)) - \pi_N(x) \in V_N$. Consequently, for every x in $\Delta, \Psi(x) - x \in V$. Then, the s.a.p of X is established. $\square$

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