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Maximal Monotonicity via Convex Analysis

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Dedicated to the memory of Simon Fitzpatrick.

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In his '23' "*Mathematische Probleme*"[1] lecture to the Paris International Congress in 1900, David Hilbert wrote *"Besides it is an error to believe that rigor in the proof is the enemy of simplicity."*

In this spirit, we use simple convex analytic methods, relying on an ingenious function due to Simon Fitzpatrick, to provide a concise proof of the maximality of the sum of two maximal monotone operators on reflexive Banach space under standard transversality conditions. Many other extension, surjectivity, convexity and local boundedness results are likewise established.

Keywords: Monotone operators, convex analysis, sandwich theorem, Fenchel duality, sum theorem, composition theorem, extension theorems, cyclic monotonicity, acyclic monotonicity, Fitzpatrick function

1. Introduction

Recall that the *domain* of an extended valued convex function, denoted $\operatorname{dom}(f)$, is the set of points with value less than $+\infty$, and that a point s is in the *core* of a set S (denoted by $s \in \operatorname{core} S$) provided that $X = \bigcup_{\lambda>0} \lambda(S - s)$. Recall that $x^* \in X^*$ is a *subgradient* of $f : X \to (-\infty, +\infty]$ at $x \in \operatorname{dom} f$ provided that $f(y) - f(x) \geq \langle x^*, y - x\rangle$ for all y in Y. The set of all subgradients of f at x is called the *subdifferential* of f at x and is denoted $\partial f(x)$. We shall need the *indicator* function $\iota_C(x)$ which is zero for x in C and $+\infty$ otherwise, the *Fenchel conjugate* $f^*(x^*) := \sup_x\{\langle x, x^*\rangle - f(x)\}$ and the *infimal convolution* $f^*\Box\frac{1}{2}\|\cdot\|_*^2(x^*) := \inf\{f^*(y^*) + \frac{1}{2}\|z^*\|_*^2 \colon x^* = y^* + z^*\}$. When f is convex and closed $x^* \in \partial f(x)$ exactly when $f(x) + f^*(x^*) = \langle x, x^*\rangle$. We recall that the *distance function* associated with a closed set C, given by $d_C(x) := \inf_{c\in C} \|x - c\|$, is convex if and only if C is. Moreover, $d_C = \iota_C\Box\|\cdot\|$.

As convenient, we shall use both $\operatorname{dom} T = D(T) := \{x \colon T(x) \neq \emptyset\}$, and $\operatorname{range} T = R(T) := T(X)$ to denote the *domain* and *range* of a multifunction. We say a multifunction $T : X \mapsto 2^{X^*}$ is *monotone* provided that for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$,

$$\langle y - x, y^* - x^*\rangle \geq 0,$$

and we say that T is *maximal monotone* if its *graph*, $\{(x, x^*) \colon x^* \in T(x)\}$, is not properly included in any other monotone graph. The subdifferential of a convex lower semicontinuous (lsc) function on a Banach space is a typical example of a maximal monotone

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[1]See the late Ben Yandell's fine account of the Hilbert Problems and their solvers in *The Honors Class*, AK Peters, 2002.

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multifunction. We reserve the notation J for the duality map

$$J(x) := \frac{1}{2}\,\partial\|x\|^2 = \{x^* \in X^* \colon \|x\|^2 = \|x^*\|^2 = \langle x, x^*\rangle\},$$

and define the convex *normal cone* to C at $x \in \operatorname{cl} C$ by

$$N_C(x) := \partial\iota_C(x).$$

All other notation is generally consistent with usage in [9, 11, 30]. Some of the side-results in this paper section are not used in the sequel and so we are somewhat sparing with such details.

Our goal is to derive many key results about maximal monotone operators entirely from the existence of subgradients and the *Sandwich theorem* given below; as much as possible using only geometric-functional-analysis tools. In Section 2 we consider general Banach spaces. In Section 3 we look at cyclic operators. In Section 4 we provide our central result on maximality of the sum in reflexive space. Section 5 looks at more applications of the technique introduced in Section 4 while limiting examples are produced in Section 6.

2. Maximality in General Banach Space

For a monotone mapping T, we associate the *Fitzpatrick function* introduced in [16]. The Fitzpatrick function is

$$\mathcal{F}_T(x, x^*) := \sup\{\langle x, y^*\rangle + \langle x^*, y\rangle - \langle y, y^*\rangle : y^* \in T(y)\},$$

which is clearly lower semicontinuous and convex as an affine supremum. Moreover,

Proposition 2.1 ([16, 11]). *For a maximal monotone operator* $T\colon X \to X^*$ *one has*

$$\mathcal{F}_T(x, x^*) \geq \langle x, x^*\rangle$$

with equality if and only if $x^* \in T(x)$. *Indeed, the equality* $\mathcal{F}_T(x, x^*) = \langle x, x^*\rangle$ *for all* $x^* \in T(x)$, *requires only monotonicity not maximality.*

Note that in general $\mathcal{F}_T$ is not useful for non-maximal operators. As an extreme example, on the real line if $T(0) = 0$ and $T(x)$ is empty otherwise, then $\mathcal{F}_T \equiv 0$. Note also that the construction in Proposition 2.1 extends to any paired vector spaces.

The idea of associating a convex function with a monotone operator and exploiting the relationship was largely neglected for many years after [7] and [16] until exploited by Penot, Simons, Simons and Zălinescu ([33, 35, 36, 38]), Burachik and Svaiter and others.

2.1. Convex Analytic Tools

The basic results that we use repeatedly follow:

Proposition 2.2 ([9, 11, 30]). *A proper lsc convex function on a Banach space is continuous throughout the core of its domain.*

Proposition 2.3 ([9, 11, 30]). *A proper lsc convex function on a Banach space has a non-empty subgradient throughout the core of its domain.*

These two basic facts lead to:

Theorem 2.4 (Hahn-Banach Sandwich, [9, 11, 30]). *Suppose f and $-g$ are lsc convex on a Banach space X and that*

$$f(x) \geq g(x),$$

for all x in X. Assume that the following constraint qualification (CQ) *holds:*

$$0 \in \operatorname{core}\left(\operatorname{dom}(f) - \operatorname{dom}(-g)\right). \tag{1}$$

Then there is an affine continuous function a such that

$$f(x) \geq a(x) \geq g(x),$$

for all x in X.

Proof. The value function $h(u) := \inf_{x \in X} f(x) - g(x-u)$ is convex and the (CQ) implies it is continuous at 0. Hence there is some $-\lambda \in \partial h(0)$, and this provides the linear part of the asserted affine separator. Indeed, we have

$$f(x) - g(u-x) \geq h(u) - h(0) \geq \lambda(u),$$

as required. $\square$

We will also refer to *constraint qualifications* like (1) as *transversality conditions* since they ensure that the sum/difference of two convex sets is large, and so resemble such conditions in differential geometry. It is an easy matter to deduce the complete *Fenchel duality theorem* (see [9, 29, 11, 38]) from Theorem 2.4 and in particular that

Corollary 2.5 ([11, 9, 29, 38]). *Suppose that f and g are convex and that* (1) *holds. Then $\partial f + \partial g = \partial (f+g)$.*

Proposition 2.6 ([37]). *For a closed convex function f and $f_J := f + \frac{1}{2}\|\cdot\|^2$ we have $\left(f + \frac{1}{2}\|\cdot\|^2\right)^* = f^* \Box \frac{1}{2}\|\cdot\|_*^2$ is everywhere continuous. Also*

$$v^* \in \partial f(v) + J(v) \Leftrightarrow f_J^*(v^*) + f_J(v) - \langle v, v^* \rangle \leq 0.$$

Edgar Asplund wrote a still-very-informative 1969 survey of those parts of convex analysis "that the author feels are important in the study of monotone mappings," [2, p. 1]. This includes averaging of norms, decomposition and differentiability results, as well as the sort of basic results we have described above.

2.2. Representative Convex Functions.

Recall that a *representative function* for a monotone operator T on X is any convex function $\mathcal{H}_T$ on $X \times X^*$ such that $\mathcal{H}_T(x, x^*) \geq \langle x, x^* \rangle$ for all x, x^*, while $\mathcal{H}_T(x, x^*) = \langle x, x^* \rangle$ when $x^* \in T(x)$. Unlike [13], we do not require $\mathcal{H}_T$ to be closed. When T is maximal, Proposition 2.1 shows $\mathcal{F}_T$ is a representative function for T, as is the convexification

$$\mathcal{P}_T(x, x^*) := \inf \left\{ \sum_{i=1}^{N} \lambda_i \langle x_i, x_i^* \rangle \colon \sum_i \lambda_i (x_i, x_i^*, 1) = (x, x^*, 1), x_i^* \in T(x_i), \lambda_i \geq 0 \right\},$$

which has the requisite properties for any monotone T, whether or not maximal:

Proposition 2.7. *For any monotone mapping T, $\mathcal{P}_T$ is a representative convex function for T.*

Proof. Directly from the definition of monotonicity we have

$$\mathcal{P}_T(x, x^*) \geq \langle x^*, y\rangle + \langle y^*, x\rangle - \langle y^*, y\rangle,$$

for $y^* \in T(y)$. Thus, for all points

$$\mathcal{P}_T(x, x^*) + \mathcal{P}_T(y, y^*) \geq \langle x^*, y\rangle + \langle y^*, x\rangle.$$

Note that by definition $\mathcal{P}_T(x, x^*) \leq \langle x^*, x\rangle$ for $x^* \in T(x)$. Hence, setting $x = y$ and $x^* = y^*$ shows $\mathcal{P}_T(x, x^*) = \langle x^*, x\rangle$ for $x^* \in T(x)$ while $\mathcal{P}_T(z, z^*) \geq \langle z^*, z\rangle$ for $(z^*, z) \in \operatorname{conv}\operatorname{graph} T$ and, also by definition, $\mathcal{P}_T(z, z^*) = +\infty$ otherwise. □

Direct calculation shows $(\mathcal{P}_T)^* = \mathcal{F}_T$ for any monotone T, [24]. This convexification originates with Simons [31] and was refined by Penot [23, Proposition 5].

2.3. Monotone Extension Formulas

We illustrate the flexibility of $\mathcal{P}$ by using it to prove a central case of the Debrunner-Flor theorem [14, 25] *without* using Brouwer's theorem.

Theorem 2.8 ([25, 33]).

(a) *Suppose T is monotone on a Banach space X with range contained in αB_{X^*}, for some $\alpha > 0$. Then for every x_0 in X there is $x_0^* \in \overline{\operatorname{conv}}^* R(T) \subset \alpha B_{X^*}$ such that (x_0, x_0^*) is monotonically related to* graph (T).

(b) *In consequence, T has a bounded monotone extension $\overline{T}$ with* $\operatorname{dom}(\overline{T}){=}X$ *and $R(\overline{T}) \subset \overline{\operatorname{conv}}^* R(T)$.*

(c) *In particular, a maximal monotone T with bounded range has* $\operatorname{dom}(T){=}X$ *and has* range (T) *connected.*

Proof. (a) It is enough, after translation, to show $x_0 = 0 \in \operatorname{dom}(T)$. Fix $\alpha > 0$ with $R(T) \subset C := \overline{\operatorname{conv}}^* R(T) \subset \alpha B_{X^*}$.

Consider

$$f_T(x) := \inf\{\mathcal{P}_T(x, x^*) : x^* \in C\}.$$

Then f_T is convex since $\mathcal{P}_T$ is. Observe that $\mathcal{P}_T(x, x^*) \geq \langle x, x^*\rangle$ and so $f_T(x) \geq \inf_{x^* \in C}\langle x, x^*\rangle \geq -\alpha\,\|x\|$ for all x in X. As $x \mapsto \inf_{x^* \in C}\langle x, x^*\rangle$ is concave and continuous the Sandwich Theorem 2.4 applies.

Thus, there exist w^* in X^* and γ in $\mathbf{R}$ with

$$\mathcal{P}_T(x, x^*) \geq f_T(x) \geq \langle x, w^*\rangle + \gamma \geq \inf_{x^* \in C}\langle x, x^*\rangle \geq -\alpha\,\|x\|$$

for all x in X and x^* in $C \subset \alpha B_{X^*}$. Setting $x = 0$ shows $\gamma \geq 0$. Now, for any (y, y^*) in the graph of T we have $\mathcal{P}_T(y, y^*) = \langle y, y^*\rangle$. Thus,

$$\langle y - 0, y^* - w^*\rangle \geq \gamma \geq 0,$$

which shows that $(0, w^*)$ is monotonically related to the graph of T. Finally, $\langle x, w^*\rangle + \gamma \geq \inf_{x^*\in C}\langle x, x^*\rangle \geq -\alpha \|x\|$ for all $x \in X$ involves three sublinear functions, and so implies that $w^* \in C \subset \alpha B_{X^*}$.

(b) Consider the set $\mathcal{E}$ of all monotone extensions of T with range in $C \subset \alpha B_{X^*}$, ordered by inclusion. By Zorn's lemma $\mathcal{E}$ admits a maximal member $\overline{T}$ and by (a) $\overline{T}$ has domain the whole space. (c) follows immediately, since T being maximal and everywhere defined is a weak-star cusco and so has a weak-star connected range. □

One may consult [17] and [33, Theorem 4.1] for other convex analytic proofs of (c). Note also that the argument in (a) extends to an unbounded set C whenever

$$x_0 \in \operatorname{core}(\operatorname{dom} f_T + \operatorname{dom} \sup_C).$$

The full Debrunner-Flor result is stated next:

Theorem 2.9 (Debrunner-Flor extension theorem, [14, 25]). *Suppose T is a monotone operator on Banach space X with $\operatorname{range} T \subset C$ with C weak-star compact and convex. Suppose also $\varphi\colon C \mapsto X$ is weak-star to norm continuous. Then there is some $c^* \in C$ with $\langle x - \varphi(c^*), x^* - c^*\rangle \geq 0$ for all $x^* \in T(x)$.*

It seems worth observing that:

Proposition 2.10. *The full Debrunner-Flor extension theorem is equivalent to Brouwer's theorem.*

Proof. An accessible derivation of Debrunner-Flor from Brouwer's theorem is given in [25]. Conversely, let g be a continuous self-map of a norm-compact convex set $K \subset \operatorname{int} B_X$ in a Euclidean space X. We apply the Debrunner-Flor extension theorem to the identity map I on B_X and to $\varphi\colon B_X \mapsto X$ given by $\varphi(x) := g(P_K x)$, where P_K is the metric projection mapping (any retraction would do). We obtain $x_0^* \in B_X$ and also $x_0 := \varphi(x_0^*) = g(P_K x_0^*) \in K$ with

$$\langle x - x_0, x - x_0^*\rangle \geq 0$$

for all $x \in B_X$. Since $x_0 \in \operatorname{int} B_X$, for $h \in X$ and small $\epsilon > 0$ we have $x_0 + \epsilon h \in B_X$ and so $\langle h, x_0 - x_0^*\rangle \geq 0$ for all $h \in X$. Thus, $x_0 = x_0^*$ and so $P_K x_0^* = P_K x_0 = x_0 = g(P_K x_0^*)$, is a fixed point of the arbitrary self-map g. □

2.4. Local Boundedness Results

We next turn to local boundedness results. Recall that an operator T is *locally bounded* around a point x if $T(B_\varepsilon(x))$ is bounded for some $\varepsilon > 0$.

Theorem 2.11 ([31, 37]). *Let X be a Banach space and let S and $T\colon X \to 2^{X^*}$ be monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(S)].$$

Then there exist $r, c > 0$ such that, for any $x \in \operatorname{dom}(T) \cap \operatorname{dom}(S)$, $t^* \in T(x)$ and $s^* \in S(x)$,

$$\max(\|t^*\|, \|s^*\|) \leq c\,(r + \|x\|)(r + \|t^* + s^*\|).$$

Proof. Consider the convex lower semicontinuous function

$$\sigma_T(x) := \sup_{z^* \in T(z)} \frac{\langle x - z, z^* \rangle}{1 + \|z\|}.$$

This is a refinement of the function [7] originally used to prove local boundedness of monotone operators [31, 37, 11]. We first show that $\operatorname{conv}\operatorname{dom}(T) \subset \operatorname{dom}\sigma_T$, and that $0 \in \operatorname{core}\bigcup_{i=1}^{\infty}[\{x\colon \sigma_S(x) \le i, \|x\| \le i\} - \{x\colon \sigma_T(x) \le i, \|x\| \le i\}]$. We now apply conventional Baire category techniques – with some care. □

Corollary 2.12 ([31, 11, 37]). *Let X be any Banach space. Suppose T is monotone and*

$$x_0 \in \operatorname{core}\operatorname{conv}\operatorname{dom}(T).$$

Then T is locally bounded around x_0.

Proof. Let $S = 0$ in Theorem 2.11 or directly apply Proposition 2.2 to σ_T. □

We can also improve Theorem 2.8.

Corollary 2.13. *A monotone mapping T with bounded range admits an everywhere defined maximal monotone extension with bounded weak-star connected range contained in $\overline{\operatorname{conv}}^* R(T)$.*

Proof. Let $\widehat{T}$ denote the extension of Theorem 2.8(b). Clearly it is everywhere locally bounded. The desired maximal monotone extension $T^*(x)$ is the operator whose graph is the norm-weak-star closure of the graph of $x \mapsto \operatorname{conv}\widehat{T}(x)$, since this is both monotone and is a norm-w* cusco. Explicitly, $T^*(x) := \cap_{\varepsilon>0}\overline{\operatorname{conv}}^*\widehat{T}(B_\varepsilon(x))$, see [11]. □

Recall that a maximal monotone mapping is *locally maximal monotone*, or *type* (FP), if $(\operatorname{graph} T^{-1}) \cap (V \times X)$ is maximal monotone in $V \times X$, for every convex open set V in X^* with $V \cap \operatorname{range} T \neq \emptyset$. Dually, a maximal monotone mapping is *maximal monotone locally* (VFP), is defined by reversing the roles of X and X^* with T instead of T^{-1}. It is known that all maximal monotone operators on a reflexive space are type (FP) and (VFP), see [17, 18, 25, 32] and Theorem 5.4, as are all subgradients of closed convex functions, [31, 32] and Theorem . It is shown in [17] that a maximal monotone operator T with $\operatorname{range} T = X^*$(resp. $\operatorname{dom} T = X$) is locally maximal monotone (resp. maximal monotone locally).

For a maximal monotone operator T we may usefully apply Corollary 2.13 to the mapping $T_n(x) := T(x) \cap n\,B_{X^*}$. Under many conditions the extension, $\widehat{T_n}$ is unique. Indeed as proven by Fitzpatrick and Phelps:

Proposition 2.14 ([16, 17]). *Suppose T is maximal monotone and suppose n is large enough so that $R(T) \cap n\operatorname{int} B_{X^*} \neq \emptyset$.*

(a) *There is a unique maximal monotone $\widehat{T_n}$ such that $T_n(x) \subset \widehat{T_n}(x) \subset nB_{X^*}$ whenever the mapping M_n defined by*

$$M_n(x) := \{x^* \in nB_{X^*}\colon \langle x^* - z^*, x - z\rangle \ge 0 \quad \text{for all } z^* \in T(z) \cap n\operatorname{int} B_{X^*}\},$$

is monotone; in which case $M_n = \widehat{T_n}$.

(*b*) *This happens whenever T is type (FP) and B_{X^*} is strictly convex, hence for any maximal monotone operator on a reflexive space in a strictly convex dual norm.*

Proof. Since $\widehat{T}_n$ exists by Corollary 2.13 and since $\widehat{T}_n(x) \subset M_n(x)$, (*a*) follows. We refer to [16, Theorem 2.2] for the fairly easy proof of (*b*). $\square$

It is reasonable to think of the sequence $\{\widehat{T}_n\}_{n\in\mathbf{N}}$ as a good non-reflexive generalization of the resolvent-based Yosida approximate [25, 11] or of Hausdorff's Lipschitz regularization of a convex function, [16, 25, 9] – especially in the (FP) case where one also shows easily that (i) $\widehat{T}_n(x) = T(x) \cap n\, B_{X^*}$ whenever $T(x) \cap \operatorname{int} n\, B_{X^*} \neq \emptyset$, and (ii) $\widehat{T}_n(x) \backslash T(x) \subset n\, S_{X^*}$, [17]. Thus, for local properties, such as differentiability, one may often replace T by some $\widehat{T}_n$ if it simplifies other matters.

2.5. Convexity of Domain and Range

We start with

Corollary 2.15 ([27, 28, 31]). *Let X be any Banach space. Suppose that T is maximal monotone with* $\operatorname{core}\operatorname{conv}\operatorname{dom}(T)$ *nonempty. Then*

$$\operatorname{core}\operatorname{conv}\operatorname{dom}(T) \;= \operatorname{int}\operatorname{conv}\operatorname{dom}(T) \subset \operatorname{dom}(T). \tag{2}$$

In consequence $\operatorname{dom}(T)$ *has both a convex closure and a convex interior.*

Proof. We first establish the inclusion in (2). Fix $x + \varepsilon\, B_X \subset \operatorname{int}\operatorname{conv}\operatorname{dom}(T)$ and, appealing to Corollary 2.12, select $M := M(x, \varepsilon) > 0$ so that $T(x + \varepsilon\, B_X) \subset M\, B_{X^*}$.

For $N > M$ define nested sets

$$T_N(x) := \{x^* : \langle x - y, x^* - y^*\rangle \geq 0, \quad \forall y^* \in T(y) \cap N\, B_{X^*}\},$$

and note these images are w^*-closed. By Theorem 2.8(*b*), the sets are non-empty, and by the next Lemma 2.16 bounded, hence w^*-compact. Observe that by maximality of T, $T(x) = \cap_N T_N(x) \neq \emptyset$, as a nested intersection, and x is in $\operatorname{dom}(T)$ as asserted.

Then $\operatorname{int}\operatorname{conv}\operatorname{dom}(T) = \operatorname{int}\operatorname{dom}(T)$ and so the final conclusion follows. $\square$

Lemma 2.16. *For $x \in \operatorname{int}\operatorname{conv}\operatorname{dom}(T)$ and N sufficiently large, $T_N(x)$ is bounded.*

Proof. A Baire category argument [25], shows for N large and $u \in 1/N\, B_X$ one has

$$x + u \in \operatorname{cl}\operatorname{conv} D_N \quad \text{where} \quad D_N := \{z \colon z \in \operatorname{dom}(T) \cap N\, B_X, T(z) \cap N\, B_{X^*} \neq \emptyset\}.$$

Now for each $x^* \in T_N(x)$, since $x + u$ lies in the closed convex hull of D_N, we have

$$\langle u, x^*\rangle \leq \sup\{\langle z - x, z^*\rangle \colon z^* \in T(z) \cap N B_{X*},\, z \in N\, B_X\} \leq 2N^2$$

and so $\|x^*\| \leq 2N^3$. $\square$

Another nice application is:

Corollary 2.17 ([37]). *Let X be any Banach space and let $S, T : X \to 2^{X^*}$ be maximal monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(S)].$$

For any $x \in \operatorname{dom}(T) \cap \operatorname{dom}(S)$, $T(x) + S(x)$ is a w^*-closed subset of X^*.

Proof. By the Krein-Smulian theorem, it suffices to use Theorem 2.11 to prove every bounded w^*-convergent net in $T(x) + S(x)$ has its limits in $T(x) + S(x)$. □

Thus, we preserve some structure – it is still open if $T + S$ must actually be maximal, see [31, 37].

Finally, a recent result by Simons [34] shows that:

Theorem 2.18 ([34]). *If S is maximal monotone and* $\operatorname{int}\operatorname{dom}(S)$ *is nonempty then*

$$\operatorname{int}\operatorname{dom}(S) = \operatorname{int}\{x\colon (x, x^*) \in \operatorname{dom}\mathcal{F}_S\}.$$

This then very neatly recovers the convexity of $\operatorname{int} D(S)$. It would be interesting to how much one can similarly deduce about $\operatorname{cl}\operatorname{dom}(S)$ – via regularization or enlargement – when $\operatorname{int}\operatorname{dom}(S)$ is empty.

For example, suppose S is *domain regularizable* meaning that for $\varepsilon > 0$, there is a maximal S_ε with $\mathcal{H}(D(S), D(S_\varepsilon)) \leq \varepsilon$ and $\operatorname{core} D(S_\varepsilon) \neq \emptyset$. Then $\overline{\operatorname{dom}}(S)$ is convex. In reflexive space we can use

$$S_\varepsilon := \left(S^{-1} + N_{\varepsilon B_X}^{-1}\right)^{-1},$$

which is maximal by Theorem 4.5. [Here $\mathcal{H}$ denotes Hausdorff distance and we assume $0 \in S(0)$.] See also Theorems 5.3 and 5.9. When $S = \partial f$ for a closed convex f this applies in general Banach space. Indeed $S_\varepsilon = \partial f_\varepsilon$ where $f_\varepsilon(x) := \inf_{\|y-x\|\leq\varepsilon} f(y)$.

3. Cyclic and Acyclic Monotone Operators

For completeness we offer a simple variational proof of the next theorem, originally due to Rockafellar. Simons' proof is well described in [25, 31].

Theorem 3.1 (Maximality of Subgradients). *Every closed convex function has a (locally) maximal monotone subgradient.*[2]

Proof. Without loss of generality we may suppose

$$\langle 0 - x^*, 0 - x\rangle \geq 0 \quad \text{for all } x^* \in \partial f(x)$$

but $0 \notin \partial f(0)$; so $f(\overline{x}) - f(0) < 0$ for some $\overline{x}$. By Zagrodny's *Approximate mean value theorem* (see [11, Thm. 3.4.6]), we find $x_n \to c \in (0, \overline{x}], x_n^* \in \partial f(x_n)$ with

$$\liminf_n \langle x_n^*, c - x_n\rangle \geq 0, \liminf_n \langle x_n^*, \overline{x}\rangle \geq f(0) - f(\overline{x}) > 0.$$

Now $c = \theta\,\overline{x}$ for some $\theta > 0$. Hence,

$$\limsup_n \langle x_n^*, x_n\rangle < 0,$$

a contradiction. □

[2]This fails in *all* incomplete normed spaces and in *some* Fréchet spaces.

We recall that for $N = 2, 3, \ldots$, a multifunction T is *N-monotone* if

$$\sum_{k=1}^{N} \langle x_k^*, x_k - x_{k-1} \rangle \geq 0$$

whenever $x_k^* \in T(x_k)$ and $x_0 = x_N$. We say T is *cyclically monotone* when T is N-monotone for all $N \in \mathbb{N}$, as hold for all convex subgradients. Then monotonicity and 2-monotonicity coincide, while it is a classical result of Rockafellar [30, 25] that in a Banach space every maximal cyclically monotone operator is the subgradient of a proper closed convex function (and conversely). We recast this result to make the parallel with the Debrunner-Flor Theorem 2.8 explicit.

Theorem 3.2 (Rockafellar, [25, 30]). *Suppose C is cyclically monotone on a Banach space X. Then C has a maximal cyclically monotone extension $\overline{C}$, which is of the form $\overline{C} = \partial f_C$ for some proper closed convex function f_C. Moreover, $R(\overline{C}) \subset \overline{\mathrm{conv}}^* R(C)$.*

Proof. We fix $x_0 \in \operatorname{dom} C, x_0^* \in C(x_0)$ and define

$$f_C(x) := \sup \left\{ \langle x_n^*, x - x_n \rangle + \sum_{k=1}^{n-1} \langle x_{k-1}^*, x_k - x_{k-1} \rangle \colon x_k^* \in C(x_k), n \in \mathbb{N} \right\},$$

where the sup is over all such chains. The proof in [26] shows that $C \subset \overline{C} := \partial f_C$.

The range assertion follows because f_C is the supremum of affine functions whose linear parts all lie in range C. This is most easily seen by writing $f_C = g_C^*$ with $g_C(x^*) := \inf\{\sum_i t_i \alpha_i \colon \sum_i t_i x_i^* = x^*, \sum_i t_i = 1, t_i > 0\}$ for appropriate α_i. $\square$

The exact relationship between $\mathcal{F}_{\partial f}$ and ∂f is quite complicated. One does always have

$$\langle x, x^* \rangle \leq \mathcal{F}_{\partial f}(x, x^*) \leq f(x) + f^*(x) \leq \mathcal{F}_{\partial f}^*(x, x^*) \leq \langle x, x^* \rangle + \iota_{\partial f}(x, x^*),$$

as shown in [5, Prop. 2.1]. Likewise, when L is linear and maximal (with dense range) then

$$\mathcal{F}_L(x, x^*) = \langle x, x^* \rangle - \inf_{z \in X} \langle z, Lz - x^* \rangle.$$

From various perspectives it is interesting to answer the following questions, [6].

Q1. When is a maximal monotone operator T the sum of a subgradient ∂f and a skew linear operator S? This is closely related to the behaviour of the function

$$\mathcal{FL}_T(x) := \int_0^1 \sup_{x^*(t) \in T(tx)} \langle x, x^*(t) \rangle \, dt,$$

defined assuming $0 \in \operatorname{core} \operatorname{dom} T$.[3] In this case, $\mathcal{FL}_T = \mathcal{FL}_{\partial f} = f$, and we call T *(fully) decomposable.*

Q2. How does one appropriately generalize the decomposition of a linear monotone operator L into a symmetric (cyclic) and a skew (acyclic) part? Viz

$$L = \frac{1}{2}(L + L^*|_X) + \frac{1}{2}(L - L^*|_X).$$

Answers to these questions may well allow one to make progress with open questions about behaviour of maximal monotone operators outside reflexive space – since any ‘bad’ properties are anticipated to originate with the skew or acyclic part.

[3]The use of $\mathcal{FL}_T$ originates in discussions I had with Fitzpatrick shortly before his death.

3.1. Cyclic-Acyclic Decompositions of Monotone Operators

We next describe Asplund's approach in [1, 2] to Question 2. We begin by observing that every 3-monotone operator such that $0 \in T(0)$ has the local property that

$$\langle x, x^* \rangle + \langle y, y^* \rangle \geq \langle x, y^* \rangle \tag{3}$$

whenever $x^* \in T(x)$ and $y^* \in T(y)$. We will call a monotone operator satisfying (3), 3^--*monotone*, and write $T \geq_N S$ when $T = S + R$ with R being N-monotone. Likewise we write $T \geq_{\omega_0} S$ when R is cyclically monotone.

Proposition 3.3. *Let N be one of $3^-, 3, 4, \ldots$, or ω_0. Consider an increasing (infinite) net of monotone operators on a Banach space X, satisfying*

$$0 \leq_N T_\alpha \leq_N T_\beta \leq_2 T,$$

whenever $\alpha < \beta \in \mathcal{A}$.

Suppose that $0 \in T_\alpha(0), 0 \in T(0)$ and that $0 \in \operatorname{core} \operatorname{dom} T$. Then

a) There is a N-monotone operator $T_\mathcal{A}$ with $T_\alpha \leq_N T_\mathcal{A} \leq_2 T$, for all $\alpha \in \mathcal{A}$.
b) If $R(T) \subset MB_{X^*}$ for some $M > 0$ then one may suppose $R(T_\mathcal{A}) \subset MB_{X^*}$.

Proof. *a*) We first give details of the single-valued case. As $0 \leq_2 T_\alpha \leq_2 T_\beta \leq_2 T$, while $T(0) = 0 = T_\alpha(0)$, we have

$$0 \leq \langle x, T_\alpha(x) \rangle \leq \langle x, T_\beta(x) \rangle \leq \langle x, T(x) \rangle,$$

for all x in $\operatorname{dom} T$. This shows that $\langle x, T_\alpha(x) \rangle$ converges as α goes to ∞.

Fix $\varepsilon > 0$ and $M > 0$ with $T(\varepsilon B_X) \subset M B_{X^*}$. We write $T_{\beta\alpha} = T_\beta - T_\alpha$ for $\beta > \alpha$, so that $\langle T_{\beta\alpha} x, x \rangle \to 0$ for $x \in \operatorname{dom} T$ as α, β go to ∞.

We appeal to (3) to obtain

$$\langle x, T_{\beta\alpha}(x) \rangle + \langle y, T_{\beta\alpha}(y) \rangle \geq \langle T_{\beta\alpha}(x), y \rangle, \tag{4}$$

for $x, y \in \operatorname{dom} T$. Also, $0 \leq \langle x, T_{\beta\alpha}(x) \rangle \leq \varepsilon^2$ for $\beta > \alpha > \gamma(x)$ for all $x \in \operatorname{dom} T$.

Now, $0 \leq \langle y, T_{\beta\alpha}(y) \rangle \leq \langle y, T(y) \rangle \leq \varepsilon M$ for $\|y\| \leq \varepsilon$. Thus, for $\|y\| \leq \varepsilon$ and $\beta > \alpha > \gamma(x)$ we have

$$\begin{aligned} \varepsilon(M + \varepsilon) &\geq \langle x, T_{\beta\alpha}(x) \rangle + \langle y, T(y) \rangle \\ &\geq \langle x, T_{\beta\alpha}(x) \rangle + \langle y, T_{\beta\alpha}(y) \rangle \\ &\geq \langle y, T_{\beta\alpha}(x) \rangle, \end{aligned} \tag{5}$$

from which we obtain $\|T_{\beta\alpha}(x)\| \leq M + \varepsilon$ for all $x \in \operatorname{dom} T$, while $\langle y, T_{\beta\alpha}(x) \rangle \to 0$ for all $y \in X$. We conclude that $\{T_\alpha(x)\}_{\alpha \in \mathcal{A}}$ is a norm-bounded weak-star Cauchy net and so weak-star convergent to the desired N-monotone limit $T_\mathcal{A}(x)$.

In the general case we may still use (3) to deduce that $T_\beta = T_\alpha + T_{\beta\alpha}$ where (i) $T_{\beta\alpha} \subset (M+\varepsilon)B_{X^*}$ and (ii) for each $t^*_{\beta\alpha} \in T_{\beta\alpha}$ one has $t^*_{\beta\alpha} \to^* 0$ as α and $\beta \to \infty$. The conclusion follows as before, but is somewhat more technical.

b) Fix $x \in X$. We again apply (3), this time to T_α to write

$$\langle Tx, x\rangle + \langle Ty, y\rangle \geq \langle T_\alpha x, x\rangle + \langle T_\alpha y, y\rangle \geq \langle T_\alpha x, y\rangle$$

for all $y \in D(T) = X$, by Theorem 2.8(c). Hence

$$\langle Tx, x\rangle + M\|y\| \geq \|T_\alpha x\| \, \|y\|,$$

for all $y \in Y$. This shows that $T_\alpha(x)$ lies in the M-ball, and since the ball is weak-star closed, so does $T_{\mathcal{A}}(x)$.

The set-valued case is entirely analogous, but more technical. Details will appear separately. □

We comment that $0 \leq_2 (-ny, nx) \leq_2 (-y, x)$ for $n \in \mathbb{N}$, shows the need for (3) in the deduction that $T_{\beta\alpha}(x)$ are equi-norm bounded. Moreover, if X is an Asplund space, the proof of Proposition 3.3 can be adjusted to show that $T_{\mathcal{A}}(x) = \text{norm}-\lim_{\alpha\to\infty} T_\alpha(x)$, [6] (the *Daniel* property). The single-valued case effectively comprises [2, Theorem 6.1].

We shall say that a maximal monotone operator A is *acyclic* or in Asplund's term *irreducible* if whenever $A = \partial g + S$ with S maximal monotone and g closed and convex then g is necessarily a linear function. We can now provide a broad extension of Asplund's original idea, [1, 2]:

Theorem 3.4 (Asplund Decomposition). *Suppose that T is a maximal monotone operator on a Banach space with* $\operatorname{dom} T$ *having non-empty interior.*

a) *Then T may be decomposed as $T = \partial f + A$, where f is closed and convex while A is acyclic.*

b) *If the range of T lies in $M\,B_{X^*}$ then f may be assumed M-Lipschitz.*

Proof. *a*) We normalize so $0 \in T(0)$ and apply Zorn's lemma to the set of cyclically monotone operators $\mathcal{C} := \{C\colon 0 \leq_{\omega_0} C \leq_2 T,\ 0 \in C(0)\}$ in the cyclic order. By Proposition 3.3 every chain in $\mathcal{C}$ has a cyclically monotone upper-bound. Consider such a maximal $\overline{C}$ with $0 \leq_{\omega_0} \overline{C} \leq_2 T$. Hence $T = \overline{C} + A$ where by construction A is acyclic. Now, $T = \overline{C} + A \subset \partial f + A$, by Rockafellar's result. Since T is maximal the decomposition is as asserted.

b) In this case we require all members of $\mathcal{C}$ to have their range in the M-ball and apply part *b*) of Proposition 3.3. Alternatively, observe that $0 \leq_{3^-} U \leq_2 T$ implies that

$$\|U(x)\| \leq \|T(x)\|$$

for all $x \in X$. It remains to observe that every M-bounded cyclically monotone operator extends to an M-Lipschitz subgradient – as an inspection of the proof of Rockafellar's result of Theorem 3.2 confirms. □

By way of application we offer:

Corollary 3.5. *Let T be an arbitrary maximal monotone operator T on a Banach space. For $\mu > 0$ one may decompose*

$$T \cap \mu B_{X^*} \subset \widehat{T_\mu} = \partial f_\mu + A_\mu,$$

where f_μ is μ-Lipschitz and A_μ is acyclic (with bounded range).

Proof. Combining Theorem 3.4 with Proposition 2.14 we deduce that the composition is as claimed. □

Note that since the acyclic part A_μ is bounded in Corollary 3.5, it is only skew and linear when T is itself cyclic. Hence, such a range bounded monotone operator is never fully decomposable in the sense of Question 1.

Theorem 3.4 and related results in [1, 6] are entirely existential: how can one prove Corollary 3.4 constructively in finite dimensions? How in general does one effectively diagnose acyclicity? What is the decomposition for such simple monotone maps such as $(x, y) \mapsto (\sinh(x) - \alpha\, y^2/2, \sinh(x) - \alpha\, x^2/2)$ which is monotone exactly for $\alpha \geq -2/\sqrt{x_0^2 - 1} \sim 0.7544\ldots$ with x_0 the smallest fixed point of coth?

Asplund comments in [2] that "nothing more is known about irreducible monotone mappings in this the simplest of cases" than the following:

Example 3.6. Let f be a C^1-complex function on an open convex subset $D \subset \mathcal{C}$. Viewed as a real function on $\mathbb{R}^2$ the following is the case

- f is monotone if and only if it satisfies
$$\mathrm{Re}\,\frac{\partial f}{\partial z} \geq \left|\frac{\partial f}{\partial \overline{z}}\right|$$
- f is a subgradient if and only if it satisfies
$$\frac{\partial f}{\partial z} \geq \left|\frac{\partial f}{\partial \overline{z}}\right|.$$

Thus, for f to be acyclic there must exist no nonconstant analytic function g with
$$\frac{\partial g}{\partial z} \geq \left|\frac{\partial g}{\partial \overline{z}}\right| \text{ and } \mathrm{Re}\,\frac{\partial (f-g)}{\partial z} \geq \left|\frac{\partial (f-g)}{\partial \overline{z}}\right|.$$

Example 3.7. Consider the maximal monotone linear mapping
$$T_\theta : (x, y) \mapsto (\cos(\theta)x - \sin(\theta)y, \cos(\theta)y + (\sin(\theta)x)$$
for $0 \leq \theta \leq \pi/2$. The methods in [1] show that for $n = 1, 2, \ldots$ the rotation mapping $T_{\pi/n}$ is n-monotone but is not $(n+1)$-monotone.

4. Maximality In Reflexive Banach Space

We begin with:

Proposition 4.1. *A monotone operator T on a reflexive Banach space is maximal if and only if the mapping $T(\cdot + x) + J$ is surjective for all x in X. [Moreover, when J and J^{-1} are both single valued, a monotone mapping T is maximal if and only if $T + J$ is surjective.]*

Proof. We prove the 'if'. The 'only if' is completed in Corollary 4.9. Assume (w, w^*) is monotonically related to the graph of T. By hypothesis, we may solve $w^* \in T(x + w) + J(x)$. Thus $w^* = t^* + j^*$ where $t^* \in T(x + w), j^* \in J(x)$. Hence
$$0 \leq \langle w - (w + x), w^* - t^*\rangle = -\langle x, w^* - t^*\rangle = -\langle x, j^*\rangle = -\|x\|^2 \leq 0.$$
Thus, $j^* = 0, x = 0$. So $w^* \in T(w)$ and we are done. □

We now prove our central result whose proof – originally very hard and due to Rockafellar [29] – has been revisited over many years culminating in the results in [31, 35, 36, 38, 11] among others:

Theorem 4.2. *Let X be a reflexive space. Let T be maximal monotone and let f be closed and convex. Suppose that*

$$0 \in \operatorname{core}\{\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(\partial f)\}.$$

Then

(*a*) $\partial f + T + J$ *is surjective.*
(*b*) $\partial f + T$ *is maximal monotone.*
(*c*) ∂f *is maximal monotone.*

Proof. (*a*) As in [35, 36, 38], we consider the Fitzpatrick function $\mathcal{F}_T(x, x^*)$ and further introduce $f_J(x) := f(x) + 1/2\|x\|^2$. Let $G(x, x^*) := -f_J(x) - f_J^*(-x^*)$. Observe that

$$\mathcal{F}_T(x, x^*) \geq \langle x, x^* \rangle \geq G(x, x^*)$$

pointwise thanks to the *Fenchel-Young inequality*

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle,$$

for all $x \in X, x^* \in X^*$, along with Proposition 2.1. Now, the constraint qualification

$$0 \in \operatorname{core}\{\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(\partial f)\}$$

assures that the *Sandwich theorem* applies to $\mathcal{F}_T \geq G$ since f_J^* is everywhere finite by Proposition 2.6.

Then there are $w \in X$ and $w^* \in X^*$ such that

$$\mathcal{F}_T(x, x^*) - G(z, z^*) \geq w(x^* - z^*) + w^*(x - z) \tag{6}$$

for all x, x^* and all z, z^*. In particular, for $x^* \in T(x)$ and for all z^*, z we have

$$\langle x - w, x^* - w^* \rangle + [f_J(z) + f_J^*(-z^*) + \langle z, z^* \rangle] \geq \langle w - z, w^* - z^* \rangle.$$

Now use the fact that $-w^* \in \operatorname{dom}(\partial f_J^*)$, by Proposition 2.6, to deduce that $-w^* \in \partial f_J(v)$ for some z and so

$$\langle v - w, x^* - w^* \rangle + [f_J(v) + f_J^*(-w^*) + \langle v, w^* \rangle] \geq \langle w - v, w^* - w^* \rangle = 0.$$

The second term on the left is zero and so $w^* \in T(w)$ by maximality. Substitution of $x = w$ and $x^* = w^*$ in (6), and rearranging yields

$$\langle w, w^* \rangle + \{\langle -z^*, w \rangle - f_J^*(-z^*)\} + \{\langle z, -w^* \rangle - f_J(z)\} \leq 0,$$

for all z, z^*. Taking the supremum over z and z^* produces $\langle w, w^* \rangle + f_J(w) + f_J^*(-w^*) \leq 0$. This shows $-w^* \in \partial f_J(w) = \partial f(w) + J(w)$ on using the sum formula for subgradients, implicit in Proposition 2.6.

Thus, $0 \in (T + \partial f_J)(w)$, and since all translations of $T + \partial f$ may be used, while the (CQ) is undisturbed by translation, $(\partial f + T)(x + \cdot) + J$ is surjective which completes (*a*). Also $\partial f + T$ is maximal by Proposition 4.1 which is (*b*). Finally, setting $T \equiv 0$ we recover the reflexive case of the maximality for a lsc convex function. □

Recall that the *normal cone* $N_C(x)$ to a closed convex set C at a point x in C is $N_C(x) = \partial\iota_C(x)$.

Corollary 4.3. *The sum of a maximal monotone operator* T *and a normal cone* N_C *on a reflexive Banach space, is maximal monotone whenever the transversality condition* $0 \in \operatorname{core}[C - \operatorname{conv}\operatorname{dom}(T)]$ *holds.*

In particular, if T is monotone and $C := \operatorname{cl}\operatorname{conv}\operatorname{dom}(T)$ has nonempty interior, then for any maximal extension $\overline{T}$ the sum $\overline{T} + N_C$ is a 'domain preserving' maximal monotone extension of T.

Corollary 4.4 ([31, 36]). *The sum of two maximal monotone operators* T_1 *and* T_2*, on a reflexive Banach space, is maximal monotone whenever the transversality condition* $0 \in \operatorname{core}[\operatorname{conv}\operatorname{dom}(T_1) - \operatorname{conv}\operatorname{dom}(T_2)]$ *holds.*

Proof. Theorem 4.2 applies to the maximal monotone product mapping $T(x,y) := (T_1(x), T_2(y))$ and the indicator function $f(x,y) = \iota_{\{x=y\}}$ of the diagonal in $X \otimes X$. Finally, check that the given transversality condition implies the needed (CQ), along the lines of Theorem 4.2. We obtain that $T + J_{X\otimes X} + \partial\iota_{\{x=y\}}$ is surjective. Thus, so is $T_1 + T_2 + 2\,J$ and we are done. □

As always in convex analysis, one may easily replace the core condition by a relativized version – with respect to the closed affine hull.

4.1. The Fitzpatrick Inequality

We record that

$$F_{\partial f}(x, x^*) \leq f(x) + f^*(x^*),$$

and that we have exploited the beautiful inequality

$$\mathcal{F}_T(x, x^*) + f(x) + f^*(-x^*) \geq 0, \qquad \forall x \in X, x^* \in X^*, \tag{7}$$

valid for *any* maximal monotone T and *any* convex function f. Also, note that $(x, x^*) \mapsto f(x) + f^*(x^*)$ is a representative function for ∂f. Correspondingly, we have the *Fitzpatrick inequality*

$$\mathcal{F}_{T_1}(x, x^*) + \mathcal{F}_{T_2}(x, -x^*) \geq 0, \qquad \forall x \in X, x^* \in X^*, \tag{8}$$

valid for *any* maximal monotone T_1, T_2. Moreover, by Proposition 2.1,

$$\mathcal{F}_T^*(x^*, x) \geq \sup_{y^* \in T(y)} \langle x, y^* \rangle + \langle x^*, y \rangle - \mathcal{F}_T(y, y^*) = \mathcal{F}_T(x, x^*). \tag{9}$$

We clearly have an extension of (8): $\mathcal{H}_T^1(x, x^*) + \mathcal{H}_S^2(x, -x^*) \geq 0$, for any representative functions $\mathcal{H}_T^1$ and $\mathcal{H}_S^2$.

Letting $\widehat{\mathcal{F}_S}(x, x^*) := \mathcal{F}_S(x, -x^*)$, we may establish:

Theorem 4.5. *Let* S *and* T *be maximal monotone on a reflexive space. Suppose that* $0 \in \operatorname{core}\{\operatorname{dom}(\mathcal{F}_T) - \operatorname{dom}(\widehat{\mathcal{F}_S})\}$ *as happens if* $0 \in \operatorname{core}\{\operatorname{conv}\operatorname{graph}(T) - \operatorname{conv}\operatorname{graph}(-S)\}$. *Then* $0 \in \operatorname{range}(T + S)$.

Proof. We apply the Fenchel duality theorem, or follow through the steps of Theorem 4.2. From either result one obtains $\mu \in X$, $\lambda \in X^*$ and $\beta \in \mathbb{R}$ such that

$$\mathcal{F}_T(x,x^*) - \langle x,\lambda\rangle - \langle \mu,x^*\rangle + \langle \mu,\lambda\rangle \geq \beta \geq -\mathcal{F}_S(y,-y^*) + \langle y,\lambda\rangle - \langle \mu,y^*\rangle - \langle \mu,\lambda\rangle,$$

for all variables x, y, x^*, y^*. Hence for $x^* \in T(x)$ and $-y^* \in S(y)$ we obtain

$$\langle x-\mu, x^*-\lambda\rangle \geq \beta \geq \langle y-\mu, y^*+\lambda\rangle.$$

If $\beta \leq 0$, we derive that $-\lambda^* \in S(\mu)$ and so $\beta = 0$; consequently, $\lambda \in T(\mu)$ and since $0 \in (T+S)(\mu)$ we are done. If $\beta \geq 0$ we argue first with T. □

Note that the graph condition in Theorem 4.5 is formally more exacting than the domain condition as shown by $\operatorname{conv}\operatorname{graph}(J_{\ell^2})$ which is the diagonal in $\ell^2 \otimes \ell^2 = \operatorname{dom}(F_{J_{\ell^2}})$, indeed $\mathcal{F}_{J_{\ell^2}}(x,x^*) = \frac{1}{4}\|x+x^*\|^2$. More interestingly, Zalinescu [40] has adapted this argument to extend results like those in [34] in the reflexive case.

4.2. Extensions to Non-reflexive Space

We let $\overline{T}$ denote the *monotone closure* of T in $X^{**} \times X^*$. That is, $x^* \in \overline{T}(x^{**})$ when

$$\inf_{y^* \in T(y)} \langle x^*-y^*, x^{**}-y\rangle \geq 0.$$

Recall that T is *type (NI)* if

$$\inf_{y^* \in T(y)} \langle x^*-y^*, x^{**}-y\rangle \leq 0$$

for all $x^{**} \in X^{**}$ and $x^* \in X^*$, see [31, 32].

We say T is of *dense type* if every pair (x^*, x^{**}) in the graph of the monotone closure of T is the limit of a net (x^*_α, x_α) with $x^*_\alpha \in T(x_\alpha)$ with $x_\alpha \rightharpoonup_* x^{**}$, $x^*_\alpha \to x^*$ and $\sup_\alpha \|x_\alpha\| < \infty$, and we write $x^* \in T_1(x^{**})$, [19, 31, 32].

Clearly, $T_1(x) \subset \overline{T}(x)$; so every dense type operator is of type (NI). We denote $\overline{\mathcal{F}}_T(G,x^*) := \mathcal{P}^*_T(x^*,G)$, viewed as a mapping on $X^{**} \times X^*$, and make the following connection with $\mathcal{F}_T$.

Proposition 4.6. *Let T be maximal monotone on a Banach space X. Then*

$$\overline{\mathcal{F}}_T(G,x^*) = \sup_{y^* \in T(y)} \langle y^*,G\rangle + \langle y,x^*\rangle - \langle y,y^*\rangle \geq \langle x^*,G\rangle$$

with equality if and only if $x^* \in \overline{T}(G)$.

In particular, $\overline{\mathcal{F}}_T|_{X\times X^*} = \mathcal{F}_T$.

Moreover, T is type (NI) if and only if $\overline{\mathcal{F}}_T$ is a representative function for $\overline{T}$.

Proof. These are left for the reader. □

Proposition 4.7 (Gossez). *The subgradient of every closed convex function f on a Banach space is of dense type. Indeed*

$$\overline{\partial f} = (\partial f)_1 = (\partial f^*)^{-1}.$$

Proof. For any closed convex f we have $(\partial f^*)^{-1} \subset \overline{\partial f}$, while – with a little effort – Golstein's theorem shows that $(\partial f^*)^{-1} \subset \partial f_1$. □

A fairly satisfactory extension of Theorem 4.2 is:

Corollary 4.8. *If T is type (NI) then*

$$\operatorname{range}(\overline{T} + \partial f^{**} + J^{**}) = X^*.$$

Proof. Follow the steps of Theorem 4.2 or Theorem 4.5 using $\mathcal{P}_T$ and and $f_J + f_J^*$ as the functions in the Fitzpatrick inequality. □

In the case that T is dense type this result originates with Gossez (see [19, 25]). We next recover the *Rockafellar-Minty surjectivity theorem*:

Corollary 4.9. *For a maximal monotone operator on a reflexive Banach space,* $\operatorname{range}(T + J) = X^*$.

Proof. Let $f \equiv 0$ in Theorem 4.2. Alternatively, on noting that

$$\mathcal{F}_J(x, x^*) \le \frac{\|x\|^2 + \|x^*\|^2}{2},$$

we may apply Theorem 4.5. □

Correspondingly, we see for an maximal monotone N of type (NI) that $\operatorname{range}(\overline{N} + J^{**}) = X^*$ which implies that $\overline{\overline{N}} = \overline{N}$; and so that $\overline{N}$ is maximal as a monotone mapping from X^{**} to X^*.

Theorem 4.10 (Fitzpatrick, Phelps, [17]). *Every locally maximal monotone operator on a Banach space has* $\operatorname{cl}\operatorname{range} T$ *convex.*

Proof. We suppose not and then may suppose by homothety that there are $\pm x^*$ in $\operatorname{cl}\operatorname{range} T$ of unit-norm but with midpoint $0 \notin \operatorname{cl}\operatorname{range} T$.

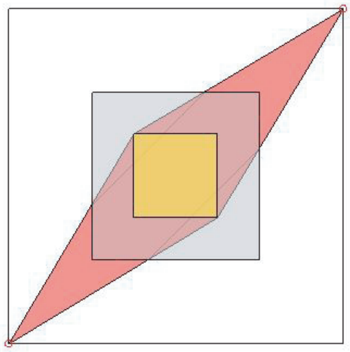
We build the equivalent dual ball $B' := \operatorname{conv}\{\pm 2x^*, \alpha B_X^*\}$ where $0 < \alpha < 1/2$ is chosen with $(\operatorname{range} T) \cap 2\alpha B_X^* = \emptyset$. We consider $\widehat{T}$ extending $T \cap B'$ as in Proposition 2.14, so that

$$\operatorname{range}\widehat{T} \subset \operatorname{cl}\operatorname{conv}\{R(T) \cap B'\} \quad \text{and} \quad \operatorname{range}\widehat{T} \setminus \operatorname{range} T \subset \operatorname{bd} B'.$$

It follows that

$$\operatorname{range}\widehat{T} \subset (R(T) \cap B') \bigcup (\operatorname{cl}\operatorname{conv}\{R(T) \cap B'\} \cap \operatorname{bd} B').$$

Hence range $\widehat{T}$ is weak-star disconnected. As $\widehat{T}$ is a weak-star cusco it has a weak-star connected range which contradicts the construction. □



B' (red), αB_{X^*} (yellow) and $2\alpha B_{X^*}$ (grey)

A dual argument shows that type (VFP) mappings have $\mathrm{cl}\,\mathrm{dom}\,(T)$ convex.

5. Further Applications

5.1. Local Maximality Revisited

We are also in a position to show why in a reflexive setting all maximal monotone operators are locally maximal, that is type (FP). We start with the following contrapositive whose simple proof is in [25].

Proposition 5.1. *Let T be a monotone operator on a Banach space X. Then T is locally maximal monotone if and only if every weak-star compact and convex set C^* in X^* with $R(T) \cap \mathrm{int}\, C^* \neq \emptyset$ is such that if $x^* \in \mathrm{int}\, C^*$ (in norm) but $x^* \notin T(x)$ there is $z^* \in T(z) \cap C^*$ with $0 > \langle z^* - x^*, z - x\rangle$.*

It is obvious that every maximal monotone operator on a reflexive space is type (D), and it is known that type (D) implies type (FP), [32, Thm. 17]. A direct proof follows.

Theorem 5.2 (Fitzpatrick-Phelps). *Every maximal monotone operator on a reflexive space is locally maximal.*

Proof. Since X is reflexive, $M := (T^{-1} + \partial\iota_{C^*})^{-1}$ is maximal, by Theorem 4.2(b), while $x^* \in \mathrm{int}\, C^* \setminus M(x)$. Since M is maximal monotone we can find $z \in \mathrm{dom}\, T, z^* \in C^*$, $z^* \in T(z)$ and $u \in N_{C^*}(z^*)$ such that $z + u \in (T + \partial\iota_{C^*})^{-1}(z^*)$ with

$$0 > \langle z^* - x^*, z - x\rangle + \langle z^* - x^*, u\rangle.$$

As the second term on the right is nonnegative, we are done. $\square$

A useful reformulation of the argument in Theorem 5.4 is given next.

Proposition 5.3 (Fitzpatrick-Phelps). *Let X be a Banach space. A maximal monotone operator T is locally maximal monotone if*

$$M(T, C^*) := \left(T^{-1} + \partial\iota_{C^*}\right)^{-1}$$

is maximal monotone on X whenever C^* is convex and weak-star compact with $R(T) \cap \mathrm{int}\, C^* \neq \emptyset$.

We shall call an operator satisfying these hypotheses *strongly locally maximal*. While every maximal monotone operator on a reflexive space is clearly strongly locally maximal, not every convex subgradient is – as Theorem 6.5 will show. However we can still use Proposition 5.1 to prove:

Theorem 5.4 (Simons). *The subgradient of every closed convex function f on a Banach space is locally maximal monotone.*

Proof. Fix $x^* \in \operatorname{int} C^* \setminus \partial g(x)$ as in the hypotheses of Proposition 5.1. Then Corollary 2.5 and Theorem 3.1 and combine to show that $M := \partial g^* + \partial \iota_{C^*} = \partial\left(g^* + \iota_{C^*}\right)$ is maximal monotone on X^*, since $\operatorname{dom} g^* \cap \operatorname{int} C^* \neq \emptyset$. Now $x \notin M(x^*)$ (since $x^* \in \operatorname{int} C^*$) so we deduce the existence of $w^{**} = z^{**} + u^{**}, z^{**} \in \partial g^*(z^*), u^{**} \in \partial \iota_C(z^*) = N_C(z^*)$ with

$$0 > \langle z^* - x^*, z^{**} - x\rangle + \langle z^* - x^*, u^{**}\rangle \geq \langle z^* - x^*, z^{**} - x\rangle.$$

By Proposition 4.7 we may select z_α^* and z_α with $z_\alpha^* \in \partial g(z_\alpha)$, $z_\alpha^* \to z^* \in C^*$, $z_\alpha \to_* z^{**}$ and $\sup_\alpha \|z_\alpha\| < \infty$. In consequence,

$$\limsup_\alpha \langle z_\alpha^* - x^*, z_\alpha - x\rangle < 0.$$

Now the observation that $z_\alpha^* \in C^*$ for large α, and an appeal to Proposition 5.1 finishes the proof of (a). □

One may similarly prove Simon' result that dense type operators are locally maximal by applying a variant of Corollary 4.8 with ∂g^* replaced by $\overline{T^{-1}}$. The corresponding results for type (VFP) appear usually to be easier. For example, T is type (VFP) if $T + N_C$ is maximal monotone for all bounded closed convex sets with $\operatorname{dom} T \cap \operatorname{int} C \neq \emptyset$. In consequence, subgradients and reflexive maximal monotones are type (VFP).

5.2. The Composition Formula

Another very useful foundational result is:

Theorem 5.5 ([38, Thm. 6]). *Suppose X and Y are Banach spaces with X reflexive, that T is a maximal monotone operator on Y, and that $A\colon X \mapsto Y$, is a bounded linear mapping. Then $T_A := A^* \circ T \circ A$ is maximal monotone on X whenever $0 \in \operatorname{core}(\operatorname{range}(A) + \operatorname{conv} \operatorname{dom} T)$.*

Proof. Monotonicity is clear. To obtain maximality, we consider the Fitzpatrick inequality (8) to write

$$f(x, x^*) + g(x, x^*) \geq 0,$$

where

$$f(x, x^*) := \inf\{\mathcal{F}_T(Ax, y^*)\colon A^* y^* = x^*\}, \quad g(x, x^*) := \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2,$$

and apply Fenchel's duality theorem [39, 30, 9, 11] – or use the Sandwich theorem directly – to deduce the existence of $\overline{x} \in X, \overline{x}^* \in X^*$ with

$$f^*(\overline{x}^*, \overline{x}) + g^*(\overline{x}^*, \overline{x}) \leq 0. \tag{10}$$

Semi-continuity of f is not needed since g is continuous throughout.

Also, the constraint qualification implies that the condition used in [11, Thm. 4.4.3] and in [23, Proposition 13] holds. Thus, applying [23, Proposition 13] – or carefully using the standard formula for the conjugate of a convex composition ([28], [11, Thm. 4.4.3]) – we have for some $\overline{y}^*$ with $A^*\overline{y}^* = \overline{x}^*$:

$$\begin{aligned} f^*(\overline{x}^*, \overline{x}) = \inf\{\mathcal{F}_T^*(A\overline{x}, y^*)\colon A^*y^* = \overline{x}^*\} &= \min\{\mathcal{F}_T^*(y^*, A\overline{x})\colon A^*y^* = \overline{x}^*\} \\ &= \mathcal{F}_T^*(\overline{y}^*, A\overline{x}) \\ &\geq \mathcal{F}_T(A\overline{x}, \overline{y}^*), \end{aligned}$$

where the last inequality follows from (9). Moreover,

$$g^*(\overline{x}^*, \overline{x}) = \frac{1}{2}\|\overline{x}\|^2 + \frac{1}{2}\|A^*\overline{y}^*\|^2.$$

Thus, (10) implies that

$$\Big\{ \mathcal{F}_T(A\overline{x}, \overline{y}^*) - \langle \overline{y}^*, A\overline{x}\rangle \Big\} + \left\{\frac{1}{2}\|\overline{x}\|^2 + \frac{1}{2}\|A^*\overline{y}^*\|^2 + \langle \overline{y}^*, A\overline{x}\rangle\right\} \leq 0,$$

and we conclude that $\overline{y}^* \in T(A\overline{x})$ and $-\overline{x}^* := -A^*\overline{y}^* \in J_X(\overline{x})$ since both bracketed terms are non-negative. Hence, $0 \in J_X(\overline{x}) + T_A(\overline{x})$.

In the same way if we start with

$$f(x, x^*) := \inf\{\mathcal{F}_T(Ax, y^*)\colon A^*y^* = x^* + x_0^*\}, \quad g(x, x^*) := \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 - \langle x, x_0^*\rangle,$$

we deduce, $x_0^* \in J_X(\overline{x}) + T_A(\overline{x})$. This applies to all domain translations of T. As in Theorem 4.2, this is sufficient to conclude T_A is maximal. □

Note that only the domain space needs to be reflexive. Application of Theorem 5.5 to $T(x, y) := (T_1(x), T_2(y))$, and $A(x) := (x, x)$ yields $T_A(x) = T_1(x) + T_2(x)$ and recovers Theorem 4.2. With a little more effort the reader can discover how to likewise embed Theorem 5.5 in Theorem 4.2. Alternatively, one may combine Theorem 4.2 and this result in one. Again, it is relatively easy to relativize this result to the intrinsic core; this is especially useful in finite dimensions, see [3]. A recent paper [12] observes that the result remains true when one assumes only that

$$\{(A^*y^*, Ax, r)\colon \mathcal{F}_T^*(Ax, y^*) \leq r\}$$

is relatively closed in $X^* \times R(A) \times \mathbb{R}$.

An important case of Theorem 5.5 is the case of a reflexive injection.

Corollary 5.6. *Let T be maximal monotone on a Banach space Y. Let ι denote the injection of a reflexive subspace $Z \subset Y$ into Y. Then $T_Z := \iota^* \circ T \circ \iota$ is maximal monotone on Z whenever $0 \in \operatorname{core}(Z + \operatorname{conv}\operatorname{dom} T)$. In particular, if $0 \in \operatorname{core}(\operatorname{conv}\operatorname{dom} T)$ then T_Z is maximal for each reflexive subspace Z.*

In this case [12] implies the result remains true when one assumes only that

$$\{(y^*|_Z, z, r)\colon \mathcal{F}_T^*(z, y^*) \leq r, z \in Z\}$$

is relatively closed in $Z^* \times Z \times \mathbb{R}$.

5.3. Monotone Variational Inequalities

We say that T is *coercive on* C if $\inf_{y^*\in(T+N_C)(y)}\langle y, y^*\rangle/\|y\| \to \infty$ as $y \in C$ goes to infinity in norm, with the convention that $\inf\emptyset = +\infty$. A *variational inequality* requests a solution $y \in C$ and $y^* \in T(y)$ to

$$\langle y^*, x - y\rangle \geq 0 \qquad \forall x \in C.$$

Equivalently this requires us to solve the set inclusion $0 \in T(y) + N_C(y)$. In Hilbert space this is also equivalent to finding a zero of the *normal mapping* $T_C(x) := T(P_C(x)) + (I - P_C)(x)$.

We denote the variational inequality by $V(T;C)$. (For general variational inequalities with T merely upper semi-continuous, proving solutions to V(T,C) is equivalent to establishing Brouwer's theorem.)

Corollary 5.7. *Suppose T is maximal monotone on a reflexive Banach space and is coercive on the closed convex set C. Suppose also that* $0 \in \operatorname{core}(C - \operatorname{conv}\operatorname{dom}(T))$. *Then $V(T,C)$ has a solution.*

Proof. Let $f := \iota_C$, the indicator function. For $n = 1, 2, 3, \cdots$, let $T_n := T + J/n$. We solve

$$0 \in (T_n + \partial\iota_C)(y_n) = (T + \partial\iota_C) + \frac{1}{n}J(y_n) \tag{11}$$

and take limits as n goes to infinity. More precisely, we observe that using our key Theorem 4.2, we find y_n in C, and $y_n^* \in (T + \partial\iota_C)(y_n), j_n^* \in J(y_n)/n$ with $y_n^* = -j_n^*$. Then

$$\langle y_n^*, y_n\rangle = -\frac{1}{n}\langle j_n^*, y_n\rangle = -\frac{1}{n}\|y_n\|^2 \leq 0$$

so coercivity of $T + \partial\iota_C$ implies that $\|y_n\|$ remains bounded and so $j_n^* \to 0$. On taking a subsequence we may assume $y_n \rightharpoonup y$. Since $T + \partial\iota_C$ is maximal monotone (again by Theorem 4.2), it is demi-closed [11]. It follows that $0 \in (T + \partial\iota_C)(y) = T(y) + N_C(y)$ as required. □

A more careful argument requires only that for some $\bar{c} \in C$

$$\inf_{y^*\in T(y)}\langle y - \bar{c}, y^*\rangle/\|y\| \to \infty$$

as $\|y\| \to \infty$, $y \in C$. Letting $C = X$ in Corollary 5.7 we deduce:

Corollary 5.8. *Every coercive maximal monotone operator on a Banach space is surjective if (and only if) the space is reflexive.*

Proof. To complete the proof we recall that, by James' theorem, surjectivity of J is equivalent to reflexivity of the corresponding space. □

We can similarly derive that $V(\overline{T}, \overline{C}^*)$ will have solution when the (CQ) holds and T is type (NI). We refer to [20, 17, 18] for more detailed results regarding coercive operators in non-reflexive space. We can also improve Corollary 2.15 in the reflexive setting.

Theorem 5.9 ([31]). *Suppose T is maximal monotone on a reflexive Banach space. Then* $\operatorname{dom}(T)$ *and range* (T) *have convex closure (and interior).*

Proof. Without loss of generality, we assume 0 is in the closure of $\operatorname{conv}\operatorname{dom}(T)$. Fix $y \in \operatorname{dom}(T)$, $y^* \in T(y)$. Theorem 4.9 applied to T/n solves $w_n^*/n + j_n^* = 0$ with $w_n^* \in T(w_n), j_n^* \in J(w_n)$, for integer $n > 0$. By monotonicity

$$\frac{1}{n}\langle y^*, y - w_n\rangle \geq \frac{1}{n}\langle w_n^*, y - w_n\rangle = \|w_n\|^2 - \langle j_n^*, y\rangle$$

where $\|w_n\|^2 = \|j_n^*\|^2 = \langle j_n^*, w_n\rangle$ and $w_n \in \operatorname{dom}(T)$. We deduce $\sup_n \|w_n\| < \infty$. Thus, (j_n^*) has a weak cluster point j^*. In particular, denoting $D := \operatorname{dom}(T)$

$$d_D^2(0) \leq \liminf_{n\to\infty} \|w_n\|^2 \leq \inf_{y\in D}\langle j^*, y\rangle = \inf_{y\in \operatorname{conv} D}\langle j^*, y\rangle \leq \|j^*\|\, d_{\operatorname{conv} D}(0) = 0.$$

We have actually shown that $\operatorname{cl}\operatorname{conv}\operatorname{dom}(T) \subset \operatorname{cl}\operatorname{dom}(T)$ and so $\operatorname{cl}\operatorname{dom}(T)$ is convex as required.

Since $\operatorname{range}(T) = \operatorname{dom}(T^{-1})$ and X^* is also reflexive we are done. $\square$

In a non-reflexive space, Theorem 4.8 applied similarly proves that $\overline{\operatorname{dom} T}$ is convex. Consequently, if as when T is type (ED) – see [32] – $\overline{\operatorname{dom}}(\overline{T})\cap \subset \overline{\operatorname{dom}}(T)$, the later is convex. Dually, it is known that every locally maximal operator T – and so every dense type operator – has $\operatorname{range}(T)$ convex, [25, Prop. 4.2].

Corollary 5.10. *Suppose T is maximal monotone on a reflexive Banach space X and is locally bounded at each point of* $\operatorname{cl}\operatorname{dom}(T)$. *Then* $\operatorname{dom}(T) = X$.

Proof. Let us first observe that $\operatorname{dom}(T)$ must be closed and so convex. By the Bishop-Phelps theorem, (see [11]), there is some boundary point $\overline{x} \in \operatorname{dom}(T)$ with a non-zero support functional $\overline{x}^*$. Then $T(\overline{x}) + [0,\infty)\,\overline{x}^*$ is monotonically related to the graph of T. By maximality $T(\overline{x}) + [0,\infty)\,\overline{x}^* = T(\overline{x})$ which is then non-empty and (linearly) unbounded. $\square$

It also seems worth noting that the techniques of §5.1 are all at heart techniques for variational equalities. We conclude this section by noting another convex approach to the *affine* monotone variational inequality (complementarity problem) on a closed convex cone S in a reflexive space. We consider the abstract quadratic program

$$0 \leq \mu := \inf\{\langle L(x) - q, x\rangle \colon Lx \geq_{S^+} q, x \geq_S 0\}, \tag{12}$$

which has a convex objective function. Suppose that (12) satisfies a constraint qualification as happens if either $L(S)+S^+ = X^*$ or if X is finite dimensional and S is polyhedral. Then there is $y \in S^{++} = S$ with

$$\mu \leq \langle L(x) - q, x\rangle + \langle Lx - q, y\rangle, \tag{13}$$

for all $x \in S$. Letting $x := y$ shows $\mu = 0$, and we have approximate solutions to the affine complementarity problem. Moreover, when L is coercive on S or in the polyhedral case, (12) is attained and we have produced a solution to the problem.

6. Limiting Examples and Constructions

It is unknown outside reflexive space whether cl dom (T) must always be convex for a maximal monotone operator, though the assumption of reflexivity in Theorem may be relaxed to requiring $R(T+J)$ is boundedly w^*-dense – as an examination of the proof will show.

We do however have the following result:

Theorem 6.1 ([8]). *The following are equivalent for a Banach space X.*

(a) *X is reflexive;*
(b) int range (∂f) *is convex for each coercive lsc convex function f on X;*
(c) int range (T) *is convex for each coercive maximal monotone mapping T.*

Proof. Suppose X is nonreflexive and $p \in X$ with $\|p\| = 5$ and $p^* \in Jp$ where J is the duality map. Define

$$f(x) := \max\left\{\frac{1}{2}\|x\|^2, \|x-p\| - 12 + \langle p^*, x\rangle, \|x+p\| - 12 - \langle p^*, x\rangle\right\}$$

for $x \in X$. By the max-formula, we have, for $x \in B_X$,

$$\partial f(p) = B_{X^*} + p^*, \quad \partial f(-p) = B_{X^*} - p^*, \quad \partial f(x) = Jx \tag{14}$$

using inequalities like $\|p-p\| - 12 + \langle p^*, p\rangle = 13 > \frac{25}{2} = \frac{1}{2}\|p\|^2$.

Moreover, $f(0) = 0$ and $f(x) > \frac{1}{2}\|x\|$ for $\|x\| > 1$, thus $\|x^*\| > \frac{1}{2}$ if $x^* \in \partial f(x)$ and $\|x\| > 1$. Combining this with (14) shows

$$\text{range}\,(\partial f) \cap \frac{1}{2}B_{X^*} = \text{range}\,(J) \cap \frac{1}{2}B_{X^*}.$$

Let U_{X^*} denote the open unit ball in X^*. Now James' theorem [31, 11] gives us points $x^* \in \frac{1}{2}U_{X^*} \setminus \text{range}\,(J)$, thus $U_{X^*} \setminus \text{range}\,(\partial f) \neq \emptyset$. However, from (14)

$$U_{X^*} \subset \text{conv}\,((p^* + U_{X^*}) \cup (-p^* + U_{X^*})) \subset \text{conv}\,\text{int}\,\text{range}\,(\partial f)$$

so range (∂f) has non-convex interior. This shows that (b) implies (a) while (c) implies (b) is clear. Finally (a) implies (c) follows from Theorem 6. □

Observe the distinct role of convexity in each direction the proof of (a) ⇔ (c). It is most often the case that one uses the same logic to establish any result of the form "Property P holds for all maximal monotone operators if and only if X is a Banach space with property Q." Another example is "Every (maximal) monotone operator T on a Banach space X is bounded on bounded subsets of int dom T iff X is finite dimensional." (See [8] for this and other like results.)

Example 6.2. The easiest explicit example, due to Fitzpatrick and Phelps (see [8]), lies in the space c_0 of null sequences endowed with the supremum norm. One may use

$$f(x) := \|x - e_1\|_\infty + \|x + e_1\|_\infty \tag{15}$$

where e_1 is first unit vector. Then

$$\text{int range}(\partial f) = \{U_{\ell_1} + e_1\} \cup \{U_{\ell_1} - e_1\}$$

$$\text{cl int range}(\partial f) = \{B_{\ell_1} + e_1\} \cup \{B_{\ell_1} - e_1\}$$

both of which are far from convex. It is instructive to compute the closure of the range of the subgradient. □

Example 6.3. Gossez [21] produces a coercive maximal monotone operator with full domain whose range has a non-convex closure, see also Example 6.4. It is of the form $2^{-n} J_{\ell^1} + S$ for some $n > 0$ and sufficiently large.

The continuous linear map $S : \ell_1 \to \ell_\infty$ is given by

$$(Sx)_n := -\sum_{k<n} x_k + \sum_{k>n} x_k, \quad \forall x = (x_k) \in \ell_1, n \in N.$$

We record that $\mp S : \ell_1 \mapsto \ell_\infty$ is a skew bounded linear operator, for which S^* is not monotone but $-S^*$ is. Hence, $-S$ is both of dense type and locally maximal monotone (also called FP) while S is in neither class, [31, 4].

Relatedly, let ι denote the injection of ℓ^1 into ℓ^∞. Then for small positive ϵ, the mapping $S_\varepsilon := \varepsilon\iota + S$ is a coercive maximal monotone operators for which $\overline{S_\varepsilon}$ fails to be coercive, see also [20]. □

Example 6.4 (Some further related results). Somewhat more abstractly, one can show that if the underlying space X is *rugged*, meaning that $\text{cl span range}\,(J - J) = X^*$, then the following are equivalent whenever T is bounded linear and maximal monotone, see [4]:

(i) T is of dense type.
(ii) $\text{cl range}\,(T + \lambda J) = X^*$, $\forall\lambda > 0$.
(iii) $\text{cl range}\,(T + \lambda J)$ is convex, $\forall\lambda > 0$.
(iv) $T + \lambda J$ is locally maximal monotone, $\forall\lambda > 0$.

It actually suffices that (ii)–(iv) hold for a sequence $\lambda_n \downarrow 0$. The equivalence of (i)–(iv) thus holds for the following rugged spaces: c_0, c, ℓ_1, ℓ_∞, $L_1[0, 1]$, $L_\infty[0, 1]$, $C[0, 1]$. In cases like c_0, or $C[0, 1]$ which contain no complemented copy of ℓ_1, a maximal monotone bounded linear T is always of dense type [4].

In particular, S in Example 6.3 is necessarily not of dense type, and so on. Also, one may use a smooth renorming of ℓ_1. This means $T + \lambda J$ is single-valued, demicontinuous. □

Fittingly, we finish with another result due implicitly to Simon Fitzpatrick. It again uses convexity twice.

Theorem 6.5 (Fitzpatrick-Phelps, [16]). *The following are equivalent for a Banach space X.*

(a) *X is reflexive;*
(b) *∂f is strongly locally maximal for each continuous convex function f on X;*
(c) *Each maximal monotone mapping T on X is strongly locally maximal.*

Proof. $(a) \Rightarrow (c)$ was proven in Theorem 5.4 while $(c) \Rightarrow (b)$ follows from Theorem 3.1. We prove $(b) \Rightarrow (a)$ by contradiction and James theorem. Select $x^* \in X^*$ such that $\|x^*\| = 1$ but $|\langle x^*, x\rangle| < 1$ whenever $\|x\| \leq 1$, and define $f(x) := |\langle x^*, x\rangle|$. Let $T := \partial f$ and $C^* := B_{X^*}$. Then $\operatorname{dom}(\partial f^* + N_{C^*}) = \{tx^* \colon |t| < 1\}$ while for $|t| < 1$, $(\partial f^* + N_{C^*})(tx^*) = \{0\}$. Thus, the graph of $M = (\partial f^* + N_C)^{-1}$ is the set $(-x^*, x^*), \times\{0\} \subset X \times X^*$, which is monotone but not closed and hence not the graph of a maximal monotone operator. $\square$

Note that we have exhibited a case of two convex functions f and $g := \iota^*_{B^*}$ on a Banach space such that $(\partial f^* + \partial g^*)^{-1}$ is maximal as a monotone mapping on X^{**} but not as a mapping restricted to X.

7. Conclusion

The *Fitzpatrick function* introduced in [16] was discovered precisely to provide a more transparent convex alternative to the earlier saddle function construction due to Krauss [22]. At the time, Fitzpatrick's interests were more centrally in the differentiation theory for convex functions and monotone operators.

The search for results relating when a maximal monotone T is single-valued to differentiability of $\mathcal{F}_T$ did not yield fruit, and he put the function aside. This is still the one area where to the best of my knowledge $\mathcal{F}_T$ has proved of little help – in part because generic properties of $\operatorname{dom}\mathcal{F}_T$ and of $\operatorname{dom}(T)$ seem poorly related.

By contrast, as we have seen the Fitzpatrick function and its relatives now provide the easiest access to a gamut of solvability and boundedness results. The clarity of the constructions also offers hope for resolving some of the most persistent open questions about maximal monotone operators such as:

1. Must $\operatorname{cl}\operatorname{dom}(T)$ always be convex? This is true for all operators of type (FPV), see [31].
2. Must $T_1 + T_2$ be maximal when $0 \in \operatorname{core}\operatorname{conv}(\operatorname{dom}(T_1) - \operatorname{dom}(T_2))$? What if both operators are locally maximal?
3. Is every locally maximal operator of dense type [32]? Is every maximal monotone operator maximal locally [32]?
4. Given a maximal monotone operator T, can one associate a convex function f_T to T in such a fashion that $T(x)$ is singleton as soon as $\partial f_T(x)$ is?
5. Are there some nonreflexive spaces, such as c_0 or the *James space* (which is codimension-one in its bidual, [15]), for which the answer to such questions can be answered in the affirmative?

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