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Aubin Criterion for Metric Regularity*

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Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

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We present a derivative criterion for metric regularity of set-valued mappings that is based on works of J.-P. Aubin and co-authors. A related implicit mapping theorem is also obtained. As applications, we first show that Aubin criterion leads directly to the known fact that the mapping describing an equality/inequality system is metrically regular if and only if the Mangasarian-Fromovitz condition holds. We also derive a new necessary and sufficient condition for strong regularity of variational inequalities over polyhedral sets. A new proof of the radius theorem for metric regularity based on Aubin criterion is given as well.

Keywords: Set-valued mappings, metric regularity, variational analysis, graphical derivative, implicit mapping theorem, variational inequality, strong regularity

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1. Introduction

This paper deals with metric regularity of set-valued mappings. Throughout, X and Y are Banach spaces. The norms of both X and Y are denoted by $\|\cdot\|$; the closed ball centered at x with radius r by $\mathbb{B}_r(x)$ and the open ball by $\mathbb{B}^\circ_r(x)$; the closed unit ball is simply $\mathbb{B}$ and the open one $\mathbb{B}^\circ$. A neighborhood of a point x is any open set containing x. The distance from a point x to a set A is denoted by $d(x, A)$. By a mapping F from X to Y we generally mean a set-valued mapping and write $F : X \rightrightarrows Y$, having its inverse F^{-1} defined as $F^{-1}(y) = \{x \mid y \in F(x)\}$ and graph $\operatorname{gph} F = \{(x, y) \mid y \in F(x)\}$. When F is single-valued (a function) we write $F : X \to Y$.

Definition 1.1. A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ if

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$(\bar{x},\bar{y}) \in \operatorname{gph} F$ and there exist a constant $\kappa > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } (x,y) \in U \times V. \tag{1}$$

The metric regularity can be identified with the finiteness of the *regularity modulus* defined as

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \inf\{\, \kappa \mid \text{there exist neighborhoods } U \text{ and } V \text{ such that (1) holds}\}.$$

The absence of metric regularity is indicated by $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \infty$.

The concept of metric regularity goes back to classical results by Banach, Lyusternik and Graves. More recently, its central role has been recognized in variational analysis for both theoretical developments such as obtaining necessary optimality conditions and also in numerically oriented studies, e.g., when deriving error bounds for solution approximations. Discussions of the property of metric regularity, its relations to other properties and characterizations by various approximations are presented in [21] and [13].

Given a mapping $F : X \rightrightarrows Y$, the *graphical (contingent) derivative* of F at $(x,y) \in \operatorname{gph} F$ is the mapping $DF(x|y) : X \rightrightarrows Y$ whose graph is the tangent cone $T_{\operatorname{gph} F}(x,y)$ to $\operatorname{gph} F$ at (x,y):

$$v \in DF(x|y)(u) \iff (u,v) \in T_{\operatorname{gph} F}(x,y).$$

Recall that the tangent cone is defined as follows: $(u,v) \in T_{\operatorname{gph} F}(x,y)$ when there exist sequences $t_n \downarrow 0$, $u_n \to u$ and $v_n \to v$ such that $y + t_n v_n \in F(x + t_n u_n)$ for all n.

The mapping $DF(x|y)$ is positively homogeneous since its graph is a cone; specifically, one has $DF(x|y)(0) \ni 0$ and $DF(x|y)(\lambda u) = \lambda DF(x|y)(u)$ for all $u \in X$ for $\lambda > 0$. The *convexified graphical derivative* $D^{\star\star}F(x|y)$ of F at x for y is defined in a similar way:

$$v \in D^{\star\star}F(x|y)(u) \iff (u,v) \in \operatorname{clco} T_{\operatorname{gph} F}(x,y)$$

where clco stands for the closed convex hull. We also use the *inner* and the *outer* “norms" (see [21], Section 9D) of a mapping $H : X \rightrightarrows Y$:

$$\|H\|^- = \sup_{x \in B} \inf_{y \in H(x)} \|y\| \quad \text{and} \quad \|H\|^+ = \sup_{x \in B} \sup_{y \in H(x)} \|y\|.$$

Outer and inner norms can be related through adjoints. For a positively homogeneous mapping $F : X \rightrightarrows Y$ the *upper adjoint* $F^{*+} : Y^* \rightrightarrows X^*$ is defined by

$$(y^*, x^*) \in \operatorname{gph} F^{*+} \iff \langle x^*, x\rangle \leq \langle y^*, y\rangle \text{ for all } (x,y) \in \operatorname{gph} F,$$

while the *lower adjoint* $F^{*-} : Y^* \rightrightarrows X^*$ is

$$(y^*, x^*) \in \operatorname{gph} F^{*-} \iff \langle x^*, x\rangle \geq \langle y^*, y\rangle \text{ for all } (x,y) \in \operatorname{gph} F,$$

where X^* and Y^* are the dual spaces of X and Y. Borwein derived in [7] the following duality relations between outer and inner norms for a sublinear mapping F with closed graph:

$$\|F\|^+ = \|F^{*+}\|^- = \|F^{*-}\|^- \quad \text{and} \quad \|F\|^- = \|F^{*+}\|^+ = \|F^{*-}\|^+.$$

Recall that a mapping $F : X \rightrightarrows Y$ is said to have a locally closed graph at $(\bar{x},\bar{y})$ when $\operatorname{gph} F \cap [\mathbb{B}_r(\bar{x}) \times \mathbb{B}_r(\bar{y})]$ is a closed set for some $r > 0$.

The central result in this paper is the following theorem:

Theorem 1.2 (Aubin criterion). *Consider two Banach spaces X and Y, and a mapping $F : X \rightrightarrows Y$ which graph is locally closed at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$\operatorname{reg} F(\bar{x}|\bar{y}) \leq \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|DF(x|y)^{-1}\|^{-}, \tag{2}$$

and hence F is metrically regular at $\bar{x}$ for $\bar{y}$ provided that

$$\limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|DF(x|y)^{-1}\|^{-} < \infty. \tag{3}$$

If X is finite dimensional, then (2) *becomes an equality,*

$$\operatorname{reg} F(\bar{x}|\bar{y}) = \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|DF(x|y)^{-1}\|^{-}, \tag{4}$$

and hence F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if (3) *holds. Moreover, when both spaces X and Y are finite dimensional one has*

$$\operatorname{reg} F(\bar{x}|\bar{y}) = \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|D^{\star\star}F(x|y)^{-1}\|^{-}. \tag{5}$$

Theorem 1.2 can be viewed as a partial extension of Theorem 5.4.3 in book [3] where a sufficient condition for the Aubin property of the inverse F^{-1} is shown, for predecessors of this result see [1] and [4]. That condition is in general weaker than (3) but, as we see here, for a finite-dimensional X is actually equivalent to it. Recall that a mapping $S : Y \rightrightarrows X$ has the Aubin property at $\bar{y}$ for $\bar{x}$ when $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ and there exist neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$e(S(y) \cap U, S(y')) \leq \kappa\|y - y'\| \quad \text{for all } y, y' \in V, \tag{6}$$

where $e(A, B) = \sup_{x\in A} d(x, B)$ is the excess from A to B. The Aubin property of a mapping S is known to be equivalent to the metric regularity of S^{-1} and was introduced in [2] under the name "pseudo-Lipschitz" continuity, it was studied in [4] in the infinite dimensional case, for more bibliographical details see [21]. Moreover, the infimum of the constant κ in (6) is equal to $\operatorname{reg} S^{-1}(\bar{x}|\bar{y})$.

The equality (5) was stated recently in [6] with a proof based on viability theory. The given here proof of (5) is inspired by the proof of Theorem 3.2.4 in [5] due to Frankowska.

The characterization of metric regularity exhibited in Theorem 1.2 complements, and in some sense also completes, results previously displayed by J.-P. Aubin and co-authors; therefore, we call it here the *Aubin criterion for metric regularity.*

In a sense "dual" to the Aubin criterion is the known Mordukhovich criterion in finite dimensions, see [17] and [21] which uses the *coderivative* $D^*F(x|y)$ defined as

$$v \in D^*F(x|y)(u) \iff (v, -u) \in N_{\operatorname{gph} F}(x, y),$$

where $N_C(x)$ is the (nonconvex, limiting) normal cone to the set C at x. The Mordukhovich criterion says that F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if

$$\|D^*F^{-1}(\bar{y}|\bar{x})\|^{+} < \infty.$$

One would expect that each of these criteria could be directly derived from the other one, and this is clearly so when $\operatorname{gph} F$ is Clarke regular, see [21], 8.40 and 11.29. In infinite dimensions, the characterizations of the metric regularity via coderivatives assume some more from the spaces, e.g., to be Asplund, see [18, 19]. When the domain space X is finite dimensional and Y is any Banach space, a necessary and sufficient condition for metric regularity in terms of the Ioffe approximate coderivative is given in [14]. This latter result is the closest to Theorem 1.2 from the literature known to the authors. We also refer to the book [15] as a source of information on criteria for metric regularity.

If $F : X \to Y$ is a bounded linear mapping, denoted $F \in L(X, Y)$, then the Aubin criterion (4) is also valid for X and Y Banach spaces and reduces to $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \|F^{-1}\|^{-}$. Equivalently, F is metrically regular (at any point) if and only if F is surjective; this covers the classical case of the Banach open mapping principle. The equivalence among metric regularity at the origin, the finiteness of the inner norm, and the surjectivity holds also for mappings acting in Banach spaces whose graphs are closed and convex cones. Specifically, we have the following result proved in [11], Example 2.1:

Proposition 1.3. *Let X and Y be Banach spaces and let $F : X \rightrightarrows Y$ be such that* $\operatorname{gph} F$ *is a closed and convex cone. Then the modulus of regularity of F at the origin satisfies*

$$\operatorname{reg} F(0\,|\,0) = \|DF(0\,|\,0)^{-1}\|^{-} = \|F^{-1}\|^{-}.$$

Moreover, $\operatorname{reg} F(0\,|\,0) < \infty$ *if and only if F is surjective and then F is metrically regular at any point in its graph.*

In Section 3 we give a proof of Theorem 1.2 by first obtaining the sufficiency part of the Aubin criterion as a corollary from a more general “implicit mapping theorem" (Theorem 2.1) in the following section, which is about the solution mapping of a generalized equation of the form

$$0 \in G(p, x), \tag{7}$$

where p is a parameter. We show that if the partial graphical derivative with respect to x of the mapping G is bounded in the sense of (3), then G has a property of “partial metric regularity."

In a related paper [16], Ledyaev and Zhu obtained an implicit mapping theorem for a general inclusion of the form (7) in terms of coderivatives in Banach spaces assumed to have Fréchet-smooth Lipschitz bump functions. Putting aside the derivative condition in our Theorem 2.1 and the coderivative condition in Theorem 3.7 of [16] which are independent from each other and can not be compared, we impose weaker conditions on the mapping G and allow for arbitrary Banach spaces X and Y.

In Section 4 we present applications of the Aubin criterion to systems of inequalities and to variational inequalities, obtaining a new characterization of strong regularity of variational inequalities over polyhedral sets. We also give a new proof[1] of the radius (Eckart-Young) theorem first proven in [11] with the help of Mordukhovich criterion; for history and recent developments, see [11], [10] and [8].

In addition to the Aubin criterion, in Section 4 we use a fundamental result in the modern nonlinear analysis, commonly known as the Lyusternik-Graves theorem, for more see, e.g.,

[1]The initially submitted version of the paper did not contain this proof; it was provided by N. Zlateva shortly before the paper went to press.

[2], [4], [11] and [13]. First, we need some terminology. For a function $g : X \to Y$ and a point $\bar{x} \in \operatorname{int} \operatorname{dom} g$, we introduce Lipschitz modulus of g at $\bar{x}$ as follows:

$$\operatorname{lip} g(\bar{x}) = \limsup_{\substack{x,x' \to \bar{x} \\ x \neq x'}} \frac{\|g(x) - g(x')\|}{\|x - x'\|}.$$

Recall that a function $g : X \to Y$ is strictly differentiable at $\bar{x} \in \operatorname{int} \operatorname{dom} g$ with a strict derivative mapping $\nabla g(\bar{x}) \in L(X, Y)$, the space of linear bounded mappings from X to Y, if

$$\operatorname{lip}(g - \nabla g(\bar{x}))(\bar{x}) = 0.$$

We use the following form of the Lyusternik-Graves theorem:

Theorem 1.4 (Lyusternik-Graves). *Let X and Y be Banach spaces and consider a mapping $\mathcal{F} : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}$ at which $\operatorname{gph} \mathcal{F}$ is locally closed. Then for any function $g : X \to Y$ which is strictly differentiable at $\bar{x}$ one has*

$$\operatorname{reg}(g + \mathcal{F})(\bar{x} \,|\, \bar{y} + g(\bar{x})) = \operatorname{reg}(\nabla g(\bar{x}) + \mathcal{F})(\bar{x} \,|\, \bar{y} + \nabla g(\bar{x})\bar{x}).$$

2. An implicit mapping theorem

In this section we study the inclusion (generalized equation)

$$0 \in G(p, x),$$

where $G : P \times X \rightrightarrows Y$, X and Y are Banach spaces, P is a metric space, $x \in X$ is the variable we are solving for and $p \in P$ is a parameter. Let us denote by $S : P \rightrightarrows X$ the *solution mapping* which associates to a value p the set of solutions

$$S(p) := \{x \in X \mid G(p, x) \ni 0\}. \tag{8}$$

We will show that the local boundedness of the partial graphical derivative of the mapping G in x, of the kind displayed in (3), implies partial metric regularity of G. The partial graphical derivative $D_x G(p, x|y)$ of G is defined as the graphical derivative of the mapping $x \mapsto G(p, x)$ with p fixed.

Theorem 2.1 (Implicit mapping theorem). *Let X and Y be Banach spaces, and let P be a metric space. Consider a mapping $G : P \times X \rightrightarrows Y$ and a point $(\bar{p}, \bar{x}, 0) \in \operatorname{gph} G$ such that the graph of G is locally closed near $(\bar{p}, \bar{x}, 0)$ and the function $p \to d(0, G(p, \bar{x}))$ is upper semicontinuous at $\bar{p}$. Then for every positive scalar c satisfying*

$$\limsup_{\substack{(p,x,y) \to (\bar{p},\bar{x},0) \\ (p,x,y) \in \operatorname{gph} G}} \|D_x G(p, x|y)^{-1}\|^- < c \tag{9}$$

there exist neighborhoods V of $\bar{p}$ and U of $\bar{x}$ such that one has

$$d(x, S(p)) \leq c d(0, G(p, x)) \quad \text{for } x \in U \text{ and } p \in V. \tag{10}$$

Proof. On the product space $Z := X \times Y$ we consider the norm

$$|||(x, y)||| := \max\{\|x\|, c\|y\|\},$$

which makes $(Z, |||\cdot|||)$ a Banach space, and on the space $P \times Z$ we introduce the metric

$$\sigma((p,z),(q,w)) := \max\{\rho(p,q), |||z-w|||\} \quad \text{for } p,q \in P,\ z,w \in Z,$$

where ρ stands for the metric of P.

A constant c satisfies (9) if and only if there exists $\eta > 0$ such that

$$\begin{aligned} &\text{for every } (p,x,y) \in \operatorname{gph} G \text{ with } \sigma((p,x,y),(\bar{p},\bar{x},0)) \le 3\eta, \\ &\text{and for every } v \in Y \text{ there exists } u \in D_x G(p,x|y)^{-1}(v) \text{ with } \|u\| \le c\|v\|. \end{aligned} \tag{11}$$

We can always choose η smaller so that the set $\operatorname{gph} G \cap \mathbb{B}_{3\eta}(\bar{p},\bar{x},0)$ is closed. Next, let us pick $\varepsilon > 0$ such that

$$c\varepsilon < 1. \tag{12}$$

In the proof we use the following lemma:

Lemma 2.2. *For η and ε as above, choose any $(p,\omega,\nu) \in \operatorname{gph} G$ with $(p,\omega,\nu) \in \mathbb{B}_\eta(\bar{p},\bar{x},0)$ and any s, $0 < s \le \varepsilon\eta$. Then for every $y' \in \mathbb{B}^\circ_s(\nu)$ there exists $\hat{x}$ with $(p,\hat{x},y') \in \operatorname{gph} G$ such that*

$$\|\hat{x} - \omega\| \le \frac{1}{\varepsilon}\|y' - \nu\|. \tag{13}$$

Proof of Lemma 2.2. Pick $(p,\omega,\nu) \in \operatorname{gph} G$ and s as required. The set $E_p := \{(x,y) | (p,x,y) \in \operatorname{gph} G \cap \mathbb{B}_{3\eta}(\bar{p},\bar{x},0)\} \subset X \times Y$ equipped with the metric induced by the norm $|||\cdot|||$ is a complete metric space. The function $V_p : E_p \to \mathbb{R}$ defined as

$$V_p(x,y) := \|y' - y\| \quad \text{for } (x,y) \in E_p$$

is continuous in its domain E_p. We apply the Ekeland variational principle to V_p for (x,y) near (ω,ν) and the ε chosen in (12) to obtain the existence of $(\hat{x},\hat{y}) \in E_p$ such that

$$V_p(\hat{x},\hat{y}) + \varepsilon|||(\omega,\nu) - (\hat{x},\hat{y})||| \le V_p(\omega,\nu) \tag{14}$$

and

$$V_p(\hat{x},\hat{y}) \le V_p(x,y) + \varepsilon|||(x,y) - (\hat{x},\hat{y})||| \quad \text{for all } (x,y) \in E_p. \tag{15}$$

The relations (14) and (15) come down as

$$\|y' - \hat{y}\| + \varepsilon|||(\omega,\nu) - (\hat{x},\hat{y})||| \le \|y' - \nu\| \tag{16}$$

and

$$\|y' - \hat{y}\| \le \|y' - y\| + \varepsilon|||(x,y) - (\hat{x},\hat{y})||| \quad \text{for all } (x,y) \in E_p. \tag{17}$$

From (16) we obtain

$$|||(\omega,\nu) - (\hat{x},\hat{y})||| \le \frac{1}{\varepsilon}\|y' - \nu\|. \tag{18}$$

Since $y' \in \mathbb{B}^\circ_s(\nu)$, we then have

$$|||(\omega,\nu) - (\hat{x},\hat{y})||| < \frac{s}{\varepsilon},$$

and hence, from the choice of (p,ω,ν),

$$\begin{aligned}\sigma((p,\hat{x},\hat{y}),(\bar{p},\bar{x},0)) &\le \sigma((p,\hat{x},\hat{y}),(p,\omega,\nu))+\sigma((p,\omega,\nu),(\bar{p},\bar{x},0))\\ &\le \eta+\|\!|(\omega,\nu)-(\hat{x},\hat{y})|\!\|<\eta+\frac{s}{\varepsilon}\le 2\eta.\end{aligned}$$

Thus, $(p,\hat{x},\hat{y})\in \operatorname{gph} G$ with $\sigma((\bar{p},\bar{x},0),(p,\hat{x},\hat{y}))\le 2\eta$, and then (11) implies that there exists $u\in X$ such that

$$y'-\hat{y}\in D_xG(p,\hat{x}\,|\,\hat{y})(u) \quad \text{and} \quad \|u\|\le c\|y'-\hat{y}\|. \tag{19}$$

By the definition of the partial graphical derivative, there exist sequences $t_n\downarrow 0$, $u_n\to u$, and $v_n\to y'-\hat{y}$ such that

$$\hat{y}+t_nv_n\in G(p,\hat{x}+t_nu_n) \text{ for all } n,$$

meaning that, for sufficiently large n, $(\hat{x}+t_nu_n,\hat{y}+t_nv_n)\in E_p$. Now, if we plug the point $(\hat{x}+t_nu_n,\hat{y}+t_nv_n)$ into (17), we obtain

$$\|y'-\hat{y}\|\le \|y'-(\hat{y}+t_nv_n)\|+\varepsilon\|\!|(\hat{x}+t_nu_n,\hat{y}+t_nv_n)-(\hat{x},\hat{y})|\!\|$$

resulting in

$$\|y'-\hat{y}\|\le (1-t_n)\|y'-\hat{y}\|+t_n\|v_n-(y'-\hat{y})\|+\varepsilon t_n\|\!|(u_n,v_n)|\!\|.$$

After obvious simplifications, this gives

$$\|y'-\hat{y}\|\le \varepsilon\|\!|(u_n,v_n)|\!\|+\|v_n-(y'-\hat{y})\|.$$

Passing to the limit with $n\to\infty$ we obtain

$$\|y'-\hat{y}\|\le \varepsilon\|\!|(u,y'-\hat{y})|\!\|$$

and hence, taking into account the second relation in (19) we conclude that

$$\|y'-\hat{y}\|\le \varepsilon c\|y'-\hat{y}\|.$$

Since by (12) $\varepsilon c<1$, we finally obtain that $y'=\hat{y}$. Then (18) yields (13) and the proof of the lemma is complete. □

We continue with the proof of Theorem 2.1. Fix $s\in(0,\varepsilon\eta/2]$. Since the function $p\to d(0,G(p,\bar{x}))$ is upper semicontinuous at $\bar{p}$, there exists $\delta>0$ such that $d(0,G(p,\bar{x}))\le s/2$ for all p with $\rho(p,\bar{p})<\delta$. Of course, we can take smaller δ, e.g., $\delta\le s/\varepsilon$. For such p we can find y such that $y\in G(p,\bar{x})$ with $\|y\|\le d(0,G(p,\bar{x}))+s/3<s$. Then we apply Lemma 2.2 with s, $y'=0$ and $(p,\omega,\nu)=(p,\bar{x},y)$ inasmuch as

$$\sigma((p,\bar{x},y),(\bar{p},\bar{x},0))=\max\{\rho(p,\bar{p}),c\|y\|\}\le \max\{\delta,cs\}\le \max\left\{\frac{s}{\varepsilon},cs\right\}=\frac{s}{\varepsilon}\le\frac{\varepsilon\eta}{\varepsilon}=\eta,$$

obtaining the existence of $\hat{x}$ such that $(p,\hat{x},0)\in\operatorname{gph} G$; that is, $\hat{x}\in S(p)$. Also, from the estimate (13) with $\omega=\bar{x}$ we have that $\hat{x}\in I\!B^{\circ}_{s/\varepsilon}(\bar{x})$.

Set $V:=I\!B^{\circ}_{\delta}(\bar{p})$, $U:=I\!B^{\circ}_{s/\varepsilon}(\bar{x})$ and pick $p\in V$ and $x\in U$. We consider two cases.

Case 1. $d(0, G(p, x)) \geq 2s$.

We just proved that there exists $\hat{x} \in S(p)$ with $\hat{x} \in I\!B^{\circ}_{s/\varepsilon}(\bar{x})$; then

$$\begin{aligned} d(x, S(p)) &\leq d(\bar{x}, S(p)) + \|x - \bar{x}\| \leq \|\bar{x} - \hat{x}\| + \|x - \bar{x}\| \\ &\leq \frac{s}{\varepsilon} + \frac{s}{\varepsilon} = \frac{2s}{\varepsilon} \leq \frac{1}{\varepsilon} d(0, G(p, x)). \end{aligned} \tag{20}$$

Case 2. $d(0, G(p, x)) < 2s$.

In this case, for a sufficiently small $\gamma > 0$ we can find $y_\gamma \in G(p, x)$ such that

$$\|y_\gamma\| \leq d(0, G(p, x)) + \gamma < 2s.$$

Then

$$c\|y_\gamma\| < 2cs \leq 2c\frac{\varepsilon\eta}{2} < \eta$$

and, hence, $(p, x, y_\gamma) \in \operatorname{gph} G$ is such that $\sigma((p, x, y_\gamma), (\bar{p}, \bar{x}, 0)) \leq \eta$. Applying Lemma 2.2 for $(p, \omega, \nu) = (p, x, y_\gamma)$, $y' = 0$ and $2s$ in place of s, we find $\hat{x}_\gamma \in S(p)$ such that

$$\|x - \hat{x}_\gamma\| \leq \frac{1}{\varepsilon}\|y_\gamma\|.$$

Then, by the choice of y_γ,

$$d(x, S(p)) \leq \|x - \hat{x}_\gamma\| \leq \frac{1}{\varepsilon}\|y_\gamma\| \leq \frac{1}{\varepsilon}(d(0, G(p, x)) + \gamma),$$

thus

$$d(x, S(p)) \leq \frac{1}{\varepsilon}(d(0, G(p, x)) + \gamma).$$

The left-hand side of this inequality does not depend on γ, hence letting $\gamma \downarrow 0$ leads to

$$d(x, S(p)) \leq \frac{1}{\varepsilon} d(0, G(p, x)). \tag{21}$$

We obtained this inequality also in the Case 1 in (20), hence it holds for any p in V and $x \in U$. Since $1/\varepsilon$ can be arbitrarily close to c, this gives us (10). □

The relation (10), obtained in Theorem 2.1 can be considered as *metric regularity of* G *with respect to* x at $(\bar{p}, \bar{x})$ for 0. Parallel to the partial metric regularity of G in x, we can define the partial Aubin property for G in p in the following way: $G : P \times X \rightrightarrows Y$ is said to have the *Aubin property with respect to* p *uniformly in* x at $(\bar{p}, \bar{x})$ for 0 if $0 \in G(\bar{p}, \bar{x})$ and there exist a constant $\kappa > 0$ and neighborhoods O of 0, Q for $\bar{p}$ and U of $\bar{x}$ such that

$$e(G(p, x) \cap O, G(p', x)) \leq \kappa\rho(p, p') \text{ for all } p, p' \in Q \text{ and } x \in U.$$

By combining this definition with (10) one obtains (see also [16], Corollary 3.9)

Proposition 2.3. *Let* $G : P \times X \rightrightarrows Y$ *be both metrically regular with respect to* x *and have the Aubin property with respect to* p *uniformly in* x *at* $(\bar{p}, \bar{x})$ *for* 0. *Then the solution mapping* S *has the Aubin property at* $\bar{p}$ *for* $\bar{x}$.

Proof. Take p, p' near $\bar{p}$ and $x \in S(p)$ near $\bar{x}$. Then we have

$$d(x, S(p')) \le \kappa' d(0, G(p', x)) \le \kappa' \kappa \rho(p, p'),$$

where κ' and κ are the constants of the assumed metric regularity and Aubin property, respectively. Since x is arbitrarily chosen in $S(p)$ near $\bar{x}$, we are done. □

3. Proof of Theorem 1.2.

For short, denote

$$d^-_{DF}(\bar{x}\,|\,\bar{y}) := \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\in \operatorname{gph} F}} \|DF(x\,|\,y)^{-1}\|^-.$$

Step 1. Proof of the inequality $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \le d^-_{DF}(\bar{x}\,|\,\bar{y})$.

If $d^-_{DF}(\bar{x}\,|\,\bar{y}) = +\infty$ there is nothing to prove. Let $d^-_{DF}(\bar{x}\,|\,\bar{y}) < \infty$. Applying Theorem 2.1 with $P = Y$ and $G(p, x) = F(x) - p$, for y in the place of p and $\bar{y} = \bar{p}$, we have that $S(y) = F^{-1}(y)$ and $d(0, G(y, x)) = d(y, F(x))$. Then for any $c > d^-_{DF}(\bar{x}\,|\,\bar{y})$ from (10) we obtain that F is metrically regular at $\bar{x}$ for $\bar{y}$ with a constant c. Thus, $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \le c$ and therefore $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \le d^-_{DF}(\bar{x}|\bar{y})$ which gives us (2).

Step 2. Proof of $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = d^-_{DF}(\bar{x}\,|\,\bar{y})$ when X is finite dimensional.

If $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = +\infty$ we are done. Let $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \infty$. Then there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)) \ \text{ whenever } \ x \in U,\ y \in V. \tag{22}$$

It is obvious that when F satisfies (22) one can choose V so small that $F^{-1}(y) \cap U \neq \emptyset$ for all $y \in V$. Pick any $y \in V$ and $x \in F^{-1}(y) \cap U$, and let $v \in I\!B$. Take a sequence $t_n \downarrow 0$ such that $y_n := y + t_n v \in V$ for all n. By (22) there exists $x_n \in F^{-1}(y + t_n v)$ such that

$$\|x - x_n\| = d(x, F^{-1}(y_n)) \le \kappa d(y_n, F(x)) \le \kappa \|y_n - y\| = \kappa t_n \|v\|.$$

For $u_n := (x_n - x)/t_n$ we obtain

$$\|u_n\| \le \kappa \|v\|; \tag{23}$$

thus the sequence u_n is bounded and hence $u_n \to u$ for a subsequence. Since $(x_n, y + t_n v) \in \operatorname{gph} F$, by the definition of the tangent cone, we obtain $(u, v) \in T_{\operatorname{gph} F}(x, y)$ and hence, by the definition of the graphical derivative, we have $u \in DF(x\,|\,y)^{-1}(v)$. From (23) it follows

$$\|DF(x\,|\,y)^{-1}\|^- \le \kappa.$$

Since $(x, y) \in \operatorname{gph} F$ is arbitrarily chosen near $(\bar{x}, \bar{y})$, we conclude that $d^-_{DF}(\bar{x}\,|\,\bar{y}) \le \kappa$. Finally, since κ can be arbitrarily close to $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ we obtain $d^-_{DF}(\bar{x}\,|\,\bar{y}) \le \operatorname{reg} F(\bar{x}\,|\,\bar{y})$. This, combined with (2), gives us (4) and Step 2 of the proof is complete.

Step 3. Proof of

$$\limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\in \operatorname{gph} F}} \|D^{\star\star} F(x\,|\,y)^{-1}\|^- = d^-_{DF}(\bar{x}\,|\,\bar{y})$$

when both X and Y are finite dimensional.

Since $\operatorname{gph} D^{\star\star}F(x\,|\,y) = \operatorname{clco}\operatorname{gph} DF(x\,|\,y)$, we have $D^{\star\star}F(x\,|\,y)^{-1}(v) \supset DF(x\,|\,y)^{-1}(v)$ for any v, which implies

$$\inf_{u\in D^{\star\star}F(x\,|\,y)^{-1}(v)} \|u\| \le \inf_{u\in DF(x\,|\,y)^{-1}(v)} \|u\|,$$

consequently

$$\|D^{\star\star}F(x\,|\,y)^{-1}\|^{-} \le \|DF(x\,|\,y)^{-1}\|^{-},$$

and then

$$\limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|D^{\star\star}F(x\,|\,y)^{-1}\|^{-} \le d^{-}_{DF}(\bar{x}\,|\,\bar{y}).$$

Therefore, we only need to prove the opposite inequality

$$d^{-}_{DF}(\bar{x}\,|\,\bar{y}) \le \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|D^{\star\star}F(x\,|\,y)^{-1}\|^{-} \tag{24}$$

If the right hand side in (24) is finite, pick λ such that

$$\limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in\operatorname{gph} F}} \|D^{\star\star}F(x\,|\,y)^{-1}\|^{-} < \lambda < +\infty.$$

Let $X\times Y$ be equipped with the Euclidian norm, and let $r>0$ be small enough to ensure that

$$\max_{v\in B} \min_{u\in D^{\star\star}F(x\,|\,y)^{-1}(v)} \|u\| \le \lambda \quad \text{ for all } (x,y)\in \operatorname{gph} F\cap I\!B_r(\bar{x},\bar{y}), \tag{25}$$

and that $\operatorname{gph} F\cap I\!B_r(\bar{x},\bar{y})$ is a closed set. We will prove that

$$\max_{v\in B} \min_{u\in DF(x\,|\,y)^{-1}(v)} \|u\| \le \lambda \quad \text{ for all } (x,y)\in \operatorname{gph} F\cap I\!B^{\circ}_r(\bar{x},\bar{y}). \tag{26}$$

Fix $v\in I\!B$. For any sets A,B denote by $d(A,B) := \inf\{\|a-b\| \mid a\in A, b\in B\}$. Let us fix $(x,y)\in \operatorname{gph} F\cap I\!B^{\circ}_r(\bar{x},\bar{y})$. Let $(u^*,v^*)\in \operatorname{gph} DF(x\,|\,y)$ and $w\in\lambda I\!B$ be such that

$$\|(w,v)-(u^*,v^*)\| = d(\lambda I\!B\times\{v\}, \operatorname{gph} DF(x\,|\,y)).$$

Observe that the point (u^*,v^*) is the unique projection of any point in the open segment $((u^*,v^*),(w,v))$ on $\operatorname{gph} DF(x\,|\,y)$. We will prove that $(u^*,v^*) = (w,v)$ and this will be enough to have (26) and hence (24).

By the definition of the graphical derivative, there exist sequences $t_n\downarrow 0$, $u_n\to u^*$, and $v_n\to v^*$ such that $y+t_nv_n\in F(x+t_nu_n)$ for all n. Let (x_n,y_n) be a point in $\operatorname{cl}\operatorname{gph} F$ which is closest to $(x,y)+\frac{t_n}{2}(u^*+w,v^*+v)$ (a projection, not necessarily unique, of the latter point on the closure of $\operatorname{gph} F$). Since $(x,y)\in\operatorname{gph} F$ we have

$$\left\|(x,y)+\frac{t_n}{2}(u^*+w,v^*+v)-(x_n,y_n)\right\| \le \frac{t_n}{2}\left\|(u^*+w,v^*+v)\right\|,$$

and hence

$$\begin{aligned}\|(x,y)-(x_n,y_n)\| &\le \left\|(x,y)+\frac{t_n}{2}(u^*+w,v^*+v)-(x_n,y_n)\right\| \\ &\quad +\frac{t_n}{2}\left\|(u^*+w,v^*+v)\right\| \le t_n\left\|(u^*+w,v^*+v)\right\|\end{aligned}$$

Thus, for n sufficiently large, we have $(x_n, y_n) \in \mathbb{B}_r^\circ(\bar{x}, \bar{y})$ and hence $(x_n, y_n) \in \operatorname{gph} F \cap \mathbb{B}_r^\circ(\bar{x}, \bar{y})$. Setting $(\bar{u}_n, \bar{v}_n) = (x_n - x, y_n - y)/t_n$, we deduce by the usual property of a projection that

$$\frac{1}{2}(u^* + w, v^* + v) - (\bar{u}_n, \bar{v}_n) \in [T_{\operatorname{gph} F}(x_n, y_n)]^0 = [\operatorname{gph} D^{\star\star}F(x_n|y_n)]^0,$$

where K^0 stands for the negative polar cone of a set K. Then, by (25), there exists $w_n \in \lambda\mathbb{B}$ such that $v \in D^{\star\star}F(x_n|y_n)(w_n)$ and from the above relation

$$\left\langle \frac{u^* + w}{2} - \bar{u}_n, w_n \right\rangle + \left\langle \frac{v^* + v}{2} - \bar{v}_n, v \right\rangle \leq 0. \tag{27}$$

We claim that $(\bar{u}_n, \bar{v}_n)$ converges to (u^*, v^*) as $n \to \infty$. Indeed,

$$\begin{aligned}
& \left\| \left(\frac{u^* + w}{2}, \frac{v^* + v}{2} \right) - (\bar{u}_n, \bar{v}_n) \right\| \\
= \; & \frac{1}{t_n} \left\| (x, y) + t_n \left(\frac{u^* + w}{2}, \frac{v^* + v}{2} \right) - (x_n, y_n) \right\| \\
\leq \; & \frac{1}{t_n} \left\| (x, y) + t_n \left(\frac{u^* + w}{2}, \frac{v^* + v}{2} \right) - (x, y) - t_n(u_n, v_n) \right\| \\
= \; & \left\| \left(\frac{u^* + w}{2}, \frac{v^* + v}{2} \right) - (u_n, v_n) \right\|.
\end{aligned}$$

Therefore, $(\bar{u}_n, \bar{v}_n)$ is a bounded sequence and then, since $y_n = y + t_n\bar{v}_n \in F(x_n) = F(x + t_n\bar{u}_n)$, every cluster point $(\bar{u}, \bar{v})$ of it belongs to $\operatorname{gph} DF(x|y)$. Moreover, $(\bar{u}, \bar{v})$ satisfies

$$\left\| \left(\frac{u^* + w}{2}, \frac{v^* + v}{2} \right) - (\bar{u}, \bar{v}) \right\| \leq \left\| \left(\frac{u^* + w}{2}, \frac{v^* + v}{2} \right) - (u^*, v^*) \right\|.$$

The above inequality together with the fact that (u^*, v^*) is the unique closest point to $\frac{1}{2}(u^* + w, v^* + v)$ in $\operatorname{gph} DF(x|y)$ implies that $(\bar{u}, \bar{v}) = (u^*, v^*)$. Our claim is proved.

Up to a subsequence, w_n satisfying (27) converges to some $\bar{w} \in \lambda\mathbb{B}$. Passing to the limit in (27) one obtains

$$\langle w - u^*, \bar{w} \rangle + \langle v - v^*, v \rangle \leq 0. \tag{28}$$

Since (w, v) is the unique closest point of (u^*, v^*) to the closed convex set $\lambda\mathbb{B} \times \{v\}$, we have

$$\langle w - u^*, w - \bar{w} \rangle \leq 0. \tag{29}$$

Finally, since (u^*, v^*) is the unique closest point to $\frac{1}{2}(u^* + w, v^* + v)$ in $\operatorname{gph} DF(x|y)$ which is a closed cone, we get

$$\langle w - u^*, u^* \rangle + \langle v - v^*, v^* \rangle = 0. \tag{30}$$

In view of (28), (29) and (30), we obtain

$$\begin{aligned}
& \|(w, v) - (u^*, v^*)\|^2 \\
= \; & \langle w - u^*, w - \bar{w} \rangle + (\langle w - u^*, \bar{w} \rangle + \langle v - v^*, v \rangle) - (\langle w - u^*, u^* \rangle + \langle v - v^*, v^* \rangle) \leq 0.
\end{aligned}$$

Hence $w = u^*$ and $v = v^*$ and the proof is complete. □

4. Applications

As a first specific application of the Aubin criterion we consider the constraint system

$$\text{Find } x \in I\!R^n \text{ such that } f_i(x) \begin{cases} = 0 & \text{for } i = 1, \ldots, r, \\ \leq 0 & \text{for } i = r+1, \ldots, m, \end{cases} \tag{31}$$

where $f_i : I\!R^n \to I\!R, i = 1, \ldots, m$. This system can also be written as the inclusion $0 \in F(x)$ with $F : I\!R^n \rightrightarrows I\!R^m$ given by

$$F(x) = f(x) + K, \tag{32}$$

where $f = (f_1, \ldots, f_m)$ and $K = \{0\}^r \times I\!R_+^{m-r}$. Let $\bar{x}$ be a solution of (31) and f be strictly differentiable at $\bar{x}$. We denote the index set of active inequality constraints at $\bar{x}$ as

$$\bar{J} = \{i \in \{r+1, \ldots, m\} \mid f_i(\bar{x}) = 0\}.$$

We will now show that Aubin criterion directly leads to the following well-known result:

Theorem 4.1. *The mapping F in (32) is metrically regular at $\bar{x}$ for 0 if and only if the Mangasarian-Fromovitz condition holds: the vectors $\nabla f_i(\bar{x})$, $i = 1, \ldots, r$ are linearly independent and also there exists $w \in I\!R^n$ such that*

$$\begin{cases} \nabla f_i(\bar{x}) w = 0 & \textit{for } i = 1, \ldots, r, \\ \nabla f_i(\bar{x}) w < 0 & \textit{for } i \in \bar{J}. \end{cases} \tag{33}$$

Proof. By the Lyusternik-Graves theorem (Theorem 1.4) with $\mathcal{F} = K$ and $g = f$, the metric regularity of the mapping F at $\bar{x}$ for 0 is equivalent to the metric regularity at $\bar{x}$ for 0 of its "partial linearization"

$$F_0(x) = f(\bar{x}) + A(x - \bar{x}) + K \quad \text{where } A = \nabla f(\bar{x}).$$

Also, by the specific form of K,

$$v \in DK(x|y)(u) \iff \begin{cases} v_i = 0 & \text{for } i \in I(y), \\ v_i \geq 0 & \text{for } i \in J(y), \end{cases}$$

where

$$I(y) = \{i \in \{1, \ldots, r\} \mid \ y_i = 0\} \quad \text{and} \quad J(y) = \{i \in \{r+1, \ldots, m\} \mid y_i = 0\}. \tag{34}$$

Then, of course, $\bar{J} = J(f(\bar{x}))$. Since $f_i(\bar{x}) < 0$ for $i \in \{r+1, \ldots, m\} \setminus \bar{J}$, we have that $y_i - f_i(x) > 0$ for all such i and for (x, y) close to $(\bar{x}, 0)$. This means that for such (x, y) the set $J(y)$ in (34) is always a subset of $\bar{J}$. Then the Aubin criterion for metric regularity of F_0 becomes the following condition: for every $I \subset \{1, \ldots, r\}$ and for every $J \subset \bar{J}$ we have:

$$\forall v \in I\!R^{I \cup J} \ \ \exists u \in I\!R^n \text{ such that } (v - Au)_i = 0 \text{ for } i \in I \text{ and } (v - Au)_i \geq 0 \text{ for } i \in J. \tag{35}$$

Assume that Mangasarian-Fromovitz condition holds and let $I \subset \{1, \ldots, r\}$ and $J \subset \bar{J}$. If either $I = \emptyset$ or $J = \emptyset$ we skip the corresponding step of the proof. Let $I \neq \emptyset$. Then the matrix $H = [\nabla f_i(\bar{x})]_{i \in I}$ is onto and hence, by the metric regularity of H there exists a constant κ such that

$$\forall v \in I\!R^I \quad \exists u \in I\!R^n \text{ such that } v - Hu = 0 \text{ and } \|u\| \le \kappa \|v\|. \tag{36}$$

This means in particular that taking v with a norm small enough we can have the corresponding u in (36) with arbitrarily small norm. Then, since $\nabla f_i(\bar{x})w < -\alpha$ for all $i \in J$ and some $\alpha > 0$, we end up having that for any $v \in I\!R^{I \cup J}$ with sufficiently small norm

$$v_i - \nabla f_i(\bar{x})(u + w) \begin{cases} = 0 & \text{for } i \in I, \\ \ge 0 & \text{for } i \in J. \end{cases} \tag{37}$$

By the positive homogeneity, from (36) and (37) we obtain (35).

Conversely, if (35) holds, then taking $I = \{1, \dots, r\}$ and $J = \emptyset$ we conclude that $\nabla f_i(\bar{x}), i = 1, \dots, r$ must be linearly independent. Next, taking $I = \{1, \dots, r\}$ and $J = \bar{J}$, for

$$v_i = \begin{cases} 0 & \text{for } i = 1, \dots, r, \\ -\varepsilon & \text{for } i \in \bar{J} \end{cases}$$

with some $\varepsilon > 0$ we obtain (33). □

Our second application is for a mapping describing the variational inequality

$$\langle f(x), u - x \rangle \ge 0 \text{ for all } u \in C, \tag{38}$$

where $f : I\!R^n \to I\!R^n$ and C a nonempty convex closed set in $I\!R^n$ that is *polyhedral.* In terms of the normal cone mapping

$$N_C(x) = \begin{cases} \{y \mid \langle y, u - x \rangle \le 0 \text{ for all } u \in C\} & \text{for } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

we can write the variational inequality (38) as the inclusion $0 \in F(x)$ where

$$F(x) = f(x) + N_C(x). \tag{39}$$

We assume that $\bar{x}$ is a solution of (38) and f is strictly differentiable at $\bar{x}$. Then, again, the Lyusternik-Graves theorem, this time with $\mathcal{F} = N_C$ and $g = f$, allows us to restrict our attention to the linearized mapping

$$F_0(x) = f(\bar{x}) + A(x - \bar{x}) + N_C(x) \quad \text{where } A = \nabla f(\bar{x}).$$

Let $[v]$ be the subspace of dimension one (or zero for $v = 0$) spanned on a vector $v \in I\!R^n$, that is, $[v] = \{\tau v \mid \tau \in I\!R\}$, and let $[v]^\perp$ be its orthogonal complement. The form of the graphical derivative of F_0 will be obtained by introducing the *critical cone* $K(x, v)$ to the set C at $x \in C$ for $v \in N_C(x)$,

$$K(x, v) = T_C(x) \cap [v]^\perp,$$

via the following lemma:

Lemma 4.2 (Reduction lemma [9]). *Let C be a convex polyhedral set in $I\!R^n$. For any $(x,v) \in \operatorname{gph} N_C$ there is a neighborhood O of the origin in $I\!R^n \times I\!R^n$ such that for $(x',v') \in O$ one has*

$$v + v' \in N_C(x + x') \iff v' \in N_{K(x,v)}(x').$$

Consequently,

$$(x',v') \in T_{\operatorname{gph} N_C}(x,v) \iff v' \in N_{K(x,v)}(x'),$$

and hence, for $(x,y) \in \operatorname{gph} F_0$ and $v = y - f(\bar{x}) - A(x - \bar{x})$, where $A = \nabla f(\bar{x})$, we have

$$DF_0(x|y)(u) = Au + N_{K(x,v)}(u). \tag{40}$$

For any cone K, a set of the form

$$F = K \cap [v]^{\perp} \text{ for some } v \in K^0,$$

where K^0 is the polar to K, is said to be a *face* of K. The largest of the faces is K itself while the smallest is the set $K \cap (-K)$ which is the largest subspace contained in K. Every polyhedral cone has finitely many faces.

Our next lemma gives the form of the of critical cones in a neighborhood of a fixed reference point. It is extracted from the proof of Theorem 2 in [9].

Lemma 4.3. *Let C be a convex polyhedral set, let $\bar{v} \in N_C(\bar{x})$ and let $\bar{K}$ be the critical cone to C at $\bar{x}$ for $\bar{v}$. Then there exists an open neighborhood O of $(\bar{x},\bar{v})$ such that for every choice of $(x,v) \in \operatorname{gph} N_C \cap O$ the corresponding critical cone $K(x,v)$ has the form*

$$K(x,v) = F_1 - F_2,$$

for some faces F_1, F_2 of $\bar{K}$ with $F_1 \supset F_2$. And conversely, for every two faces F_1, F_2 of $\bar{K}$ with $F_1 \supset F_2$ and every neighborhood O of $(\bar{x},\bar{v})$ there exists $(x,v) \in O$ such that $K(x,v) = F_1 - F_2$.

It was proved in [9], Theorem 1, that the metric regularity of a mapping F of the form (39) with a polyhedral set C implies a sharper property called *strong regularity*. A mapping $F : X \rightrightarrows Y$ is said to be strongly regular at $\bar{x}$ for $\bar{y}$ is it is metrically regular there and, in addition, the graphical localization of its inverse F^{-1} near $(\bar{y},\bar{x})$ is single-valued. In other words, F is strongly regular at $\bar{x}$ for $\bar{y}$ when there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that the mapping $V \ni y \mapsto F^{-1}(y) \cap U$ is a Lipschitz continuous function.

We are now ready to apply the Aubin criterion to obtain a new necessary and sufficient condition for strong regularity of variational inequalities over polyhedral sets, which complements the criterion given in [9], Theorem 2:

Theorem 4.4. *The variational inequality mapping* (39) *is strongly regular at $\bar{x}$ for $\bar{y}$ if and only if for all choices of faces F_1 and F_2 of the critical cone $\bar{K}$ to the set C at $\bar{x}$ for $\bar{v} = \bar{y} - f(\bar{x})$, with $F_1 \supset F_2$, the following condition holds:*

$$\forall v \in I\!R^n \quad \exists u \in F_1 - F_2 \text{ such that } (v - Au) \in (F_1 - F_2)^0 \text{ and } v - Au \perp u.$$

Proof. According to Aubin criterion given in Theorem 1.2, the mapping F_0 (and, hence F) is metrically regular, and hence strongly regular, if and only if the limsup of the inner norms of the graphical derivatives is finite. The form of the graphical derivative of F_0 is given in (40) while in Lemma 4.3 it is shown that there are finitely many critical cones near the reference point $(\bar{x}, \bar{v})$ that are to be taken into account, and these cones are given by faces of $\bar{K}$ in a way described in this lemma. Hence, for any choice of faces F_1 and F_2 of $\bar{K}$ with $F_1 \supset F_2$ it is enough to ensure that $\|A + N_{F_1 - F_2}\|^-$ is finite. For any cone K

$$v \in N_K(x) \iff x \in K, \quad v \in K^0, \quad x \perp v.$$

It remains to observe that the inner norm of the mapping $A + N_{F_1 - F_2}$ will be finite if and only if the condition claimed in the theorem holds. □

Our last aplication of Aubin criterion is a new proof of the radius theorem first proved in [11].

Theorem 4.5. *Let X and Y be finite-dimensional linear normed spaces and let $F : X \rightrightarrows Y$ has closed graph locally near $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$\inf_{G \in L(X,Y)} \{\|G\| \mid F + G \textit{ is not metrically regular at } \bar{x} \textit{ for } \bar{y} + G(\bar{x})\} = \frac{1}{\operatorname{reg} F(\bar{x}|\bar{y})}.$$

Moreover, the infimum is unchanged if taken with respect to linear mappings G of rank 1, but also remains unchanged when the perturbations G are locally Lipschitz continuous functions with $\|G\|$ replaced by the Lipschitz modulus $\operatorname{lip} G(\bar{x})$ of G at $\bar{x}$.

Proof. The general perturbation inequality derived in [11], Corollary 3.4, yields (also in infinite dimensions) the estimate

$$\inf_{G : X \to Y} \{\operatorname{lip} G(\bar{x}) \mid F + G \text{ is not metrically regular at } \bar{x} \text{ for } \bar{y} + G(\bar{x})\} \geq \frac{1}{\operatorname{reg} F(\bar{x}|\bar{y})}. \tag{41}$$

It remains to show the opposite inequality. The limit cases are easy to handle, since if $\operatorname{reg} F(\bar{x}|\bar{y}) = \infty$ we have nothing to prove, and if $\operatorname{reg} F(\bar{x}|\bar{y}) = 0$, then by the general perturbation inequality (41), which also holds in this case, we obtain the claimed equality.

Let now $0 < \operatorname{reg} F(\bar{x}|\bar{y}) < \infty$. By Theorem 1.2 we have $\operatorname{reg} F(\bar{x}|\bar{y}) = d^-_{DF}(\bar{x}|\bar{y}) = d^-_{D^{\star\star}F}(\bar{x}|\bar{y})$ where $d^-_{DF}(\bar{x}|\bar{y})$ is defined in the beginning of Section 3 while $d^-_{D^{\star\star}F}(\bar{x}|\bar{y})$ is defined in the same way with DF replaced by $D^{\star\star}F$.

Take a sequence of positive reals $\varepsilon_k \to 0$. Then for any k there exists $(x_k, y_k) \in \operatorname{gph} F$ with $(x_k, y_k) \to (\bar{x}, \bar{y})$ and

$$d^-_{D^{\star\star}F}(\bar{x}|\bar{y}) + \varepsilon_k \geq \|D^{\star\star}F(x_k|y_k)^{-1}\|^- \geq d^-_{D^{\star\star}F}(\bar{x}|\bar{y}) - \varepsilon_k > 0.$$

For short, set $H_k := D^{\star\star}F(x_k|y_k)$; then H_k is a sublinear mapping with closed graph. For $S_k := H_k^{*+}$ the norm duality gives us $\|H_k^{-1}\|^- = \|S_k^{-1}\|^+$.

For each k choose a positive real r_k which satisfies $\|S_k^{-1}\|^+ - \varepsilon_k < 1/r_k < \|S_k^{-1}\|^+$. From the last inequality there must exist $(\hat{y}_k, \hat{x}_k) \in \operatorname{gph} S_k$ with $\|\hat{x}_k\| = 1$ and $\|S_k^{-1}\|^+ \geq \|\hat{y}_k\| >$

$1/r_k$. Pick $y_k^* \in Y$ with $\langle \hat{y}_k, y_k^* \rangle = \|\hat{y}_k\|$ and $\|y_k^*\| = 1$ and define the rank-one mapping $\hat{G}_k \in L(Y, X)$ as

$$\hat{G}_k(y) := -\frac{\langle y, y_k^* \rangle}{\|\hat{y}_k\|} \hat{x}_k.$$

Then $\hat{G}_k(\hat{y}_k) = -\hat{x}_k$ and hence $(S_k + \hat{G}_k)(\hat{y}_k) = S_k(\hat{y}_k) + \hat{G}_k(\hat{y}_k) = S_k(\hat{y}_k) - \hat{x}_k \ni 0$. Therefore, $\hat{y}_k \in (S_k + \hat{G}_k)^{-1}(0)$ and since $\hat{y}_k \neq 0$, by [11], Proposition 2.5,

$$\|S_k + \hat{G}_k\|^+ = \infty. \tag{42}$$

Note that $\|\hat{G}_k\| = \|\hat{x}_k\| / \|\hat{y}_k\| = 1/\|\hat{y}_k\| < r_k$.

Since the sequences $\hat{y}_k$, $\hat{x}_k$ and y_k^* are bounded, we can extract from them subsequences converging respectively to $\hat{y}$, $\hat{x}$ and y^*; the limits then satisfy $\|\hat{y}\| = d^-_{D^{\star\star}F}(\bar{x}\,|\,\bar{y})$, $\|\hat{x}\| = 1$ and $\|y^*\| = 1$. Define the rank-one mapping $\hat{G} \in L(Y, X)$ as

$$\hat{G}(y) := -\frac{\langle y, y^* \rangle}{\|\hat{y}\|} \hat{x}.$$

Then we have $\|\hat{G}\| \leq 1/d^-_{D^{\star\star}F}(\bar{x}\,|\,\bar{y})$ and $\|\hat{G}_k - \hat{G}\| \to 0$.

Denote $G := (\hat{G})^*$ and suppose that $F + G$ is metrically regular at $\bar{x}$ for $\bar{y} + G(\bar{x})$. Then Theorem 1.2 yields that for some finite positive constant c and for k sufficiently large we have

$$c > \|D^{\star\star}(F + G)(x_k | y_k + G(x_k))^{-1}\|^- = \|(D^{\star\star}F(x_k | y_k) + G)^{-1}\|^-,$$

which, by norm duality and the equality $G^* = ((\hat{G})^*)^* = \hat{G}$, is equivalent to

$$c > \|([D^{\star\star}F(x_k|y_k) + G]^{*+})^{-1}\|^+ = \|[D^{\star\star}F(x_k|y_k)^{*+} + G^*]^{-1}\|^+ = \|(S_k + \hat{G})^{-1}\|^+. \tag{43}$$

We apply the following lemma, which is a reformulation of a result by Robinson [20], see also [8]:

Lemma 4.6. *For a sublinear mapping $H : X \to Y$ with closed graph and for $B \in L(X, Y)$, if $[\|H^{-1}\|^+]^{-1} \geq \|B\|$, then*

$$\|(H + B)^{-1}\|^+ \leq [[\|H^{-1}\|^+]^{-1} - \|B\|]^{-1}.$$

Now we are ready to complete the proof of Theorem 4.5. Take k sufficiently large such that $\|\hat{G} - \hat{G}_k\| \leq 1/(2c)$ and (x_k, y_k) that satisfies (43). Setting $P_k := S_k + \hat{G}$ and $B_k := \hat{G}_k - \hat{G}$ we have that $[\|P_k^{-1}\|^+]^{-1} \geq 1/c > 1/(2c) \geq \|B_k\|$. By Lemma 4.6 we obtain

$$\|S_k + \hat{G}_k\|^+ = \|P_k + B_k\|^+ \leq [[\|P_k^{-1}\|^+]^{-1} - \|B_k\|]^{-1} \leq 2c < \infty,$$

which contradicts (42). Hence, $F + G$ is not metrically regular at $\bar{x}$ for $\bar{y} + G(\bar{x})$. Remembering that $\|G\| = \|\hat{G}\| \leq 1/\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ we complete the proof. □

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