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Spreading Control Applied to Target Problems

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This paper investigates the problem of control with target condition for systems governed by semilinear PDEs. It presents a unified approach based on the concept of spreading control under speed constraints. It is shown that a convenient choice of the lower bound of the speed yields a control generating a spread which reaches the state constraints at terminal time. Numerical examples are studied.

1. Introduction and statement of the problem

Spreading controls [5, 6] stand for controls which generate spreads in systems governed by partial differential equations. The main motivation for stating them, stems from spatially distributed processes where actually expansion phenomena in space usually arize. Illustrative examples are proliferation of cancer cells in biomedicine and desertification in a vegetation dynamics.

Recent studies [9] clearly demonstrate that the problem of existence of feedback spreading control (FSC) laws may be investigated by a set-valued approach in the context of monotonicity with respect to a preorder. They show that FSC laws are selections of maps defined by tangential conditions.

More recent results [2, 8] take under consideration the notion of the speed of a spread that they firstly define, and show how FSC laws which generate spreads either slower or quicker than a desired given speed can be examined in the same setting.

This paper continues the search, started in [9], for FSC laws and their applications. Its main objective is to use the established results in order to study the problem of control with state constraints at terminal time for systems governed by semilinear parabolic equations.

Let $\Omega \subset \mathbb{R}^n$ ($n = 1; 2$ or 3) be an open and bounded domain with sufficiently smooth boundary $\partial\Omega$. Assume $-A$ to be an unbounded densely defined linear operator which generates a $\mathcal{C}_0$ analytic [3, 10] semigroup $(S(t))_{t\geq 0}$ on $Z = L^2(\Omega, \mathbb{R}^k)$ (for an integer k) and consider the following semilinear control system,

$$\frac{\partial z}{\partial t} + Az = \varphi(z, v) \quad \text{in } \Omega \times (0, \infty[\,, \tag{1}$$

with initial data

$$z(x, 0) = z_0(x) \quad \text{in } \Omega. \tag{2}$$

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where $z_0 \in \mathrm{dom}(A)$ and φ denotes a nonlinear operator which maps $\mathcal{D} \times V$ into Z, with V another Hilbert space and $\mathcal{D}$ a closed subset of Z. For $t_f > 0$ and a measurable function $\bar{v} : [0, t_f[\to V$, we denote by $z(\cdot, \bar{v})$ a solution, when it exists, of system (1)-(2) on the interval $[0, t_f[$.

Now, let $\xi : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$. The control problem we have to deal with in this paper is stated as follows,

$$\left| \begin{array}{l} \text{Find a control } v : [0, t_f] \to V \text{ such that:} \\ \text{system } (1) - (2) \text{ has a solution } z(\cdot, v) \text{ and} \\ \xi(z(x, t_f, v), z'(x, t_f, v)) \leq 0 \text{ for a.e. } x \in \Omega, \end{array} \right. \tag{3}$$

where $z(x, t_f, v)$ denotes the value of $z(t_f, v)$ at x and $z'(x, t_f, v)$ its partial derivative with respect to x.

A set-valued map Q on $\mathcal{D}$ is said to be lower semicontinuous whenever for each $z_0 \in \mathcal{D}$ and any sequence of elements $z_n \in \mathcal{D}$ converging to z_0, for every $y_0 \in Q(z_0)$, there exists a sequence of elements $y_n \in Q(z_n)$ which converges to y_0. See for instance [1, 4].

The directional derivative [7] of a mapping $\alpha : \mathcal{D} \to \mathbb{R}$ is defined by,

$$d\alpha(z)y \doteq \liminf_{h \downarrow 0} \frac{\alpha(z + hy) - \alpha(z)}{h} \quad \text{for each } y \in Z.$$

The organization of this paper is as follows. In Section 2 we provide brief backgrounds on feedback spreading control, including the notion of the speed of a spread. The purpose of Section 3 is to solve the control problem (3). Section 4 is devoted to a numerical example.

2. On feedback spreading control

Next we present definitions and recent results on feedback spreading control. Let $\omega : \mathcal{D} \subset Z \to 2^{\Omega}$ be the map by which spreads will be designed in system (1)-(2). Then a measurable function $\bar{v} : [0, t_f) \to V$ is called a spreading control with respect to ω if:

$$\left| \begin{array}{l} z(t, \bar{v}) \in \mathcal{D} \quad \text{for all } t \in [0, t_f) \quad \text{and} \\ (\omega(z(t, \bar{v})))_{0 \leq t < t_f} \text{ is non-decreasing.} \end{array} \right. \tag{4}$$

Thereby a feedback spreading control law can be defined as follows.

Definition 2.1. The mapping $\varsigma : \mathcal{D} \to V$ is said to be a feedback spreading control (for short *FSC*) law with respect to ω if, for all z_0 in $\mathcal{D}$, $v = \varsigma(z)$ defines a spreading control for system (1)-(2), i.e. satisfying conditions (4).

As established in [9], FSC laws may be investigated in the framework of set-valued analysis. In fact, there is shown that these laws have to produce monotone solutions of system (1)-(2) with respect to a preorder which involves the map ω. Then FSC laws are derived by making selections of tangential maps as in the following way: For each couple $(y, z) \in Z \times \mathcal{D}$ consider the tangential condition,

$$\left| \begin{array}{l} \forall\, \delta > 0, \exists\, 0 < h < \delta \quad \text{and} \quad \|p\| \leq \delta \quad \text{such that} \\ S(h)z + h(y + p) \in \mathcal{D} \quad \text{and} \\ \omega(S(h)z + h(y + p)) \supset \omega(z). \end{array} \right. \tag{5}$$

Then define the set-valued maps $\mathcal{T}_\omega$ and $\mathcal{F}_\omega$ for each $z \in \mathcal{D}$ as follows,

$$\mathcal{T}_\omega(z) \doteq \{y \in Z \mid \text{ Eq. (5) holds with } (y, z)\}$$

and

$$\mathcal{F}_\omega(z) \doteq \{v \in V \mid \varphi(z, v) \in \mathcal{T}_\omega(z)\}. \tag{6}$$

Hence we may present the following basic result which characterizes FSC laws.

Theorem 2.2. *Let $\varsigma : \mathcal{D} \to V$ be a measurable function and assume that,*

(i) *The semigroup $S(\cdot)$ is compact.*
(ii) *The set $\Sigma_\omega \doteq \{(y, z) \in \mathcal{D}^2 \mid \omega(y) \supset \omega(z)\}$ is closed.*
(iii) *$\varphi(\cdot, \varsigma(\cdot))$ sends convergent sequences of $\mathcal{D}$ into weakly convergent ones in Z.*

Then ς is a FSC *law with respect to ω iff ς is a selection of the map $\mathcal{F}_\omega$ given by* (6)*, i.e. $\varsigma(z) \in \mathcal{F}_\omega(z)$ for each $z \in \mathcal{D}$.*

Proof. See [9, Theorem 3.1]. □

Remark 2.3. We stress that an FSC law ζ derives a spreading control $\bar{v} = \zeta(\bar{z}(\cdot))$ for all initial states $z_0 \in \mathcal{D}$. Note the dependence of $\bar{v}$ upon z_0 because $\bar{z}$ depends of z_0, contrary to the law ζ.

Remark 2.4. In general, parabolicity of the linear part of the system implies the compactness of the semigroup $S(\cdot)$ which is required in condition (i).

In a natural way, the speed of the spread $(\omega(z(t, \bar{v})))_t$ which is generated by a spreading control $\bar{v}$ may be set for each $t \in [0, t_f)$ as follows,

$$\text{speed}(t, \bar{v}) \doteq \liminf_{h \downarrow O} \frac{\mu(\omega(z(t + h, \bar{v}))) - \mu(\omega(z(t, \bar{v})))}{h} \geq 0,$$

where μ satisfies,

$$\left| \begin{array}{l} \mu : \mathcal{M}_\Omega \to \mathbb{R}^+ \quad \text{and} \\ \sigma_1 \subset \sigma_2 \Rightarrow \mu(\sigma_1) \leq \mu(\sigma_2) \end{array} \right. \tag{7}$$

where $\mathcal{M}_\Omega$ denotes the set of measurable subsets of Ω. For an FSC law ζ, the speed of the spread generated by the control $\bar{v} = \zeta(\bar{z})$ will be computed by using the formula shown in [8],

$$\text{speed}(t, \bar{v}) = \theta(\varphi(\bar{z}(t), \bar{v}(t)), \bar{z}(t)), \tag{8}$$

where $\bar{z} = z(\cdot, \bar{v})$ and θ is given by,

$$\theta(y, z) \doteq \liminf_{h \downarrow 0, \|p\| \to 0} \frac{\mu(\omega(S(h)z + h(y + p)) - \mu(\omega(z))}{h}, \tag{9}$$

for each $z \in \mathcal{D}$ and $y \in \mathcal{T}_\omega(z)$. In addition, when $\mathcal{D}$ and $\mu_\omega \doteq \mu \circ \omega$ are convex we have,

$$\theta(y, z) = d\mu_\omega(z)(y - Az),$$

for each $y \in \mathcal{T}_\omega(z)$ and $z \in \mathcal{D} \cap \text{dom}(A)$. For sake of notation we define the spread speed functional as follows,

$$\rho(z, v) \doteq \theta(\varphi(z, v), z) \quad \text{for each } z \in \mathcal{D}, v \in \mathcal{F}_\omega(z). \tag{10}$$

Thus, the speed of the spread generated by a spreading control law $\bar{v}$ can be measured, at each time t, by the number $\rho(z(t,\bar{v}),\bar{v}(t))$. Note that it is dependent on the functional μ.

Let ν be a non-negative measurable function on $\mathcal{D}$ and define for each $z \in \mathcal{D}$ the following maps by,

$$\mathcal{T}_\omega^\nu(z) \doteq \{y \in \mathcal{T}_\omega(z) \mid \theta(y,z) \geq \nu(z)\}, \tag{11}$$

and

$$\mathcal{F}_\omega^\nu(z) \doteq \{v \in V \mid \varphi(z,v) \in \mathcal{T}_\omega^\nu(z)\}. \tag{12}$$

It follows that any selection of the map $\mathcal{F}_\omega^\nu$ which satisfies assumptions of Theorem 2.2 provides an FSC law which produces a spread having a speed greater than ν. Conditions of existence of such a law can be listed in the following result.

Theorem 2.5. *Further assumptions* (i) *and* (ii) *of Theorem* 2.2, *assume that,*

(iv) ω^{-1} *has convex values, i.e.* $\{z \in \mathcal{D} \mid x \in \omega(z)\}$ *is convex for all* $x \in \Omega$.
(v) $\mathcal{T}_\omega$ *is a lower semicontinuous map.*
(vi) *For each sequences* $(z_n)_n \subset \mathcal{D}$ *and* $(y_n)_n \subset Z$ *such that* $y_n \in \mathcal{T}_\omega(z_n)$ *for every* n, *we have,*

$$\begin{array}{l} z_n \to z \ (strong) \\ y_n \to y \ (weak) \end{array} \Longrightarrow y \in \mathcal{T}_\omega(z) \quad and \quad \theta(y_n,z_n) \to \theta(y,z).$$

(vii) *For each* $z \in \mathcal{D}$ *and* $y \in \mathcal{T}_\omega(z)$, *there exists* $v \in V$ *such that* $\varphi(z,v) = y$.
$(viii)$ φ *is continuous.*
(ix) μ_ω *has a directional derivative.*
(x) *For all* $z \in \mathcal{D}$, *there exists* $y \in \mathcal{T}_\omega(z)$ *such that* $\theta(y,z) > \nu(z)$.
(xi) ν *is upper semicontinuous.*

Then there is an FSC law ζ *(with respect to* ω*) which satisfies*

$$\rho(z,\zeta(z)) \geq \nu(z) \quad \text{for each } z \in \mathcal{D}.$$

Proof. See [8, Section 5] for a detailed proof which can be outlined as follows,

(a) Because of (x), the map $\mathcal{T}_\omega^\nu$ of (11) has non empty values.
(b) We show that the map $\mathcal{T}_\omega^\nu$ is lower semicontinuous by proving [1, 4] that the functional,

$$\kappa : z \in \mathcal{S} \to d(y_0,\mathcal{T}_\omega^\nu(z))^2 = \min_{\substack{y\in\mathcal{T}_\omega(z) \\ \nu(z)-\theta(y,z)\leq 0}} \|y_0 - y\|^2$$

is upper semicontinuous for each $y_0 \in Y$. For that end the following expression is used,

$$\kappa(z) = \sup_{\lambda\geq 0} \inf_{y\in\mathcal{T}_\omega(z)} \{\|y_0 - y\|^2 + \lambda(\nu(z) - \theta(y,z))\}$$

(c) $\mathcal{T}_\omega^\nu$ is lower semicontinuous and has closed convex values due to (i) and (ii). Then, thanks to Michael's selection Theorem [4] (any lsc closed convex valued map has a continuous selection), the map $\mathcal{T}_\omega^\nu$ admits a continuous selection $y(\cdot)$. Now, we can use condition (vii) to build an FSC law $\varsigma_\nu : \mathcal{D} \to V$ in such a manner that $\varphi(z,\varsigma_\nu(z)) = y(z)$ for each $z \in \mathcal{S}$.

□

Following the works [1, 11] on viability algorithms, a convenient use of the expressions (5) and (6) has led the authors [2] to state the following algorithm.

Algorithm 2.6. Let $h > 0$ and $N \in \mathbb{N}$ such that $h = t_f/N$.

I. Initialize $r = z_0$ and $\sigma = \omega(z_0)$.

II. Iterate for $k = 1$ to N.

II.a Find v such that:

$$\omega(S(h)r + h\varphi(r,v)) \supset \sigma \quad \text{and} \quad \rho(r,v) \geq \nu(r). \tag{13}$$

II.b Let $v_k = v$ and $z_k = S(h)r + h\varphi(r,v)$.

II.c Put $r = z_k$, $\sigma = \omega(z_k)$ and go to II.a.

III. At each time $t_k = kh$,

the approximated state is z_k,
the spreading control is v_k,
the generated spread is $(\omega(z_k))_k$.

Convergence of Algorithm 2.6 means that:

$$\left| \begin{array}{ll} (a) & v^h \to v_s \text{ in } L^2(0,t_f,V) \text{ when } h \to 0, \\ (b) & v_s \text{ is a spreading control,} \\ (c) & z^h \to z(\cdot, v_s) \text{ when } h \to 0, \end{array} \right.$$

where v^h and z^h are given as follows,

$$(v^h(t), z^h(t)) \doteq \left| \begin{array}{ll} (v_1, z_1) & \text{on } [0,h)\,, \\ \vdots & \\ (v_N, z_N) & \text{on } [(N-1)h, t_f]\,. \end{array} \right.$$

Although Algorithm 2.6 has been computationally tested successfully [2], its convergence according to the above sense, is not yet proved rigorously. Note the key sequence (13) by which an approximate value of the spreading control is derived by selection.

3. Control with terminal target

We turn next to study the reachability problem (3) by restating it as spreading control problem with speed constraints. Let the map ω of the previous section be given as follows,

$$\omega(z) \doteq \{x \in \Omega \mid \xi(z(x), z'(x)) \leq 0\} \quad \text{for each } z \in \mathcal{D} \tag{14}$$

where $\mathcal{D}$ denotes, for instance, a closed convex subset of $H_0^1(\Omega)$. It follows that Problem (3) may reduce to seek a control v in such a manner that,

$$\lambda(\Omega \setminus \omega(z(t_f, v))) = 0.$$

This gives rise to the following result.

Theorem 3.1. *Consider system* (1)-(2) *and let* $\omega(z_0) = \omega_0 \neq \emptyset$. *Assume that conditions of Theorem* 2.5 *are satisfied except for* (x) *and* (xi). *Furthermore suppose the following be held,* (*xii*) *For all* $z \in \mathcal{D}$, *there exists* $y \in \mathcal{T}_\omega(z)$ *such that*

$$\theta(y, z) > \frac{\mu(\Omega) - \mu(\omega_0)}{t_f}.$$

where μ *and* θ *are given respectively as in* (7) *and* (9).

(*xiii*) μ *is such that*

$$\left| \begin{array}{l} \sigma \subset \Omega \quad \text{and} \\ \mu(\sigma) = \mu(\Omega) \end{array} \right. \Rightarrow \lambda(\Omega \setminus \sigma) = 0$$

where λ *denotes the Lebesgue measure on* Ω.

Then there is an FSC law ζ_μ *which solves Problem* (3).

Proof. Let ν be defined as follows,

$$\nu(z) = \frac{\mu(\Omega) - \mu(\omega_0)}{t_f} \quad \text{for each } z \in \mathcal{D},$$

then it follows that conditions (x) and (xi) in Theorem 2.5 are well verified. Therefore all of the assumptions in that theorem are satisfied and thereby there is an FSC law, say ζ_μ, which satisfies,

$$\rho(z, \zeta_\mu(z)) \geq \nu(z) \quad \text{for each } z \in \mathcal{D}$$

Now, denote by $\bar{z}$ a solution of system (1)-(2) with $v = \zeta_\mu(z)$ and $\bar{v} = \zeta_\mu(\bar{z})$. Then we get,

$$\mu_\omega(\bar{z}(t_f)) \geq \mu_\omega(\bar{z}(0)) + \int_0^{t_f} d\mu_\omega(\bar{z}(t))\dot{\bar{z}}(t)dt$$

and by considering (1)-(2) and (10) we get,

$$\mu(\omega(\bar{z}(t_f))) \geq \mu(\omega_0) + \int_0^{t_f} \rho(\bar{z}(t), \bar{v}(t))dt$$

It follows that,

$$\mu(\omega(\bar{z}(t_f))) \geq \mu(\omega_0) + t_f \frac{\mu(\Omega) - \mu(\omega_0)}{t_f},$$

and hence $\mu(\omega(\bar{z}(t_f))) \geq \mu(\Omega)$. Now, by using ($xiii$) we get,

$$\lambda(\Omega \setminus \omega(\bar{z}(t_f))) = 0,$$

and thus we have,

$$\xi(z(x, t_f, \bar{v}), z'(x, t_f, \bar{v})) \leq 0 \quad \text{for a.e. } x \in \Omega$$

ending the proof. □

Remark 3.2. Note that an instance of a function μ can be provided by any measure on Ω which obeys to $\mu(A) = 0 \Rightarrow \lambda(A) = 0$ so as condition ($xiii$) holds. Nevertheless, as required by (ix), it is needed that the function $\mu_\omega = \mu \circ \omega$ has a directional derivative.

Remark 3.3. It is of interest to notice that the FSC law ζ_μ the existence of which is proven in Theorem 3.1 depends upon the number $(\mu(\Omega) - \mu(\omega_0))/t_f$ and therefore we can conclude what follows: (a) it depends only upon $\mu(\Omega) - \mu(\omega_0)$ and t_f, and will work for any initial state which corresponds to ω_0 by the map ω. (b) For an initial state which is different from z_0, the law ζ_μ derives a spreading control, though it does not solve necessarily Problem (3). (c) The functional μ only has a technical role. Note the fact that it does not appear in the formulation of the problem.

4. A numerical example

Let $\Omega = (-1;1) \times (-1;1)$ and consider the following semilinear parabolic equation,

$$\frac{\partial z}{\partial t} - \operatorname{div}(D\nabla z) = z^2 + v(t)g(x) \quad \text{on } \Omega \times [0, t_f)\,, \tag{15}$$

with Dirichlet boundary conditions,

$$z(t)\mid_{\partial\Omega} = 0 \quad \text{on } [0, t_f)\,, \tag{16}$$

and initial data,

$$z(0) = z_0 \quad \text{on } \Omega. \tag{17}$$

where D, g, z_0 map Ω into $\mathbb{R}$. D and g stand, respectively, for the diffusion coefficient and the actuator. We know [3] that the linear operator $-A : z \to \operatorname{div}(D\nabla z)$ with domain $\operatorname{dom}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ generates a compact analytic $\mathcal{C}_0$ semigroup on $Z = L^2(\Omega)$ in accord with condition (i).

Let the function ξ which expresses the constraints in (3), be as follows,

$$\xi(s,t) = m - s \quad \text{for each } s, t \in \mathbb{R}.$$

for $m > 0$, so as the map ω of (14) is given by

$$\omega(z) = \{x \in \Omega \mid z(x) \geq m\} \tag{18}$$

for each $z \in \mathcal{D}$. In Table 4.1 we list all the data regards the two numerical examples we are

	Example 1	Example 2
$D(x_1,x_2)$	2	0.7
$g(x_1,x_2)$	$4 + x_1x_2$	$\exp(x_1 + x_2)$
$z_0(x_1,x_2)$	$(1-x_1^2)(1-x_2^2)$	$\mathbf{1}_{\{(x_1-0.6)^2+(0.4+x_2)^4 \leq 0.3\}}$
(t_f, h, m)	$(0.1, 0.0025, 0.9)$	$(0.1, 0.1/30, 0.5)$

Table 4.1: The data

going to treat. The results of Example 1 are presented in Figure 4.1 and Figure 4.2 which contain respectively the computed spread $(\omega(z(t))_{0\leq t\leq t_f}$ at four times and the spreading control on the interval $[0, t_f]$. While Example 2 is illustrated in the same manner in Figure 4.3 and Figure 4.4. We emphasize that the values of h above are the smaller amongst a set of values taken under consideration in order to get convergence of Algorithm 2.6.

Next, we proceed to examine sequence II.a of Algorithm 2.6. According to expression (18), the inclusion (13) can be rewritten for each $r \in Z, \sigma \subset \Omega$ and $v \in \mathbb{R}$, as follows,

$$S(h)r + h\varphi(r,v) \geq m \quad \text{on } \sigma,$$

Then, by (15) we get,

$$S(h)r + h(r^2 + vg) \geq m \quad \text{on } \sigma,$$

or equivalently,

$$vg \geq \frac{m - S(h)r - hr^2}{h} \quad \text{on } \sigma$$

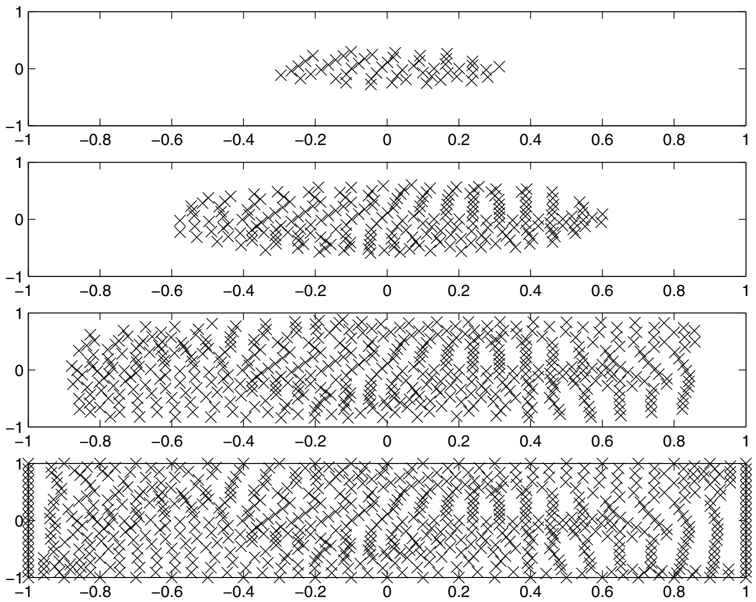

Figure 4.1: *The spread provided by Example* 1 *(see Table* 4.1*) in up to down order at times* $0, 0.015, 0.7$ *and terminal time* 0.1*.*

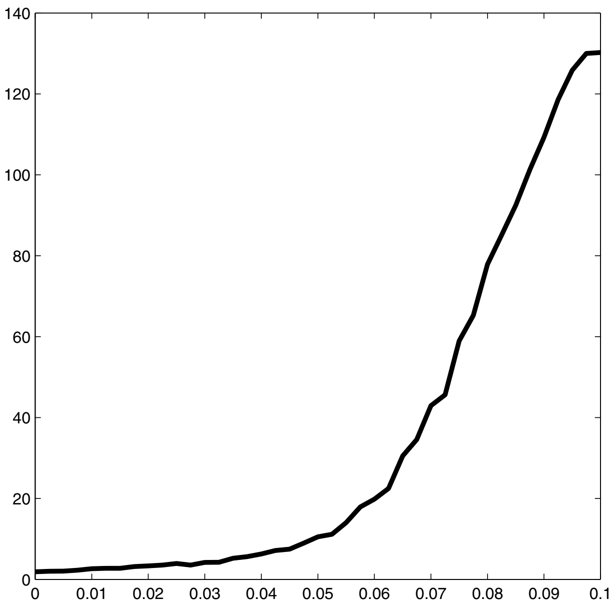

Figure 4.2: *The spreading control in Example* 1

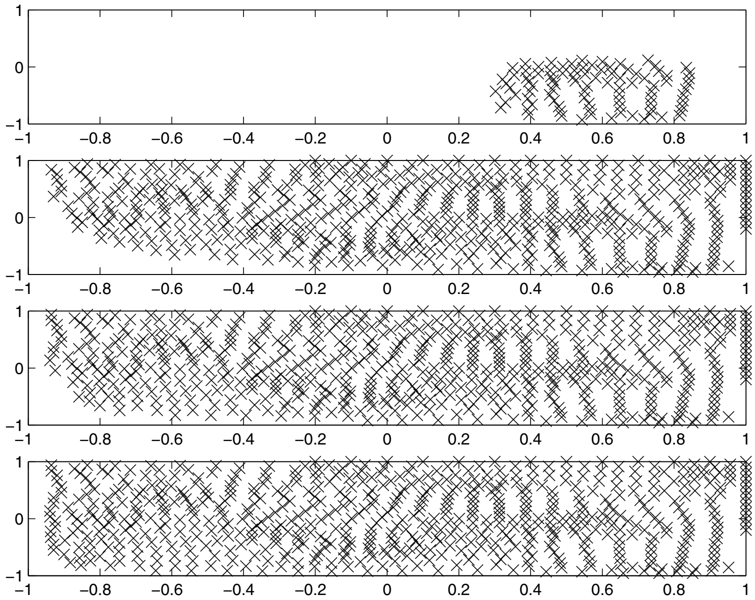

Figure 4.3: *The spread provided by Example* 2 *in up to down order at times* $0, 0.033, 0.05$ *and terminal time* 0.1.

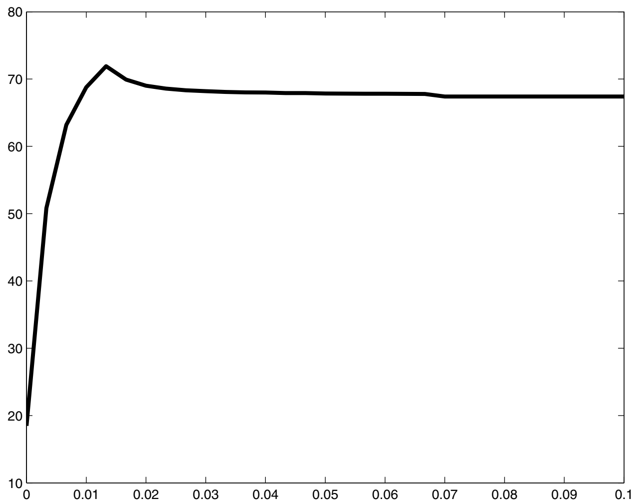

Figure 4.4: *The spreading control in Example* 2

Now, by assuming that the actuator g is satisfying, as in Table 4.1, the following condition,

$$g(x_1, x_2) > 0 \quad \text{on } \Omega \tag{19}$$

it follows that,

$$v \geq \sup_{\sigma} \frac{m - S(h)r - hr^2}{hg} \tag{20}$$

In other words, an approximate spreading control will be provided by a suitable selection of the set-valued map

$$z \in \mathcal{D} \to \mathcal{F}_h(\omega(z), z),$$

where

$$\mathcal{F}_h(\sigma, r) \doteq \left[\sup_{\sigma} \frac{m - S(h)r - hr^2}{hg}, +\infty\right).$$

for each $\sigma \subset \Omega$ and $r \in L^2(\Omega)$.

5. Conclusion

In the present paper, spreading control techniques have been applied to control with target condition in semilinear parabolic systems. The novelty is that a unified approach is provided along with an easy implementable algorithm. The latter leads to an FSC law, a solution to the problem, which generates a spread whose speed is greater than $(\mu(\Omega) - \mu(\omega_0))/t_f$ where ω_0 stands for the zone where the constraints initially hold. A consequence is that, through its speed, the spread can be controlled in such a manner that the target condition is reached at terminal time. Thus the speed generated by that FSC law increases with $\mu(\Omega) - \mu(\omega_0)$ and decreases regards t_f.

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