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Local Integration of Prox-Regular Functions in Hilbert Spaces

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In this paper we show that prox-regular functions are locally uniquely determined by their subgradients i.e. if two functions are prox-regular at $\bar{x}$ for $\bar{v}$, then in a neighborhood of $(\bar{x}, \bar{v})$, the functions differ by an additive constant. The class of prox-regular functions includes all convex functions, all qualified convexly composite functions (i.e. with an appropriate constraint qualification) and all pln functions. This result represents an improvement over previous results since the class of prox-regular functions is strictly bigger than the class of pln functions (an example is provided in this paper).

Keywords: Prox-regular, primal-lower-nice, pln, regularization, nonsmooth analysis, integration of subgradients, subgradient mappings, subgradient localization, amenable functions, proximal subgradients, Moreau envelopes, proximal mapping.

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1. Introduction and main result

A fundamental problem in nonsmooth analysis is to identify functions that can be recovered up to an additive constant, from the knowledge of their subgradients. More precisely, a function f is deemed integrable if whenever $\partial_{\#} g(x) = \partial_{\#} f(x)$ for all x then f and g differ only by an additive constant. Here $\partial_{\#}$ refers to a subdifferential, which can be taken in many different ways. In this paper $\partial f(x)$ denotes the set of limiting proximal subgradients; see [8].

Probably the most well known and the oldest result in this area concerns convex functions. If two l.s.c. convex functions (defined on Banach spaces) have the same subgradients, then they differ by a constant; see Rockafellar [21]. The result is also valid for locally Lipschitzian functions that are upper regular, semismooth or separably regular functions; see [11]. However the result fails for arbitrary locally Lipschitzian functions.

Example 1.1 (Benoist [1]). For every countable dense set $D \subset I\!R$, there exists in-

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finitely many Lipschitzian functions f, differing by more than a constant, such that

$$\partial_p f(x) = \begin{cases} (-1, +1) & \text{if } x \in D, \\ \emptyset & \text{if } x \notin D. \end{cases}$$

Here $\partial_p f(x)$ denotes the set of proximal subgradients ($v \in \partial_p f(x)$ when for all y, $f(y) \geq f(x) + < v, y - x > -t\|y - x\|^2$ for some $t > 0$).

The first work outside the field of locally Lipschitzian functions was done by Poliquin for the primal-lower-nice (pln) functions. If two functions are pln at $\bar{x}$ and have the same subgradients, then on a neighborhood of $\bar{x}$ the functions differ by a constant; see Poliquin [16]. Note that the class of pln functions includes all convex functions and all convexly composite functions with an appropriate constraint qualification. Later this result was extended to Hilbert spaces by Thibault and Zagrodny [23] and additional results were provided in Bernard, Thibault and Zagrodny [2]. Other work on the subject inlude [3], [11], [20], [24] and [25].

Poliquin and Rockafellar in [17] introduced the class of prox-regular functions (a definition is given in the next paragraph). These functions are more general than the pln functions, yet admit effective generalizations of many of the subdifferential properties of extended-valued convex functions. For more on prox-regular functions and their rich subdifferential properties see, [4], [5], [6], [7], [9], [10], [12], [15], [17], [18], [19].

Let X be a real Hilbert space, $f : X \to \overline{I\!R} := I\!R \cup \{\pm\infty\}$ be a lower semicontinuous (l.s.c.) function and ∂f denotes the set of limiting proximal subgradients of f. Then f is said to be *prox-regular* at $\bar{x}$, a point where f is finite, for the subgradient $\bar{v} \in \partial f(\bar{x})$, if there exist parameters $\varepsilon > 0$ and $r \geq 0$ such that for every point $(x, v) \in \operatorname{gph} \partial f$ obeying $\|x - \bar{x}\| < \varepsilon$, $|f(x) - f(\bar{x})| < \varepsilon$, and $\|v - \bar{v}\| < \varepsilon$, one has the local estimate

$$f(x') \geq f(x) + \langle v, x' - x \rangle - \frac{r}{2}\|x' - x\|^2 \text{ for all } x' \in I\!B(\bar{x}; \varepsilon).$$

To obtain the integration result in this paper we need not only that the functions be prox-regular but we need to know that the function values are close when the points are close. This is called subdifferential continuity. A function $f : X \to \overline{I\!R}$ is *subdifferentially continuous at $\bar{x}$ for $\bar{v}$*, where $\bar{v} \in \partial f(\bar{x})$, if for every $\delta > 0$ there exist $\varepsilon > 0$ such that $|f(x) - f(\bar{x})| < \delta$ whenever $\|x - \bar{x}\| < \varepsilon$ and $\|v - \bar{v}\| < \varepsilon$ with $v \in \partial f(x)$. If this holds for all $\bar{v} \in \partial f(\bar{x})$, f is said to be *subdifferentially continuous at $\bar{x}$.*

In this paper, we prove, in an arbitrary Hilbert space, that if two functions, which have the same limiting subgradients locally, are prox-regular and subdifferentially continuous relative to a pair $(\bar{x}, \bar{v})$ then the functions differ by a constant in a local neighborhood of $(\bar{x}, \bar{v})$.

Theorem 1.2. *Let X be a Hilbert space. Let $f_i : X \to \overline{I\!R}$ be prox-regular at $\bar{x}$ for $\bar{v} \in \partial f_i(\bar{x})$, $i = 1, 2$. Assume that f_1 and f_2 have the same limiting subgradients in a neighborhood of $\bar{x}$, and that f_1 is subdifferentially continuous at $\bar{x}$ for $\bar{v}$. Then f_2 is subdifferentially continuous at $\bar{x}$ for $\bar{v}$ and there is a k in $I\!R$ and $\varepsilon > 0$ such that $f_1(x) = f_2(x) + k$ for all x with the properties that $\|x - \bar{x}\| \leq \varepsilon$ and x has a subgradient v ($v \in \partial f(x)$) with $\|v - \bar{v}\| \leq \varepsilon$.*

The following examples show that the sudifferentially continuous assumption in Theorem 1.2 is necessary.

Example 1.3 (necessity of subdifferential continuity). Let

$$f_1(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0, \end{cases} \qquad f_2(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 2 & \text{if } x > 0, \end{cases}$$

then

$$\partial f_1(x) = \partial_p f_1(x) = \partial f_2(x) = \partial_p f_2(x) = \begin{cases} \{0\} & \text{if } x \ne 0, \\ [0, \infty) & \text{if } x = 0. \end{cases}$$

These two functions are prox-regular but not subdifferentially continuous at $\bar{x} = 0$ for $\bar{v} = 0$. We see that they do not differ by a constant in any neighborhood of $(\bar{x}, \bar{v})$.

The following example shows that Theorem 1.2 covers a much broader class of functions than that of pln functions and provides an example that illustrates that not only does x have to be close to $\bar{x}$ but it has to have a subgradient that is close to $\bar{v}$ (note that the functions values are close since we have subdifferential continuity).

Example 1.4 (necessity of the closeness of the subgradients). Let

$$f_1(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \sqrt{x} & \text{if } x > 0, \end{cases} \qquad f_2(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 + \sqrt{x} & \text{if } x > 0, \end{cases}$$

then

$$\partial f_1(x) = \partial_p f_1(x) = \partial f_2(x) = \partial_p f_2(x) = \begin{cases} \{0\} & \text{if } x < 0, \\ [0, \infty) & \text{if } x = 0, \\ \frac{1}{2\sqrt{x}} & \text{if } x > 0. \end{cases}$$

First, we claim that both f_1 and f_2 are prox-regular and subdifferentially continuous at $\bar{x} = 0$ for $\bar{v} = 0$. To see this , take $\varepsilon = \frac{1}{4}$ and for $i = 1, 2$, let T_i be the f_i-attentive ε-localization T_i of ∂f_i at $(\bar{x}, \bar{v})$. Recall that for a function f, the f-attentive ε-localization T of ∂f at $(\bar{x}, \bar{v})$ is

$$T(x) = \begin{cases} \{v \in \partial f(x) \mid \|v - \bar{v}\| < \varepsilon\} & \text{if } \|x - \bar{x}\| < \varepsilon \text{ and } |f(x) - f(\bar{x})| < \varepsilon, \\ \emptyset & \text{otherwise.} \end{cases}$$

An easy calculation gives,

$$T_i(x) = \begin{cases} \{0\} & \text{if } -\frac{1}{4} < x < 0, \\ [0, \frac{1}{4}) & \text{if } x = 0, \\ \emptyset & \text{if } 0 < x < \frac{1}{4}. \end{cases}$$

Then the prox-regularity of f_i, $i = 1, 2$, follows from the monotonicity of T_i. Indeed Theorem 3.4 of [4] in Hilbert spaces and Theorem 3.2 of [17] in finite dimensional spaces state that a function is prox-regular if and only if an f-attentive ε-localization of the subgradient mapping is r-monotone i.e. the localization plus r-times the identity mapping is monotone (obviously if the localization itself is monotone then it is r-monotone). Since f_1 is continuous it remains to verify that f_2 is subdifferentially continuous at $\bar{x} = 0$ for $\bar{v} = 0$. Indeed, for any sequence $(x_n, v_n) \to (0, 0)$ with $v_n \in \partial f_2(x_n)$ eventually we have $f_2(x_n) = 0 = f_2(0)$. Thus, f_2 too is subdifferentially continuous at $\bar{x} = 0$ for $\bar{v} = 0$. Yet f_1 and f_2 differ by different constants on any neighborhood of $\bar{x} = 0$. However, when we restrict to points x that also have a close subgradient i.e, with $\varepsilon = \frac{1}{4}$, if $\|x - \bar{x}\| < \varepsilon$ but $\|v - \bar{v}\| < \varepsilon$ with $v \in \partial f_1(x) = \partial f_2(x)$, then such x has to be in $(-\varepsilon, 0]$ and we have $f_1(x) = 0 = f_2(x)$ for all x in $(-\varepsilon, 0]$.

This example also reveals that Theorem 1.2 covers a much broader class of functions than that of pln functions. For this, we only have to verify that f_1 is not pln at $\bar{x} = 0$. Here we make use of a corresponding subgradient characterization available for pln functions.

Theorem 1.5 (Levy, Poliquin and Thibault [13], Corollary 2.3). *Let $f : X \to \overline{I\!R}$ be a l.s.c. function that is finite at $\bar{x}$. The following are equivalent:*

(a) *f is primal-lower-nice at $\bar{x}$.*

(b) *There exist positive constants ε, c and R such that*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -r\|x_1 - x_2\|^2$$

whenever $v_i \in \partial_p f(x_i)$, $\|v_i\| \leq cr$, $r \geq R$ and $\|x_i - \bar{x}\| \leq \varepsilon$, $i = 1, 2$.

If f_1 were pln at $\bar{x} = 0$ then there would be constants ε, c and R as in Theorem 1.5. Then for any $r > R$, consider the mapping T formed by adding r times the identity to the subgradient mapping of f_1, (Note that the displayed inequality in (b) of Theorem 1.5 amounts to saying that T is monotone).

$$T(x) := \frac{1}{2\sqrt{x}} + rx \text{ for } x \in (0, \varepsilon).$$

The critical points of T are given by $T'(x) = -\frac{1}{4x^{\frac{3}{2}}} + r = 0$, and attained at $x_m := \frac{1}{(4r)^{\frac{2}{3}}}$. Since $T''(x) = \frac{3}{8x^{\frac{5}{2}}} > 0$, x_m is a local minimum for T. Now restrict the subgradients of f_1 such that $\frac{1}{2\sqrt{x}} \leq cr$, i.e, $x_0 := \frac{1}{4c^2r^2} \leq x$. Then for T to be monotone on $[x_0, \varepsilon)$, x_m has to be less than or equal to x_0. This requires that $r^2 \leq \frac{1}{2c^3}$. But, for the large values of r this is impossible and this contradicts the monotonicity of T required by Theorem 1.5. This confirms that f_1 is not pln at $\bar{x} = 0$.

2. Proof of the main result

In the proof of our main result we use the Moreau envelopes and Proximal mappings to go from one function to another. Recall that for a proper, l.s.c. function $f : X \to \overline{I\!R}$ and parameter value $\lambda > 0$, the *Moreau envelope* function, e_λ and the *proximal mapping*, P_λ are defined by

$$e_\lambda(x) := \inf_{x'} \left\{ f(x') + \frac{1}{2\lambda} \|x' - x\|^2 \right\},$$

$$P_\lambda(x) := \operatorname*{argmin}_{x'} \left\{ f(x') + \frac{1}{2\lambda}\|x' - x\|^2 \right\}.$$

Let $f : X \to \overline{I\!R}$ be prox-regular at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$. Then $\bar{v}$ is actually a proximal subgradient of f at $\bar{x}$. In order to simplify our analysis, without any loss of generality, we normalize to the case $\bar{x} = 0$, $\bar{v} = 0$ and $f(0) = 0$. Since our primary interest of f and ∂f depend only on the local geometry of epi f around $(\bar{x}, f(\bar{x}))$, we may further, if necessary, add to f the indicator of some ball with center at $\bar{x}$ to make dom f be bounded. By taking the radius of that ball small enough we can get the quadratic inequality for $\bar{v} \in \partial_p f(\bar{x})$ to hold for all x. Thus we work under the baseline assumptions that

$$\left.\begin{array}{l} f \text{ is locally l.s.c. at } 0 \text{ with } f(0) = 0, \text{ and} \\ r > 0 \text{ is such that } f(x) \geq -\frac{r}{2}\|x\|^2 \text{ for all } x \end{array}\right\} \tag{1}$$

which imply that

$$e_\lambda(0) = 0 \text{ and } P_\lambda(0) = \{0\} \text{ when } \lambda \in (0, 1/r). \tag{2}$$

First we extend some of the results of Propositions 4.2 and 4.3 of [17] to Hilbert spaces. First, we require a lemma.

Lemma 2.1 ([17], Lemma 4.1)). *Under assumptions* (1), *consider any* $\lambda \in (0, 1/r)$ *and let* $\mu = (1 - \lambda r)^{-1}$. *For any* $\rho > 0$,

$$f(x') + \frac{1}{2\lambda}\|x' - x\|^2 \leq e_\lambda(x) + \rho \quad \Longrightarrow \quad \begin{cases} \|x'\| \leq 2\mu\|x\| + \sqrt{2\lambda\mu\rho}, \\ f(x') \leq \frac{1}{2\lambda}\|x\|^2 + \rho, \\ f(x') \geq -\frac{r}{2}\left(2\mu\|x\| + \sqrt{2\lambda\mu\rho}\right)^2. \end{cases} \tag{3}$$

Proof. The same proof of [17], Lemma 4.1 can be carried over to this Hilbertian case, since the only requirement there was the norm be given by an inner product. □

The results in the next two propositions can be found in [4] and [7].

Proposition 2.2. *Under assumptions* (1), *consider any* $\lambda \in (0, 1/r)$. *For any* $\varepsilon > 0$ *there is a neighborhood* V_λ *of* $\bar{x} = 0$ *such that*

(*a*) e_λ *is Lipschitz continuous on* V_λ *with constant* ε *and bounded below by a quadratic function,*

(*b*) $\|x'\| < \varepsilon$, $|f(x')| < \varepsilon$ *and* $\lambda^{-1}\|x - x'\| < \varepsilon$ *for all* $x' \in P_\lambda(x)$ *when* $x \in V_\lambda$.

Proof (from [7]). (The proof given here differs from that of [4]. It also differs from the proof given in Poliquin and Rockafellar [17], Proposition 4.2 (a) and (c) because the argument given there relies on the existence of minimizers of a l.s.c. function over a compact set, which is not true in the case of Hilbert spaces). Let $\mu = (1 - \lambda r)^{-1}$ and $\varepsilon' \in (0, \varepsilon)$. Choose $\delta_\lambda > 0$ and $\rho > 0$ small enough that $(2\varepsilon' + 3\delta_\lambda)/\lambda \leq \varepsilon$ and

$$2\mu\delta_\lambda + \sqrt{2\lambda\mu\rho} \leq \varepsilon', \quad \frac{1}{2\lambda}\delta_\lambda^2 + \rho \leq \varepsilon', \quad \frac{r}{2}\left(2\mu\delta_\lambda + \sqrt{2\lambda\mu\rho}\right)^2 \leq \varepsilon', \quad \frac{\delta_\lambda(1 + 2\mu)}{\lambda} \leq \varepsilon',$$

and let $V_\lambda := \{x \mid \|x\| \le \delta_\lambda\}$ and $C := \{x \mid \|x\| \le \varepsilon'\}$.
(a) Let x and y belong to V_λ. For any $\rho > 0$, by the definition of $e_\lambda(y)$ as an infimum, there exists x' such that

$$f(x') + \frac{1}{2\lambda}\|x' - y\|^2 \le e_\lambda(y) + \rho.$$

Then by Lemma 2.1 we have $\|x'\| \le 2\mu\|x\| + \sqrt{2\lambda\mu\rho} \le 2\mu\delta_\lambda + \sqrt{2\lambda\mu\rho} \le \varepsilon'$, thus $x' \in C$. We have

$$\begin{aligned} e_\lambda(x) - e_\lambda(y) &\le f(x') + \frac{1}{2\lambda}\|x' - x\|^2 - f(x') - \frac{1}{2\lambda}\|x' - y\|^2 + \rho \\ &= \frac{1}{2\lambda}\|x - y\|^2 - \frac{1}{\lambda}\langle x - y, x' - y\rangle + \rho \\ &\le \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{\lambda}\|x - y\|\|x' - y\| + \rho \\ &\le K\|x - y\| + \rho, \end{aligned} \tag{4}$$

where K is chosen so that $K := (1/\lambda)\sup\{\|x\| + 2\|z - x\|;\, x \in V_\lambda,\, z \in C\} < \infty$. Indeed, we have $K \ge (1/\lambda)\{\|y\| + 2\|x' - y\|\}$ for all $y \in V_\lambda$ and $x' \in C$ and hence

$$\|x - y\|K \ge \frac{1}{\lambda}\{\|x - y\|\|y\| + 2\|x - y\|\|x' - y\|\}.$$

We also have that $\|x - y\|K \ge \frac{1}{\lambda}\|x\|\|x - y\|$ because $K \ge \frac{1}{\lambda}\|x\|$ for all x in V_λ. In adding these inequalities together, we get the inequality in (4):

$$\begin{aligned} \|x - y\|K &\ge \frac{1}{2\lambda}\{\|x - y\|(\|x\| + \|y\|) + 2\|x - y\|\|x' - y\|\} \\ &\ge \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{\lambda}\|x - y\|\|x' - y\|. \end{aligned}$$

And this constant K cannot be bigger than ε:

$$K = \frac{1}{\lambda}\sup\{\|x\| + 2\|z - x\|;\, x \in V_\lambda, z \in C\} \tag{5}$$

$$\le \frac{1}{\lambda}\sup\{\|x\| + 2(\|z\| + \|x\|);\, x \in V_\lambda, z \in C\} \tag{6}$$

$$\le \frac{1}{\lambda}(3\delta_\lambda + 2\varepsilon') \le \varepsilon. \tag{7}$$

Reversing the roles of x and y, and then letting $\rho \searrow 0$ in (4) shows that e_λ is Lipschitz of rank ε on V_λ.
The asserted lower bound for e_λ follows from

$$\begin{aligned} e_\lambda(x) &= \inf_{x'}\{f(x') + \frac{1}{2\lambda}\|x' - x\|^2\} \\ &\ge \inf_{x'}\{-\frac{r}{2}\|x'\|^2 + \frac{1}{2\lambda}\|x' - x\|^2\} \\ &= \frac{\frac{1}{2\lambda}\frac{r}{2}}{\frac{1}{2\lambda} - \frac{r}{2}}\|x\|^2 \\ &= -\frac{r}{2(1 - r\lambda)}\|x\|^2. \end{aligned}$$

(b) When $x' \in P_\lambda(x)$, then Lemma 2.1 is true for every $\rho > 0$ which implies

$$\begin{aligned} \|x'\| &\le 2\mu\|x\| \le 2\mu\delta_\lambda \le \varepsilon' < \varepsilon, \\ f(x') &\le \frac{1}{2\lambda}\|x\|^2 \le \frac{1}{2\lambda}\delta_\lambda^2 \le \varepsilon' < \varepsilon, \\ f(x') &\ge -\frac{r}{2}(2\mu\|x\|)^2 \ge -2r\mu^2\delta_\lambda^2 \ge -\varepsilon' > -\varepsilon, \end{aligned}$$

and also

$$\begin{aligned} \frac{1}{\lambda}\|x - x'\| &\le \frac{1}{\lambda}(\|x\| + \|x'\|) \\ &\le \frac{1}{\lambda}(1+2\mu)\|x\| \\ &\le \frac{\delta_\lambda}{\lambda}(1+2\mu) \le \varepsilon' < \varepsilon. \end{aligned}$$

□

Proposition 2.3. *Under assumptions* (1) *there exists for each* $\lambda \in (0, 1/r)$ *a neighborhood* V_λ *of* $\bar{x} = 0$ *on which*

$x' \in P_\lambda(x) \Longrightarrow \lambda^{-1}(x - x') \in \partial f(x')$, *i.e.*, $x' \in (I + \lambda\partial f)^{-1}(x)$.

Proof. Recall that the existence of a proximal subgradient at x' corresponds to the existence of a "local quadratic support" to f at x'. When $x' \in P_\lambda(x)$ we have

$$f(x'') + \frac{1}{2\lambda}\|x'' - x\|^2 \ \ge\ f(x') + \frac{1}{2\lambda}\|x' - x\|^2 \text{ for all } x'',$$

so that $f(x'') - f(x') \ge q(x'')$ for the quadratic function $q(x'') = \big(\|x' - x\|^2 - \|x'' - x\|^2\big)/2\lambda$. We have $q(x') = 0$ and $Dq(x') = \lambda^{-1}(x - x')$, so q forms a local quadratic support to f at x'. Thus $\lambda^{-1}(x - x') \in \partial_p f(x')$. □

When we assume f to be prox-regular, the above propositions with the r-monotonicity of the localization of the subgradient mapping entail the $\mathcal{C}^{1+}$ smoothness of e_λ and the local single-valuedness of P_λ as seen by the next theorem. The finite-dimensional case can be found in [17], Theorem 4.4. The extension to Hilbert spaces is given in [4] Proposition 5.3.

Theorem 2.4 ([17], Prop. 5.3). *Suppose that* f *is prox-regular at* $\bar{x} = 0$ *for* $\bar{v} = 0$ *with respect to* ε *and* r*, in particular with* (1) *holding. Let* T *be the* f*-attentive* ε*-localization of* ∂f *around* $(0,0)$*. Then for each* $\lambda \in (0, 1/r)$ *there is a neighborhood* V_λ *of* $\bar{x} = 0$ *such that, on* V_λ*, the mapping* P_λ *is single-valued and Lipschitz continuous with constant* $1/(1 - \lambda r)$ *and*

$$P_\lambda(x) \ = \ (I + \lambda T)^{-1}(x) \ = \ [\textit{singleton}],$$

while the function e_λ *is of class* $\mathcal{C}^{1+}$ *with* $De_\lambda(0) = 0$ *and*

$$De_\lambda(x) = \frac{x - P_\lambda(x)}{\lambda} = \lambda^{-1}\left[I - [I + \lambda T]^{-1}\right](x).$$

We can now prove our main result.

Proof of Theorem 1.2. Without loss of generality we normalize to the case $\bar{x} = 0$, $\bar{v} = 0$ with

$$\left.\begin{array}{l} f_i \text{ is locally l.s.c. at } 0 \text{ with } f_i(0) = 0, \text{ and } r > 0 \\ \text{is such that } f_i(x) \geq -\frac{r}{2}\|x\|^2 \text{ for all } x, \text{ and } i = 1,2 \end{array}\right\} \tag{5}$$

which imply that

$$e_\lambda^i(0) = 0 \text{ and } P_\lambda^i(0) = \{0\} \text{ when } \lambda \in (0, 1/r) \text{ and } i = 1,2, \tag{6}$$

where e_λ^i and P_λ^i are the Moreau envelope function and the proximal mapping of f_i, respectively.
We may further assume that there exists $\varepsilon > 0$ such that f_1 and f_2 are prox-regular at $\bar{x} = 0$ for $\bar{v} = 0$ with respect to the same r with (5) holding. For $i = 1,2$ let T_i be the f_i-attentive ε-localization T_i of ∂f_i around $(0,0)$. We can assume that $T_i + rI$ (where i is the identity mapping) is monotone; see [4] Theorem 3.4). Then, by Theorem 2.4, for each $\lambda \in (0, 1/r)$ and $i = 1,2$ there exists $\delta_\lambda > 0$ such that, on $V_\lambda := \{x\,;\, \|x\| < \delta_\lambda\}$, the mappings P_λ^i are single-valued and Lipschitz continuous with constant $1/(1 - \lambda r)$ and

$$P_\lambda^i(x) \;=\; (I + \lambda T_i)^{-1}(x) \;= [\text{singleton}], \tag{7}$$

while the functions e_λ^i is of class $\mathcal{C}^{1+}$ with $De_\lambda^i(0) = 0$ and

$$De_\lambda^i(x) = \frac{x - P_\lambda^i(x)}{\lambda} = \lambda^{-1}\left[I - [I + \lambda T_i]^{-1}\right](x), \tag{8}$$

and the properties in Propositions 2.2 and 2.3 hold.

Decreasing ε further if necessary, we can arrange that f_1 and f_2 have the same subgradients on $\varepsilon I\!\!B$, where $\varepsilon > 0$ comes from the definition of prox-regularity of f_i.

Claim 1. For each $\lambda \in (0, 1/r)$, we have $P_\lambda^1(x) = P_\lambda^2(x) = [\text{singleton}]$, and $e_\lambda^1(x) = e_\lambda^2(x)$ on V_λ.

Proof of Claim 1. First notice that the proximal mappings P_λ^i, $i = 1,2$ are single-valued on V_λ by (7). Let any x in V_λ and $x_i = P_\lambda^i(x)$, $i = 1,2$. Then by Propositions 2.2(b) and 2.3 we have $\|x_1\| < \varepsilon$, $|f_1(x_1)| < \varepsilon$ and $\|v_1\| < \varepsilon$, where $v_1 = \frac{1}{\lambda}(x - x_1) \in \partial f_1(x_1)$. With the same reasoning $x_2 = P_\lambda^2(x)$ gives $\|x_2\| < \varepsilon$ and $\|v_2\| < \varepsilon$, where $v_2 = \frac{1}{\lambda}(x - x_2) \in \partial f_2(x_2)$. Since $\|x_2\| < \varepsilon$ we have $v_2 \in \partial f_2(x_2) = \partial f_1(x_2)$. Since f_1 is subdifferentially continuous at $\bar{x} = 0$, we may also assume that $|f_1(x_2)| < \varepsilon$. The monotonicity of $T_i + rI$ implies, for the pairs (x_1, v_1) and (x_2, v_2), that

$$\left\langle \left[\frac{x - x_1}{\lambda}\right] - \left[\frac{x - x_2}{\lambda}\right],\, x_1 - x_2 \right\rangle \geq -r\|x_1 - x_2\|^2,$$

hence $-\lambda^{-1}\|x_1 - x_2\|^2 \geq -r\|x_1 - x_2\|^2$. Then $(1 - \lambda r)\|x_1 - x_2\|^2 \leq 0$, so $x_1 = x_2$. Therefore, we have $P_\lambda^1(x) = P_\lambda^2(x)$ and by (8), $De_\lambda^1(x) = De_\lambda^2(x)$ on V_λ. Thus we conclude $e_\lambda^1(x) = e_\lambda^2(x)$ since $e_\lambda^i(0) = 0$ when $\lambda \in (0, 1/r)$ and $i = 1,2$ by (6).

Claim 2. For all x in $\operatorname{dom} \partial f_1 \cap (\delta_\lambda/4) I\!\!B$ and v in $\partial f_1(x)$ with $0 < \delta_\lambda/4 < \varepsilon$ and $\lambda < 3$ such that $\|v\| < (\delta_\lambda/4) < \varepsilon$ we have $P_\lambda^1(z_\lambda) = P_\lambda^2(z_\lambda) = \{x\}$, where $z_\lambda = x + \lambda v$.

Proof of Claim 2. Take any x in $\operatorname{dom} \partial f_1 \cap (\delta_\lambda/4) I\!\!B$ and restrict $\lambda < 3$. Then

$$\|z_\lambda\| \leq \|x\| + \lambda\|v\| < \frac{\delta_\lambda}{4} + \lambda\frac{\delta_\lambda}{4} = (1 + \lambda)\frac{\delta_\lambda}{4} < 4(\frac{\delta_\lambda}{4}) = \delta_\lambda,$$

so z_λ belongs to V_λ.

Let $\tilde{x}$ be an element of $P^1_\lambda(z_\lambda) = P^2_\lambda(z_\lambda)$ (equality due to Claim 1). Then by Propositions 2.2(b) and 2.3 we have $\|\tilde{x}\| < \varepsilon$, $|f_1(\tilde{x})| < \varepsilon$ and $\|\tilde{v}\| < \varepsilon$, where $\tilde{v} = \frac{1}{\lambda}(z_\lambda - \tilde{x}) \in \partial f_1(\tilde{x})$. By our hypothesis $v = \frac{z_\lambda - x}{\lambda} \in \partial f_1(x)$ with $\|v\| < (\delta_\lambda/4) < \varepsilon$ and $\|x\| < (\delta_\lambda/4) < \varepsilon$. Since f_1 is subdifferentially continuous at $\bar{x} = 0$, we may also assume that $|f_1(x)| < \varepsilon$. Again the monotonicity of $T_i + rI$ gives for the pairs $(\tilde{x}, \tilde{v})$ and (x, v) that

$$\left\langle \left[\frac{z_\lambda - \tilde{x}}{\lambda}\right] - \left[\frac{z_\lambda - x}{\lambda}\right],\, \tilde{x} - x \right\rangle \geq -r\left\|\tilde{x} - x\right\|^2,$$

hence $-\lambda^{-1}\left\|\tilde{x} - x\right\|^2 \geq -r\left\|\tilde{x} - x\right\|^2$. Then $(1 - \lambda r)\|\tilde{x} - x\|^2 \leq 0$, so $\tilde{x} = x$. Thus we have $P^1_\lambda(z_\lambda) = P^2_\lambda(z_\lambda) = \{x\}$ as claimed.

Claim 3. If x belongs to $\operatorname{dom} \partial f_1$ with x near $\bar{x} = 0$ and x has a subgradient $v \in \partial f_1(x) = \partial f_2(x)$ close to $\bar{v} = 0$, then $f_1(x) = f_2(x)$.

Proof of Claim 3. For $\lambda < 3$, take any x in $\operatorname{dom} \partial f_1 \cap (\delta_\lambda/4)I\!B$ and v in $\partial f_1(x)$ with $\|v\| < (\delta_\lambda/4) < \varepsilon$. We then have $z_\lambda = x + \lambda y$ in V_λ. Then by Claims 1 and 2, we get $P^1_\lambda(z_\lambda) = P^2_\lambda(z_\lambda) = \{x\}$ and $e^1_\lambda(z_\lambda) = e^2_\lambda(z_\lambda)$. This means

$$f_1(x) + \frac{1}{2\lambda}\|x - z_\lambda\|^2 \;=\; f_2(x) + \frac{1}{2\lambda}\|x - z_\lambda\|^2,$$

and hence $f_1(x) = f_2(x)$. This completes the Claim and hence the Theorem. $\square$

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