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A Set Evolution Approach to the Control of Uncertain Systems with Discrete-Time Measurement

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Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

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We investigate here a continuous time minimization problem in the presence of disturbances in the dynamics. The only information available to the controller is an incomplete observation of the state space at times given in advance. Also, the initial state is not supposed to be perfectly known. The corresponding control problem can be understood as a dynamic game of Min-Max type where the controller wants to minimize the cost - by choosing a strategy depending on a discrete-time incomplete measurement - against the worst case of disturbance and initial state. Our main goal is to pass from imperfect information in the *measurement space* to perfect information in the *estimation space*, hence we introduce a second problem based on estimation sets on the state. We prove that the value functions of both problems are equal. Finally, we provide a characterization of the value function through a system of Hamilton-Jacobi equations and inequalities in terms of Dini derivatives.

Keywords: Uncertain systems, optimal control, set evolution, dynamic games, dynamic programming, viability theory, discrete-time measurement, imperfect measurement

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Introduction

Consider the following minimization problem

$$\text{minimize } g(x(T)) \tag{1}$$

subject to

$$x'(t) = f(x(t), u(t), v(t)), \tag{2}$$

where $x \in I\!R^n$ is the state variable, $u(\cdot) : [t_0, T] \to U$ the control, and $v(\cdot) : [t_0, T] \to V$ a disturbance in the dynamics. We can consider u and v as two players acting on the system. The parameter $T > 0$ defines a prescribed fixed time horizon, while the initial moment is t_0, and we suppose that a discretization $\{t_1, t_2, ..., t_N\}$ of $[t_0, T]$ is given in advance and satisfies

$$t_0 < t_1 < t_2 < ... < t_N = T.$$

The initial state $x(t_0)$ is known only through the estimation

$$x(t_0) = e \in E_0, \tag{3}$$

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and the information available to the controller is not the current state, but only a sequence of discrete-time measurements $(y_k)_{k\in[1,N]}$ such that:

$$y_k = h(x(t_k)), \quad t_k \in \{t_1, t_2, ..., t_N\} \tag{4}$$

where $h(\cdot) : \mathbb{R}^n \to Y$ is given, Y being a subset of $\mathbb{R}^p$, $p \leq n$. The main concern of the paper is the optimal control problem of system (2) where the controller wants to minimize - by choosing $u(\cdot)$ - the cost (1) against the worst case of disturbance $v(\cdot)$ and initial state $e \in E_0$. The only informations available to the controller are the initial estimation E_0 and the measurement $(y_k)_{k\in[1,N]}$. Hence, the control performance is understood in Min-Max sense, which yields a dynamic game with a discrete-time incomplete measurement. In other words, the disturbance can be viewed as a first player (choosing an open-loop control $v(\cdot)$), while the controller u can be viewed as a second player playing a strategy over the on-line measurements $(y_k)_{k\in[1,N]}$.

Controlling problems with uncertainties via the differential game approach has already been studied in several papers, both in continuous and discrete-time (see e.g. [5, 6, 8, 12, 15, 16, 19]). In the case of perfect measurement, one can associate a finite dimensional Hamilton-Jacobi-Bellman-Isaacs equation to the optimal control problem. But, according to [5, 6, 8] and [15], this equation becomes infinite dimensional in the case of imperfect measurement. However, it has been proved in [5] that the problem with incomplete measurement can be reformulated as a problem with complete information in the *information space*, which is still infinite dimensional. In other works (cf. [10, 14, 20, 29]), the authors have provided examples where the problem could be reformulated with complete information in the *estimation space*. Generally, the estimation space is also infinite dimensional, but in some cases, one can equivalently replace it with a finite dimensional space. For instance, using the certainty equivalence principle found in [8, 9] for specific games with incomplete state information leads to a finite dimensional H-J-I equation as in [28]. Let us also quote [25, 24, 30] where Viability theory [1] is used to deal with uncertain state information qualitative problems.

Nevertheless, none of the mentioned work combine continuous systems with discrete measurements. A typical case of discrete measurements in a continuous control context consists in a planar pursuit game where the light is alternatively put on and off at times fixed in advance. When the light is on, the player knows some of the coordinates of the current state, but when the light is off the player is supposed to be "blind". This example, derived from [28], has also good applications in the real world. GPS position update for autonomous vehicles is one of them: Autonomous vehicles operate in continuous mode while the GPS information is updated only every second. Another interesting application arises when voluntarily choosing discrete-time measurements over continuous measurements. For example, when dealing with autonomous vehicles, energy saving becomes a real issue and making use of the detectors at low sampling rates helps saving energy. Therefore, one could eval! uate the cost (1) for different sampling rates - or discretizations $\{t_1, t_2, ..., t_N\}$ - to find out which is the smallest one keeping the cost under a certain threshold.

As suggested by the literature (see [10, 14, 20, 27, 29]), we shall start from the incomplete information problem and reformulate it as a complete information problem. According to the information known to the controller, the control synthesis should be first considered in a feedback form which may depend on the current and past values of the discrete-time measurement $(y_k)_{i\in[1,N]}$, but not on v nor e. The corresponding feedback law is a function $\gamma(\cdot,\cdot) : \mathbb{R}^+ \times Y^N \to U$ called the *ouput feedback strategy*. Usually, the guaranteed value obtained by using a given output feedback strategy γ is

$$I_1(t_0, E_0; \gamma) := \sup_{e,v(\cdot)} g\left(x(T, t_0, e, \gamma, v(\cdot))\right), \tag{5}$$

$x(\cdot, t_0, e, \gamma, v(\cdot))$ being the trajectory of system (2) corresponding to control $u(\cdot) := \gamma(\cdot,\cdot)$, disturbance $v(\cdot)$ and initial condition (3). Here, instead of considering single trajectories $x(\cdot)$, we rather take a set-evolution approach such as in [25, 20, 27]. We consider the tube of all possible trajectories of (2) corresponding to a given strategy γ and measurement $(y_k)_{k\in\boldsymbol{N}}$. This tube, called the *estimation tube* and denoted by $E(\cdot, t_0, E_0, \gamma, (y_k)_{k\in\boldsymbol{N}})$, represents the set evolution of E_0 for given strategy γ and measurement $(y_k)_{k\in\boldsymbol{N}}$, and takes into account all possible disturbances $v(\cdot)$. Two related theories have been developed to describe the dynamics of the set-evolution: The approach of the 'funnel equations' as in [18, 31] and the framework of the mutational equations [2, 3]. However, in the context of our paper, both approaches are equivalent.

Let us now explain how this paper is organized. After the preliminaries of Section 1, we introduce in Section 1.2 the estimation tubes. In the case of output feedback strategies, we give in Section 1.3 a reformulation of the value function I_1 involving estimation tubes instead of single trajectories, and we prove the corresponding dynamic programming principle. In Section 1.4, we investigate feedback controls depending only on the current estimation $E(\cdot)$ instead of the whole history of the measurement. Moreover, we eliminate the need for a continuous trajectory $x(\cdot)$ in the formulation of the problem. This leads to a new dynamic game with state $E(t)$ for which full information is available, and we prove in Theorem 1.6 that the corresponding value function $I_2(t_0, E_0)$ coincides with the value function $I_1(t_0, E_0)$ of the initial output feedback formulation. This is the main result of our ! paper. Then, we prove in Section 2 the Lipschitz continuity of the value function, and we provide a new characterization through a system of mixed Hamilton-Jacobi equations and inequalities in terms of Dini derivatives. Finally, as a concluding paragraph, we discuss in Section 3 some possible extensions of the present work.

1. Output Feedback and Estimation Feedback Formulations

1.1. Hypotheses and Notations

In the sequel, control system (2) is supposed to satisfy the following standing suppositions:

Condition 1.1.

1. *$f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ has the form*

$$f(x, u, v) = f_0(x, v) + f_1(x)u$$

where $f_0(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f_1(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are continuous, locally Lipschitz in x uniformly with respect to the other variables (L_f being their Lipschitz constant). U and V are convex compact subsets of $\mathbb{R}^l$ and $\mathbb{R}^m$ respectively. They are bounded by $M > 0$: $U \subset M\mathbf{B}_{R^l}$ and $V \subset M\mathbf{B}_{R^m}$.

2. *$f_0(\cdot, V)$ has convex compact values.*

3. *f satisfies the linear growth condition*

$$\exists c > 0,\ \forall x \in I\!R^n,\ \forall u \in U,\ \forall v \in V, \quad ||f(x,u,v)|| \leq c(1 + ||x||)$$

where $\mathbf{B}_{R^l}$ and $\mathbf{B}_{R^m}$ represent respectively the unit ball in $I\!R^l$ and the unit ball in $I\!R^m$. As a consequence to Condition 1.1, the set-valued map $F(\cdot,\cdot) : I\!R^n \times U \to I\!R^n$ defined by

$$F(x,u) := f(x,u,V), \quad \forall x \in I\!R^n,\ \forall u \in U \tag{6}$$

has convex compact values.

Throughout this paper, the unit ball in $I\!R^n$ will be denoted by $\mathbf{B}$, while $\mathrm{comp}(I\!R^n)$ denotes the collection of all the compact sets of $I\!R^n$. For any two sets $A, B \in \mathrm{comp}(I\!R^n)$, we denote by $H^+(A,B) := \min\{\epsilon \geq 0 \mid A \subset B + \epsilon\mathbf{B}\}$ the Hausdorff semidistance from A to B and by $H(A,B) := \max\{H^+(A,B), H^+(B,A)\}$ the Hausdorff distance between A and B.

We denote by $\mathcal{U}_{[t_0,T]}$ and $\mathcal{V}_{[t_0,T]}$ respectively the sets of all measurable open-loop admissible controls $u(\cdot) : [t_0,T] \to U$ and disturbance $v(\cdot) : [t_0,T] \to V$. Similarly, $\mathcal{X}_{[t_0,T]}$ denotes the set of all continuous fonctions $x(\cdot) : [t_0,T] \to I\!R^n$. If i is an integer, we denote $i \in [j,k]$ when $i \in \{j, j+1, ..., k\}$. We denote by Y^N the set of all sequences $(y_k)_{k\in[1,N]}$ of measurements on $[t_1,T]$. If $h^{-1}(y)$ denotes the set of all states $x \in I\!R^n$ compatible with measurement y, we suppose that

Condition 1.2. *For every $y \in Y$, the set $h^{-1}(y) := \{x \in I\!R^n \mid y = h(x)\}$ is nonempty and closed.*

Note that h can also be used to characterize E_0. For example, one can fix $y_0 \in Y$ and choose $E_0 = \{e \in I\!R^n \mid h(e) = y_0\}$.

1.2. Construction of the Estimation Tubes

Fix an open-loop control $u(\cdot)$ and consider all possible trajectories of system (2), (3). On $[t_0,t_1)$, the uncertainty due to the disturbance v gives rise to the following differential inclusion

$$x'(t) \in F(x(t),u(t)), \quad a.e.\ t \in [t_0,t_1), \tag{7}$$

where F is the set-valued map defined by (6). For any given control $u(\cdot)$, the set of all possible states of system (2), (3) at time $t \in [t_0,t_1)$ is called the *reachable set of system* (3)*,* (7) *at time* t, denoted by

$$R_{F,u}[E_0](t) := \{x(t) \mid \exists e \in E_0,\ \exists v(\cdot),\ x(\cdot) \text{ solution to system (2), (3)}\}. \tag{8}$$

Similarly, for any given control $u(\cdot)$, the set of all possible trajectories of system (2), (3) on $[t_0,t_1)$ is a time depending tube $E(\cdot,t_0,E_0,u(\cdot))$ called the *solution tube of system* (7) *in* $I\!R^n$ *starting from* E_0. It provides the deterministic estimation of the trajectory on $[t_0,t_1)$[1]:

$$x(t) \in E(t,t_0,E_0,u(\cdot)) := R_{F,u}[E_0](t-t_0), \quad a.e.\ t \in [t_0,t_1).$$

For $t = t_1$, the measurement y_1 is given, thus the state $x(t_1)$ belongs to

$$E(t_1,t_0,E_0,u(\cdot),y_1) = R_{F,u}[E_0](t_1-t_0) \cap h^{-1}(y_1),$$

[1]This formulation of the set-evolution is equivalent to what both the funnel equations and the mutational equations would give in the particular collection $\mathrm{comp}(I\!R^n)$.

and, similarly, for any measurement y_k corresponding to time t_k, we have

$$E(t_k, t_{k-1}, E(t_{k-1}), u(\cdot), y_k) = R_{F,u}[E(t_{k-1})](t_k - t_{k-1}) \cap h^{-1}(y_k).$$

Define $\tau(\cdot)$ and $\sigma(\cdot)$ by:

$$t \in [t_k, t_{k+1}) \Rightarrow \tau(t) := t_k, \quad \text{and} \quad t \in [t_k, t_{k+1}) \Rightarrow \sigma(t) := t_{k+1}.$$

Then, for given control $u(\cdot)$ and measurement $(y_k)_{k\in[1,N]}$, the initial set

$$E(t_0) = E_0 \tag{9}$$

"propagates" according to the equation

$$\forall t \in (t_0, T), \quad \begin{cases} E(t, t_0, E_0, u(\cdot), (y_k)_{k\in[1,N]}) = R_{F,u}[E(\tau(t))](t - \tau(t)), \text{ if } t \notin \{t_1, ..., t_{N-1}\} \\ E(\sigma(t), t_0, E_0, u(\cdot), (y_k)_{k\in[1,N]}) = R_{F,u}[E(\tau(t))](\sigma(t) - \tau(t)) \cap h^{-1}(y_j) \end{cases} \tag{10}$$

where y_j is the measurement at time $t_j := \sigma(t)$.

1.3. Output Feedback Formulation

We call *ouptut feedback* strategy (or merely *strategy*) on $[t_0, T]$ any mapping $\gamma : [t_0, T] \times Y^N \to U$ such that $\gamma(t_0, (y_k)_{k\in[1,N]})$ is independent of $(y_k)_{k\in[1,N]}$, and for any $t \in (t_0, T)$, if two given sequences $(y_k^1)_{k\in[1,N]}$ and $(y_k^2)_{k\in[1,N]}$ coincide for all k such that $t_k \le \tau(t)$, then $\gamma(t, (y_k^1)_{k\in[1,N]}) = \gamma(t, (y_k^2)_{k\in[1,N]})$. These strategies are a particular case of the so-called "Varayia-Roxin-Elliott-Kalton non-anticipative strategies" (see e.g. Appendix B5 in [7] and Section 1 in [11]).

Given a pair (t_0, E_0), an output feedback $\gamma : [t_0, T] \times Y^N \to U$, and a "disturbance" $(e, v(\cdot)) \in E_0 \times \mathcal{V}_{[t_0,T)}$, the following system (11)-(13) has a unique solution $(u(\cdot), x(\cdot), (y_k)_{k\in[1,N]}) \in \mathcal{U}_{[t_0,T]} \times \mathcal{X}_{[t_0,T]} \times Y^N$:

$$\begin{aligned} u(t) &= \gamma(t, (y_k)_{k\in[1,i]}) && (11)\\ x'(t) &= f(x(t), u(t), v(t)), \quad x(t_0) = e && (12)\\ y_k &= h(x(t_k)) && (13) \end{aligned}$$

where i is such that $\tau(t) = t_i$. Hence, the corresponding guaranteed result obtained by using the strategy γ is given by (5), and

$$I_1(t_0, E_0) := \inf_{\gamma} I_1(t_0, E_0; \gamma) \tag{14}$$

is the minimal guaranteed value that can be achieved starting from the set E_0 at time t_0 over all possible strategies on $[t_0, T]$. In the sequel, we shall refer to system (11)-(13) as system **S1**:

$$\mathbf{S1} \quad \begin{cases} u(t) = \gamma(t, (y_k)_{k\in[1,i]}) \\ x'(t) = f(x(t), u(t), v(t)), \quad x(t_0) = e \\ y_k = h(x(t_k)) \end{cases}$$

and we refer to problem (1), (11)-(13) as problem **P1**:

$$\mathbf{P1} \quad \left\{ \min_{\substack{x(\cdot) \text{ solution} \\ \text{to } \mathbf{S1}}} g(x(T)) \right.$$

Then, any solution $(u(\cdot), x(\cdot), (y_k)_{i\in[1,N]})$ of system **S1** is associated with a unique tube $E(\cdot, t_0, E_0, u(\cdot), (y_k)_{i\in[1,N]})$ by (9)-(10) and we have the following lemma:

Lemma 1.3. *Fix $E_0 \in \mathrm{comp}(I\!R^n)$. If we denote by $E(\cdot, t_0, E_0, \gamma, (y_k)_{k\in[1,N]})$ the tube associated by (9)-(10) to a given solution of system **S1**, then*

$$I_1(t_0, E_0) = \inf_{\gamma} \sup_{(y_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \gamma, (y_k)))$$

where G is defined as follows:

$$\forall Z \in \mathrm{comp}(I\!R^n), \quad \begin{cases} G(Z) := \sup_{z\in Z} g(z) \text{ if } Z \neq \emptyset \\ G(\emptyset) := -\infty \text{ else.} \end{cases} \tag{15}$$

Proof. Let us denote

$$I_{1'}(t_0, E_0) := \inf_{\gamma} \sup_{(y_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \gamma, (y_k))) \tag{16}$$

We want to prove $I_1 = I_{1'}$.

First, fix $\gamma : [t_0, T] \times Y^N \to U$, $e \in E_0$, and $v(\cdot) \in \mathcal{V}_{[t_0,T]}$. Let $x(\cdot, t_0, e, \gamma, v(\cdot))$ be the corresponding solution to system **S1**, and $(y_k)_{k\in[1,N]}$ the corresponding measure. Denote by $E(\cdot, t_0, E_0, \gamma, (y_k)_{k\in[1,N]})$ the tube associated to γ and $(y_k)_{k\in[1,N]}$ by (9)-(10). For any $t \in [0, T]$, we obviously have $x(t, t_0, e, \gamma, v(\cdot)) \in E(t, t_0, E_0, \gamma, (y_k))$, thus

$$g(x(T, t_0, e, \gamma, v(\cdot))) \leq G(E(T, t_0, E_0, \gamma, (y_k)_{k\in[1,N]})),$$

hence

$$g(x(T, t_0, e, \gamma, v(\cdot))) \leq \sup_{(y_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \gamma, (y_k))),$$

so that taking the supremum over all possible e and $v(\cdot)$ leads to

$$\sup_{e,v(\cdot)} g(x(T, t_0, e, \gamma, v(\cdot))) \leq \sup_{(y_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \gamma, (y_k))),$$

for any γ. Take the infimum over all possible strategies γ, then (14) and (16) give $I_1 \leq I_{1'}$.

Conversely, fix an output feedback strategy $\gamma : [t_0, T] \times Y^N \to U$ and a measurement $(y_k)_{k\in[1,N]}$ such that the corresponding tube $E(\cdot, t_0, E_0, \gamma, (y_k)_{k\in[1,N]})$ solution to system **S1** and (9)-(10) is non-degenerate (i.e. $E(T, t_0, E_0, \gamma, (y_k)_{k\in[1,N]}) \neq \emptyset$). Then, there exists $x_N \in E(T, t_0, E_0, \gamma, (y_k)_{k\in[1,N]})$ such that

$$g(x_N) = G(E(T, t_0, E_0, \gamma, (y_k)_{k\in[1,N]})),$$

and, according to (8), (10) there exist $v_{N-1}(\cdot) : [t_{N-1}, T] \to V$ and $x_{N-1} \in E(t_{N-1}, t_0, E_0, \gamma, (y_k)_{k\in[1,N-1]})$ such that

$$x_N = x(T, t_{N-1}, x_{N-1}, \gamma, v_{N-1}(\cdot)).$$

We can repeat the process recursively to define a sequence of disturbances $(v_0(\cdot), ..., v_{N-1}(\cdot))$ and a sequence of states $(x_{N-1}, ..., x_1, e)$ with $e \in E_0$ such that

$$x_N = x(T, t_0, e, \gamma, (v_0(\cdot), ..., v_{N-1}(\cdot))).$$

So we have

$$G(E(T,t_0,E_0,\gamma,(y_k)_{k\in[1,N]}))=g(x(T,t_0,e,\gamma,(v_0(\cdot),...,v_{N-1}(\cdot)))),$$

thus

$$G(E(T,t_0,E_0,\gamma,(y_k)_{k\in[1,N]}))\le\sup_{e,v(\cdot)}g(x(T,t_0,e,\gamma,v(\cdot)))$$

for any γ, $(y_k)_{k\in[1,N]}$, hence

$$\sup_{(y_k)_{k\in[1,N]}}G(E(T,t_0,E_0,\gamma,(y_k)))\le\sup_{e,v(\cdot)}g(x(T,t_0,e,\gamma,v(\cdot)),$$

so finally $I_{1'}\le I_1$, and $I_{1'}=I_1$. □

Now we prove the following dynamic programming equation:

Proposition 1.4. *The value function of problem* **P1** *satisfies:*

$$I_1(t_0,E_0)=\inf_{\gamma}\sup_{(y_i)_{i\in[1,j]}}I_1(t,E(t,t_0,E_0,\gamma,(y_i))),\quad\forall t\in[t_0,T],$$

with j such that $t_j=\tau(t)$.

Proof. First, fix $\epsilon>0$. According to Lemma 1.3, there exists $\gamma^\epsilon:[t_0,T]\times Y^N\to U$ such that

$$I_1(t_0,E_0)\le\sup_{(y_k)_{k\in[1,N]}}G(E(T,t_0,E_0,\gamma^\epsilon,(y_k)))\le I_1(t_0,E_0)+\epsilon.$$

Let $t\in[t_0,T]$ be given and let us denote by $t_j:=\tau(t)$ the last measurement time before t. We have:

$$\begin{aligned}
I_1(t_0,E_0)\ &\ge\ \sup_{(y_i)_{i\in[1,j]}}\sup_{(y_k)_{k\in[j+1,N]}}G(E(T,t,E(t,t_0,E_0,\gamma^\epsilon,(y_i)),\gamma^\epsilon,(y_k)))-\epsilon\\
&\ge\ \sup_{(y_i)_{i\in[1,j]}}\inf_{\gamma}\sup_{(y_k)_{k\in[j,N]}}G(E(T,t,E(t,t_0,E_0,\gamma^\epsilon,(y_i)),\gamma,(y_k)))-\epsilon\\
&\ge\ \sup_{(y_i)_{i\in[1,j]}}I_1(t,E(t,t_0,E_0,\gamma^\epsilon,(y_i))-\epsilon\\
&\ge\ \inf_{\gamma^\epsilon}\sup_{(y_i)_{i\in[1,j]}}I_1(t,E(t,t_0,E_0,\gamma^\epsilon,(y_i))-\epsilon
\end{aligned}$$

ϵ being arbitrary, we have

$$I_1(t_0,E_0)\ge\inf_{\gamma}\sup_{(y_i)_{i\in[1,j]}}I_1(t,E(t,t_0,E_0,\gamma,(y_i)).$$

Conversely, fix $\epsilon>0$ and $t\in[t_0,T]$ and denote by $t_j:=\tau(t)$. From Lemma 1.3 we deduce the following: For any $K\in\mathrm{comp}(\mathbb{R}^n)$, there exists $\gamma_K:[t,T]\times Y^{N-j}\to U$ such that

$$\sup_{(y_k)_{k\in[j+1,N]}}G(E(T,t,K,\gamma_K,(y_k))\le I_1(t,K)+\epsilon.\tag{17}$$

Now, let $\gamma:[t_0,T]\times Y^N\to U$ be a given strategy and let us denote by $E(t):=E(t,t_0,E_0,\gamma,(y_i)_{i\in[1,j]})$. We define a new strategy $\tilde\gamma:[t_0,T]\times Y\to U$ by

$$\forall(y_k)_{k\in[1,N]}\in Y^N,\quad\tilde\gamma(s,(y_k))=\begin{cases}\gamma(s,(y_k)) & \forall s\in[t_0,t]\\ \gamma_{E(t)}(s,(y_k)) & \forall s\in[t,T]\end{cases}$$

then we have

$$G(E(T,t_0,E_0,\tilde{\gamma},(y_k)_{k\in[1,N]}) = G(E(T,t,E(t),\gamma_{E(t)},(y_k)_{k\in[j+1,N]}). \tag{18}$$

From (17), we infer:

$$\sup_{(y_k)_{k\in[j+1,N]}} G(E(T,t,E(t),\gamma_{E(t)},(y_k)) \le I_1(t,E(t)) + \epsilon.$$

Thus, by taking the supremum over all possible measurements $(y_i)_{i\in[1,j]}$, we have

$$\sup_{(y_k)_{k\in[1,N]}} G(E(T,t,E(t),\gamma_{E(t)},(y_k)) \le \sup_{(y_i)_{i\in[1,j]}} I_1(t,E(t)) + \epsilon.$$

Then, (18) gives

$$\sup_{(y_k)_{k\in[1,N]}} G(E(T,t_0,E_0,\tilde{\gamma},(y_k)) \le \sup_{(y_i)_{i\in[1,j]}} I_1(t,E(t)) + \epsilon.$$

Hence, for any given strategy $\tilde{\gamma}$ we obtain:

$$I_1(t_0,E_0) \le I_1(t_0,E_0;\tilde{\gamma}) \le \sup_{(y_i)_{i\in[1,j]}} I_1(t,E(t)) + \epsilon.$$

By taking the infimum over all possible strategies γ, we have

$$I_1(t_0,E_0) \le \inf_{\gamma} \sup_{(y_i)_{i\in[1,j]}} I_1(t,E(t,t_0,E_0,\gamma,(y_i)_{i\in[1,j]})) + \epsilon,$$

which gives the wished result, $\epsilon > 0$ being arbitrary. □

1.4. Estimation Feedback Formulation

The drawback of the output feedback formulation lies in its dependency on the whole history of the measurement $(y_k)_{k\in[1,N]}$. We shall now see that without loss of performance one may use control strategies depending only on the current estimation of the state $E(t)$ determined by (9)-(10).

We call *estimation feedback* (we still use the term "strategy") any mapping $(t,E) \mapsto \phi(t,E) \in U$ defined on $[t_0,T] \times \mathrm{comp}(\mathbb{R}^n)$, measurable in t. Every such strategy makes use of the exact estimation (10) at every measurement time $t_k \in \{t_0,...,t_{N-1}\}$. Namely, substituting $y_k = h(x(t_k))$ in (10), we obtain the following closed-loop system for an initial set $E_0 \in \mathrm{comp}(\mathbb{R}^n)$:

$$\begin{aligned}
u(t) &= \phi(t,E(\tau(t))) && (19)\\
x'(t) &= f(x(t),u(t),v(t)), \quad x(t_0) = e \in E_0 && (20)\\
E(t) &= R_{F,u}[E(\tau(t))](t-\tau(t)), \text{ if } t \notin \{t_1,...,t_{N-1}\} && (21)\\
E(\sigma(t)) &= R_{F,u}[E(\tau(t))](\sigma(t)-\tau(t)) \cap h^{-1}(h(x(\sigma(t)))) && (22)\\
E(t_0) &= E_0 && (23)
\end{aligned}$$

The corresponding guaranteed result obtained by using the strategy ϕ is:

$$I_{2'}(t_0,E_0;\phi) := \sup_{\substack{E(\cdot)\text{ solution to}\\ (19)\text{-}(23)}} G(E(T)) \tag{24}$$

and

$$I_{2'}(t_0, E_0) := \inf_{\phi} I_{2'}(t_0, E_0; \phi) \tag{25}$$

is the minimal guaranteed value that can be achieved starting from the set E_0 at time t_0 over all possible estimation feedbacks on $[t_0, T]$.

We can prove that the value of the output feedback problem coincides with that of the estimation feedback problem, namely $I_1(t_0, E_0) = I_{2'}(t_0, E_0)$. However, the above estimation feedback formulation still requires a continuous trajectory $x(\cdot)$ corresponding to a disturbance $v(\cdot)$ through (20). In order to deal with this particular issue, we introduce an "improved" estimation feedback formulation by replacing (20) with the inclusion

$$x(t_k) \in R_{F,u}[E(t_{k-1})](t_k - t_{k-1}), \quad k \in [1, N]. \tag{26}$$

In the sequel, we shall refer to system (19), (21)-(23) and (26) as system **S2**:

$$\textbf{S2} \quad \left\{ \begin{array}{rcl} u(t) & = & \phi(t, E(\tau(t))) \\ E(t) & = & R_{F,u}[E(\tau(t))](t - \tau(t)), \text{ if } t \notin \{t_0, ..., t_N\} \\ x(\sigma(t)) & \in & R_{F,u}[E(\tau(t))](\sigma(t) - \tau(t)) \\ E(\sigma(t)) & = & R_{F,u}[E(\tau(t))](\sigma(t) - \tau(t)) \cap h^{-1}(h(x(\sigma(t)))) \\ E(t_0) & = & E_0 \end{array} \right.$$

and we shall refer to the corresponding minimization problem as problem **P2**:

$$\textbf{P2} \quad \left\{ \min_{\substack{E(\cdot) \text{ solution} \\ \text{to } \textbf{S2}}} G(E(T)) \right.$$

where G is defined by (15). Then, for any given strategy ϕ, a tube $E(\cdot, t_0, E_0, \phi, (x_k)_{k\in[1,N]})$ solution to system **S2** is determined by the choice of a sequence $(x_k)_{k\in[1,N]}$ such that $x_k := x(t_k)$ satisfies (26) at every measurement time t_k. In other words, the state $x(t_k)$ is now allowed to "jump" at every measurement step to the "worst" point in the reachable set, and the continuous trajectory $x(\cdot)$ is eliminated from consideration. The new value function is now

$$I_2(t_0, E_0) := \inf_{\phi} \sup_{\substack{E(\cdot) \text{ solution to} \\ \text{system S2}}} G(E(T, t_0, E_0, \phi, (x_k)_{k\in[1,N]})) \tag{27}$$

and we prove the following proposition:

Proposition 1.5. *I_2 satisfies the dynamic programming:*

$$I_2(t_0, E_0) = \inf_{\phi} \sup_{\substack{E(\cdot) \text{ solution to} \\ \text{system S2}}} I_2(t, E(t, t_0, E_0, \phi, (x_i)_{i\in[1,j]}))), \quad \forall t \in [t_0, T],$$

j being such that $\tau(t) = t_j$.

Proof. A tube solution to system **S2** is determined by the choice of a strategy ϕ and a sequence $(x_k)_{k\in[1,N]}$ such that $x_k := x(t_k)$ satisfies (26) at every measurement time t_k. Hence, (27) can be written the following way:

$$I_2(t_0, E_0) := \inf_{\phi} \sup_{(x_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \phi, (x_k)))$$

Thus, for any given $\epsilon > 0$. there exists $\phi^\epsilon : [t_0, T] \times \mathrm{comp}(\mathbb{R}^n) \to U$ such that

$$I_2(t_0, E_0) \leq \sup_{(x_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \phi^\epsilon, (x_k))) \leq I_2(t_0, E_0) + \epsilon.$$

Fix $t \in (t_0, T)$, and denote by $t_j := \tau(t)$ the latest measurement time before t. We have:

$$\begin{aligned}
I_2(t_0, E_0) &\geq \sup_{(x_i)_{i\in[1,j]}} \sup_{(x_k)_{k\in[j+1,N]}} G(E(T, t, E(t, t_0, E_0, \phi^\epsilon, (x_i)), \phi^\epsilon, (x_k))) - \epsilon \\
&\geq \sup_{(x_i)_{i\in[1,j]}} \inf_{\phi} \sup_{(x_k)_{k\in[j+1,N]}} G(E(T, t, E(t, t_0, E_0, \phi^\epsilon, (x_i)), \phi, (x_k))) - \epsilon \\
&\geq \sup_{(x_i)_{i\in[1,j]}} I_2(t, E(t, t_0, E_0, \phi^\epsilon, (x_i)) - \epsilon \\
&\geq \inf_{\phi} \sup_{(x_i)_{i\in[1,j]}} I_2(t, E(t, t_0, E_0, \phi, (x_i)) - \epsilon,
\end{aligned}$$

as the result is true for any $\epsilon > 0$, we have

$$I_2(t_0, E_0) \geq \inf_{\phi_{|[t_0,t]}} \sup_{(x_i)_{i\in[0,j]}} I_2(t, E(t, t_0, E_0, \phi, (x_i)).$$

On the other hand, fix $\epsilon > 0$ and $t > 0$ and denote by $t_j := \tau(t)$ the last measurement time before t. From (27) we deduce the following: For any K in $\mathrm{comp}(\mathbb{R}^n)$, there exists $\phi_K : [t, T] \times \mathrm{comp}(\mathbb{R}^n) \to U$ such that

$$\sup_{(x_k)_{k\in[j+1,N]}} G(E(T, t, K, \phi_K, (x_k)) \leq I_2(t, K) + \epsilon. \tag{28}$$

Now, let $\phi : [t_0, T] \times \mathrm{comp}(\mathbb{R}^n) \to U$ be a given strategy and let us denote by $E(t) := E(t, t_0, E_0, \phi, (x_i)_{i\in[1,j]})$. We define a new strategy $\tilde{\phi} : [t_0, T] \times \mathrm{comp}(\mathbb{R}^n) \to U$ by

$$\forall E \in \mathrm{comp}(\mathbb{R}^n), \quad \tilde{\phi}(s, E) = \begin{cases} \phi(s, E) & \forall s \in [0, t] \\ \phi_{E(t)}(s, E) & \forall s \in [t, T] \end{cases}$$

then we have

$$G(E(T, t_0, E_0, \tilde{\phi}, (x_k)_{k\in[1,N]}) = G(E(T, t, E(t), \phi_{E(t)}, (x_k)_{k\in[j+1,N]}), \tag{29}$$

and we deduce the following from (28)

$$\sup_{(x_k)_{k\in[j+1,N]}} G(E(T, t, E(t), \phi_{E(t)}, (x_k)) \leq I_2(t, E(t)) + \epsilon.$$

Thus, by taking the supremum over all possible states $(x_i)_{i\in[1,j]}$, we have

$$\sup_{(x_k)_{k\in[1,N]}} G(E(T, t, E(t), \phi_{E(t)}, (x_k)) \leq \sup_{(x_i)_{i\in[1,j]}} I_2(t, E(t)) + \epsilon.$$

Then, (29) gives

$$\sup_{(x_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \tilde{\phi}, (x_k)) \leq \sup_{(x_i)_{i\in[1,j]}} I_2(t, E(t)) + \epsilon,$$

so that

$$\inf_{\phi} \sup_{(x_k)_{k\in[1,N]}} G(E(T,t_0,E_0,\phi,(x_k)) \leq \sup_{(x_i)_{i\in[1,j]}} I_2(t,E(t)) + \epsilon,$$

and we have

$$I_2(t_0,E_0) \leq \inf_{\phi} \sup_{(x_i)_{i\in[1,j]}} I_2(t,E(t)) + \epsilon$$

which holds true for any given $\epsilon > 0$ and completes the proof. □

Here is a key result of this paper:

Theorem 1.6. *The value function of problem* **P1** *in the class of output feedback strategies, and the value function of problem* **P2** *in the class of estimation feedback strategies coincide.*

Proof. Fix $E_0 \in \mathrm{comp}(\mathbb{R}^n)$. We obviously have $I_1(T,E_0) = I_2(T,E_0) = G(E_0)$. Then, considering the problem on $[t_{N-1},T]$ with initial condition $E(t_{N-1}) := E_0$, we prove that $I_1(t_{N-1},E_0) = I_2(t_{N-1},E_0)$: Fix $\epsilon > 0$, according to Lemma 1.3 there exists γ^ϵ : $[t_{N-1},T] \times Y \to U$ such that

$$I_1(t_{N-1},E_0) \leq \sup_{y_N} G(E(T,t_{N-1},E_0,\gamma^\epsilon,y_N)) \leq I_1(t_{N-1},E_0) + \epsilon.$$

By (10), we have

$$\sup_{y_N} G(R_{F,\gamma^\epsilon}[E_0](T-t_{N-1}) \cap h^{-1}(y_N)) \leq I_1(t_{N-1},E_0) + \epsilon.$$

In the previous inequality, we can restrict the sup to those $y_N \in h(R_{F,\gamma^\epsilon}[E_0](T-t_{N-1}))$ with no loss of generality. Moreover, t_{N-1} being the initial time, the first measurement involved here is y_N. Hence, γ^ϵ can be understood as an open-loop control on $[t_{N-1},T)$. Also, one can easily to prove that the reachable set $R_{F,\gamma^\epsilon}[E_0](T-t_{N-1})$ at time T does not depend on the value $\gamma^\epsilon(T,y_N)$ of the control at that time. Hence, for any estimation feedback strategy $\phi : [t_{N-1},T] \times \mathrm{comp}(\mathbb{R}^n) \to U$ such that $\phi(t,E_0) := \gamma^\epsilon(t)$ on $[t_{N-1},T)$. Note that $\phi(T,\cdot)$ and $\phi(\cdot,Z)$ with $Z \neq E_0$ can take any value. We have $R_{F,\gamma^\epsilon}[E_0](T-t_{N-1}) = R_{F,\phi}[E_0](T-t_{N-1})$, thus

$$\sup_{x_N \in R_{F,\phi}[E_0](T-t_{N-1})} G\left(R_{F,\phi}[E_0](T-t_{N-1}) \cap h^{-1}(h(x_N))\right) \leq I_1(t_{N-1},E_0) + \epsilon.$$

By (22), and taking the infimum over all possible strategies ϕ, we have

$$\inf_{\phi} \sup_{\substack{E(\cdot)\text{ solution to}\\ \text{system S2}}} G(E(T,t_{N-1},E_0,\phi,x_N)) \leq I_1(t_{N-1},E_0) + \epsilon.$$

By (27), this leads to $I_2(t_{N-1},E_0) \leq I_1(t_{N-1},E_0)$. Conversely, according to (27), for any given $\epsilon > 0$, there exists $\phi^\epsilon : [t_{N-1},T] \times \mathrm{comp}(\mathbb{R}^n) \to U$ such that

$$I_2(t_{N-1},E_0) \leq \sup_{x_N} G(E(T,t_{N-1},E_0,\phi^\epsilon,x_N)) \leq I_2(t_{N-1},E_0) + \epsilon,$$

where $x_N := x(T)$ satisfies (26) at time T. From (22), we deduce the following:

$$\sup_{x_N} G\left(R_{F,\phi^\epsilon}[E_0](T-t_{N-1}) \cap h^{-1}(h(x_N))\right) \leq I_2(t_0,E_0) + \epsilon.$$

For any output feedback strategy $\gamma : [t_{N-1}, T] \times Y \to U$ such that $\gamma(t) = \phi^\epsilon(t, E_0)$ on $[t_{N-1}, T)$, we have $R_{F,\phi^\epsilon}[E_0](T - t_{N-1}) = R_{F,\gamma}[E_0](T - t_{N-1})$. Note that $\gamma(T, \cdot)$ can take any value. Denote $y_N := h(x_N)$, we have

$$\sup_{y_N} G\left(R_{F,\gamma}[E_0](T - t_{N-1}) \cap h^{-1}(y_N)\right) \leq I_2(t_0, E_0) + \epsilon.$$

By (10), we have:

$$\sup_{y_N} G\left(E(T, E_0, t_{N-1}, \gamma, y_N\right) \leq I_2(t_0, E_0) + \epsilon.$$

Taking the infimum over all possible strategies γ and using Lemma 1.3, we have $I_1(t_{N-1}, E_0) \leq I_2(t_{N-1}, E_0)$. This finally leads to $I_1(t_{N-1}, E_0) = I_2(t_{N-1}, E_0)$ for any $E_0 \in \mathrm{comp}(I\!R^n)$.

Then, consider the problem on $[t_{N-2}, T]$ with initial condition $E(t_{N-2}) := E_0$. Fix an output feedback strategy $\gamma : [t_{N-2}, t_{N-1}] \times Y \to U$, and let $\phi : [t_{N-2}, t_{N-1}] \times \mathrm{comp}(I\!R^n) \to U$ be an estimation feedback strategy such that $\phi(t, E_0) = \gamma(t)$ on $[t_{N-2}, t_{N-1})$. Note that $\phi(t_{N-1}, \cdot)$ and $\phi(\cdot, Z)$ with $Z \neq E_0$ can take any value. We have $R_{F,\phi}[E_0](t_{N-1} - t_{N-2}) = R_{F,\gamma}[E_0](t_{N-1} - t_{N-2})$ and we deduce the following from (22) and from Proposition 1.5:

$$I_2(t_{N-2}, E_0) \leq \sup_{x_{N-1}} I_2\left(t_{N-1}, R_{F,\phi}[E_0](t_{N-1} - t_{N-2}) \cap h^{-1}(h(x_{N-2}))\right).$$

Denote $y_{N-1} := h(x_{N-1})$, according to (10) we have

$$I_2(t_{N-2}, E_0) \leq \sup_{y_{N-1}} I_2\left(t_{N-1}, E(t_{N-1}, E(t_{N-1}, t_{N-2}, E_0, \gamma, y_{N-1})\right),$$

which holds true for any γ. Because $I_1(t_{N-1}, E) = I_2(t_{N-1}, E)$ for any $E \in \mathrm{comp}(I\!R^n)$, we have

$$I_2(t_{N-2}, E_0) \leq \sup_{y_{N-1}} I_1\left(t_{N-1}, E(t_{N-1}, E(t_{N-1}, t_{N-2}, E_0, \gamma, y_{N-1})\right).$$

Hence, taking the infimum over all possible strategies γ and using Proposition 1.4, we deduce the following: $I_2(t_{N-2}, E_0) \leq I_1(t_{N-2}, E_0)$. Conversely, fix $\epsilon > 0$, according to Proposition 1.5, there exists $\phi^\epsilon : [t_{N-2}, t_{N-1}] \times \mathrm{comp}(I\!R^n) \to U$ such that

$$I_2(t_{N-2}, E_0) \leq \sup_{x_{N-1}} I_2(t_{N-1}, E(t_{N-1}, t_{N-2}, E_0, \phi^\epsilon, x_{N-1})) \leq I_2(t_{N-2}, E_0) + \epsilon.$$

Following the same idea as above, we denote $y_{N-1} := h(x_{N-1})$. For any output feedback strategy $\gamma : [t_{N-2}, t_{N-1}] \times Y \to U$ such that $\gamma(t) := \phi^\epsilon(t, E_0)$ on $[t_{N-2}, t_{N-1})$. Note that $\gamma(t_{N-1}, \cdot)$ can take any value. We have

$$\sup_{y_{N-1}} I_2(t_{N-1}, E(t_{N-1}, t_{N-2}, E_0, \gamma, y_{N-1})) \leq I_2(t_{N-2}, E_0) + \epsilon.$$

Moreover, $I_1(t_{N-1}, E) = I_2(t_{N-1}, E)$ for any $E \in \mathrm{comp}(I\!R^n)$, hence

$$\sup_{y_{N-1}} I_1(t_{N-1}, E(t_{N-1}, t_{N-2}, E_0, \gamma, y_{N-1})) \leq I_2(t_{N-2}, E_0) + \epsilon,$$

and, taking the infimum over all possible strategies γ, we deduce from Proposition 1.4 the following: $I_1(t_{N-2}, E_0) \leq I_2(t_{N-2}, E_0))$. So, we have $I_2(t_{N-2}, E_0) = I_1(t_{N-2}, E_0)$, and we can obtain the final result by an easy iteration. $\square$

2. The Value Function

2.1. Regularity of the Value Function

Proposition 2.1. *If $g(\cdot)$ is a Lipschitz function, then $I_2(\cdot,\cdot)$ is locally Lipschitz on any intervall $[t_k, t_{k+1})$ with $k \in [0, N-1]$, namely: For any $E \in \mathrm{comp}(I\!R^n)$, there exists $L_E > 0$ such that for any $k \in [0, N-1]$, for any $(t,s) \in [t_k, t_{k+1}) \times [t_k, t_{k+1})$, and for any $E' \in \mathrm{comp}(I\!R^n)$, we have*

$$|I_2(t,E) - I_2(s,E')| \leq L_E(H(E,E') + |t-s|).$$

Proof. Fix $k \in [0, N-1]$, $t \in [t_k, t_{k+1})$, and E, $E' \in \mathrm{comp}(I\!R^n)$. We first prove that there exists $L_{I_2} > 0$, independent from t, E, E' such that $|I_2(t,E) - I_2(t,E')| \leq L_{I_2} H(E,E')$. Let $\phi(\cdot,\cdot)$ be a given estimation feedback. According to (27), for any $\epsilon > 0$ there exists $\phi^\epsilon(\cdot,\cdot) : [t,T] \times \mathrm{comp}(I\!R^n) \to U$, independent from ϕ, such that[2]

$$I_2(t,E) - I_2(t,E') \leq \sup_{\substack{E(\cdot)\text{ solution to}\\ \text{system S2}}} G\left(E(T,t,E,\phi)\right) - \sup_{\substack{E_\epsilon(\cdot)\text{ solution to}\\ \text{system S2}}} G\left(E_\epsilon(T,t,E',\phi^\epsilon)\right) + \epsilon.$$

Then, for any estimation feedback $\phi(\cdot,\cdot)$ and for any $E_\epsilon(\cdot)$ solution to system **S2** corresponding to ϕ^ϵ, we have

$$I_2(t,E) - I_2(t,E') \leq \sup_{\substack{E(\cdot)\text{ solution to}\\ \text{system S2}}} G(E(T,t,E,\phi)) - G(E_\epsilon(T,t,E',\phi^\epsilon)) + \epsilon. \tag{30}$$

The idea here is to choose both an estimation feedback ϕ and a tube $E_\epsilon(\cdot,t,E',\phi^\epsilon)$ corresponding to ϕ^ϵ, such that any tube $E(\cdot,t,E,\phi)$ of system **S2** corresponding to the chosen ϕ stays "close enough" to $E_\epsilon(\cdot,t,E',\phi^\epsilon)$. Typically, on $[t, t_{k+1})$, we choose $\phi(\cdot,\cdot)$ by setting

$$\phi(s,E) := \phi^\epsilon(s,E'), \quad \forall s \in [t, t_{k+1}),$$

Note that $\phi(\cdot,Z)$ with $Z \neq E$ can take any value on $[t, t_{k+1})$. Denote by $u(\cdot) := \phi(\cdot,E)$. Fix $s \in [t, t_{k+1})$ and consider $a \in R_{F,u}[E](s)$. Then, there exist a perturbation $v(\cdot)$ and a trajectory $x^0(\cdot)$ of system (2) such that $x^0(s) = a$ and $x^0(t) = e \in E$. Let $e' \in E'$ be such that $|e - e'| \leq H(E,E')$. Then we consider the trajectory $x^1(\cdot)$ of system (2) starting from e' for the same $u(\cdot)$ and $v(\cdot)$ as above, and we denote by $b := x^1(s)$. Thus we have:

$$|a-b| \leq e^{L_f(s-t)}|e-e'| \leq e^{L_f(s-t)}H(E,E')$$

and

$$R_{F,u}[E](s-t) \subset R_{F,u}[E'](s-t) + e^{L_f(s-t)}H(E,E')\mathbf{B}.$$

Interchanging a and b, we finally obtain

$$H\left(E(s,t,E,\phi), E_\epsilon(s,t,E',\phi^\epsilon)\right) \leq e^{L_f s}H(E,E'), \quad \forall s \in [t, t_{k+1}).$$

For $x^0_{k+1} \in R_{F,u}[E](t_{k+1}-t)$, we choose $\phi(\cdot,\cdot)$ on $[t_{k+1}, t_{k+2})$ by setting:

$$\begin{aligned} & \phi(s, R_{F,u}[E](t_{k+1}-t) \cap h^{-1}h(x^0_{k+1})) \\ := \;& \phi^\epsilon(t, R_{F,u}[E'](t_{k+1}-t) \cap h^{-1}h(x^1_{k+1})), \quad \forall s \in [t_{k+1}, t_{k+2}), \end{aligned}$$

[2]In this proof, for the convenience of the reader, we do not explicit the dependency of the tubes on the state $(x_k)_{k\in[1,N]}$. Hence, for a given strategy ϕ, the corresponding tube starting from E at time t will be denoted here by $E(\cdot,t,E,\phi)$.

where x^1_{k+1} is such that $x^1_{k+1} \in R_{F,u}[E'](t_{k+1}-t)$ and $|x^1_{k+1}-x^0_{k+1}| \leq e^{L_f(t_{k+1}-t)}H(E,E')$. Note that for $Z \neq R_{F,u}[E](t_{k+1}) \cap h^{-1}h(x^0_{k+1})$, one can take any value for $\phi(t,Z)$ on $[t_{k+1},t_{k+2})$. Then, similarly to what has been done on $[t,t_{k+1}]$, we fix $s \in [t_{k+1},t_{k+2})$. For any $a \in R_{F,u}[R_{F,u}[E](t_{k+1}-t) \cap h^{-1}h(x^0_{k+1})](s-t_{k+1})$, there exist a perturbation $v(\cdot)$ and a trajectory $x^0(\cdot)$ of system (2) on $[t_{k+1},t_{k+2})$ such that $x^0(s)=a$ and $x^0(t_{k+1})=x^0_{k+1}$. Then, we consider the trajectory $x^1(\cdot)$ of the same system on $[t_{k+1},t_{k+2})$ starting from $x^1(t_{k+1}) := x^1_{k+1}$. We denote by $b := x^1(s)$ and we have

$$|a-b| \leq e^{L_f(s-t_{k+1})}|x^0_{k+1}-x^1_{k+1}| \leq e^{L_f(s-t_{k+1})}e^{L_f(t_{k+1}-t)}H(E,E')$$

so, one easily deduces that

$$\begin{aligned} & R_{F,u}[R_{F,u}[E](t_{k+1}-t) \cap h^{-1}h(x^0_{k+1})](s-t_{k+1}) \\ \subset \;& R_{F,u}[R_{F,u}[E'](t_{k+1}-t) \cap h^{-1}h(x^1_{k+1})](s-t_{k+1}) + e^{L_f(s-t_{k+1})}H(E,E')\mathbf{B}, \end{aligned}$$

and, interchanging a and b, we finally have

$$H\left(E(s,t,E,\phi), E_\epsilon(s,t,E',\phi^\epsilon)\right) \leq e^{L_f(s-t)}H(E,E'), \quad \forall s \in [t_{k+1},t_{k+2}).$$

By iterating, we define a tube $E_\epsilon(\cdot,t,E',\phi^\epsilon)$ and an estimation feedback ϕ such that for any corresponding tube $E(\cdot,t,E,\phi)$ solution to system **S2** we have

$$H\left(E(s,t,E,\phi), E_\epsilon(s,t,E',\phi^\epsilon)\right) \leq e^{L_f(s-t)}H(E,E'), \quad \forall s \in [t,T].$$

Moreover, g being Lipschitz, it is easy to prove that G is Lipschitz too. Hence, we can deduce the following result from (30):

$$|I_2(t,E) - I_2(t,E')| \leq C(H(E,E')),$$

C being a constant. Now, fix $s \in (t,t_{k+1})$, we have:

$$\begin{aligned} |I_2(t,E) - I_2(s,E')| &\leq |I_2(t,E) - I_2(t,E')| + |I_2(t,E') - I_2(s,E')| \\ &\leq CH(E,E') + |I_2(t,E') - I_2(s,E')| \\ &\leq CH(E,E') + |\inf_\phi \sup_{\substack{E(\cdot) \text{ solution to} \\ \text{system S2}}} I_2\left(s, E(s,t,E',\phi)\right) - I_2(s,E')| \\ &\leq C\left(H(E,E') + \inf_\phi \sup_{\substack{E(\cdot) \text{ solution to} \\ \text{system S2}}} H\left(E(s,t,E',\phi),E'\right)\right). \end{aligned}$$

Fix a strategy $\phi : [t,s] \times \mathrm{comp}(\mathbb{R}^n) \to U$ and a corresponding tube $E(\cdot,t,E',\phi)$ solution to system **S2**. As $s \in (t,t_{k+1}) \subset [t_k,t_{k+1})$, no measurement is involved, and we have $E(s,t,E',\phi) = R_{F,\phi}[E'](s-t)$. Then, the growth condition in Condition 1.1 implies in a standard way that

$$H\left(E(s,t,E',\phi),E'\right) \leq 2c(1+|E'|)|s-t|,$$

hence the result. $\square$

2.2. Characterization of the value function

Let $J(\cdot,\cdot) : [t_0, T] \times \mathrm{comp}(\mathbb{R}^n) \to \mathbb{R}$ be a Lipschtiz function, let $E \in \mathrm{comp}(\mathbb{R}^n)$ be fixed and $F : E \to \mathrm{comp}(\mathbb{R}^n)$ be a set-valued map defined on E.

Definition 2.2. We define the lower Dini derivative of J at E in the direction of F as

$$D^- J(E; F) := \liminf_{h\to 0+} \inf \left\{ \frac{J(E') - J(E)}{h} \mid E' \in \mathrm{comp}(\mathbb{R}^n),\ (Id + hF)(E) \subset E' \right\}.$$

where Id is the identity mapping. We have the following characterization of the value function:

Theorem 2.3. *Suppose that $g(\cdot)$ is Lipschitz and consider the following system:*

$$\begin{cases} \inf\limits_{u\in U} D^- I_2(t_0, E; (1, f(\cdot, u, V))) \leq 0 \\ I_2(t_1^-, E) = \sup\limits_{x_1\in E} I_2(t_1, E \cap h^{-1}h(x_1)) \\ s \cdots \\ \inf\limits_{u\in U} D^- I_2(t_{N-2}, E); (1, f(\cdot, u, V))) \leq 0 \\ I_2(t_{N-1}^-, E) = \sup\limits_{x_{N-1}\in E} I_2(t_{N-1}, E \cap h^{-1}h(x_{N-1})) \\ \inf\limits_{u\in U} D^- I_2(t_{N-1}, E; (1, f(\cdot, u, V))) \leq 0 \end{cases} \tag{31}$$

with the side condition

$$I_2(T, E) = G(E) \quad \forall E \in \mathrm{comp}(\mathbb{R}^n). \tag{32}$$

Then, the optimal value function I_2 is the minimal solution to (31)-(32) *that is Lipschitz on every intervall* $[t_k, t_{k+1})$ *with* $k \in [0, N-1]$.

Proof. First, we consider the problem on $[t_0, t_1)$. To prove the result, we shall use the dynamic programming principle. Fix (t_0, E_0) and consider a sequence $h_k \to 0+$ such that $t_k := t_0 + h_k \in [t_0, t_1)$ for all $k \in \mathbb{N}$. Thus, according to Proposition 1.5, for any $k \in \mathbb{N}$, there exists ϕ_k such that:

$$I_2(t_0, E_0) \geq I_2(t_k, E(t_k, t_0, E_0, \phi_k(E_0, \cdot), (x_k)_{k\in[1,N]})) - \frac{h_k}{k}. \tag{33}$$

Denote

$$\bar{u}_k := \frac{1}{h_k} \int_{t_0}^{t_k} \phi_k(E_0, s) ds.$$

Since U is compact, there exists $\bar{u} \in U$ such that $\beta_k := |\bar{u}_k - \bar{u}| \to 0$ up to a subsequence. Then, according to the structural and the convexity assumptions from Condition 1.1, one can prove in a standard way that there is a constant C such that:

$$H\left(E\left(t_k, t_0, E_0, \phi_k(E_0, \cdot), (x_k)_{k\in[1,N]}\right), \left(I + h_k f(\cdot, \bar{u}, V)\right)(E_0)\right) \leq C h_k (h_k + \beta_k).$$

Thus, we have

$$(I + h_k f(\cdot, \bar{u}, V))(E_0) \subset E\left(t_k, t_0, E_0, \phi_k(E_0, \cdot), (x_k)_{k\in[1,N]}\right) + C h_k (h_k + \beta_k)\mathbf{B}.$$

Moreover, (33) implies

$$I_2\left(t_k, E(t_k, t_0, E_0, \phi_k(E_0, \cdot), (x_k)_{k\in[1,N]})\right) - I_2(t_0, E_0) \leq \frac{h_k}{k} \to 0,$$

such that we have

$$D^- I_2\left(t_0, E_0; (1, f(\cdot, \bar{u}, V))\right) \leq 0,$$

and

$$\inf_{u\in U} D^- I_2\left(t_0, E_0; (1, f(\cdot, u, V))\right) \leq 0.$$

Now, we consider the problem at measurement time t_1. By using the dynamic programming between t_1^- and t_1, for any tube $E(\cdot)$ solution to system **S2**, we obtain the following:

$$I_2(t_1^-, E(t_1^-)) = \sup_{x_1\in E(t_1^-)} I_2\left(t_1, E(t_1^-) \cap h^{-1}h(x_1)\right)$$

and we can repeat the process recursively on every intervall $[t_i, t_{i+1})$ and at every moment t_i so that I_2 is finally solution to system (31).

Now, let J be a solution to (31). Then, Epi J is viable on $[t_0, t_1)$ for the system (33) with target $\{t_1\} \times$ Epi J, thus for any E_0, there exist a control ϕ and a corresponding tube $E(\cdot)$ defined on $[t_0, t_1)$ such that

$$J(t_1^-, E(t_1^-)) \leq J(t_0, E_0),$$

and

$$J(t_1^-, E(t_1^-)) = \sup_{x_1\in E(t_1^-)} J\left(t_1, E(t_1^-) \cap h^{-1}h(x_1)\right).$$

For any arbitrary $x_1 \in E(t_1^-)$, we have $J(t_1^-, E(t_1^-)) \leq J(t_1, E(t_1^-) \cap h^{-1}h(x_1))$. Moreover, Epi J is also viable on $[t_1, t_2)$ for the system (33) with target $\{t_2\} \times$ Epi J, thus for any E_1, there exists ϕ and $E(\cdot)$ defined on $[t_1, t_2)$ such that

$$J\left(t_2^-, E(t_2^-)\right) \leq J(t_1, E_1).$$

By taking $E_1 := E(t_1^-) \cap h^{-1}h(x_1)$, we obtain $J(t_2^-, E(t_2^-)) \leq J(t_0, E_0)$, and we can repeat the process recursively to finally obtain

$$G(E(T)) = J(T, E(T)) \leq J(t_0, E_0),$$

$E(\cdot)$ being any tube associated to a certain control ϕ. In other words, ϕ is fixed by construction and one can freely choose the sequence $\{x_1...x_N\}$ which determines tube $E(\cdot)$. Then, by taking the supremum over all possible tubes $E(\cdot)$ corresponding to that given ϕ and starting from E_0, we have

$$\sup_{\substack{E(\cdot)\text{ solution to}\\ \text{system S2}}} G(E(T)) \leq J(t_0, E_0)$$

and we finally take the infimum over all possible strategies ϕ to get

$$I_2(t_0, E_0) = \inf_{\phi} \sup_{\substack{E(\cdot)\text{ solution to}\\ \text{system S2}}} G(E(T)) \leq J(t_0, E_0)$$

which completes the proof. □

3. Conclusion

First, let us underline the fact that the present work can be easily extended to the case of set evolutions in a prescribed collection of sets (see e.g. [25, 26]) which is better suited for further numerical applications.

Also, a possible extension of the approach we have considered here would be to introduce a noise in the measurement. Namely, one could replace the initial incomplete measurement (4) with the following noisy measurement:

$$y_k = h(x(t_k), w_k), \quad t_k \in \{t_1, t_2, ..., t_N\}$$

where $w_k \in W$ is a disturbance. Then, the corresponding value function of the control problem would become

$$I(t_0, E_0) := \inf_{\phi} \sup_{\substack{E(\cdot) \text{ solution to} \\ \text{system S2}}} \sup_{(w_k)_{k\in[1,N]}} G(E(T, t_0, E_0, \phi, (x_k)_{k\in[1,N]}, (w_k)_{k\in[1,N]})),$$

and one could easily adapt the results of this paper to the case of a noisy measurement.

Then, this work could also be easily adapted to integral cost or minimal time problems. Actually, an integral cost problem with fixed end time can equivalently be written as a terminal cost problem for an "extended system". For example, for the following problem:

$$\text{minimize} \int_{t_0}^{T} g(x(s))ds,$$

one can extend System (2) in the following way:

$$\begin{cases} x'(t) &= f(x(t), u(t), v(t)) \\ z'(t) &= g(x(t)) \end{cases}$$

then, the new problem becomes

$$\text{minimize } z(T),$$

which is equivalent to the problem considered in this paper as soon as g satisfies Condition 1 namely, it is continuous, locally Lipschitz, and satisfies the linear growth condition.

Similarly, in order to include minimal time problems in our approach, one could generalize it by considering a time-dependent cost $g(x, t)$ and by introducing a set $M \subset [t_0, T] \times I\!R^n$ which will determine the termination time $T(E(\cdot))$ of the control process in the following way: $T(E(\cdot)) := \min\{t \geq t_0 \mid (t, E(t)) \in M\}$. In the case of a fixed end time, we would have $M = \{T\} \times I\!R^n$. In the case of a minimal time problem, we would have $g(x, t) = t$.

Finally, another interesting problem would be to study some numerical schemes from which we may be able to derive discrete approximated controls, as optimal strategies for the player u cannot be characterized from the knowledge of the value function (writing the Pontryaguin maximum principle in this case is too difficult). Such numerical schemes have already been successfully introduced in the fully discrete time case (see [20]), and they are being investigated in the fully continuous time. Hence, the present work is a first step towards numerical schemes in the case of continuous time systems with discrete time measurements, but this will be investigated in a future work.

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