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# Regularity of Minimizers and of Adjoint States in Optimal Control under State Constraints

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*Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.*

This paper is devoted to regularity of minimizers and adjoint states for the Bolza optimal control problem under state constraints. It is well known that the adjoint state of the Pontryagin maximum principle may be discontinuous whenever the optimal trajectory lies partially on the boundary of constraints. Still we prove that if the associated Hamiltonian $H(t, x, \cdot)$ is differentiable and the constraints are sleek, then every optimal trajectory is continuously differentiable. Moreover if for all $x$ on the boundary of constraints, $\frac{\partial H}{\partial p}(t, x, \cdot)$ is strictly monotone in directions normal at $x$ to the set of constraints, then the adjoint state is also continuous on interior of its interval of definition. Finally, we identify a class of constraints for which the adjoint state is absolutely continuous or even Lipschitz on this open interval. This allows us to derive necessary conditions for optimality in the form of variational differential inequalities, maximum principle and modified transversality conditions. We also provide sufficient conditions for Lipschitz continuity of optimal controls and for normality of the maximum principle.

*Keywords:* Optimal control, state constraints, constrained maximum principle, smooth minimizers, normal necessary conditions, absolutely continuous costate, Lipschitz optimal control

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## 1. Introduction

Consider a control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \qquad \text{a.e. in } [0,1] \tag{1}$$

under state constraints

$$x(t) \in K \qquad \text{for all } t \in [0,1], \tag{2}$$

where $U$ is a set-valued map from $[0,1]$ into subsets of a complete separable metric space $\mathcal{Z}$, $f : [0,1] \times \mathbf{R}^n \times \mathcal{Z} \to I\!\!R^n$ and $K$ is a closed subset of $I\!\!R^n$.

Denote by $S^K_{[0,1]}$ the set of all absolutely continuous solutions to (1) satisfying state constraints (2).

In this paper we investigate regularity of minimizers for the Bolza optimal control problem under state constraints

$$\min \left\{ \varphi(x(0), x(1)) + \int_0^1 L(t, x(t), u(t))dt \mid x \in S^K_{[0,1]},\ (x(0), x(1)) \in K_1 \right\}, \tag{3}$$

where $\varphi : I\!\!R^n \times I\!\!R^n \to I\!\!R$, $L : [0,1] \times I\!\!R^n \times \mathcal{Z} \to I\!\!R$ and $K_1 \subset I\!\!R^n \times I\!\!R^n$ are given.

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Consider an optimal trajectory/control pair $(z, \bar{u})$. Under some regularity assumptions on data it satisfies the following first order necessary conditions for optimality: there exist $\lambda \in \{0, 1\}$, an absolutely continuous $p : [0, 1] \to \mathbb{R}^n$ and a mapping $\psi : [0, 1] \to \mathbb{R}^n$, $\psi \in NBV([0, 1])$ (space of normalized functions with bounded variation on $[0, 1]$) not vanishing simultaneously such that

*i*) for a positive Radon measure $\mu$ on $[0, 1]$ and a Borel measurable $\nu(\cdot) : [0, 1] \to \mathbb{R}^n$ satisfying $\nu(s) \in N_K(z(s)) \cap B$ $\mu$-almost everywhere (where $N_K(z(s))$ denotes the normal cone to $K$ at $z(s)$) we have $\psi(t) = \int_{[0,t]} \nu(s) d\mu(s)$ for all $t \in (0, 1]$,

*ii*) $p(\cdot)$ is a solution to the adjoint system

$$-p'(s) = \frac{\partial f}{\partial x}(s, z(s), \bar{u}(s))^*(p(s) + \psi(s)) - \lambda \frac{\partial L}{\partial x}(s, z(s), \bar{u}(s)) \text{ a.e.} \tag{4}$$

satisfying the maximum principle

$$\begin{aligned} & \langle p(s) + \psi(s), z'(s) \rangle - \lambda L(s, z(s), z'(s)) \\ = \; & \max_{u \in U(s)} \left( \langle p(s) + \psi(s), f(s, z(s), u) \rangle - \lambda L(s, z(s), u) \right) \text{ a.e.} \end{aligned}$$

and the transversality condition

$$(p(0), -p(1) - \psi(1)) \in \lambda \nabla \varphi(z(0), z(1)) + N_{K_1}(z(0), z(1)).$$

The above necessary conditions are called normal if $\lambda = 1$.

The two Hamiltonians $H : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{H} : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \times \{0, 1\} \to \mathbb{R}$ associated to the above Bolza problem are defined by

$$H(t, x, p) = \sup_{u \in U(t)} (\langle p, f(t, x, u) \rangle - L(t, x, u)), \tag{5}$$

$$\mathcal{H}(t, x, p, \lambda) = \sup_{u \in U(t)} (\langle p, f(t, x, u) \rangle - \lambda L(t, x, u)). \tag{6}$$

Notice that the maximum principle can be written as

$$\mathcal{H}(s, z(s), p(s) + \psi(s), \lambda) = \langle p(s) + \psi(s), z'(s) \rangle - \lambda L(s, z(s), z'(s)) \text{ a.e. in } [0, 1].$$

For all $(t, x) \in [0, 1] \times \mathbb{R}^n$, $H(t, x, \cdot)$ is convex and

$$\partial_p H(t, x, p) = \{ f(t, x, u) \mid H(t, x, p) = \langle p, f(t, x, u) \rangle - L(t, x, u), \; u \in U(t) \}, \tag{7}$$

where $\partial_p H(t, x, p)$ denotes the subdifferential of convex analysis of $H(t, x, \cdot)$ at $p$. Thus in the normal case, i.e. with $\lambda = 1$, the optimal trajectory satisfies

$$z'(t) \in \partial_p H(t, z(t), p(t) + \psi(t)) \text{ a.e. in } [0, 1].$$

When there is no state and end point constraints, i.e. $K = \mathbb{R}^n$ and $K_1 = \mathbb{R}^n \times \mathbb{R}^n$, then $\lambda = 1$, $\psi = 0$. If in addition the Hamiltonian $H$ is differentiable in the last variable, then the above inclusion allows to deduce regularity of the derivative $z'$ from regularity of $\frac{\partial H}{\partial p}$. Indeed if $\frac{\partial H}{\partial p}$ is continuous (respectively locally Lipschitz), then $z'$ is continuous (respectively absolutely continuous). This fails to be true in general because in the constrained case

$$z'(t) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) \text{ a.e. in } [0, 1] \tag{8}$$

and $\psi$ may be discontinuous.

In this paper we focus our attention on the problem of regularity of optimal solutions and of mapping $\psi$ for sleek $K$ in the normal case, i.e. $\lambda = 1$. (The state constraints $K$ are sleek, if for every $x \in K$ the contingent cone to $K$ at $x$ coincides with Clarke's tangent cone to $K$ at $x$). We show that if $H$ is continuous, then the function

$$(0,1) \ni t \mapsto H(t, z(t), p(t) + \psi(t))$$

is continuous, even though $\psi$ may be discontinuous. If in addition $H(t, z(t), \cdot)$ is differentiable, then $z \in C^1$ and the mapping

$$(0,1) \ni t \mapsto \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t))$$

is continuous (we prove this result without assuming continuity of $\frac{\partial H}{\partial p}$). Moreover

$$\langle \psi(t) - \psi(t-), z'(t) \rangle = 0, \quad \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t-)) \ \ \forall\, t \in (0,1).$$

This implies that jumps of $\psi$ occur only in the directions that are orthogonal to derivatives of $z$ (see Theorem 3.4). Furthermore, we prove continuity of $\psi$ on $(0,1)$ provided that $\frac{\partial H}{\partial p}(t, z(t), \cdot)$ is strictly monotone in the directions normal to constraints at $z(t)$: for all $p \neq q \in I\!\!R^n$ such that $p - q \in N_K(z(t))$

$$H(t, z(t), p) = H(t, z(t), q) \implies \left\langle \frac{\partial H}{\partial p}(t, z(t), p) - \frac{\partial H}{\partial p}(t, z(t), q), p - q \right\rangle > 0.$$

We also show that $\psi$ is absolutely continuous on $(0,1)$ if the following two conditions $a$) and $b$) are fulfilled:

*a*) for some closed sets $K_j$ with $C^{1,1}_{loc}$–boundary, $K = \cap_{j=1}^m K_j$ and

$$0 \notin co\{n_j(x) \mid j \in I(x)\}, \tag{9}$$

where $I(x)$ denotes the set of active indices at $x$ and $n_j(x)$ the outward unit normal to $K_j$ at $x$, i.e. $j \in I(x)$ if and only if $x \in \partial K_j$,

*b*) $H$ is continuous, $\frac{\partial H}{\partial p}$ is locally Lipschitz and for every $r > 0$ there exists $k_r > 0$ such that for all $t \in [0,1]$

$$\begin{aligned} & p,\ q \in rB \ \ \&\ \ p - q \in N_K(z(t)) \\ \implies & \left\langle \frac{\partial H}{\partial p}(t, z(t), p) - \frac{\partial H}{\partial p}(t, z(t), q), p - q \right\rangle \geq k_r |p - q|^2. \end{aligned}$$

In the other words, $\frac{\partial H}{\partial p}(t, z(t), \cdot)$ is strictly monotone with respect to $N_K(z(t))$.

Moreover, if $p$ is Lipschitz, then also $\psi$ is Lipschitz on $(0,1)$ and $z'$ is Lipschitz on $[0,1]$ (see a more general Theorem 4.2). To obtain this result we use some ideas of proofs from [12, 18], but we impose a monotonicity assumption on the Hamiltonian with respect to normals to constraints (instead of supposing the strict convexity of the Lagrangian) and consider control systems that are not affine with respect to controls. In Example 3.6 of Section 3 we discuss relations of our assumptions to those from [18].

If for a trajectory/control pair $(z, \bar{u})$ the normal necessary conditions $i) - ii)$ hold true with absolutely continuous $\psi$ and $a)$ is verified, then, setting $q := p + \psi$, we deduce that there exists an *absolutely continuous* mapping $q : [0,1] \to I\!R^n$ satisfying the differential variational inequalities

$$-q'(s) \in \frac{\partial f}{\partial x}(s, z(s), \bar{u}(s))^* q(s) - \frac{\partial L}{\partial x}(s, z(s), \bar{u}(s)) - N_K(z(s)) \text{ a.e. in } [0,1], \tag{10}$$

the transversality condition

$$(q(0), -q(1)) \in \nabla\varphi(z(0), z(1)) + N_{K_1}(z(0), z(1)) + N_K(z(0)) \times N_K(z(1)) \tag{11}$$

and the maximum principle

$$\langle q(s), z'(s)\rangle - L(s, z(s), \bar{u}(s)) = \max_{u \in U(s)} (\langle q(s), f(s, z(s), u)\rangle - L(s, z(s), u)) \quad \text{a.e. in } [0,1]. \tag{12}$$

The Maximum Principle with regular costate was proved by Gamkrelidze in [11] for smooth optimal trajectories. Then a number of papers were written on this subject without restrictions imposed on $z$, but using measures in the definition of costate (see for instance [9]). We refer to [14, 19] for extended discussions on the constrained maximum principle and further references and to [1, 13] for the Russian bibliography on the subject.

In this paper we go another way around. We impose some assumptions on the Hamiltonian and constraints to deduce absolute continuity of $\psi$ and smoothness of the optimal trajectory $z$.

Regularity of $\psi$ can be used for further investigation of smoothness of the corresponding optimal control $\bar{u}$. Indeed, if for all $q \in I\!R^n$ there exists exactly one $u(t, q)$ such that $H(t, z(t), q) = \langle q, f(t, z(t), u(t, q))\rangle - L(t, z(t), u(t, q))$, then, by the maximum principle, $\bar{u}(t) = u(t, p(t) + \psi(t))$ for almost all $t$. Thus regularity of $\bar{u}$ depends on regularity of the mapping $u(\cdot, \cdot)$ and $p + \psi$. For instance Lipschitz continuity of $\psi$ on $(0, 1)$ implied Lipschitz continuity of minimizing controls in [12, Hager] for linear control systems, convex Lagrangian and convex state constraints, in [8, Dontchev & Hager] for the LQR problem under affine state constraints and in [16, Malanowski] for both control system and Lagrangian nonlinear with respect to the state. Very recently Shvartsman and Vinter [18] considered the case of fully nonlinear state constraints

$$K = \{x \mid h_j(x) \le 0,\ j = 1, ..., m\}$$

with $h_j \in C^{1,1}_{loc}$ (actually in their paper $h_j$ are also time dependent). In their work the system is supposed to be affine with respect to controls and the Lagrangian $L(t, x, \cdot)$ is smooth and strictly convex. Under various sets of conditions they show that the above mapping $u(\cdot, \cdot)$ is locally Lipschitz and $p$, $\psi$ are Lipschitz, implying that the optimal control is Lipschitz continuous.

The representation appearing in condition $a)$ stated above has the advantage of not implying assumptions depending on a "parameterization" of $K$ by $h_j$. We neither require the affine dependence of system on controls but ask instead some smoothness of the Hamiltonian $H$ to state sufficient conditions for Lipschitz continuity of $\psi$ on $(0, 1)$. Then Lipschitz continuity of the optimal control $\bar{u}$ follows from the local Lipschitz continuity of the above mapping $u(\cdot,\cdot)$.

To prove regularity of optimal solutions and adjoint states, we shall use equation (8) and the jump conditions

$$\psi(0+) \in N_K(z(0)), \quad \psi(t) - \psi(t-) \in N_K(z(t)) \quad \forall\, t \in (0,1]$$

that follow from some versions of the constrained maximum principle (see [5, 6, 14]). They hold true for instance when the set-valued map $t \rightsquigarrow N_K(z(t))$ has closed graph. Since the adjoint system is never used in this paper (except Theorem 5.2), results of Sections 3, 4 and 5 can be applied with various maximum principles, including their non smooth versions (see [1, 14, 19]) provided that the jump conditions hold true.

The outline of the paper is as follows. Section 2 is devoted to some preliminaries and Section 3 to $C^1-$ regularity of minimizers and continuity of $\psi$. In Section 4 the absolute (and Lipschitz) continuity of the multiplier $\psi$ and $C^{1,1}-$ regularity of minimizers are investigated. As applications we provide sufficient conditions for Lipschitz continuity of optimal controls and derive the maximum principle (10)-(12) in Section 5. Finally in Section 6 new sufficient conditions for normality of some maximum principles are proposed.

## 2. Preliminaries

Let $X$ be a real Banach space, $B$ denote the closed unit ball in $X$. A set $C \subset X$ is called a cone if it is nonempty and for all $\lambda \geq 0$ and $v \in C$ we have $\lambda v \in C$. The negative polar cone of $C$ is defined by

$$C^- = \{x^* \in X^* \mid \quad \langle x^*, x\rangle \leq 0, \quad \forall\, x \in C\},$$

where $X^*$ denotes the dual of $X$. The positive polar cone of $C$ is $C^+ = -C^-$.

**Lemma 2.1.** *Let $X$ be a Banach space, $C_i \subset X$, $i = 1,2$ be closed convex cones and assume that for some $x_0 \in C_2$ and $\varepsilon > 0$ we have $x_0 + \varepsilon B \subset C_1$. Set $M = 1 + |x_0|/\varepsilon$. Then for every $n \in (C_1 \cap C_2)^+$ with $|n| = 1$ and all $n_i \in C_i^+$ such that $n = n_1 + n_2$ we have $|n_i| \leq M$.*

**Remark.** The above assumptions imply that $int(C_1) \cap C_2 \neq \emptyset$. Thus $(C_1 \cap C_2)^+ = C_1^+ + C_2^+$. For this reason for every $n \in (C_1 \cap C_2)^+$ there exist $n_i \in C_i^+$ such that $n = n_1 + n_2$. Lemma 2.1 provides an estimate for any such decomposition of $n$.

**Proof.** By our assumptions $\varepsilon B \subset C_2 \cap |x_0|B - C_1 \cap (|x_0| + \varepsilon)B$. Hence $B \subset C_2 \cap MB - C_1 \cap MB$. Let $n,\ n_i,\ i = 1,2$ be as in the statement of the lemma and $w \in X,\ |w| = 1$. Then for some $w_i \in C_i$ with $|w_i| \leq M$ we have $w = w_2 - w_1$ and

$$\langle n_1, w\rangle = \langle n_1, w_2 - w_1\rangle \leq \langle n_1, w_2\rangle = \langle n - n_2, w_2\rangle \leq \langle n, w_2\rangle \leq M.$$

Hence $|n_1| \leq M$. The estimate of $n_2$ follows by the same arguments and is omitted. □

From the above lemma, using an induction argument, we deduce the following corollary.

**Corollary 2.2.** *Let $X$ be a Banach space, $C_i \subset X$, $i = 1, ..., m$ be closed convex cones and assume that for some $x_0 \in C_m$, $x_0 + \varepsilon B \subset \bigcap_{i=1}^{m-1} int(C_i)$. Set $M = 1 + |x_0|/\varepsilon$. Then for every $n \in (\bigcap_{i=1}^{m} C_i)^+$ with $|n| = 1$ and all $n_i \in C_i^+$ such that $n = n_1 + n_2 + ... + n_m$ we have $|n_i| \leq M^{m+1-i}$ for all $1 \leq i \leq m$.*

**Definition 2.3.** Let $K \subset I\!R^n$ be closed and $x \in K$.

i) The contingent cone to $K$ at $x$ is defined by

$$T_K(x) = \{v \in \mathbf{R}^n | \quad \liminf_{h \to 0+} \frac{dist(x + hv, K)}{h} = 0\}.$$

ii) Clarke's tangent cone to $K$ at $x$ is defined by

$$C_K(x) = \{v \in \mathbf{R}^n \mid \quad \lim_{h \to 0+,\, x' \to_K x} \frac{dist(x' + hv, K)}{h} = 0\},$$

where $\to_K$ denotes the convergence in $K$.

iii) The negative polar $N_K(x) := C_K(x)^-$ is called the normal cone to the set $K$ at $x \in K$.

iv) $K$ is called sleek if for all $x \in K$, $T_K(x) = C_K(x)$.

It is known that every convex set is sleek. For other examples of sleek sets see [2]. If $K$ is sleek, then the set-valued map $K \ni x \rightsquigarrow T_K(x)$ is lower semicontinuous and the map $K \ni x \rightsquigarrow N_K(x)$ has closed graph.

**Definition 2.4.** Let $z : [0, 1] \to I\!R^n$ be a Lipschitz function. For every $t \in [0, 1]$ denote by

$$\partial^* z(t) = \text{Limsup}_{s \to t}\{z'(s)\}.$$

where Limsup denotes the upper set-valued limit.

(See for instance [2] for the corresponding definition). If $\partial^* z(t)$ is a singleton, then $z$ is differentiable at $t$ and $\{z'(t)\} = \partial^* z(t)$ (see [4]).

Recall that any function $f : [0, 1] \to \mathbf{R}^n$ of bounded variation on $[0, 1]$ has right and left limits $f(0+)$ and $f(1-)$.

The space $NBV([0, 1])$ (Normalized Bounded Variations) is the space of functions $f$ of bounded variation on $[0, 1]$, which are continuous from the right on $(0, 1)$ and such that $f(0) = 0$. The norm of $f \in NBV([0, 1])$ is the total variation of $f$ on $[0, 1]$ denoted by $\|f\|_{TV}$. If $\beta \in C([0, 1])^*$, then there exists a unique $f \in NBV([0, 1])$ such that for all $\varphi \in C([0, 1])$, $\beta(\varphi) = \int_0^1 \varphi(s)df(s)$ (the Stieltjes integral) and $\|\beta\| = \|f\|_{TV}$ (see for instance [15, p. 113]). Conversely, every $f \in NBV([0, 1])$ defines an element $\beta_f \in C([0, 1])^*$ by setting $\beta_f(\varphi) = \int_0^1 \varphi(s)df(s)$ for all $\varphi \in C([0, 1])$.

**Proposition 2.5.** *Let $K$ be closed and $z : [0, 1] \to K$ be so that the set-valued map $t \rightsquigarrow N_K(z(t))$ has closed graph. Let $g \in NBV([0, 1])$ be such that for some scalar positive Radon measure $\mu$ on $[0, 1]$ and a selection $\nu(s) \in N_K(z(s)) \cap B$ $\mu - a.e.$ we have $g(t) = \int_{[0,t]} \nu(s)d\mu(s)$ for all $t \in (0, 1]$. Then*

$$g(0+) \in N_K(z(0)) \;\;\&\;\; g(t) - g(t-) \in N_K(z(t)) \;\; \forall\, t \in (0, 1].$$

**Remark.** For all $s > 0$ define $\omega(s) = \mu([0, s])$. From the proof given bellow it follows that $g(0+) \in N_K(z(0)) \cap \omega(0+)B$ and $g(t) - g(t-) \in N_K(z(t)) \cap (\omega(t) - \omega(t-))B$. □

**Proof.** Set $c_0 := \lim_{t \to 0+} \mu([0, t])$, $c_t = \lim_{\delta \to 0+} \mu((t - \delta, t])$. From our assumptions it follows that the set-valued map $t \rightsquigarrow G(t) := N_K(z(t)) \cap B$ is upper semicontinuous. Thus for every $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that for all $t \in [0, \delta]$, $G(t) \subset G(0) + \varepsilon B$. By the very definition of the integral and convexity of the closed set $G(0) + \varepsilon B$, $g(t) = \int_{[0,t]} \nu(s) d\mu(s) \in \mu([0, t])(G(0) + \varepsilon B)$. Taking the limit when $t \to 0+$, and then when $\varepsilon \to 0+$ we obtain $g(0+) \in c_0 G(0) \subset N_K(z(0))$. Fix $t \in (0, 1]$ and $\varepsilon > 0$ and let $0 < \delta < \varepsilon$ be such that for all $s \in [t - \delta, t]$, $G(s) \subset G(t) + \varepsilon B$. Then $g(t) - g(t - \delta) \in \mu((t - \delta, t])(G(t) + \varepsilon B)$. Taking the limit when $\delta \to 0+$, and then when $\varepsilon \to 0+$ we get $g(t) - g(t-) \in c_t G(t) \subset N_K(z(t))$. □

In this paper when we say measurable or almost everywhere without refereeing to a precise measure, we always mean the Lebesgue measure.

## 3. $C^1$-Minimizers and Continuity of the Adjoint State

Consider closed subsets $K \subset \mathbb{R}^n$ and $K_1 \subset \mathbb{R}^n \times \mathbb{R}^n$, a complete separable metric space $\mathcal{Z}$, a measurable set-valued map $U : [0, 1] \rightsquigarrow \mathcal{Z}$ with nonempty closed values, mappings $f : [0, 1] \times \mathbf{R}^n \times \mathcal{Z} \to \mathbf{R}^n$, $L : [0, 1] \times \mathbf{R}^n \times \mathcal{Z} \to \mathbf{R}$ which are measurable with respect to the first variable, continuous with respect to the second and third variables and a locally Lipschitz $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Denote by $\partial\varphi$ the generalized gradient of $\varphi$ ([4]).

**Definition 3.1.** A trajectory/control pair $(z, \bar{u})$ of (1), (2) with $(z(0), z(1)) \in K_1$ satisfies the constrained maximum principle for problem (3) if there exist $\lambda \in \{0, 1\}$, $\psi \in NBV([0, 1])$ and an absolutely continuous function $p(\cdot) : [0, 1] \to \mathbf{R}^n$ not vanishing simultaneously such that

$$(p(0), -(p(1) + \psi(1))) \in \lambda \partial\varphi(z(0), z(1)) + N_{K_1}(z(0), z(1)), \tag{13}$$

for almost all $s \in [0, 1]$

$$\langle p(s) + \psi(s), z'(s) \rangle - \lambda L(s, z(s), \bar{u}(s)) = \max_{u \in U(s)} (\langle p(s) + \psi(s), f(s, z(s), u) \rangle - \lambda L(s, z(s), u)) \tag{14}$$

and

$$\psi(0+) \in N_K(z(0)), \quad \psi(t) - \psi(t-) \in N_K(z(t)), \quad \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \quad \forall\, t \in (0, 1] \tag{15}$$

for a positive (scalar) Radon measure $\mu$ on $[0, 1]$ and a Borel measurable $\nu(\cdot) : [0, 1] \to \mathbf{R}^n$ satisfying

$$\nu(s) \in N_K(z(s)) \cap B \quad \mu - a.e. \tag{16}$$

The constrained maximum principle is called normal if $\lambda = 1$.

**Remark.** Notice that we did not invoke the adjoint system in the above definition. In fact it will not be needed in this paper. On the other hand, many non smooth maximum principles that exist in the literature differ just in the adjoint system. In this way results of this paper may be applied with any maximum principle under state constraints, including non smooth versions (see for instance [1, 14, 19]), provided $\psi$ is right continuous and the jump conditions (15) hold true. This implies that the maximum principles of [19] have to be written in a slightly different way (with right continuous instead of left continuous multipliers). Notice that (15) implies that if for some $t \in (0,1)$, $z(t) \in Int(K)$, then $\psi$ is continuous at $t$. □

In this section we investigate smoothness of trajectories satisfying the constrained maximum principle using the Hamiltonian $H : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by (5).

**Lemma 3.2.** *Let $K \subset \mathbb{R}^n$ be a closed set, $z : [0,1] \to K$ be a Lipschitz function and $t \in [0,1]$ be so that $z(t) \in \partial K$. Then for every $n \in T_K(z(t))^-$ we have*

*i)* *if $z$ is differentiable at $t \in (0,1)$, then $\langle n, z'(t)\rangle = 0$.*

*ii)* *for every set $A \subset [0,1]$ of zero measure there exist $s_i \to t$, $s_i \notin A$, $s_i \le t$ and $t_i \to t$, $t_i \notin A$, $t_i \ge t$ such that*

$$t > 0 \implies \lim_{i\to\infty} \langle n, z'(s_i)\rangle \ge 0 \quad \& \quad t < 1 \implies \lim_{i\to\infty} \langle n, z'(t_i)\rangle \le 0.$$

**Proof.** Fix $n \in T_K(z(t))^-$. Since $z([0,1]) \subset K$, for all $t \in (0,1)$ such that $z$ is differentiable at $t$ we have $\pm z'(t) \in T_K(z(t))$. Hence $\langle n, z'(t)\rangle = 0$. To show *ii)* we prove only the first inequality when $t > 0$ because the proof of the second one follows by similar arguments. Indeed assume for a moment that there exists $\varepsilon > 0$ such that for almost all $\tau < t$ sufficiently close to $t$, $\langle n, z'(\tau)\rangle \le -\varepsilon$. Integrating, we get $\langle n, z(s) - z(t)\rangle \ge \varepsilon(t-s)$ for all $s < t$ near $t$. Let $s_i \to t-$, be such that for some $v \in \mathbb{R}^n$, $\lim_{i\to\infty} z(s_i) - z(t)/(t - s_i) = v$. Then $\langle n, v\rangle \ge \varepsilon$. On the other hand $v \in T_K(z(t))$ and therefore $\langle n, v\rangle \le 0$. The obtained contradiction ends the proof. □

Let $(z, \bar{u})$ be a trajectory control pair of (1) and let $F$ be defined by

$$F(t,x) := \{(L(t,x,u) + v, f(t,x,u)) \mid u \in U(t),\ v \ge 0\}. \tag{17}$$

**Theorem 3.3.** *Assume that $(z, \bar{u})$ satisfies the constrained maximum principle with some $\lambda$, $p$, $\psi$, that $z$ is Lipschitz and $K$ is sleek.*

*i)* *If $\lambda = 1$, graph$(F)$ is closed and $H$ is continuous on graph$(z) \times \mathbb{R}^n$, then the function*

$$[0,1] \ni t \mapsto H(t, z(t), p(t) + \psi(t))$$

*is continuous on $(0,1)$ and upper semicontinuous at the end points 0, 1.*

*ii)* *If $\lambda = 0$, $\mathcal{H}$ is continuous on graph$(z) \times \mathbb{R}^n \times \{0\}$ and the set-valued map $t \rightsquigarrow f(t, z(t), U(t))$ has a closed graph, then the function*

$$[0,1] \ni t \mapsto \mathcal{H}(t, z(t), p(t) + \psi(t), 0)$$

*is continuous on $(0,1)$ and upper semicontinuous at the end points 0, 1.*

**Proof.** We only prove the first statement. Set $\phi(t) = H(t, z(t), p(t) + \psi(t))$. Since $\psi$ is right continuous on $(0,1)$, $\phi$ is right continuous on $(0,1)$. Fix $0 \le t \le 1$. Define $n := \psi(t) - \psi(t-) \in N_K(z(t))$ if $t > 0$ and $n = \psi(0+)$ otherwise. Since $K$ is sleek, $N_K(z(t)) = T_K(z(t))^-$.

By (14) and Lemma 3.2 if $t < 1$, then there exist $t_i \to t,\ t_i \geq t$ such that

$$H(t_i, z(t_i), p(t_i) + \psi(t_i)) = \langle p(t_i) + \psi(t_i), z'(t_i)\rangle - L(t_i, z(t_i), u(t_i))$$

and $\lim_{i\to\infty}\langle n, z'(t_i)\rangle \leq 0$. Using that $p,\ H$ are continuous, $\psi$ is bounded, taking a subsequence and keeping the same notations, we may assume that for some $u \in U(t),\ v_0 \geq 0$

$$\lim_{i\to\infty} z'(t_i) = f(t, z(t), u), \quad \lim_{i\to\infty} L(t_i, z(t_i), u(t_i)) = L(t, z(t), u) + v_0.$$

Thus

$$\phi(t+) = \langle n, f(t, z(t), u)\rangle + \langle p(t) + \psi(t+) - n, f(t, z(t), u)\rangle - L(t, z(t), u) - v_0.$$

It follows that $\phi(0+) \leq \phi(0)$ and if $0 < t < 1$, then $\phi(t) \leq \phi(t-)$. Thus $\phi$ is upper semicontinuous at zero. Similarly, if $t > 0$, then for some $v \in U(t),\ v_1 \geq 0$

$$\phi(t-) \leq \langle p(t) + \psi(t), f(t, z(t), v)\rangle - L(t, z(t), v) - v_1 \leq \phi(t).$$

Consequently, $\phi$ is continuous on $(0, 1)$ and upper semicontinuous at 1. □

**Remark.** *i*) In general there is no continuity of $\phi$ at the end points. Indeed consider the problem

$$\min\{x(1) \mid x' = u,\ u \in [-1, 1],\ x(0) = 0,\ x(t) \geq 0\}.$$

Then $x \equiv 0$ is an optimal solution. Using for instance [6] we obtain a normal constrained maximum principle with $p \equiv -1 - \psi(1)$ on $[0, 1]$. By (14) $\psi \equiv 1 + \psi(1)$ on $(0, 1)$. Thus $H(1, 0, p(1) + \psi(1)) = 1$ and $H(t, 0, p(t) + \psi(t)) = 0$ for all $t \in (0, 1)$. So there is no continuity at the final time. For this example for the initial time we may, or may not have continuity at zero. Indeed for $\lambda = 1,\ p \equiv k,\ \mu = k\delta_0 + \delta_1,\ \nu \equiv -1,\ \psi \equiv -k$ on $(0, 1)$ and $\psi(1) = -k - 1$, the triple $(\lambda, \psi, p)$ satisfies the constrained maximum principle for every $k \geq 0$. If $k > 0$, then $H(0, 0, p(0)) = k$. So $\phi$ is continuous at zero if $k = 0$ and discontinuous otherwise.

*ii*) Suppose that statement *i*) or *ii*) of Theorem 3.3 holds true and define a function $\psi_1$ of bounded variation by $\psi_1 = \psi - \psi(0+)$ on $(0, 1)$, $\psi_1(0) = 0$ and $\psi_1(1) = \psi(1-) - \psi(0+)$ and the absolutely continuous function $p_1 = p + \psi(0+)$. Then $\psi_1$ is continuous at the end points. Furthermore $t \to \mathcal{H}(t, z(t), p_1(t) + \psi_1(t), \lambda)$ is continuous on $[0, 1]$ and (14) holds true with $p,\ \psi$ replaced by $p_1,\ \psi_1$. The transversality condition becomes then

$$(p_1(0), -(p_1(1) + \psi_1(1))) \in \lambda\partial\varphi(z(0), z(1)) + N_{K_1}(z(0), z(1)) + N_K(z(0)) \times N_K(z(1)).$$

When $K_1 = Q_1 \times Q_2$ for some closed subsets $Q_i$ of $\mathbb{R}^n$, then, under appropriate assumptions,

$$N_{Q_1\times Q_2}(z(0), z(1)) + N_K(z(0)) \times N_K(z(1)) = N_{Q_1\cap K}(z(0)) \times N_{Q_2\cap K}(z(1)))$$

(see for instance [2]) and so the transversality condition may be written as

$$(p_1(0), -(p_1(1) + \psi_1(1))) \in \lambda\partial\varphi(z(0), z(1)) + N_{Q_1\cap K}(z(0)) \times N_{Q_2\cap K}(z(1))).$$

In this way Theorem 3.3 may be linked to the "jump and transversality conditions" derived as part of necessary conditions in [1, Theorem 1]. In [1] however the authors used in the transversality condition cones of limiting normals to $Q_1 \cap K$ and to $Q_2 \cap K$ instead of Clarke's normal cones. We would like to underline here that Theorem 3.3 concerns any triple $(\lambda, p, \psi)$ satisfying the maximum principle, while in [1, Theorem 1] it is shown that there exists such a triple. □

**Theorem 3.4.** *Assume that $(z, \bar{u})$ satisfies the normal constrained maximum principle with some $p$, $\psi$, that $z$ is Lipschitz, $K$ is sleek, $graph(F)$ is closed, $H$ is continuous on $graph(z) \times \mathbb{R}^n$ and $H(t, z(t), \cdot)$ is differentiable for all $t \in [0,1]$. Then $z \in C^1([0,1])$, the mapping*

$$(0,1) \ni t \mapsto \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t))$$

*is continuous and $z'(t) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t))$ for every $t \in (0,1)$. Furthermore*

$$\langle \psi(t) - \psi(t-), z'(t) \rangle = 0 \ \ \forall\, t \in (0,1), \ \ \langle \psi(0+), z'(0) \rangle \le 0, \ \ \langle \psi(1) - \psi(1-), z'(1) \rangle \ge 0$$

*and for a measurable function $u(t) \in U(t)$ such that $u = \bar{u}$ almost everywhere*

$$z'(t) = f(t, z(t), u(t)) \ \ \forall \ \ t \in [0,1],$$

$$H(t, z(t), p(t) + \psi(t+)) = \langle p(t) + \psi(t+), f(t, z(t), u(t)) \rangle - L(t, z(t), u(t)) \ \ \forall \ \ t \in [0,1),$$

$$H(1, z(1), p(1) + \psi(1-)) = \langle p(1) + \psi(1-), f(1, z(1), u(1)) \rangle - L(1, z(1), u(1)).$$

*Furthermore,*

$$\frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t-)) \ \ \forall \ \ t \in (0,1). \tag{18}$$

**Remark.** We underline that we do not assume continuity of $\frac{\partial H}{\partial p}$ with respect to $t$ and $x$ in the above theorem. □

**Proof.** Define the subset $\mathcal{D} \subset [0,1]$ of full measure by

$$\mathcal{D} = \{s \in [0,1] \mid z'(s) = f(s, z(s), \bar{u}(s)) \text{ and (14) holds true}\}$$

and fix $0 \le t \le 1$. We claim that there exist $w, \ v \in U(t)$ such that

$$H(t, z(t), p(t) + \psi(t-)) = \langle p(t) + \psi(t-), f(t, z(t), w) \rangle - L(t, z(t), w) \ \text{ if } \ t > 0,$$

$$H(t, z(t), p(t) + \psi(t+)) = \langle p(t) + \psi(t+), f(t, z(t), v) \rangle - L(t, z(t), v) \ \text{ if } \ t < 1$$

and that $f(t, z(t), w) = f(t, z(t), v)$, $L(t, z(t), w) = L(t, z(t), v)$ whenever $t \in (0,1)$.

Indeed assume first that $t > 0$. Let $\mathcal{D} \ni t_i \mapsto t-$ be such that $z'(t_i)$ converge to some $\zeta$. Then, by continuity of $H$ and closedness of $graph(F)$, for some $w \in U(t)$, $\zeta = f(t, z(t), w)$ and

$$H(t, z(t), p(t) + \psi(t-)) = \langle p(t) + \psi(t-), f(t, z(t), w) \rangle - L(t, z(t), w).$$

Since $H(t, z(t), \cdot)$ is differentiable, we get $\frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t-)) = f(t, z(t), w)$. Consequently,

$$\text{Limsup}_{s \to t-,\, s \in \mathcal{D}} \{z'(s)\} = \left\{ \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t-)) \right\}. \tag{19}$$

According to Lemma 3.2 applied with $A = [0,1] \backslash \mathcal{D}$,

$$\langle \psi(t) - \psi(t-), f(t, z(t), w) \rangle \ge 0.$$

Similarly if $t < 1$ and $\mathcal{D} \ni t_i \mapsto t+$ are such that $z'(t_i)$ converge to some $\eta$, then for some $v \in U(t)$, $f(t, z(t), v) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t+))$ and

$$H(t, z(t), p(t) + \psi(t+)) = \langle p(t) + \psi(t+), f(t, z(t), v)\rangle - L(t, z(t), v).$$

Consequently,

$$\text{Limsup}_{s \to t+,\, s \in \mathcal{D}} \{z'(s)\} = \left\{ \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t+)) \right\}. \tag{20}$$

By Theorem 3.3, if $0 < t < 1$, then

$$\begin{aligned} & H(t, z(t), p(t) + \psi(t)) = H(t, z(t), p(t) + \psi(t-)) \\ = \; & \langle p(t) + \psi(t), f(t, z(t), w)\rangle - \langle \psi(t) - \psi(t-), f(t, z(t), w)\rangle - L(t, z(t), w) \\ \leq \; & H(t, z(t), p(t) + \psi(t)). \end{aligned}$$

This implies that $\langle \psi(t) - \psi(t-), f(t, z(t), w)\rangle = 0$ and

$$H(t, z(t), p(t) + \psi(t)) = \langle p(t) + \psi(t), f(t, z(t), w)\rangle - L(t, z(t), w).$$

Since $\psi(t+) = \psi(t)$ for $t \in (0, 1)$, we deduce from (7) that $f(t, z(t), w) = f(t, z(t), v)$, $L(t, z(t), w) = L(t, z(t), v)$ and (18) follows.

By [4], $\partial z(t) = co\,\text{Limsup}_{s \to t,\, s \in \mathcal{D}}\{z'(s)\}$. So (18), (19) and (20) imply that $\partial^* z(t)$ is a singleton for all $t \in [0, 1]$. Hence $z$ is differentiable and $z'$ is continuous on $[0, 1]$. Let $V(t) \subset U(t)$ be such that for every $u \in V(t)$,

$$t \in [0, 1) \implies H(t, z(t), p(t) + \psi(t+)) = \langle p(t) + \psi(t+), f(t, z(t), u)\rangle - L(t, z(t), u)$$

and

$$t = 1 \implies H(1, z(1), p(1) + \psi(1-)) = \langle p(1) + \psi(1-), f(1, z(1), u)\rangle - L(1, z(1), u).$$

Then $V$ is measurable and has closed nonempty images. Consider a measurable selection $u(t) \in V(t)$ for all $t \in [0, 1]$ such that $u = \bar{u}$ a.e. Then $\frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t+)) = f(t, z(t), u(t))$ for all $t \in [0, 1)$ and $z'(t) = f(t, z(t), u(t))$ for all $t \in [0, 1]$, $\langle \psi(t) - \psi(t-), z'(t)\rangle = 0$ for every $t \in (0, 1)$ and, since $z([0, 1]) \subset K$, by (15), $\langle \psi(0+), z'(0)\rangle \leq 0$, $\langle \psi(1) - \psi(1-), z'(1)\rangle \geq 0$. □

Theorems 3.3, 3.4 do not exclude discontinuity of $\psi$. Still Theorem 3.4 implies $C^1-$regularity of an optimal solution. It can be also used to deduce continuity of $\psi$ under some additional assumptions on the Hamiltonian $H$.

**Corollary 3.5.** *Under all the assumptions of Theorem* 3.4, *suppose in addition that for every* $t \in (0, 1)$ *and* $p,\ q \in I\!\!R^n$,

$$\left( p - q \in N_K(z(t)),\ H(t, z(t), p) = H(t, z(t), q),\ \frac{\partial H}{\partial p}(t, z(t), p) = \frac{\partial H}{\partial p}(t, z(t), q) = z'(t) \right) \\ \implies p = q.$$

*Then $\psi$ is continuous in $(0,1)$.*

*In particular, if $\frac{\partial H}{\partial p}(t,z(t),\cdot)$ is strictly monotone in the directions normal to $K$ at $z(t)$: for every $t \in (0,1)$ and all $p \neq q \in \mathbb{R}^n$ such that $p-q \in N_K(z(t))$ and $H(t,z(t),p) = H(t,z(t),q)$ we have*

$$\left\langle \frac{\partial H}{\partial p}(t,z(t),p) - \frac{\partial H}{\partial p}(t,z(t),q), p-q \right\rangle > 0,$$

*then $\psi$ is continuous in $(0,1)$.*

**Remark.** Naturally, in the above corollary the points of interest are those where $z(t) \in \partial K$, since $N_K(x) = \{0\}$ for all $x \in Int(K)$. Let $u(t,p) \in U(t)$ denote a maximizer in (5) for $x = z(t)$. In terms of mappings $f$, $L$ the assumption of the above corollary means that if $z(t) \in \partial K$ and for some $p$, $q \in \mathbb{R}^n$ such that $p-q \in N_K(z(t))$ we have $(L,f)(t,z(t),u(t,p)) = (L,f)(t,z(t),u(t,q))$ and $z'(t) = f(t,z(t),u(t,p))$, then $p=q$.

If $H(t,z(t),\cdot)$ is twice continuously differentiable and $\frac{\partial^2 H}{\partial p^2}(t,z(t),\cdot) > 0$, then $\frac{\partial H}{\partial p}(t,z(t),\cdot)$ satisfies the assumption of Corollary 3.5. □

**Proof of Corollary 3.5.** By Theorem 3.4 for all $t \in (0,1)$

$$\frac{\partial H}{\partial p}(t,z(t),p(t)+\psi(t)) = \frac{\partial H}{\partial p}(t,z(t),p(t)+\psi(t-)) = z'(t),$$

and

$$H(t,z(t),p(t)+\psi(t)) = H(t,z(t),p(t)+\psi(t-)).$$

Since $\psi(t)-\psi(t-) \in N_K(z(t))$, the proof follows. □

**Example 3.6.** Let $K$ be a closed sleek subset of $\mathbb{R}^n$, $K_1$ be a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$, $d : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, $g : [0,1] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be continuous, $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Assume that $L(t,x,\cdot)$ is convex and satisfies the Tonelli condition

$$L(t,x,u) \geq \Theta(|u|), \ \ \forall\, t \in [0,1], \ \forall\, x \in K,$$

where $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ has a superlinear growth.

Set $U(t) = \mathbb{R}^m$, $f(t,x,u) = d(t,x) + g(t,x)u$ and consider the associated constrained Bolza problem (1)-(3). The Hamiltonian $H$ is defined by

$$H(t,x,p) = \max_{u\in R^m}(\langle p, d(t,x) + g(t,x)u\rangle - L(t,x,u)).$$

Let $(z,\bar{u})$ be a trajectory/control pair of (1), (2). Assume that for all $t \in [0,1]$, $L(t,z(t),\cdot)$ is differentiable and that $\frac{\partial L}{\partial u}(t,z(t),\cdot)$ is monotone in the following sense:

$$u \neq v \in \mathbb{R}^m \ \ \Rightarrow \ \ \left\langle \frac{\partial L}{\partial u}(t,z(t),u) - \frac{\partial L}{\partial u}(t,z(t),v), u-v \right\rangle > 0.$$

We claim that $H$ and $\frac{\partial H}{\partial p}$ are continuous on $graph(z) \times \mathbb{R}^n$. Indeed, fix $t \in [0,1]$ and $p_1$, $p_2 \in \mathbb{R}^n$. By Tonelli's condition, there exist $u_i \in \mathbb{R}^m$ such that

$$H(t,z(t),p_i) = \langle p_i, d(t,z(t)) + g(t,z(t))u_i\rangle - L(t,z(t),u_i),$$

$i = 1, 2$. Then $\frac{\partial L}{\partial u}(t, z(t), u_i) = g(t, z(t))^* p_i$ and

$$\langle p_1 - p_2, g(t, z(t))(u_1 - u_2)\rangle = \left\langle \frac{\partial L}{\partial u}(t, z(t), u_1) - \frac{\partial L}{\partial u}(t, z(t), u_2), u_1 - u_2 \right\rangle.$$

By the monotonicity assumption this implies that $u_1 = u_2$ whenever $p_1 = p_2$. Hence for every $p \in I\!R^n$ there exists exactly one $u(t, p)$ satisfying $H(t, z(t), p) = \langle p, d(t, z(t)) + g(t, z(t))u(t, p)\rangle - L(t, z(t), u(t, p))$. From (7) we deduce that $H(t, z(t), \cdot)$ is differentiable and $\frac{\partial H}{\partial p}(t, z(t), p) = d(t, z(t)) + g(t, z(t))u(t, p)$ for all $t \in [0, 1]$ and $p \in I\!R^n$. Using the Tonelli condition and continuity of $d$, $g$, $L$, it is not difficult to show that $u(\cdot, \cdot)$ is continuous on $[0, 1] \times I\!R^n$. So also $H$ and $\frac{\partial H}{\partial p}$ are continuous on $graph(z) \times I\!R^n$.

Assume in addition that for every $t \in [0, 1]$ such that $z(t) \in \partial K$ we have

$$N_K(z(t)) \cap kernel(g(t, z(t))^*) = \{0\}. \tag{21}$$

Notice that this implies that for a constant $\rho > 0$ and all $t \in [0, 1]$ and $n \in N_K(z(t))$, $|g(t, z(t))^* n| \geq \rho |n|$. We claim that for every $t \in [0, 1]$ and all $p_1 \neq p_2 \in I\!R^n$ such that $p_1 - p_2 \in N_K(z(t))$ we have $u(t, p_1) \neq u(t, p_2)$. Indeed, assume that for some $p_i$, $i = 1, 2$, $u(t, p_1) = u(t, p_2)$ and $p_1 - p_2 \in N_K(z(t))$. Since $u(t, p_i)$ maximizes the function $I\!R^m \ni u \mapsto \langle p_i, g(t, z(t))u\rangle - L(t, z(t), u)$, we deduce that $g(t, z(t))^*(p_1 - p_2) = 0$ and, by (21), $p_1 = p_2$ proving our claim.

We next show that the assumption of Corollary 3.5 holds true. Indeed for all $p \neq q \in I\!R^n$ such that $p - q \in N_K(z(t))$

$$\left\langle \frac{\partial H}{\partial p}(t, z(t), p) - \frac{\partial H}{\partial p}(t, z(t), q), p - q \right\rangle = \langle g(t, z(t))(u(t, p) - u(t, q)), p - q\rangle$$
$$= \left\langle u(t, p) - u(t, q), \frac{\partial L}{\partial u}(t, z(t), u(t, p)) - \frac{\partial L}{\partial u}(t, z(t), u(t, q)) \right\rangle > 0.$$

Therefore, if $(z, \bar{u})$ satisfies the normal constrained maximum principle with some $p$, $\psi$, then, by Corollary 3.5, $\psi$ is continuous on $(0, 1)$.

If moreover

$$\begin{cases} \forall\, r > 0,\ \exists\, c_r,\ k_r > 0,\ \ \forall\, t \in [0, 1],\ \dfrac{\partial L}{\partial u}(t, z(t), \cdot) \ \text{is}\ c_r - \text{Lipschitz on}\ B(0, r) \\ \forall\, u, v \in rB,\ \left\langle \dfrac{\partial L}{\partial u}(t, z(t), u) - \dfrac{\partial L}{\partial u}(t, z(t), v), u - v \right\rangle \geq k_r |u - v|^2, \end{cases} \tag{22}$$

then a stronger monotonicity condition holds true: for every $r > 0$, there exists a constant $l_r > 0$

$$\forall\, p,\ q \in rB \text{ with } p - q \in N_K(z(t)),\ \left\langle \frac{\partial H}{\partial p}(t, z(t), p) - \frac{\partial H}{\partial p}(t, z(t), q), p - q \right\rangle \geq l_r |p - q|^2. \tag{23}$$

Indeed, by the Tonelli condition, for all $r > 0$ there exists $R > 0$ such that for every

$p \in rB,\ |u(t,p)| \le R$. Hence for all $p,\ q \in rB$,

$$\begin{aligned} & \left\langle \frac{\partial H}{\partial p}(t,z(t),p) - \frac{\partial H}{\partial p}(t,z(t),q), p-q \right\rangle \\ \ge\ & \left\langle u(t,p)-u(t,q), \frac{\partial L}{\partial u}(t,z(t),u(t,p)) - \frac{\partial L}{\partial u}(t,z(t),u(t,q)) \right\rangle \\ \ge\ & k_R|u(t,p)-u(t,q)|^2 \ge \frac{k_R}{c_R^2}|\frac{\partial L}{\partial u}(t,z(t),u(t,p)) - \frac{\partial L}{\partial u}(t,z(t),u(t,q))|^2 \\ =\ & \frac{k_R}{c_R^2}|g(t,z(t))^*(p-q)|^2 \ge \frac{\rho k_R}{c_R^2}|p-q|^2. \end{aligned}$$

Setting $l_r = \frac{\rho k_R}{c_R^2}$ we obtain the strong monotonicity condition (23).

**Example 3.7.** This example corresponds to the control system studied in [18], where the authors considered a different set of constraints (time dependent inequality constraints).

Let $d,\ g,\ L,\ K,\ K_1,\ \varphi$ be as in Example 3.6, but this time the set-valued map $t \rightsquigarrow U(t)$ is lower semicontinuous with closed graph and convex images. The Hamiltonian is given by

$$H(t,x,p) = \max_{u \in U(t)} (\langle p, d(t,x) + g(t,x)u \rangle - L(t,x,u)).$$

Assume that a trajectory/control pair $(z,\bar{u})$ satisfies the normal constrained maximum principle with some $p,\ \psi$ and that $\frac{\partial L}{\partial u}(t,z(t),\cdot)$ satisfies the monotonicity assumption of Example 3.6 on $I\!R^m$.

We claim that for all $p \in I\!R^n$ there exists at most one element $u(t,p) \in U(t)$ satisfying $H(t,z(t),p) = \langle p, d(t,z(t)) + g(t,z(t))u(t,p)\rangle - L(t,z(t),u(t,p))$.

Indeed fix $p,\ q \in I\!R^m$ and observe that

$$\left\langle \frac{\partial L}{\partial u}(t,z(t),u(t,p)), w \right\rangle \ge \langle g(t,z(t))^* p, w\rangle, \ \ \forall\, w \in T_{U(t)}(u(t,p))$$

and

$$\left\langle \frac{\partial L}{\partial u}(t,z(t),u(t,q)), w \right\rangle \ge \langle g(t,z(t))^* q, w\rangle, \ \ \forall\, w \in T_{U(t)}(u(t,q)).$$

Thus for some $\eta_p \in N_{U(t)}(u(t,p)),\ \eta_q \in N_{U(t)}(u(t,q))$ we have

$$g(t,z(t))^* p = \frac{\partial L}{\partial u}(t,z(t),u(t,p)) + \eta_p$$

and

$$g(t,z(t))^* q = \frac{\partial L}{\partial u}(t,z(t),u(t,q)) + \eta_q$$

Consequently,

$$\begin{aligned} & \left\langle \frac{\partial H}{\partial p}(t,z(t),p) - \frac{\partial H}{\partial p}(t,z(t),q), p-q \right\rangle = \langle p-q, g(t,z(t))(u(t,p)-u(t,q))\rangle \\ =\ & \langle \eta_p - \eta_q, u(t,p)-u(t,q)\rangle + \langle \frac{\partial L}{\partial u}(t,z(t),u(t,p)) - \frac{\partial L}{\partial u}(t,z(t),u(t,q)), u(t,p)-u(t,q)\rangle \\ \ge\ & \langle \frac{\partial L}{\partial u}(t,z(t),u(t,p)) - \frac{\partial L}{\partial u}(t,z(t),u(t,q)), u(t,p)-u(t,q)\rangle, \end{aligned}$$

because $\langle \eta_p, u(t,p)-u(t,q)\rangle \geq 0$ and $\langle -\eta_q, u(t,p)-u(t,q)\rangle \geq 0$. In particular, if $p=q$ then $u(t,p)=u(t,q)$. This and Tonelli's condition imply that for every $p \in I\!\!R^n$ there exists exactly one $u(t,p)$ satisfying $H(t,z(t),p) = \langle p, d(t,z(t))+g(t,z(t))u(t,p)\rangle - L(t,z(t),u(t,p))$. From (7) we deduce that $H(t,z(t),\cdot)$ is differentiable and $\frac{\partial H}{\partial p}(t,z(t),p) = d(t,z(t)) + g(t,z(t))u(t,p)$ for all $t \in [0,1]$ and $p \in I\!\!R^n$. Notice that the last inequality also implies that if $g(t,z(t))u(t,p) = g(t,z(t))u(t,q)$, then $u(t,p) = u(t,q)$. Using the Tonelli condition, regularity of $U$ and continuity of $d$, $g$, $L$, it is not difficult to show that $u(\cdot,\cdot)$ is continuous on $[0,1]\times I\!\!R^n$. So also $H$ and $\frac{\partial H}{\partial p}$ are continuous on $graph(z)\times I\!\!R^n$. Since for almost all $t\in[0,1]$, $z'(t) = \frac{\partial H}{\partial p}(t,z(t),p(t)+\psi(t))$, from continuity of $\frac{\partial H}{\partial p}$ on $graph(z)\times I\!\!R^n$ and boundedness of $p$, $\psi$, $z$ we deduce that $z' \in L^\infty$ and so $z$ is Lipschitz continuous. By Theorem 3.4, $z \in C^1$ and $g(t,z(t))u(t,p(t)+\psi(t)) = g(t,z(t))u(t,p(t)+\psi(t-))$ for all $t \in (0,1)$. Therefore, $u(t,p(t)+\psi(t)) = u(t,p(t)+\psi(t-))$ for all $t\in(0,1)$. So $u(\cdot,p(\cdot)+\psi(\cdot))$ is continuous on $(0,1)$.

Set $u_0(0) = u(0,p(0)+\psi(0+))$, $u_0(1) = u(1,p(1)+\psi(1-))$ and $u_0(t) = u(t,p(t)+\psi(t))$ for all $t\in(0,1)$ ($u_0(\cdot)$ corresponds to the control $u(\cdot)$ from the statement of Theorem 3.4). Notice that if $u(t,p)\neq u(t,q)$ whenever $0\neq p-q\in N_K(z(t))$, then, by Corollary 3.5, $\psi$ continuous on $(0,1)$ .

Assume (21) and that for all $t\in[0,1]$

$$g(t,z(t))^*(N_K(z(t)))\cap span(N_{U(t)}(u_0(t))) = \{0\}. \tag{24}$$

In the difference with Example 3.6, the condition imposed on $N_K(z(t))$ depends on the control $u_0(t)$.

Assumptions (21) and (24) together are of the same nature as Hypothesis (H6) in [18]. To prove continuity of $\psi$ fix $t\in(0,1)$ such that $z(t)\in\partial K$.

Let $p,\ q \in I\!\!R^m$ be such that $p-q\in N_K(z(t))$ and let $\eta_p,\ \eta_q$ be as above. If $\frac{\partial H}{\partial p}(t,z(t),p) = \frac{\partial H}{\partial p}(t,z(t),q) = z'(t)$, then $g(t,z(t))u(t,p) = g(t,z(t))u(t,q) = g(t,z(t))u_0(t)$ and thus $u(t,p)=u(t,q)=u_0(t)$. This implies that

$$g(t,z(t))^*(p-q) = \eta_p - \eta_q \in span(N_{U(t)}(u_0(t))).$$

From (24) we deduce that $p=q$. Therefore, by Corollary 3.5, $\psi$ is continuous on $(0,1)$.

If moreover (22) is satisfied and for some $\varepsilon>0$ and for cones $C_\varepsilon(t),\ C_0(t)$ defined by

$$C_\varepsilon(t) := \cup_{u\in U(t)\cap B(u_0(t),\varepsilon)} N_{U(t)}(u),\ \ C_0(t) := N_{U(t)}(u_0(t))$$

we have

$$\alpha := \inf_{t\in[0,1]}\ \inf_{n\in N_K(z(t))\cap S^{n-1}} dist(g(t,z(t))^*n, C_\varepsilon(t)-C_0(t)) > 0, \tag{25}$$

then a strong monotonicity condition holds true. Namely for every $r>0$, there exists a constant $l_r>0$ such that for all $t\in[0,1]$ and $q\in rB$ satisfying $\frac{\partial H}{\partial p}(t,z(t),q) = z'(t)$

$$(p-q\in N_K(z(t))\cap\varepsilon B) \implies \left\langle \frac{\partial H}{\partial p}(t,z(t),p) - \frac{\partial H}{\partial p}(t,z(t),q), p-q\right\rangle \geq l_r|p-q|^2. \tag{26}$$

Indeed, for all $n \in N_K(z(t))$, $dist(g(t,z(t))^* n, \overline{C_\varepsilon(t) - C_0(t)}) \geq \alpha |n|$. Thus, for some $c_R > 0$, $k_R > 0$ depending only on $r$, $\varepsilon$ and for all $p, q \in (r+\varepsilon)B$,

$$\begin{aligned} & \left\langle \frac{\partial H}{\partial p}(t,z(t),p) - \frac{\partial H}{\partial p}(t,z(t),q), p-q \right\rangle \\ \geq\ & \langle \frac{\partial L}{\partial u}(t,z(t),u(t,p)) - \frac{\partial L}{\partial u}(t,z(t),u(t,q)), u(t,p) - u(t,q) \rangle \\ \geq\ & k_R |u(t,p) - u(t,q)|^2 \geq \frac{k_R}{c_R^2} |\frac{\partial L}{\partial u}(t,z(t),u(t,p)) - \frac{\partial L}{\partial u}(t,z(t),u(t,q))|^2 \\ =\ & \frac{k_R}{c_R^2} |g(t,z(t))^*(p-q) - \eta_p + \eta_q|^2. \end{aligned}$$

Hence, if in addition $|q| \leq r$, $\frac{\partial H}{\partial p}(t,z(t),q) = z'(t)$ and $p - q \in N_K(z(t)) \cap \varepsilon B$, we get

$$\left\langle \frac{\partial H}{\partial p}(t,z(t),p) - \frac{\partial H}{\partial p}(t,z(t),q), p-q \right\rangle \geq \frac{\alpha^2 k_R}{c_R^2} |p-q|^2.$$

## 4. Absolute Continuity of Adjoint States

Let $Q \subset I\!R^n$ be a closed set. We say that its boundary $\partial Q \in C^{1,1}_{loc}$ if for every $x \in \partial Q$ there exists $\delta > 0$ such that the signed distance defined by

$$h(x) = \begin{cases} -dist(x, \partial Q) & \forall\, x \in Q \\ \ \ dist(x, \partial Q) & \text{otherwise} \end{cases}$$

is of class $C^{1,1}$ on $x + \delta B$. By [7] this is equivalent to the assumption: $\partial Q$ is a $C^{1,1}-$manifold with a positive reach. In this section we assume that the set of state constraints $K$ satisfies the following requirements:

$$K = \cap_{j=1}^m K_j,\ K_j \text{ is closed},\ \partial K_j \in C^{1,1}_{loc} \tag{27}$$

and

$$0 \notin co\{n_j(x) \mid j \in I(x)\},\ \ \forall\, x \in \partial K, \tag{28}$$

where $n_j(x)$ denotes the outward unit normal to $K_j$ at $x \in \partial K_j$ and $I(x)$ denotes the set of all indices that are active at $x$, i.e. $j \in I(x)$ if and only if $x \in \partial K_j$. Notice that (28) implies that

$$\forall\, r > 0,\ \exists\, \rho_r > 0 \ \text{ such that } \ \min_{v \in B} \max_{j \in I(x)} \langle n_j(x), v \rangle \leq -\rho_r \quad \forall\, x \in \partial K \cap rB. \tag{29}$$

Thus assumption (28) and [2, Chapter 4] imply that $K$ is sleek, $T_K(x) = \bigcap_{j=1}^m T_{K_j}(x)$ and for every $j \in I(x)$, $T_{K_j}(x) = \{v \mid \langle n_j(x), v\rangle \leq 0\}$, while for every $j \notin I(x)$, $T_{K_j}(x) = I\!R^n$ and

$$N_K(x) = \Sigma_{j \in I(x)} N_{K_j}(x) = \Sigma_{j=1}^m N_{K_j}(x). \tag{30}$$

Let $(z, \bar{u})$ be trajectory/control pair of (1), (2) and $F$ be defined by (17).

**Proposition 4.1.** *Assume* (27), (28) *and let* $r = \|z\|_\infty$ *and* $\rho = \rho_r$ *be as in* (29). *Let* $t \in [0,1]$, $\beta_j \geq 0$, $j = 1, ..., m$ *and* $n = \Sigma_{j \in I(z(t))} \beta_j n_j(z(t))$. *Then* $|n| \geq \rho \Sigma_{j \in I(z(t))} \beta_j$.

**Proof.** By (29) there exists $v(t) \in B$ such that $|n| \geq \langle n, -v(t)\rangle \geq \rho\Sigma_{j\in I(z(t))}\beta_j$. □

**Theorem 4.2.** *Assume that graph(F) is closed and let $(z, \bar{u})$ satisfy the normal constrained maximum principle with some $p$, $\psi$ and state constraints $K$ as in (27), (28). Define $\Gamma := graph(z) \times \mathbb{R}^n$ and assume that $H$ is continuous on $\Gamma$, $\frac{\partial H}{\partial p}$ is locally Lipschitz on $\Gamma$ and is as in Corollary 3.5, and for every $r > 0$ there exist $k_r > 0$, $\bar{\varepsilon} > 0$ such that for all $t \in [0,1]$ and $q \in rB$ satisfying $\frac{\partial H}{\partial p}(t, z(t), q) = z'(t)$ we have*

$$(p - q \in N_K(z(t)) \cap \bar{\varepsilon}B) \implies \left\langle \frac{\partial H}{\partial p}(t, z(t), p) - \frac{\partial H}{\partial p}(t, z(t), q), p - q \right\rangle \geq k_r|p - q|^2. \quad (31)$$

*Then $\psi$ is absolutely continuous on $(0,1)$ and $z'$ is absolutely continuous on $[0,1]$. Furthermore, if $p(\cdot)$ is Lipschitz, then $\psi$ is Lipschitz on $(0,1)$ and $z'$ is Lipschitz on $[0,1]$.*

**Remark.** *i*) If $p$ satisfies the adjoint equation (4), then it is Lipschitz provided the mapping $(L'_x, f'_x)(\cdot, z(\cdot), \bar{u}(\cdot)) \in L^\infty(0,1)$. Alternatively it may be assumed that the Hamiltonian $H(t, \cdot, q)$ is locally Lipschitz on a neighborhood of *graph*$(z)$ in the following sense: for some $\varepsilon > 0$ and for every $r > 0$ there exists $c_r > 0$ such that for all $t \in [0,1]$, $q \in rB$, $H(t, \cdot, q)$ is $c_r-$Lipschitz on $z(t) + \varepsilon B$. Then it is not difficult to check that any $p$ solving (4) verifies for almost all $t \in [0,1]$,

$$-p'(t) \in \partial_- H_x(t, z(t), p(t) + \psi(t)) \subset c_r B,$$

where $\partial_- H_x(t, x, q)$ denotes the subdifferential of $H(t, \cdot, q)$ at $x$ (see for instance [2] for the definition of subdifferential). Setting $r = \|p + \psi\|_\infty$ we deduce that $\|p'\|_\infty \leq c_r$.

*ii*) For the Bolza problem considered in Example 3.7 of previous section requirements imposed at the end guarantee that assumption (31) is satisfied. □

**Proof.** Since $(z, \bar{u})$ satisfies (14), $z'(t) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t))$ a.e. On the other hand $\frac{\partial H}{\partial p}$ is continuous on $\Gamma$ and $\psi$ is bounded. Thus $z' \in L^\infty$ and so $z$ is Lipschitz. Let $c$ denote a Lipschitz constant of $z$ and $c_1 = \|z\|_\infty$. From Theorems 3.3 and 3.4 we know that $z \in C^1$, that for all $t \in (0,1)$

$$H(t, z(t), p(t) + \psi(t)) = H(t, z(t), p(t) + \psi(t-))$$

and from Corollary 3.5 that $\psi$ is continuous on $(0,1)$. Define $\psi_1 \in C([0,1])$ by $\psi_1 = \psi$ on $(0,1)$, $\psi_1(0) = \psi(0+)$, $\psi_1(1) = \psi(1-)$. Then $\psi_1$ is uniformly continuous on $[0,1]$ and thus $\psi$ is uniformly continuous on $(0,1)$. Let $\bar{\delta} > 0$ be so that $|\psi(t) - \psi(s)| < \frac{\bar{\varepsilon}}{2}$ whenever $|t - s| < \bar{\delta}$, $t, s \in (0,1)$. Since $\partial K_j \in C^{1,1}_{loc}$ and $z$ is Lipschitz

$$\exists\, L > 0,\ \forall\, s,\ t \in [0,1],\ \forall\, j \in I(z(s)) \cap I(z(t)),\ \ |n_j(z(s)) - n_j(z(t))| \leq L|s - t|. \quad (32)$$

By (15) $\psi(t) = \int_{[0,t]} \nu(s) d\mu(s)$, where $\nu(s) \in N_K(z(s)) \cap B$ $\mu-$a.e. Let $\mathcal{B}$ denote the $\sigma-$algebra of Borel subsets of $[0,1]$ and $\Sigma$ be the smallest $\sigma-$algebra containing $\mathcal{B}$ and all subsets of Borel sets of zero $\mu-$measure. If $A \in \mathcal{B}$ and $\mu(A) = 0$, then for every $A' \subset A$ set $\mu(A') = 0$. Then $\mu$ has a unique extension on $\Sigma$ denoted again by $\mu$ and $([0,1], \Sigma, \mu)$ is a complete finite measure space.

By (30), $\nu(s) \in \Sigma_{j=1}^m N_{K_j}(z(s))$ $\mu-$ a.e. Since $N_{K_j}(z(s)) = \mathbb{R}_+ n_j(z(s))$ for all $j \in I(z(s))$ and $N_{K_j}(z(s)) = 0$ otherwise, we deduce that the set-valued map $s \rightsquigarrow N_{K_j}(z(s))$ is Borel measurable and thus it is also $\mu-$measurable. By the Filippov theorem (see for instance [2, Chapter 8]), there exist $\mu-$measurable selections $\nu_j(s) \in N_{K_j}(z(s))$ for all $s \in [0,1]$ such that

$$\nu(s) = \Sigma_{j=1}^m \nu_j(s) \;\; \mu - a.e.$$

From (28) we deduce that there exist $\rho > 0$, $v(x) \in B$ such that for all $x \in \partial K \cap c_1 B$ and every $j \in I(x)$ we have $\langle n_j(x), v(x)\rangle \leq -\rho$. Consequently $B(v(x), \rho/2) \in Int(T_{K_j}(x))$ for every $j \in I(x)$. By Corollary 2.2 there exists $M > 0$ depending only on $\rho$ such that $\lambda_j(s) := |\nu_j(s)| \leq M \;\mu-$a.e. Define

$$\theta_j(s) = \begin{cases} n_j(z(s)) & \text{if } j \in I(z(s)) \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check that $\theta_j$ is Borel measurable for every $j$. Hence it is also $\mu-$measurable. Then

$$\nu(\tau) = \Sigma_{j=1}^m \lambda_j(\tau)\theta_j(\tau), \; 0 \leq \lambda_j(\tau) \leq M \;\; \mu - \text{a.e.} \tag{33}$$

and if $\lambda_j(\tau) \neq 0$ then $\theta_j(\tau) = n_j(z(\tau))$. Notice that $\theta_j$ is $L-$Lipschitz on the set $\{s \mid z(s) \in \partial K_j\}$. Our aim is to show that there exists $d > 0$ such that for all $0 < t < s < 1$ with $s - t < \delta := \min\{\bar{\delta}, \bar{\varepsilon}/2(L+1)(mM\mu([0,1]) + 1\}$

$$|\psi(s) - \psi(t)| \leq d \int_t^s (1 + |p'(\tau)|)d\tau. \tag{34}$$

We first prove this inequality for a particular choice of $t, s$. Let $0 < t < s < 1$ be such that

$$s - t < \delta, \;\; I(z(s)) \subset I(z(t)) \;\text{ and if for some }\; \tau \in (t,s), \;\; j \in I(z(\tau)), \;\text{ then }\; j \in I(z(s)). \tag{35}$$

Points $0 < t < s < 1$ satisfying (35) form a subset of compatible end points introduced in [12] and used in [18], because we imposed an additional requirement $I(z(s)) \subset I(z(t))$. Set $\Delta(t,s) := \Sigma_{j=1}^m \int_{(t,s]} \lambda_j(\tau)d\mu(\tau)$ and observe that $\Delta(t,s) = \Sigma_{j \in I(z(t))} \int_{(t,s]} \lambda_j(\tau)d\mu(\tau)$ and

$$\Delta(t,s) \leq mM\mu([0,1]).$$

Furthermore for every $j \in I(z(s))$

$$\left| \int_{(t,s]} \lambda_j(\tau)d\mu(\tau)\theta_j(s) - \int_{(t,s]} \lambda_j(\tau)\theta_j(\tau)d\mu(\tau) \right| \leq L(s-t) \int_{(t,s]} \lambda_j(\tau)d\mu(\tau).$$

By Lemma 3.2 if $z(\tau) \in \partial K_j$ for some $\tau \in (0,1)$, then $\langle \theta_j(\tau), z'(\tau)\rangle = 0$.

By (35), $\int_{(t,s]} \lambda_j(\tau)d\mu(\tau) \,\langle \theta_j(s), z'(s)\rangle = 0$ and therefore

$$\left\langle \int_{(t,s]} \lambda_j(\tau)\theta_j(\tau)d\mu(\tau), z'(s) \right\rangle \leq Lc(s-t) \int_{(t,s]} \lambda_j(\tau)d\mu(\tau).$$

Summing up and using (33) we get

$$\langle \psi(s) - \psi(t), z'(s)\rangle \leq Lc(s-t)\Delta(t,s).$$

On the other hand $z'(s) = \frac{\partial H}{\partial p}(s, z(s), p(s)+\psi(s))$. Thus, for a constant $C$ depending only on Lipschitz constant of $z$ and Lipschitz constant of $\frac{\partial H}{\partial p}$ on $graph(z) \times (\|p\|_\infty + \|\psi\|_\infty)B$, we have

$$\left| z'(s) - \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(s)) \right| \le C(s - t + |p(t) - p(s)|).$$

Consequently, for $C_1 := Lc + C$

$$\left\langle \psi(s) - \psi(t), \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(s)) \right\rangle \le C_1 \Delta(t, s) \int_t^s (1 + |p'(\tau)|) d\tau. \tag{36}$$

Notice that for every $j \in I(z(t))$

$$\left| \int_{(t,s]} \lambda_j(\tau)\theta_j(\tau) d\mu(\tau) - \int_{(t,s]} \lambda_j(\tau) d\mu(\tau) \theta_j(t) \right| \le L(s-t) \int_{(t,s]} \lambda_j(\tau) d\mu(\tau). \tag{37}$$

Thus

$$\left| \psi(s) - \psi(t) - \sum_{j=1}^{m} \theta_j(t) \int_{(t,s]} \lambda_j(\tau) d\mu(\tau) \right| \le L(s-t)\Delta(t, s) \tag{38}$$

and

$$\begin{aligned} &\left| \left\langle \int_{(t,s]} \lambda_j(\tau)\theta_j(\tau) d\mu(\tau), z'(t) \right\rangle - \int_{(t,s]} \lambda_j(\tau) d\mu(\tau) \left\langle \theta_j(t), z'(t) \right\rangle \right| \\ \le \;\; & Lc(s-t) \int_{(t,s]} \lambda_j(\tau) d\mu(\tau). \end{aligned}$$

Since for every $j \in I(z(t))$, $\langle \theta_j(t), z'(t) \rangle = 0$ we proved that

$$\left\langle \int_{(t,s]} \lambda_j(\tau)\theta_j(\tau) d\mu(\tau), z'(t) \right\rangle \ge -Lc(s-t) \int_{(t,s]} \lambda_j(\tau) d\mu(\tau).$$

Summing up and using that $z'(t) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t))$ we obtain

$$\left\langle \psi(s) - \psi(t), \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) \right\rangle \ge -Lc(s-t)\Delta(t, s). \tag{39}$$

Then (36) and (39) imply that

$$\left\{ \begin{array}{l} \quad \langle \psi(s) - \psi(t), \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(s)) - \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) \rangle \\ \le 2C_1 \Delta(t, s) \int_t^s (1 + |p'(\tau)|) d\tau. \end{array} \right. \tag{40}$$

Setting $n(t, s) := \sum_{j=1}^m \theta_j(t) \int_{(t,s]} \lambda_j(\tau) d\mu(\tau) \in N_K(z(t))$ and using (31) and (38), we obtain that for some $L_1 > 0$, $k > 0$ independent from $t, s$

$$\begin{aligned} &\left\langle \psi(s) - \psi(t), \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(s)) - \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) \right\rangle \\ \ge \;\; & \left\langle n(t, s), \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t) + n(t, s)) - \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t)) \right\rangle \\ & \quad - L_1(s-t)\Delta(t, s) \\ \ge \;\; & k|n(t, s)|^2 - L_1(s-t)\Delta(t, s). \end{aligned} \tag{41}$$

By Proposition 4.1

$$|n(t,s)| \geq \rho\, \Sigma_{j\in I(z(t))} \int_{(t,s]} \lambda_j(\tau)d\mu(\tau) = \rho\Delta(t,s).$$

This and (40), (41) imply that

$$\Delta(t,s)^2 \leq \frac{1}{k\rho^2}(L_1 + 2C_1)\Delta(t,s)\int_t^s (1 + |p'(\tau)|)d\tau.$$

Setting $d = \frac{1}{k\rho^2}(L_1 + 2C_1)$ we obtain (34) for all $0 < t < s < 1$ satisfying (35), where $d$ is independent from $s, t$ .

To show (34) for all $0 < t < s < 1$ with $s - t < \delta$ we first prove the following claim.

**Claim.** For every $t \in (0,1)$ and $\varepsilon_0 > 0$ there exists $0 < \varepsilon < \varepsilon_0$ such that

$$|\psi(t+\varepsilon) - \psi(t)| \leq d\int_t^{t+\varepsilon} (1 + |p'(\tau)|)d\tau. \tag{42}$$

Denote by $\sharp I(z(t))$ the number of elements in $I(z(t))$. We prove this claim using an induction argument similar to the one of [18]. If $\sharp I(z(t)) = 0$, then for every $s > t$ sufficiently close to $t$, $z(s) \in int(K)$ and therefore $\psi$ is constant on a neighborhood of $t$. Assume that we already proved our claim for some integer $0 \leq k < m$ and all $t \in (0,1)$ with $\sharp I(z(t)) \leq k$. Let $t \in (0,1)$ be such that $\sharp I(z(t)) = k+1$. Since $z$ is continuous for all $\tau > t$ sufficiently close to $t$, $I(z(\tau)) \subset I(z(t))$. There are two cases to be considered.

*Case 1.* There exist $t_i \to t+$ such that $\sharp I(z(t_i)) = k+1$. Then for all large $i$, $I(z(t_i)) = I(z(t))$ and for all $\tau \in (t, t_i]$, $I(z(\tau)) \subset I(z(t_i))$. Then (35) holds true with $s = t_i$ and (42) follows with $\varepsilon = t_i - t < \varepsilon_0$.

*Case 2.* There exists $0 < \varepsilon < \varepsilon_0$ such that $t+\varepsilon < 1$ and $\sharp I(z(s)) \leq k$ for every $s \in (t, t+\varepsilon]$. Consider any $t_i \to t+$ and let

$$r_i = \sup\{s \in [t_i, t+\varepsilon] \mid |\psi(s) - \psi(t_i)| \leq d\int_{t_i}^s (1 + |p'(\tau)|)d\tau\}.$$

Since $\sharp I(z(t_i)) \leq k$ we have $r_i > t_i$. Then $|\psi(r_i) - \psi(t_i)| \leq d\int_{t_i}^{r_i}(1+|p'(\tau)|)d\tau$, by continuity of $\psi$. Assume for a moment that $r_i < t+\varepsilon$. Since $\sharp I(z(r_i)) \leq k$, by the induction, there exists $\varepsilon_1 > 0$ such that $r_i + \varepsilon_1 < t + \varepsilon$ and $|\psi(r_i + \varepsilon_1) - \psi(r_i)| \leq d\int_{r_i}^{r_i+\varepsilon_1}(1 + |p'(\tau)|)d\tau$. But then

$$|\psi(r_i + \varepsilon_1) - \psi(t_i)| \leq |\psi(r_i + \varepsilon_1) - \psi(r_i)| + |\psi(r_i) - \psi(t_i)| \leq d\int_{t_i}^{r_i+\varepsilon_1} (1 + |p'(\tau)|)d\tau$$

contradicting the choice of $r_i$. Thus $r_i = t+\varepsilon$ and $|\psi(t+\varepsilon) - \psi(t_i)| \leq d\int_{t_i}^{t+\varepsilon}(1+|p'(\tau)|)d\tau$. Taking the limit when $i \to \infty$ we get (42) and our claim follows also for $\sharp I(z(t)) = k+1$.

Fix any $0 < t < s < 1$ with $s - t < \delta$ and let $t_0 = \sup\{r \in [t,s] \mid |\psi(r) - \psi(t)| \leq d\int_t^r (1 + |p'(\tau)|)d\tau\}$. By the last claim $t_0 > t$. We next prove that $t_0 = s$. Indeed, assume for a moment that $t_0 < s$. By continuity of $\psi$, $|\psi(t_0) - \psi(t)| \leq d\int_t^{t_0}(1 + |p'(\tau)|)d\tau$. We already know that there exists $\varepsilon > 0$ such that (42) is satisfied with $t$ replaced by $t_0$ and $t_0 + \varepsilon \leq s$. Therefore

$$|\psi(t_0 + \varepsilon) - \psi(t)| \leq |\psi(t_0 + \varepsilon) - \psi(t_0)| + |\psi(t_0) - \psi(t)| \leq d \int_t^{t_0+\varepsilon} (1 + |p'(\tau)|)d\tau.$$

But this contradicts the choice of $t_0$ and therefore (34) holds true for all $0 < t < s < 1$ with $s - t < \delta$.

Notice that (34) implies that $\psi$ is absolutely continuous on $(0, 1)$. Then, by Theorem 3.4, for all $t \in (0, 1)$, $z'(t) = \frac{\partial H}{\partial p}(t, z(t), p(t) + \psi(t))$ and therefore $z'$ is absolutely continuous on $(0, 1)$. Since $z \in C^1([0, 1])$ we deduce that $z'$ is absolutely continuous on $[0, 1]$.

Inequality (34) also implies that if $p(\cdot)$ is Lipschitz, then $\psi$ is Lipschitz on $(0, 1)$. Then also $z'$ is Lipschitz, by Theorem 3.4. The proof is complete. □

## 5. Applications

Theorem 4.2 can be used to prove regularity of optimal controls and to obtain necessary optimality conditions in the form of variational inequalities.

**Corollary 5.1.** *Under all the assumptions of Theorem* 4.2 *suppose that for every* $(t, x, p) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$ *the supremum in* (5) *is attained by exactly one* $u(t, x, p) \in U(t)$ *and that* $u(\cdot, \cdot, \cdot)$ *is locally Lipschitz on* $\Gamma$. *Then there exists an absolutely continuous selection* $u_{ac}(t) \in U(t)$ *such that* $u_{ac}(t) = \bar{u}(t)$ *almost everywhere. Furthermore, if* $p(\cdot)$ *is Lipschitz on* $[0, 1]$, *then* $u_{ac}$ *may be taken Lipschitz.*

**Remark.** The problem investigated in [18] is so that $u(\cdot, \cdot, \cdot)$ is locally Lipschitz. □

**Proof.** Set $u_{ac}(t) = u(t, z(t), p(t) + \psi(t))$ for $t \in (0, 1)$, $u_{ac}(0) = u(0, z(0), p(0) + \psi(0+))$, $u_{ac}(1) = u(1, z(1), p(1) + \psi(1-))$ . Then $u_{ac}$ is absolutely continuous (respectively Lipschitz if $p$ is Lipschitz). From (14) we deduce that $u_{ac}(t) = \bar{u}(t)$ for a.e. $t \in [0, 1]$ □

**Theorem 5.2.** *Let a trajectory/control pair* $(z, \bar{u})$ *satisfy the normal constrained maximum principle with some* $p$, $\psi$. *Suppose that for almost all* $t \in [0, 1]$, $(L, f)(t, \cdot, \bar{u}(t))$ *is differentiable at* $z(t)$ *and that* $p$ *solves the adjoint system* (4). *Under all the assumptions of Theorem* 4.2 *there exists an absolutely continuous mapping* $q : [0, 1] \to \mathbb{R}^n$ *such that* (10)-(12) *hold true with* $\nabla\varphi(z(0), z(1))$ *replaced by* $\partial\varphi(z(0), z(1))$.

**Proof.** Set $q(t) = p(t) + \psi(t)$ for $t \in (0, 1)$, $q(0) = p(0) + \psi(0+)$ and $q(1) = p(1) + \psi(1-)$. Then $q$ is continuous at the end points and therefore, by Theorem 4.2, it is absolutely continuous. From (14) we deduce (12) and from (13), (15) we obtain (11).

To prove (10) denote by $S^{n-1}$ the unit sphere in $\mathbb{R}^n$. Since $K$ is sleek, the set-valued map $s \rightsquigarrow N_K(z(s)) \cap S^{n-1}$ is upper semicontinuous. Fix $t \in [0, 1)$ such that $\psi$ is differentiable at $t$. We claim that

$$\psi'(t) \in N_K(z(t)). \tag{43}$$

Indeed if $z(t) \in Int(K)$, then $\psi'(t) = 0 \in N_K(z(t))$. Assume next that $z(t) \in \partial K$ and define for all $\varepsilon > 0$ the convex cone

$$\Gamma(\varepsilon) = \bigcup_{\lambda \geq 0} \lambda \overline{co}\, (N_K(z(t)) \cap S^{n-1} + \varepsilon B).$$

By (28) the normal cone $N_K(z(t))$ is pointed, that is $0 \notin \overline{co}\,(N_K(z(t)) \cap S^{n-1})$. Thus $\Gamma(\varepsilon)$ is closed when $\varepsilon > 0$ is small enough and $\cap_{\varepsilon>0}\Gamma(\varepsilon) = N_K(z(t))$. Fix a sufficiently small $\varepsilon > 0$ and let $\delta > 0$ be such that for every $s \in [t, t+\delta]$, $N_K(z(s)) \cap S^{n-1} \subset N_K(z(t)) \cap S^{n-1} + \varepsilon B$. Define $\lambda(s) = |\nu(s)|$,

$$n(s) := \begin{cases} \dfrac{\nu(s)}{|\nu(s)|} & \text{if } \nu(s) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\nu(s) \in N_K(z(s)) \cap B$ $\mu$−a.e., for all $0 < h < \delta$

$$\psi(t+h) - \psi(t) = \int_{(t,t+h]} \nu(s) d\mu(s) = \int_{(t,t+h]} n(s)\lambda(s) d\mu(s) \in \int_{(t,t+h]} \Gamma(\varepsilon) d\mu(s) \subset \Gamma(\varepsilon)$$

Dividing by $h$ and taking the limit yields $\psi'(t) \in \Gamma(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we proved (43).

Let $t \in (0,1)$ be such that (4) holds true at $t$ and $\psi$ is differentiable at $t$. Then the equality

$$q'(t) = p'(t) + \psi'(t) = -\frac{\partial f}{\partial x}(t, z(t), \bar{u}(t))^* q(t) + \frac{\partial L}{\partial x}(t, z(t), \bar{u}(t)) + \psi'(t)$$

and (43) imply (10). □

## 6. Normality of the Maximum Principle

Consider closed subsets $K \subset I\!R^n$ and $K_1 \subset I\!R^n \times I\!R^n$, a complete separable metric space $\mathcal{Z}$, a measurable set-valued map $U : [0,1] \rightsquigarrow \mathcal{Z}$ with nonempty closed values, mappings $f : [0,1] \times \mathbf{R}^n \times \mathcal{Z} \to \mathbf{R}^n$, $L : [0,1] \times \mathbf{R}^n \times \mathcal{Z} \to \mathbf{R}$ which are measurable with respect to the first variable, continuous with respect to the second and third variables and a locally Lipschitz $\varphi : I\!R^n \times I\!R^n \to I\!R$.

Let $(z, \bar{u})$ satisfy the constrained maximum principle with some $\lambda$, $p$, $\psi$ not simultaneously equal to zero and assume in addition that the following adjoint equation is verified

$$-p'(t) = A(t)^*(p(t) + \psi(t)) - \lambda\pi(s) \;\text{ a.e. in } [0,1], \tag{44}$$

where $A : [0,1] \to \mathcal{L}(I\!R^n, I\!R^n)$ is a measurable matrix valued mapping and $\pi : [0,1] \to I\!R^n$.

The above equation arises in some smooth and non smooth versions of the maximum principle (see [19]) that make use of the generalized Jacobian of $(L, f)(t, \cdot, \bar{u}(t))$ at $z(t)$.

**Lemma 6.1.** *Assume that* $||A(\cdot)|| \in L^1(0,1)$. *If there exists an absolutely continuous solution* $w$ *to the viability problem*

$$\begin{cases} w' &= A(t)w + v(t), \;\; v(t) \in T_{\overline{co}\,(f(t,z(t),U(t)))}(z'(t)) \\ w(t) &\in Int(C_K(z(t))), \;\; \forall\, t \in [0,1] \\ (w(0), w(1)) &\in Int(C_{K_1}(z(0), z(1))), \end{cases} \tag{45}$$

*then* $\lambda = 1$.

**Proof.** Since $Int(C_K(z(t))) \neq \emptyset$ for all $t \in [0,1]$, from [2, Chapter 4] we know that $[0,1] \ni t \rightsquigarrow C_K(z(t))$ is lower semicontinuous.

Define

$$\mathcal{C} = \{w \in C([0,1]) \mid w(t) \in Int(C_K(z(t)))\ \forall\, t \in [0,1]\},$$

$$\mathcal{C}_1 = \{w \in C([0,1]) \mid (w(0), w(1)) \in Int(C_{K_1}(z(0), z(1)))\},$$

$$S = \{w \in W^{1,1}([0,1]) \mid w'(t) \in A(t)w(t) + T_{\overline{co}\,(f(t,z(t),U(t)))}(z'(t)) \text{ a.e. in } [0,1]\}$$

and let $\bar{S}$ denote the closure of $S$ in $C([0,1])$.

Since for all $t \in [0,1]$, $Int(C_K(z(t))) \neq \emptyset$ it follows from [6] that $Int(\mathcal{C}) \neq \emptyset$. It is also clear that $Int(\mathcal{C}_1) \neq \emptyset$ and $C_K(z(0)) \times C_K(z(1)) - C_{K_1}(z(0), z(1)) = \mathbb{R}^n \times \mathbb{R}^n$. Assume for a moment that $\lambda = 0$. Set $\eta = -p(1) - \psi(1)$. Then $(p(0), \eta) \in N_{K_1}(z(0), z(1))$. Since for $w \in C([0,1])$ we have $\int_0^1 w(s)d\psi(s) = \int_{[0,1]} w(s)\nu(s)d\mu(s)$, it follows that for every $w \in \mathcal{C} \cap \mathcal{C}_1$, $\int_{[0,1]} w(s)d\psi(s) + \langle (p(0), \eta), (w(0), w(1))\rangle \leq 0$. On the other hand, by the adjoint equation and the maximum principle, for every $w \in S$ we have $\int_0^1 (p'w + pw' + \psi w')(s)ds \leq 0$. Thus $\langle p(1), w(1)\rangle - \langle p(0), w(0)\rangle + \int_0^1 \psi(s)w'(s)ds \leq 0$. Integrating by parts we get $\langle (p(0), \eta), (w(0), w(1))\rangle + \int_0^1 w(s)d\psi(s) \geq 0$.

Since $\bar{S} \cap (Int(\mathcal{C} \cap \mathcal{C}_1)) \neq \emptyset$, we deduce from the above two inequalities that for all $w \in C([0,1])$, $\langle (p(0), \eta), (w(0), w(1))\rangle + \int_0^1 w(s)d\psi(s) = 0$. This holds in particular for all absolutely continuous functions on $[0,1]$. Integrating by parts we obtain that for every $w \in W^{1,1}([0,1])$ $\langle (p(0), -p(1)), (w(0), w(1))\rangle - \int_0^1 \psi(s)w'(s)ds = 0$. By the DuBois-Reymond lemma and right continuity of $\psi$ for some $c \in \mathbb{R}^n$, $\psi = c$ on $(0,1)$. Thus $\langle -p(1) - c, w(1)\rangle + \langle p(0) + c, w(0)\rangle = 0$ for all $w \in W^{1,1}([0,1])$ and $-p(1) = \psi(1-)$, $p(0) = -\psi(0+)$. Hence for all $(v_1, v_2) \in C_{K_1}(z(0), z(1))$, $(w_1, w_2) \in C_K(z(0)) \times C_K(z(1))$ we have $\langle (p(0), \eta), (v_1, v_2) - (w_1, w_2)\rangle \leq 0$, implying that $p(0) = 0$, $\eta = 0$ and that $c = 0$. This and the adjoint equation yield $p \equiv 0$ and therefore $\psi(1) = -\eta = 0$. So $p = \psi = 0$. The obtained contradiction ends the proof. □

Our next results concern sufficient conditions for the existence of solutions to (45). From now on we assume that $K_1 = Q_0 \times Q_1$ for some closed subsets $Q_i$ of $\mathbb{R}^n$, $i = 0, 1$ and that the set of state constraints $K$ is as in Section 4, i.e.

$$K = \cap_{j=1}^m K_j,\ K_j \text{ is closed, } \partial K_j \in C^{1,1}_{loc}$$

and for all $x \in \partial K$

$$0 \notin co\{n_j(x) \mid j \in I(x)\},$$

where $n_j(x)$ denotes the outward unit normal to $K_j$ at $x \in \partial K_j$ and $I(x)$ denotes the set of all active indices at $x$. Notice that this yields

$$\forall\, x \in \partial K,\ Int(C_K(x)) = Int(T_K(x)) = \{v \in \mathbb{R}^n \mid \max_{j \in I(x)} \langle n_j(x), v\rangle < 0\} \tag{46}$$

and therefore $K$ is sleek. Furthermore, as we already noticed, this also implies (29).

Let $d_j$ denote the signed distance function associated to the set $K_j$. Then $n_j(x) = \nabla d_j(x)$ whenever $x \in \partial K$.

**Theorem 6.2.** *Let $x : [0,1] \to K$ be a Lipschitz continuous function, $\gamma : [0,1]\times I\!R^n \to I\!R^n$ be a Carathéodory function such that for some $l > 0$ , $\gamma(t,\cdot)$ is $l-$Lipschitz and $\gamma(\cdot,0) = 0$.*

*Let $\eta > 0$, $\rho > 0$ and $v \in L^\infty(0,1;I\!R^n)$ be so that for all $t \in [0,1]$ and $j$ satisfying $x(t) \in \partial K_j + \eta B$*

$$\langle \nabla d_j(x(t)), v(t)\rangle < -\rho.$$

*Then for every $0 \neq \bar{w}_0 \in Int\,(T_K(x(0)))$ there exists a solution $w$ to the viability problem*

$$\left\{\begin{array}{lcl} w'(t) & \in & \gamma(t,w(t)) + I\!R_+(v(t)-x'(t)) \text{ almost everywhere in } [0,1] \\ w(0) & = & \bar{w}_0 \\ w(t) & \in & Int\,(T_K(x(t))) \text{ for all } t\in[0,1]. \end{array}\right. \tag{47}$$

**Corollary 6.3.** *Let $(z,\bar{u})$ satisfy the constrained maximum principle with some $(\lambda,p,\psi)$, $p$ satisfy the adjoint equation (44) and $z$ be Lipschitz. Suppose that $||A(\cdot)|| \in L^\infty(0,1)$ and that $Int(C_{Q_0}(z(0))) \cap Int(T_K(z(0))) \neq \emptyset$. Further assume that for some $\eta > 0$, $\rho > 0$, $M > 0$ and every $t \in [0,1]$ there exists $u_t \in U(t)$ such that for all $j$ satisfying $z(t) \in \partial K_j + \eta B$ we have*

$$\langle \nabla d_j(z(t)), f(t,z(t),u_t)\rangle \le -\rho, \quad |f(t,z(t),u_t)| \le M.$$

*Then for every $0 \neq w_0 \in Int(C_{Q_0}(z(0))) \cap Int(T_K(z(0)))$ there exists an absolutely continuous solution $w$ to the viability problem*

$$\left\{\begin{array}{rcl} w' & = & A(t)w + v(t), \quad v(t) \in T_{\overline{co}\,(f(t,z(t),U(t)))}(z'(t)) \\ w(t) & \in & Int\,(T_K(z(t))), \quad \forall\, t\in[0,1] \\ w(0) & = & w_0. \end{array}\right. \tag{48}$$

*Consequently if $z(1) \in Int(Q_1)$, then in the maximum principle $\lambda = 1$.*

**Proof.** By the measurable selection theorem there exists a measurable selection $u(t) \in U(t)$ such that for $t \in [0,1]$ and every $j$ such that $z(t) \in \partial K_j + \eta B$ we have $\langle \nabla d_j(z(t)), f(t,z(t),u(t))\rangle \le -\rho$ and $|f(t,z(t),u(t))| \le M$. The result then follows from Theorem 6.2 with $v(t) = f(t,z(t),u(t))$ if $z(t) \in \cup_j \partial K_j + \eta B$ and $v(t) = z'(t)$ otherwise, because $I\!R_+(v(t)-z'(t)) \in T_{\overline{co}\,(f(t,z(t),U(t)))}(z'(t))$.

A similar result, but with the initial point replaced by the end point is provided by the following corollary.

**Corollary 6.4.** *Let $(z,\bar{u})$ satisfy the constrained maximum principle with some $(\lambda,p,\psi)$, $p$ satisfy the adjoint equation (44) and $z$ be Lipschitz. Suppose that $||A(\cdot)|| \in L^\infty(0,1)$ and that $Int(C_{Q_1}(z(1))) \cap Int(T_K(z(1))) \neq \emptyset$. Further assume that for some $\eta > 0$, $\rho > 0$, $M > 0$ and every $t \in [0,1]$ there exists $u_t \in U(t)$ such that for all $j$ satisfying $z(t) \in \partial K_j + \eta B$ we have*

$$\langle \nabla d_j(z(t)), f(t,z(t),u_t)\rangle \ge \rho, \ |f(t,z(t),u_t)| \le M.$$

*Then for every $0 \neq w_1 \in Int(C_{Q_1}(z(1))) \cap Int(T_K(z(1)))$ there exists an absolutely continuous solution $w$ to the constrained linear control system*

$$\left\{\begin{array}{rcl} w' & = & A(t)w + v(t), \quad v(t) \in T_{\overline{co}\,(f(t,z(t),U(t)))}(z'(t)) \\ w(t) & \in & Int\,(T_K(z(t))), \quad \forall\, t\in[0,1] \\ w(1) & = & w_1. \end{array}\right. \tag{49}$$

*Consequently if $z(0) \in Int(Q_0)$, then in the maximum principle $\lambda = 1$.*

**Proof.** By the measurable selection theorem there exists a measurable selection $u(t) \in U(t)$ such that for $t \in [0,1]$ and $j$ satisfying $z(t) \in \partial K_j + \eta B$ we have $\langle \nabla d_j(z(t)), f(t,z(t), u(t))\rangle \geq \rho$, $|f(t,z(t),u(t))| \leq M$. Set $x(t) = z(1-t)$, $v(t) = f(1-t, z(1-t), u(1-t))$ if $z(1-t) \in \cup_j \partial K_j + \eta B$ and $v(t) = z'(1-t)$ otherwise. By Theorem 6.2 applied with $x(t) = z(1-t)$, the differential inclusion

$$\begin{cases} w' & \in & -A(1-t)w + I\!R_+(-v(t) + z'(1-t)), \\ w(t) & \in & Int\,(T_K(z(1-t))), \ \ \forall\, t \in [0,1] \\ w(0) & = & w_1. \end{cases}$$

has a solution $\bar{w} \in W^{1,1}([0,1])$. Then $w(t) := \bar{w}(1-t)$ solves (49).

**Proof of Theorem 6.2.** Fix any $0 \neq \bar{w}_0 \in Int(T_K(x(0)))$. If $x([0,1]) \subset Int(K)$, then the solution $w(\cdot)$ to $w' = \gamma(s,w)$ starting at $\bar{w}_0$ is as required. So the proof continues under the assumption that $x([0,1]) \cap \partial K \neq \emptyset$.

Let $c > 0$ denote the Lipschitz constant of $x(\cdot)$. Define the closed set $\Gamma_j := \{t \in [0,1] \mid x(t) \in \partial K_j + \eta B\}$, $\xi_j(t) = \nabla d_j(x(t))$ for all $t \in \Gamma_j$ and let $k > 0$ be so that for all $j$, $|\xi_j'(t)| \leq k$ for a.e. $t \in \Gamma_j$. Finally set $G(t) := T_K(x(t))$,

$$M = ||v(\cdot)||_\infty, \quad \chi = max\left\{\frac{1}{\eta}, \frac{k+l+1}{\rho}\left(l + (c+M)\frac{k+l+1}{\rho}\right)\right\}.$$

Then $\frac{1}{\chi} \leq \eta$.

If $x(t) \in \partial K$ denote by $J(t) = I(x(t))$ the set of all active indices at $x(t)$. Since $x$ is continuous and $K$ is a finite intersection of closed sets, for every $t \in (0,1)$ with $x(t) \in \partial K$ there exists $\bar{\varepsilon}_t > 0$ such that for all $s \in (t - \bar{\varepsilon}_t, t + \bar{\varepsilon}_t) \subset (0,1)$ we have $J(s) \subset J(t)$ and for every $j \in J(t)$, $x((t-\bar{\varepsilon}_t, t+\bar{\varepsilon}_t)) \subset \partial K_j + \frac{1}{2\chi}B$. If $x(0) \in \partial K$, then consider $\varepsilon_0 > 0$ such that for all $s \in [0, \varepsilon_0)$, $J(s) \subset J(0)$ and for every $j \in J(0)$, $x([0,\varepsilon_0)) \subset \partial K_j + \frac{1}{2\chi}B$. If $x(0) \in Int(K)$, then set $\varepsilon_0 = 0$ and $[0,0) = \emptyset$. If $x(1) \in \partial K$, then consider $\varepsilon_f > 0$ such that for all $s \in (1-\varepsilon_f, 1]$, $J(s) \subset J(1)$ and for every $j \in J(1)$, $x((1-\varepsilon_f, 1]) \subset \partial K_j + \frac{1}{2\chi}B$. If $x(1) \in Int(K)$, then set $\varepsilon_f = 0$ and $(1,1] = \emptyset$. Then

$$\bigcup_{t\in(0,1),\ x(t)\in\partial K} (t-\bar{\varepsilon}_t, t+\bar{\varepsilon}_t) \cup [0,\varepsilon_0) \cup (1-\varepsilon_f, 1]$$

covers the compact set $\{t \in [0,1] \mid x(t) \in \partial K\}$. Consider a finite subcovering

$$\bigcup_{i=1}^{\bar{r}} (\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i) \cup [0,\varepsilon_0) \cup (1-\varepsilon_f, 1].$$

Renumbering and keeping the same notations we may assume that $\bar{t}_1 \leq .... \leq \bar{t}_{\bar{r}}$. If for some $1 \leq i < j \leq \bar{r}$ we have $(\bar{t}_j - \bar{\varepsilon}_j, \bar{t}_j + \bar{\varepsilon}_j) \subseteq (\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i)$, then we remove $(\bar{t}_j - \bar{\varepsilon}_j, \bar{t}_j + \bar{\varepsilon}_j)$ from this subcovering. While if $(\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i) \subsetneq (\bar{t}_j - \bar{\varepsilon}_j, \bar{t}_j + \bar{\varepsilon}_j)$ then we remove $(\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i)$ from this subcovering.

Similarly if for some $1 \le i \le \bar{r}$, $(\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i) \subset [0, \varepsilon_0)$ or $(\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i) \subset (1 - \varepsilon_f, 1]$, then we remove $(\bar{t}_i - \bar{\varepsilon}_i, \bar{t}_i + \bar{\varepsilon}_i)$ from our subcovering. In this way we obtain a new finite subcovering

$$\bigcup_{i=1}^{r} (t_i - \varepsilon_i, t_i + \varepsilon_i) \cup [0, \varepsilon_0) \cup (1 - \varepsilon_f, 1]$$

satisfying $0 < t_1 < ... < t_r < 1$ and $\varepsilon_0 < t_1 + \varepsilon_1 < ... < t_r + \varepsilon_r < 1$.

Step 1. Let $t_0 = \inf_{x(s) \in \partial K} s$.

If $t_0 = 0$, then we claim that the solution $w$ to

$$w' = \gamma(s, w) + \frac{k + l + 1}{\rho} |w| (v(s) - x'(s)), \quad w(0) = \bar{w}_0. \tag{50}$$

is so that for every $T \in [0, \varepsilon_0)$, $w(T) \in Int(G(T))$. Indeed for all $T \in [0, \varepsilon_0)$ satisfying $x(T) \in \partial K$ and all $j \in J(T) \subset J(0)$ we have

$$\langle \xi_j(T), w(T) \rangle = \langle \xi_j(0), \bar{w}_0 \rangle + \int_0^T \langle \xi_j, w \rangle'(s) ds < \int_0^T \langle \xi_j, w \rangle'(s) ds.$$

Using that $\gamma(s, \cdot)$ is $l-$Lipschitz and $|\xi_j(\cdot)|$ is $k-$Lipschitz and its norm is bounded by one we get

$$\begin{aligned}
\langle \xi_j(T), w(T) \rangle &< \int_0^T k |w(s)| ds + \int_0^T \langle \xi_j(s), w'(s) \rangle ds \\
&\le \int_0^T (k + l) |w(s)| ds + \int_0^T \frac{k + l + 1}{\rho} |w(s)| \langle \xi_j(s), v(s) \rangle ds \\
&\quad - \int_0^T \frac{k + l + 1}{\rho} (d_j \circ x)'(s) |w(s)| ds.
\end{aligned}$$

Hence, by the choice of $v(\cdot)$, integrating by parts the expression $\int_0^T (d_j \circ x)'(s) |w(s)| ds$ and using that $d_j(x(0)) = d_j(x(T)) = 0$, we obtain the following inequalities

$$\begin{aligned}
\langle \xi_j(T), w(T) \rangle &< -\int_0^T |w(s)| ds + \frac{k + l + 1}{\rho} \int_0^T |d_j(x(s))| \cdot |w'(s)| ds \\
&\le -\int_0^T |w(s)| ds + \frac{k + l + 1}{2\chi\rho} \int_0^T \left( l + (c + M) \frac{k + l + 1}{\rho} \right) |w(s)| ds.
\end{aligned}$$

and, by the choice of $\chi$,

$$\langle \xi_j(T), w(T) \rangle < -\frac{1}{2} \int_0^T |w(s)| ds < 0.$$

This holds true for all $j \in J(T)$ and our claim follows from (46).

Step 2. We define here $w$ solving $w'(s) \in \gamma(s, w(s)) + \mathbb{R}_+(v(s) - x'(s))$ on $[0, t_1]$ in such way that $w(0) = \bar{w}_0$, $w(s) \in Int(G(s))$ for all $s \in [0, t_1]$. In view of (46) it is enough to construct $w(\cdot)$ such that for all $t \in (0, t_1]$ satisfying $x(t) \in \partial K$ and every $j \in J(t)$, $\langle \xi_j(t), w(t) \rangle < 0$.

*Case 1.* $t_0 \in (t_1 - \varepsilon_1, t_1 + \varepsilon_1)$. Consider the solution $w$ to the system $w' = \gamma(s, w), w(0) = \bar{w}_0$ defined on $[0, t_1 - \varepsilon_1]$. Then for all $s \in [0, t_1 - \varepsilon_1]$, $w(s) \in Int(G(s)) = I\!R^n$. Set $w_1 = w(t_1 - \varepsilon_1) \neq 0$. Consider next the solution $w : [t_1 - \varepsilon_1, t_1] \to I\!R^n$ to the system

$$w' = \gamma(s, w) + \frac{k + l + 1}{\rho} |w| (v(s) - x'(s)) + \frac{2|w_1|}{\rho(t_0 - t_1 + \varepsilon_1)} (v(s) - x'(s)) \tag{51}$$

such that $w(t_1 - \varepsilon_1) = w_1$. Then $w(s) \in Int\,(G(s)) = \mathbf{R}^n$ for all $s \in [0, t_0)$. Fix any $T \in [t_0, t_1]$ with $x(T) \in \partial K$. We show next that for all $j \in J(T)$, $\langle \xi_j(T), w(T) \rangle < 0$. Let $j \in J(T) \subset J(t_1)$. Since $\gamma(s, \cdot)$ is $l-$Lipschitz, $|\xi_j(\cdot)|$ is $k-$Lipschitz on $[t_1 - \varepsilon_1, T]$ and its norm is bounded by one,

$$\begin{aligned}
&\langle \xi_j(T), w(T) \rangle = \langle \xi_j(t_1 - \varepsilon_1), w_1 \rangle + \int_{t_1 - \varepsilon_1}^{T} \langle \xi_j, w \rangle'(s) ds \\
\leq\ & |w_1| + \int_{t_1 - \varepsilon_1}^{T} (k|w(s)| + \langle \xi_j(s), w'(s) \rangle) ds \\
\leq\ & \int_{t_1 - \varepsilon_1}^{T} (k + l)|w(s)| ds \\
& + \int_{t_1 - \varepsilon_1}^{T} \left( \frac{k + l + 1}{\rho} |w(s)| \langle \xi_j(s), v(s) \rangle + \frac{2|w_1|}{\rho(t_0 - t_1 + \varepsilon_1)} \langle \xi_j(s), v(s) \rangle \right) ds \\
& + |w_1| - \int_{t_1 - \varepsilon_1}^{T} \left( \frac{k + l + 1}{\rho} (d_j \circ x)'(s) |w(s)| + \frac{2|w_1|}{\rho(t_0 - t_1 + \varepsilon_1)} (d_j \circ x)'(s) \right) ds.
\end{aligned}$$

Hence, by the choice of $v(\cdot)$, integrating by parts the expression $\int_{t_1 - \varepsilon_1}^{T} (d_j \circ x)'(s)|w(s)| ds$ and using that $d_j(x(t_1 - \varepsilon_1)) \leq 0$, $d_j(x(T)) = 0$ we obtain the following inequalities

$$\begin{aligned}
& \langle \xi_j(T), w(T) \rangle \\
\leq\ & -\int_{t_1 - \varepsilon_1}^{T} |w(s)| ds + \frac{k + l + 1}{\rho} \int_{t_1 - \varepsilon_1}^{T} |d_j(x(s))| \cdot |w'(s)| ds + |w_1| - \frac{2|w_1|(T - t_1 + \varepsilon_1)}{t_0 - t_1 + \varepsilon_1} \\
\leq\ & -\int_{t_1 - \varepsilon_1}^{T} |w(s)| ds + |w_1| - \frac{2|w_1|(T - t_1 + \varepsilon_1)}{t_0 - t_1 + \varepsilon_1} \\
& + \frac{k + l + 1}{2\chi\rho} \left( \int_{t_1 - \varepsilon_1}^{T} \left( l + (c + M) \frac{k + l + 1}{\rho} \right) |w(s)| ds + \frac{2|w_1|(c + M)(T - t_1 + \varepsilon_1)}{\rho(t_0 - t_1 + \varepsilon_1)} \right).
\end{aligned}$$

By the choice of $\chi$, using that $T - t_1 + \varepsilon_1 \geq t_0 - t_1 + \varepsilon_1$, we derive

$$\langle \xi_j(T), w(T) \rangle \leq -\frac{1}{2} \int_{t_1 - \varepsilon_1}^{T} |w(s)| ds + |w_1| - \frac{|w_1|(T - t_1 + \varepsilon_1)}{t_0 - t_1 + \varepsilon_1} < 0.$$

This holds true for all $j \in J(T)$ and therefore $w(T) \in Int(G(T))$.

*Case 2.* $t_0 = 0$ and $\varepsilon_0 \leq t_1 - \varepsilon_1$. Then, by our construction, $x([\varepsilon_0, t_1 - \varepsilon_1]) \subset Int(K)$. Let $w$ denote the solution to (50) defined on $[0, \varepsilon_0]$. We already know that $w(s) \in Int(G(s))$ for all $s \in [0, \varepsilon_0)$. Furthermore, $w(\varepsilon_0) \in Int(G(\varepsilon_0)) = I\!R^n$.

Set $y_0 = w(\varepsilon_0) \neq 0$. We extend $w$ on the time interval $[\varepsilon_0, t_1 - \varepsilon_1]$ by the solution to the system $w' = \gamma(s, w)$, $w(\varepsilon_0) = y_0$. Since $x([\varepsilon_0, t_1 - \varepsilon_1]) \subset Int(K)$ we know that $w(s) \in Int(G(s))$ for all $s \in [\varepsilon_0, t_1 - \varepsilon_1]$. Set $w_1 = w(t_1 - \varepsilon_1) \neq 0$. Let $t = \inf\{s \in [t_1 - \varepsilon_1, t_1] \mid x(s) \in \partial K\}$.

Consider the solution $w : [t_1 - \varepsilon_1, t_1] \to I\!\!R^n$ to (51) with $t_0$ replaced by $t$ such that $w(t_1 - \varepsilon_1) = w_1$. Then $w(s) \in Int(G(s)) = I\!\!R^n$ for all $s \in [t_1 - \varepsilon_1, t)$.

Exactly as in Case 1 we check that for every $T \in [t, t_1]$ such that $x(T) \in \partial K$ and for all $j \in J(T)$, $\langle \xi_j(T), w(T)\rangle < 0$. Thus $w(T) \in Int(G(T))$.

*Case 3.* $t_0 = 0$ and $\varepsilon_0 > t_1$. Then, by Step 1, the solution $w : [0, t_1] \to I\!\!R^n$ to (50) is as required.

*Case 4.* $t_0 = 0$ and $t_1 \geq \varepsilon_0 > t_1 - \varepsilon_1$. If in addition $x((t_1 - \varepsilon_1, \varepsilon_0)) \subset Int(K)$, then consider any $t_1 - \varepsilon_1 < s_0 < s_1 < \varepsilon_0$ and the solution $w$ to (50) defined on $[0, s_0]$. We already know that $w(s) \in Int(G(s))$ for all $s \in [0, s_0]$. Set $w_1 = w(s_0) \neq 0$. We extend $w$ by the solution $w : [s_0, t_1] \to I\!\!R^n$ to

$$w' = \gamma(s, w) + \frac{k + l + 1}{\rho}|w|(v(s) - x'(s)) + \frac{2|w_1|}{\rho(s_1 - s_0)}(v(s) - x'(s)) \tag{52}$$

such that $w(s_0) = w_1$. Then for all $T \in (s_0, s_1)$, $w(T) \in Int(G(T)) = I\!\!R^n$. Exactly as in Case 1 we get that for all $T \in [s_1, t_1]$, $w(T) \in Int(G(T))$.

If this additional requirement fails, then there exists $s_0 \in (t_1 - \varepsilon_1, \varepsilon_0)$ such that $x(s_0) \in \partial K$, then consider $s_0 < s_1 < \varepsilon_0$ such that for all $s \in [s_0, s_1]$, $J(s) \subset J(s_0)$. Let $w$ denote the solution to (50) on $[0, s_0]$ and set $w_1 = w(s_0) \neq 0$. By Step 1, $w(s) \in Int(G(s))$ for all $s \in [0, s_0)$.

Consider next the solution $w : [s_0, t_1] \to I\!\!R^n$ to (52) such that $w(s_0) = w_1$. Then, performing the same estimates as in Step 1, we show that for all $T \in (s_0, s_1]$ with $x(T) \in \partial K$ and every $j \in J(T) \subset J(s_0)$

$$\begin{array}{rl} & \langle \xi_j(T), w(T)\rangle \\ \leq & \langle \xi_j(s_0), w(s_0)\rangle - \displaystyle\int_{s_0}^T |w(s)|ds + \frac{k+l+1}{\rho}\int_{s_0}^T |d_j(x(s))| \cdot |w'(s)|ds - \frac{2|w_1|(T - s_0)}{s_1 - s_0} \\ < & \langle \xi_j(s_0), w(s_0)\rangle - \displaystyle\frac{1}{2}\int_{s_0}^T |w(s)|ds < 0. \end{array}$$

Since $j \in J(T)$ is arbitrary, $w(T) \in Int(G(T))$.

Pick any $T \in (s_1, t_1)$. Then applying the same arguments as in Case 1 we deduce that $w(T) \in Int(G(T))$.

<u>Step 3</u>. We continue by the induction argument. Let us assume that for some $1 \leq i < r$ we already defined $w : [0, t_i] \to I\!\!R^n$ solving the viability problem

$$w'(s) \in \gamma(s, w(s)) + I\!\!R_+(v(s) - x'(s)),\ w(s) \in Int(G(s)),\ w(0) = \bar{w}_0$$

on $[0, t_i]$.

Set $\bar{w}_i = w(t_i)$. Then we proceed exactly in the same way as in Steps 1, 2, Cases 2-4, replacing $\bar{w}_0$ by $\bar{w}_i$, the initial time 0 by $t_i$ and $\varepsilon_0$ by $t_i + \varepsilon_i$ to extend $w$ on the time interval $[t_i, t_{i+1}]$.

In this way we obtain an extension on the time interval $[0, t_r]$. If $\varepsilon_f > 0$, then setting $t_{r+1} = 1$ we apply the same arguments to extend $w$ on the time interval $[t_r, t_{r+1}]$.

If $\varepsilon_f = 0$, then $x([t_r + \varepsilon_r, 1]) \subset Int(K)$. Set $w_1 = w(t_r)$ and consider the solution $w$ to

$$w' = \gamma(s, w) + \frac{k + l + 1}{\rho} |w|(v(s) - x'(s)) \tag{53}$$

satisfying $w(t_r) = w_1 \neq 0$. Then as in Step 1 we deduce that for all $s \in [t_r, t_r + \varepsilon_r)$, $w(s) \in Int(G(s))$. Since for all $T \in [t_r + \varepsilon_r, 1]$, $G(T) = I\!R^n$, the proof is complete. □

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