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On Non-Enlargeable and Fully Enlargeable Monotone Operators

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Dedicated to the memory of Simon Fitzpatrick.

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In this paper we consider a family of enlargements of maximal monotone operators in a reflexive Banach space. Each enlargement, depending on a parameter $\varepsilon \geq 0$, is a continuous point-to-set mapping $E(\varepsilon, x)$ whose graph contains the graph of the given operator T. The enlargements are also continuous in ε, and they coincide with T for $\varepsilon = 0$. The family contains both a maximal and a minimal enlargement, denoted as T^e and T^{se} respectively. We address the following questions:
a) which are the operators which are not enlarged by T^e, i.e., such that $T(\cdot) = T^e(\varepsilon, \cdot)$ for some $\varepsilon > 0$?
b) same as a) but for T^{se} instead of T^e.
c) Which operators are fully enlargeable by T^e, in the sense that for all x and all $\varepsilon > 0$ there exists $\delta > 0$ such that all points whose distance to $T(x)$ is less than δ belong to $T^e(\varepsilon, x)$?
We prove that the operators not enlarged by T^e are precisely the point-to-point affine operators with skew symmetric linear part; those not enlarged by T^{se} are the point-to-point and affine operators, and the operators fully enlarged by T^e are those operators T whose Fitzpatrick function is continuous in its second argument at pairs belonging to the graph of T.

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1. Introduction

Let X be a Banach space and X^* its topological dual. A point-to-set mapping $A : X \rightrightarrows X^*$ is *outer semicontinuous* (in the sense of Kuratowski-Painlevé), if and only if $u \in T(x)$ whenever x is the limit of a sequence $\{x_n\} \subset X$ and u is the limit of a sequence $\{u_n\} \subset X^*$, satisfying $u_n \in A(x_n)$ for all n. Outer semicontinuity of A is equivalent to closedness of its graph $\text{Gph}(A) := \{(x, v) \in X \times X^* : v \in A(x)\}$. A is *inner semicontinuous* (also in the Kuratowski-Painlevé's sense), if and only if whenever $x \in X$ is the limit of a sequence $\{x_n\} \subset X$ and u belongs to $A(x)$, there exists a sequence $\{u_n\} \subset X^*$, convergent to u,

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such that $u_n \in A(x_n)$ for all large enough n. A is *continuous* when it is both inner and outer semicontinuous.

It is well known that a maximal monotone point-to-set operator $T : X \rightrightarrows X^*$ is always outer semicontinuous, because maximality implies closedness of the graph. On the other hand, maximal monotone operators fail in general to be inner semicontinuous, and hence continuous. This fact suggests the convenience of embedding a point-to-set maximal monotone operator T in a family of operators with larger graphs, of the form $E : \mathbb{R}_+ \times X \rightrightarrows X^*$, such that $T(x) \subset E(\varepsilon, x)$ for all $x \in X$ and all $\varepsilon > 0$, and $E(\varepsilon, \cdot)$ has better continuity properties than T for $\varepsilon > 0$. Assuming that E is also continuous in its first argument, and that $E(0, x) = T(x)$ for all $x \in X$, $E(\varepsilon, \cdot)$ will be, for ε small, a good approximation of T, better behaved than it. Such an E is said to be an *enlargement* of T. A particular choice of E, denoted as $T^e(\varepsilon, x)$ (when the operator being enlarged is itself denoted as T), was introduced in [2], with the following definition:

$$T^e(\varepsilon, x) = \{v \in X^* : \langle x - y, v - u\rangle \geq -\varepsilon \ \forall \ (y, u) \in \mathrm{Gph}(T)\}, \tag{1}$$

where $\mathrm{Gph}(T) = \{(y, u) \in X \times X^* : u \in T(y)\}$. It follows from the maximal monotonicity of T that $T^e(0, \cdot) = T(\cdot)$. It is also immediate that $T^e(\varepsilon_1, x) \subset T^e(\varepsilon_2, x)$ whenever $\varepsilon_1 \leq \varepsilon_2$.

The enlargement T^e was successfully used for developing inexact versions of the proximal point method with nonquadratic regularizations (see [2]), extensions of Korpelevich method for solving variational inequalities with point-to-set operators (see [7]), bundle-type methods for finding zeroes of point-to-set maximal monotone operators (see [4]), and for introducing less demanding error criteria in classical proximal point methods (see [11]). It has also been used for theoretical, rather than algorithmical purposes, e.g. for defining a new concept of extended sum of monotone operators (see [9]).

The enlargement T^e is indeed continuous in x for all fixed $\varepsilon > 0$. In fact it is jointly continuous in ε and x in the set $\mathbb{R}_{++} \times \mathrm{int}(D(T))$, where $\mathbb{R}_{++}$ is the strictly positive halfline and $\mathrm{int}(D(T))$ denotes the interior of the domain $D(T)$ of T, which is itself defined as $D(T) = \{x \in X : T(x) \neq \emptyset\}$ (see [5]).

Furthermore, a so called *transportation formula* holds, saying that if u belongs to $T(y)$ and v belongs to $T(x)$, then, for all $\alpha \in [0, 1]$, $\alpha v + (1-\alpha)u$ belongs to $T^e(\varepsilon, \alpha x + (1-\alpha)y)$, with $\varepsilon = 2\alpha(1 - \alpha)\langle y - x, u - v\rangle$. This formula allows for effective construction of points in $T^e(\varepsilon, x)$ given suitable elements in the image through T of points close to x. The formula can be improved by starting with points in $T^e(\delta, x)$ and $T^e(\eta, y)$ instead of $T(x)$ and $T(y)$. The improved result says that, for all $\alpha \in [0, 1]$, all $\delta, \eta \geq 0$, all $x, y \in D(T)$, all $v \in T^e(\delta, x)$ and all $u \in T^e(\eta, y)$, the point $\alpha v + (1 - \alpha)u$ belongs to $T^e(\varepsilon, \alpha x + (1 - \alpha)y)$, with $\varepsilon = \alpha\delta + (1 - \alpha)\eta + 2\alpha(1 - \alpha)\langle y - x, u - v\rangle$. These results were proved in [3] for the case of Hilbert spaces, and extended to Banach spaces in [5].

The transportation formula turns out to be a key property in several of the applications mentioned above. This fact led to the idea of studying enlargements from a more abstract point of view, i.e., considering all enlargements which satisfy certain selected properties. This was done in [12], where the family of enlargements $\mathbb{E}(T)$ of a maximal monotone operator T is defined in the following way:

Definition 1.1. Given $T : X \rightrightarrows X^*$, an enlargement $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ belongs to the family $\mathbb{E}(T)$ if and only if

i) $T(x) \subset E(\varepsilon, x)$ for all $\varepsilon \geq 0$ and all $x \in X$,

ii) For all $x \in X$, $E(\varepsilon_1, x) \subset E(\varepsilon_2, x)$ whenever $\varepsilon_1 \leq \varepsilon_2$,

iii) The transportation formula holds, i.e., for all $\alpha \in [0,1]$, all $\delta, \eta \geq 0$, all $x, y \in D(T)$, all $v \in E(\delta, x)$ and all $u \in E(\eta, y)$, the point $\alpha v + (1-\alpha)y$ belongs to $E(\varepsilon, \alpha x + (1-\alpha)y)$, with $\varepsilon = \alpha\delta + (1-\alpha)\eta + 2\alpha(1-\alpha)\langle y - x, u - v\rangle$.

It has been proved in [12] that the enlargement T^e defined by (1) is the largest member of the family $\mathbb{E}(T)$, in the sense that $\mathrm{Gph}(E) \subset \mathrm{Gph}(T^e)$ for all $E \in \mathbb{E}(T)$.

The enlargements in $\mathbb{E}(T)$ share many of the properties of T^e, including its continuity. In fact, it has also been proved in [12] if $T : X \rightrightarrows X^*$ is a maximal monotone operator and E belongs to $\mathbb{E}(T)$, then E is continuous in $\mathbb{R}_{++} \times \mathrm{int}(D(T))$.

We will also be interested in enlargements with closed graphs. Let $\mathbb{E}_c(T)$ the set of enlargements $E \in \mathbb{E}(T)$ such that $\mathrm{Gph}(T)$ is closed. It is immediate that T^e belongs to $\mathbb{E}_c(T)$, and is in fact the largest enlargement in $\mathbb{E}_c(T)$. The family $\mathbb{E}_c(T)$ has also a smallest element, namely the enlargement T^{se} defined as

$$T^{se}(\varepsilon, x) = \cap_{E \in \mathbb{E}_c(T)} E(\varepsilon, x). \tag{2}$$

From now on, $B(x, \rho)$ will denote the closed ball with center at x and radius ρ.

In this paper we address the following questions:

a) Which are the operators which are *non-enlargeable* by T^e, in the sense that $T^e(\varepsilon, x) = T(x)$ for all $x \in D(T)$ and some $\varepsilon > 0$?

b) Which are the operators which are *non-enlargeable* by T^{se}, with a meaning similar to the one in a)?

c) Which are the operators which are *fully enlargeable* by T^e, in the sense that for all $x \in D(T)$ and all $\varepsilon > 0$ there exists $\delta > 0$ (depending in general on x and ε), such that $T(x) + B(0, \delta) \subset T^e(\varepsilon, x)$?

d) Which are the operators which are *fully enlargeable* by T^{se}, with a meaning similar to the one in c)?

The relevance of these questions is rather obvious. In connection with a) and b), if an operator T is non-enlargeable in the announced sense, then the approximation effect attempted by the embedding of T in T^e (or T^{se}) is not attained at all (i.e., we are approximating T by itself, which is rather superfluous). It is true that this will happen only when T itself is rather well behaved, but, if the algorithms based upon the enlargements are indeed effective, one should expect that non-enlargeable operators will be simple enough, so that problems involving them (e.g. finding their zeroes), are such that no iterative numerical methods of the kind mentioned above are necessary for their solution.

On the other hand, fully enlargeable operators are those for which the enlargements under consideration work better: the enlargement is such that any point close enough to the set $T(x)$ will belong to $T^e(\varepsilon, x)$ (or $T^{se}(\varepsilon, x)$), for a suitable ε. In order for these enlargements to be "robust" enough, it is desirable that the class of fully enlargeable operators be quite comprehensive, containing all operators whose behavior, in some specific sense, is good enough.

We will give rather satisfactory answers to the first three of these questions. In connection with a), we will prove that the class of maximal monotone operators whose domain has nonempty interior and which are non-enlargeable by T^e consists precisely of those which are point-to-point in the whole space X, affine (i.e., of the form $T(x) = Lx + z$, for some linear $L : X \to X^*$ and some fixed $z \in X^*$), and skew-symmetric (i.e., such that $L + L^* = 0$, where L^* is the adjoint operator of L). Similarly, we will answer b) by establishing that the maximal monotone operators whose domain have nonempty interior and which are non-enlargeable by T^{se} are precisely the affine ones. These operators are indeed rather simple, for the purposes considered in the previous paragraph. These results will be presented in Section 2.

In order to announce our results related to c) above, we need some additional machinery. We start with the Fitzpatrick function associated to a maximal monotone operator, introduced in [6]. Given a maximal monotone operator $T : X \rightrightarrows X^*$, the *Fitzpatrick function* $\phi_T : X \times X^* \to \mathbb{R} \cup \{\infty\}$ is defined as

$$\phi_T(x, v) = \langle x, v\rangle + \sup_{(y,u)\in \mathrm{Gph}(T)} \langle x - y, u - v\rangle. \tag{3}$$

We also need the following definition.

Definition 1.2. A monotone operator T is said to be *strictly* $^{\infty}$*monotone* (to be read as “strictly monotone at infinity") if and only if there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that *liminf*$_{t\to\infty}\varphi(t)/t > 0$ and $\langle x - y, v - u\rangle \ge \varphi(\|x - y\|)$ for all $(x, v), (y, u) \in$ *Gph*(T).

We will answer question c) by giving a characterization of those maximal monotone operators which are fully enlargeable by T^e: they are precisely those operators such that $\phi_T(x, \cdot)$ is continuous at any point v belonging to $T(x)$, with ϕ_T as in (3). Also, we have a characterization of those operators which are fully enlarged by T^e in a uniform way, in the sense that the radius δ such that $T(x) + B(0, \delta) \subset T^e(\varepsilon, x)$ depends on ε but not on x: the class of these operators coincides precisely with the class of maximal monotone operators which are strictly $^{\infty}$monotone. This class can also be characterized in terms of some local continuity properties of the function $\beta_T(x, v) := \phi_T(x, v) - \langle x, v\rangle$ (see Theorem 3.3 for details). Again, this class of operators is comprehensive enough so as to ensure the robustness of the enlargement T^e. These results will be presented in Section 3.

We do not have yet satisfactory answers to question d), which is left as an open problem, deserving further study.

2. On non-enlargeable operators

We will say that a maximal monotone operator T is *non-enlargeable* by an enlargement $E \in \mathbb{E}(T)$ if there exists $\varepsilon > 0$ such that $E(\varepsilon, x) = T(x)$ for all $x \in X$ (we will see later on that when T is non-enlargeable by E then the equality above holds indeed for all $\varepsilon \ge 0$). In this section we study the class of maximal monotone operators which are non-enlargeable by either T^e or T^{se}.

We will proceed in three stages, which will be dealt with in the next three subsections: first we will establish that affine operators with skew-symmetric linear part are non-enlargeable by T^e, then we will prove that all affine operators are non-enlargeable by T^{se}, and finally we will see that operators whose domain has nonempty interior and which are non-enlargeable by any $E \in \mathbb{E}(T)$ are affine (in all cases, it is understood that all the operators under consideration are maximal monotone).

2.1. Affine operators with skew-symmetric linear part are non-enlargeable by T^e

We start with some definitions. Let X be a reflexive Banach space and $L : X \to X^*$ a continuous linear operator. The *adjoint* operator $L^* : X \to X^*$ is defined by $\langle x', Lx\rangle = \langle x, L^*x'\rangle$ for all $x, x' \in X$. The operator L is said to be

- *positive semidefinite* when $\langle Lx, x\rangle \geq 0$ for all $x \in X$,
- *skew-symmetric* when $L + L^* \equiv 0$.

We also remind that the *range* $R(A)$ of a point-to-set mapping $A : X \rightrightarrows X^*$ is defined as $\cup_{x\in X} A(x)$.

We need now a couple of introductory technical results.

Lemma 2.1. *Let X be a reflexive Banach space. Consider a positive semidefinite linear operator $S : X \to X^*$ with closed range $R(S)$. For a fixed $u \in X^*$, define $\psi : X \to \mathbb{R}$ as $\psi(y) := 1/2\langle Sy, y\rangle - \langle u, y\rangle$. Then*

i) *if $u \notin R(S)$, then* $\operatorname{argmin}_X\psi = \emptyset$ *and* $\inf_X \psi = -\infty$.

ii) *If $u \in R(S)$, then* $\emptyset \neq \operatorname{argmin}_X\psi = S^{-1}(u) := \{x \in X : S(x) = u\}$.

Proof. *i)* Since $u \notin R(S)$ and $R(S)$ is closed and convex, there exists $\bar y \neq 0$, $\bar y \in X^{**} = X$ such that $\langle \bar y, u\rangle = 1$ and $\langle \bar y, Sy\rangle \leq 0$ for all $y \in X$. Since S is positive semidefinite, it holds that $\langle \bar y, S\bar y\rangle = 0$. Then $\psi(\lambda\bar y) := (1/2)\lambda^2\langle S\bar y, \bar y\rangle - \lambda\langle u, \bar y\rangle = -\lambda$, which tends to $-\infty$ when $\lambda \uparrow \infty$. Thus, $\operatorname{argmin}_X\psi = \emptyset$ and $\inf_X \psi = -\infty$.
ii) Observe that $\operatorname{argmin}_X\psi = \{z : \nabla\psi(z) = 0\} = \{z : S(z) = u\} = S^{-1}(u)$. This set is nonempty, since $u \in R(S)$. □

Next, we explicitly compute the enlargement T^e for the case in which T is affine.

Proposition 2.2. *Let X be a reflexive Banach space and $T : X \to X^*$ a maximal monotone operator of the form $T(x) = Lx+z$, where L is linear (and hence positive semidefinite, by monotonicity of T) and $z \in X^*$. Then*

$$T^e(\varepsilon, x) = \{Lx + z + (L + L^*)w : \langle(L + L^*)w, w\rangle \leq 2\varepsilon\}. \tag{4}$$

Proof. Call $V(\varepsilon, x)$ the right hand side of (4). We prove first that

$$T^e(\varepsilon, x) \subset V(\varepsilon, x). \tag{5}$$

Write an element $w \in T^e(\varepsilon, x)$ as $w = Lx + z + u$, for some $u \in X^*$. We must prove that there exists some $\bar w$ such that $(L + L^*)\bar w = u$ and $\langle(L + L^*)\bar w, \bar w\rangle \leq 2\varepsilon$. The definition of T^e implies that for all $y \in X$, it holds that

$$-\varepsilon \leq \langle w - (Ly + z), x - y\rangle = \langle L(x - y), x - y\rangle + \langle u, x - y\rangle.$$

Rearranging the expression above, we get that

$$-\varepsilon \leq (1/2)\langle(L + L^*)(y - x), y - x\rangle - \langle u, y - x\rangle = \psi(y - x),$$

with ψ as in Lemma 2.1. Since ψ is bounded below, we get that $u \in R(L + L^*)$ and $\operatorname{argmin}_X\psi = (L+L^*)^{-1}u$. Thus, for every $\bar x \in x + (L+L^*)^{-1}u$ we have $-\varepsilon \leq \psi(\bar x - x) \leq \psi(x' - x)$ for all $x' \in X$. In other words,

$$-\varepsilon \leq (1/2)\langle(L + L^*)(\bar x - x), \bar x - x\rangle - \langle u, \bar x - x\rangle = -(1/2)\langle(L + L^*)(\bar x - x), \bar x - x\rangle,$$

using the fact that $(L+L^*)(\bar{x}-x)=u$. Thus, $\langle (L+L^*)(\bar{x}-x), \bar{x}-x\rangle \le 2\varepsilon$, and hence there exists $\bar{w} := (\bar{x}-x)$ such that $u=(L+L^*)\bar{w}$ with $\langle (L+L^*)\bar{w},\bar{w}\rangle \le 2\varepsilon$. We have proved that (5) holds. We proceed to prove the converse inclusion.

Take $u = Lx + z + (L+L^*)w \in V(\varepsilon, x)$. We must prove that u belongs to $T^e(\varepsilon, x)$, i.e., that

$$-\varepsilon \le \langle x-y, u-T(y)\rangle \tag{6}$$

for all $y \in X$. Note that

$$\begin{aligned}
\langle x-y, u-T(y)\rangle &= \langle x-y, Lx+z+(L+L^*)w-(Ly+z)\rangle \\
&= \langle x-y, L(x-y)+(L+L^*)w\rangle \\
&= \frac{1}{2}\langle x-y, (L+L^*)(x-y) + \langle x-y, (L+L^*)w\rangle \\
&= \frac{1}{2}\langle x-y+w, (L+L^*)(x-y+w)\rangle - \frac{1}{2}\langle (L+L^*)w, w\rangle. \qquad (7)
\end{aligned}$$

It follows easily from the monotonicity of $T(x)=Lx+z$ that the first term in the rightmost expression in (7) is nonnegative. The second term is greater than or equal to $-\varepsilon$, as a consequence of the definition of $V(\varepsilon,x)$, since $w \in V(\varepsilon,x)$. Hence, (6) holds for all $y \in X$, i.e., u belongs to $T^e(\varepsilon,x)$. It follows that $V(\varepsilon,x)\subset T^e(\varepsilon,x)$, completing the proof. □

An immediate consequence of Proposition 2.2 is the characterization of those linear operators T which cannot be actually enlarged by T^e.

Corollary 2.3. *Let X be a reflexive Banach space, and $T: X\to X^*$ a maximal monotone operator defined as $T(x)=Lx+z$, with $z\in X^*$ and $L: X\to X^*$ linear and continuous. Then the following statements are equivalent:*

i) *L is skew-symmetric.*
ii) *$T^e(\varepsilon,x)=T(x)$ for every $\varepsilon>0$.*
iii) *$T^e(\bar{\varepsilon},x)=T(x)$ for some $\bar{\varepsilon}>0$.*

Proof. The fact that *i)* implies *ii)* follows from Proposition 2.2. Since *ii)* implies *iii)* trivially, it suffices to prove that *iii)* implies *i)*. Assume that *iii)* holds. The operator $S := L+L^*$ is self-adjoint, and positive semidefinite by maximal monotonicity of T. Under these circumstances, it is well known that if $\langle Sx,x\rangle=0$ then $x=0$. We claim that $Sx=0$ for all $x\in X$. Otherwise, there exists $w\notin \mathrm{Ker}(S)$, so that $\langle Sw,w\rangle>0$. Let $\bar{w} := \sqrt{\frac{2\bar{\varepsilon}}{\langle Sw,w\rangle}}\,w$. Then, $S\bar{w}\neq 0$ and

$$\langle S\bar{w},\bar{w}\rangle \le 2\bar{\varepsilon}. \tag{8}$$

In view of (8) and Proposition 2.2, $T(x)+S\bar{w}$ belongs to $T^e(\bar{\varepsilon},x)$, in contradiction with *iii)*, which states that the unique element of $T^e(\bar{\varepsilon},x)$ is $T(x)$. It follows that $L+L^*=0$ and *i)* holds. □

2.2. Affine operators are non-enlargeable by T^{se}

We will compute next T^{se} for an affine operator T.

Proposition 2.4. *Let X be a reflexive Banach space, and $T : X \to X^*$ be defined as $T(x) = Lx + z$, with $z \in X^*$ and $L : X \to X^*$ linear and continuous. Then $T^{se}(\varepsilon, x) = T(x)$ for all $\varepsilon \geq 0$.*

Proof. For $\varepsilon \geq 0$, consider the enlargement $\bar{E}(\varepsilon, x)$ defined as $\bar{E}(\varepsilon, x) = T(x)$ for all $x \in X$. We claim that $\bar{E}$ belongs to $\mathbb{E}(T)$. Items i) and ii) in Definition 1.1 are trivially satisfied, so that it suffices to check iii). Note that in this case we have $E(\delta, x) = Lx + z$, $E(\eta, y) = Ly + z$ for all $\delta, \eta \geq 0$ and all $x, y \in X$, so that the statement in iii) reduces to

$$\alpha(Lx + z) + (1 - \alpha)(Ly + z) = L(\alpha x + (1 - \alpha)y) + z,$$

which follows immediately from the linearity of L. Thus, $\bar{E}$ belongs to $\mathbb{E}_c(T)$. Since $\bar{E}(\varepsilon, x) = T(x) \subset E(\varepsilon, x)$ for all $\varepsilon \geq 0$ and all $x \in X$ as a consequence of Definition 1.1 i), we conclude that

$$\bar{E}(\varepsilon, x) = \cap_{E \in \mathbb{E}_c(T)} E(\varepsilon, x) = T^{se}(\varepsilon, x)$$

for all $\varepsilon \geq 0$ and all $x \in X$, which gives the result. □

2.3. Most non-enlargeable operators are affine

In order to complete our analysis of non-enlargeable operators, it remains to be proved that if T is non-enlargeable by an arbitrary $E \in \mathbb{E}(T)$ and $\text{int}(D(T)) \neq \emptyset$ then T is point-to-point and affine. We start by showing that if $E(\bar{\varepsilon}, \cdot) = T(\cdot)$ for some $\bar{\varepsilon} > 0$, then $E(\varepsilon, \cdot) = T(\cdot)$ for all $\varepsilon \geq 0$.

Lemma 2.5. *Let $E \in \mathbb{E}(T)$. If there exists $\bar{\varepsilon} > 0$ such that $E(\bar{\varepsilon}, x) = T(x)$ for all $x \in D(T)$, then the same property holds for every $\varepsilon > 0$.*

Proof. Take $x \in D(T)$ and define

$$\varepsilon(x) := \sup\{\varepsilon \geq 0 : E(\varepsilon, x) = T(x)\}.$$

By assumption, $\varepsilon(x) \geq \bar{\varepsilon} > 0$. Observe that Definition 1.1 ii) implies that $E(\eta, x) = T(x)$ for all $\eta < \varepsilon(x)$. The conclusion of the lemma clearly holds if $\varepsilon(x) = \infty$ for every $x \in D(T)$. Suppose that there exists $x \in D(T)$ for which $\varepsilon(x) < \infty$. Fix $\varepsilon > \varepsilon(x)$. By definition of $\varepsilon(x)$, we can find $w \in E(\varepsilon, x)$ such that $w \notin T(x)$. Consider $\bar{v}$:=argmin$\{\|v - w\| : v \in T(x)\}$ and take $\alpha \in (0, 1)$ such that $0 < \alpha\varepsilon < \varepsilon(x)$. By Definition 1.1 iii) we have that

$$\bar{v}' := \alpha w + (1 - \alpha)\bar{v} \in E(\alpha\varepsilon, x) = T(x).$$

Note that $\|\bar{v}' - w\| = (1 - \alpha)\|\bar{v} - w\| < \|\bar{v} - w\|$, which contradicts the definition of $\bar{v}$. Hence, $\varepsilon(x) = \infty$ for every $x \in D(T)$ and the lemma is proved. □

In order to continue, we need several results from the theory of maximal monotone operators. First, we recall the definition of normalizing operator. Given a convex set $C \subset X$, the *normalizing operator* $N_C : X \rightrightarrows X^*$, defined as

$$N_C(x) = \begin{cases} \{v \in X^* : \langle v, y - x\rangle \leq 0 \ \ \forall y \in C\} & \text{if } x \in C \\ \emptyset & \text{otherwise,} \end{cases} \tag{9}$$

is maximal monotone, because it is in fact the subdifferential of the convex function $I_C : X \to \mathbb{R} \cup \{\infty\}$, defined as $I_C(x) = 0$ if $x \in C$, $I_C(x) = \infty$ otherwise.

We start with the following three results, established in [10].

Theorem 2.6. *Let X be a Banach space with dual X^*, and $T : X \rightrightarrows X^*$ a monotone operator, such that* $\operatorname{int}(D(T)) \neq \emptyset$.

i) $\operatorname{int}(D(T))$ *is a nonempty convex set whose closure is* $\overline{D(T)}$.

ii) *if* $\overline{D(T)} = X$, *then* $D(T) = X$.

Theorem 2.7. *Let X be a Banach space with dual X^*. If $T : X \rightrightarrows X^*$ is maximal monotone, then T is locally bounded at any point $x \in \operatorname{int}(D(T))$.*

Theorem 2.8. *Let X be a Banach space with dual X^*, and $T : X \rightrightarrows X^*$ a monotone operator. If T is maximal and $\operatorname{int}(D(T)) \neq \emptyset$ then, for all z in the set $D(T) \setminus \operatorname{int}(D(T))$,*

a) *there exists a nonzero $w \in N_{D(T)}(z)$, with $N_{D(T)}$ as defined in* (9),

b) $T(z) + N_{D(T)}(z) \subset T(z)$,

c) *$T(z)$ is not bounded.*

The following result establishes demi-outer semicontinuity of maximal monotone operators. We start with the pertinent definitions.

Definition 2.9.

a) A sequence $\{(x^n, v^n)\} \subset X \times X^*$ *demi-converges* to (x, v) if one of the situations below holds:
   i) $w\text{-}\lim_n x^n = x$ and $s\text{-}\lim_n v^n = v$, or
   ii) $s\text{-}\lim_n x^n = x$ and $w^*\text{-}\lim_n v^n = v$,
   where $s\text{-}\lim_n$, $w\text{-}\lim_n$ and $w^*\text{-}\lim_n$ refer to limits of sequences in the strong, the weak and the weak* topology respectively.

b) A subset $W \subset X \times X^*$ is *demi-closed* if it contains all limits of demi-converging sequences $\{(x_n, v_n)\} \subset W$.

c) A point-to-set mapping $A : X \rightrightarrows X^*$ is *demi-outer semicontinuous* if its graph is demi-closed, i.e., whenever $\{x_n\}$ converges weakly (respectively strongly) to x, $\{v_n\}$ converges strongly (respectively weakly*) to v, and $v_n \in T(x_n)$ for all n, it holds that $v \in T(x)$.

Proposition 2.10. *If X is a Banach space and $T : X \rightrightarrows X^*$ is a maximal monotone operator, then T is demi-outer semicontinuous.*

Proof. See Proposition 4.3 in [12]. □

Now, we present a result on continuity of marginal functions constructed from point-to-set mappings.

Proposition 2.11. *Let X and Y be metric spaces and $F : X \rightrightarrows Y$ a point-to-set mapping. Given f :*$\operatorname{Gph}(F) \to \mathbb{R}$, *define $g : X \to \mathbb{R} \cup \{\infty\}$ as $g(x) := \sup_{y \in F(x)} f(x, y)$. Take $x \in D(F)$.*

a) *If F is inner semicontinuous at x and f is lower semicontinuous at every $(x, y) \in \{x\} \times F(x)$, then g is lower semicontinuous at x.*

b) *If X is a reflexive Banach space, $Y = X^*$, F is outer semicontinuous with respect to the weak* topology and locally bounded at x, and f is upper semicontinuous at any $(x, y) \in \{x\} \times F(x)$ with respect to the weak* topology, then g is upper semicontinuous at x.*

Proof. *a*) See Theorem 1.4.16 in [1].

b) In order to prove that g is upper semicontinuous at x, we must prove that for all sequence $\{x_n\}$ converging to x, it holds that

$$g(x) \geq \text{limsup}_n g(x_n). \tag{10}$$

Assume that the inequality above does not hold, and so there exists $\gamma \in \mathbb{R}$ such that

$$g(x) < \gamma < \text{limsup}_n g(x_n) = \inf_n \sup_{k \geq n} g(x_k). \tag{11}$$

It follows from (11) that for all n there exists $k_n \geq n$ and $x_{k_n} \in X$ such that

$$g(x) < \gamma < g(x_{k_n}). \tag{12}$$

We conclude from (12) and the definition of g that for all $n \in \mathbb{N}$ there exists $y_{k_n} \in F(x_{k_n})$ verifying

$$g(x) < \gamma < f(x_{k_n}, y_{k_n}). \tag{13}$$

Since F is locally bounded at x, there exist a neighborhood V of x and a radius $\rho > 0$ such that $F(V) \subset B(0, \rho)$. Take $\bar{n} \in \mathbb{N}$ such that x_n belongs to V for all $n \geq \bar{n}$. Thus, for all $n \geq \bar{n}$, we have $k_n \geq \bar{n}$, and henceforth $x_{k_n} \in V$, yielding $x_{k_n} \in V$ for $n \geq \bar{n}$. Therefore $y_{k_n} \in B(0, \rho)$ for all $n \geq \bar{n}$, and invoking Bourbaki-Alaoglu's Theorem (see e.g. [8], Vol. I, p. 248), we can assume without loss of generality (i.e., refining $\{y_{k_n}\}$ if needed) that $\{y_{k_n}\}$ is weakly* convergent to some $y \in B(0, \rho)$. Since F is outer semicontinuous with respect to the weak* topology, it follows that $y \in F(x)$. Using this fact, upper semicontinuity of f with respect to the weak* topology and (13), we get

$$f(x, y) \leq g(x) < \gamma \leq \limsup_j f(x_{k_{n_j}}, y_{k_{n_j}}) \leq f(x, y),$$

which is a contradiction, so that (10) holds. □

We continue with a result which relates single-valuedness with inner semicontinuity, whose proof requires the previous two propositions.

Proposition 2.12. *Let* $T : X \rightrightarrows X^*$ *be a maximal monotone operator such that* $\text{int}(D(T)) \neq \emptyset$. *Then*

i) T *is not inner semicontinuous at any boundary point of* $D(T)$.

ii) $T(x)$ *is a singleton if and only if* T *is inner semicontinuous at* x.

Proof. *i*) Let $x \in D(T)$ be a boundary point of $D(T)$. Theorem 2.8 *a*) implies that there exists a nonzero element $w \in N_{D(T)}(x)$. Therefore,

$$D(T) \subset \{z \in X : \langle w, z \rangle \leq \langle w, x \rangle\}. \tag{14}$$

By Theorem 2.6 *i*), $\text{int}(D(T))$ is a nonempty convex set and, taking interiors in both sides of (14), we get

$$\text{int}(D(T)) \subset \{z \in X : \langle w, z \rangle < \langle w, x \rangle\}. \tag{15}$$

Fix $y \in \text{int}(D(T))$. By (15), there exists $\alpha < 0$ such that $\langle w, y - x \rangle < \alpha$. Suppose now that T is inner semicontinuous at x and fix $u \in T(x)$. Since $T(x) + N_{D(T)}(x) \subset T(x)$ by

Theorem 2.8 *b*), we conclude that $u + w \in T(x)$. Define now $x_n := (1 - 1/n)x + (1/n)y$. Clearly, $\lim_{n\to\infty} x_n = x$. Since x belongs to $D(T)$ and y belongs to $\mathrm{int}(D(T))$, x_n belongs to $\mathrm{int}(D(T))$ for all n (see e.g. Theorem 2.23(b) in [13]). By inner semicontinuity of T at x, there exists a sequence $\{v_n\}$ such that $v_n \in T(x_n)$ for all n and $\{v_n\}$ converges weakly* to $u + w \in T(x)$. Therefore,

$$0 \leq \langle v_n - u, x_n - x\rangle = \frac{1}{n}\langle v_n - u, y - x\rangle = \frac{1}{n}\langle v_n - (u + w), y - x\rangle + \frac{1}{n}\langle w, y - x\rangle,$$

so that

$$0 \leq \langle v_n - (u + w), y - x\rangle + \langle w, y - x\rangle. \tag{16}$$

Taking now limits for $n \to \infty$ in (16) and using the fact that $\{v_n\}$ converges weakly* to $u + w$, we get $0 \leq \langle w, y - x\rangle < \alpha < 0$, a contradiction, which implies that T is not inner semicontinuous at any point x in the boundary of $D(T)$.

ii) Assume first that $T(x)$ is a singleton. Note that $x \in \mathrm{int}(D(T))$ by Theorem 2.8 *c*), so that there exists $\rho > 0$ such that $T(B(x, \rho))$ is contained in a weakly* compact set B^*. Let $\{x_n\} \subset D(T)$ be a sequence converging to x. Without loss of generality, we can assume that $\{x_n\} \subset B(x, \rho)$. Take a sequence $\{v_n\} \subset X^*$ such that $v_n \in T(x_n)$ for all n. Since $\{x_n\} \subset B(x, \rho)$, it holds that $\{v_n\} \subset B^*$. By Bourbaki-Alaoglu's Theorem, $\{v_n\}$ has weak* accumulation points. Since T is demi-closed by Proposition 2.10, all such accumulation points belong to $T(x)$. Using now the fact that $T(x)$ is a singleton, we conclude that $\{v_n\}$ has a unique weak* accumulation point, which must be $T(x)$. Since $\{v_n\}$ is bounded, the whole sequence is weakly convergent to $T(x)$. This proves the inner-semicontinuity of T.

Conversely, assume that T is inner semicontinuous at x, and suppose that there exist $v_1, v_2 \in T(x)$, with $w := v_1 - v_2 \neq 0$. By definition of the norm in X^*, there exists $z \in X$ such that $\|z\| = 1$ and $\langle w, z\rangle \geq \|w\|/2 > 0$. Define now $g : D(T) \to \mathbb{R} \cup \{\infty\}$ as $g(y) := \sup_{u\in Ty}\langle u, z\rangle$. We claim that g is continuous at x. In order to prove this claim, we wil use Proposition 2.11 *a*) and *b*), with $F := T$ and f :Gph$(T) \to \mathbb{R}$ defined as $f(y, u) := \langle u, z\rangle$. It is easy to check that f is strong-weak* continuous at every point $(x, v) \in \{x\}\times T(x)$. Since T is inner semicontinuous at x, we can apply Proposition 2.11 *a*) for concluding that g is lower semicontinuous at x. Now we observe that, as a consequence of Theorem 2.7, T is locally bounded at x because, since T is inner semicontinuous at x, by item *i*) of this proposition x cannot be a boundary point of $D(T)$, and hence $x \in \mathrm{int}(D(T))$. We are within the hypotheses Proposition 2.11 *b*), which implies upper-semicontinuity of g. Therefore, the claim holds and g is continuous at x. Since T is locally bounded at x, there exists $\rho > 0$ such that $T(B(x, \rho))$ is contained in a weak* compact set. Take $t \in (-\rho, \rho)$, so that $T(x + tz) \subset T(B(x, \rho))$. For all $u \in T(x + tz)$ and all $v \in T(x)$ we have that

$$0 \leq \langle u - v, (x + tz) - x\rangle = t\langle u - v, z\rangle. \tag{17}$$

Using (17) for a fixed $t =: s \in (-\rho, 0)$ and $v = v_2 \in T(x)$, we get $\langle v_2, z\rangle \geq \langle u, z\rangle$. Since this inequality holds for every $u \in T(x + sz)$ we obtain

$$\langle v_2, z\rangle \geq g(x + sz). \tag{18}$$

Using again (17) for a fixed $t =: s' \in (0, \rho)$ and $v = v_1 \in T(x)$, we get

$$\langle v_1, z\rangle \leq \langle u, z\rangle \leq g(x + s'z). \tag{19}$$

By (18), (19) and continuity of g at x, we have

$$0 = \lim_{s\uparrow 0\ s'\downarrow 0} g(x+s'z) - g(x+sz) \geq \langle v_1, z\rangle - \langle v_2, z\rangle = \langle w, z\rangle \geq \frac{\|w\|}{2} > 0,$$

a contradiction, which implies that $T(x)$ is a singleton. □

Next we prove that maximal monotone point-to-point operators $T : X \to X^*$ have open domains and are continuous.

Proposition 2.13. *Let X be a Banach space and $T : X \to X^*$ a point-to-point monotone operator such that* $\operatorname{int}(D(T)) \neq \emptyset$. *If T is maximal, then $D(T)$ is open and T is continuous with respect to the strong topology in X and the weak* topology in X^* at every point of $D(T)$.*

Proof. Assume that there exists a point $x \in D(T)$ which is at the boundary of $D(T)$. Then $T(x)$ is unbounded by Theorem 2.8 *c*), contradicting the fact that T is point-to-point. Therefore, $D(T) \subset \operatorname{int}(D(T))$ and hence $D(T)$ is open. Fix $x \in D(T)$. Since T is point-to-point, continuity of T at x is equivalent to inner-semicontinuity of T at x, which follows from Proposition 2.12 *ii*) and the fact that $T(x)$ is a singleton. □

We continue with a technical lemma, which gives sufficient conditions for a maximal monotone operator to be affine.

Lemma 2.14.

i) *Let X and Y be vector spaces. If $G : X \to Y$ satisfies*

$$G(\alpha x + (1-\alpha)y) = \alpha G(x) + (1-\alpha)G(y) \tag{20}$$

for all $x, y \in X$ and all $\alpha \in [0,1]$, then the same property holds for all $\alpha \in \mathbb{R}$. In this case, G is affine, i.e., there exists a linear function $L : X \to Y$ and an element $\bar{y} \in Y$ such that $G(\cdot) = L(\cdot) + \bar{y}$.

ii) *Let $T : X \to X^*$ be a maximal monotone point-to-point operator such that* $\operatorname{int}(D(T)) \neq \emptyset$. *If*

$$T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y),$$

for all $x, y \in D(T)$ and all $\alpha \in [0,1]$, then T is affine and $D(T) = X$.

Proof. *i*) is well known and easy to prove. For *ii*), note that Proposition 2.13 implies that $D(T)$ is open. Without loss of generality, we can assume that $0 \in D(T)$. Otherwise, take a fixed $x_0 \in D(T)$ and consider $T_0 := T(\cdot + x_0)$. The operator T_0 also satisfies the assumption on affine combinations, and its domain is the (open) set $\{y - x_0 : y \in D(T)\}$. It is also clear that $D(T_0) = X$ if and only if $D(T) = X$. Thus, we can suppose that $0 \in D(T)$. Since $D(T)$ is open, for every $x \in X$ there exists $\lambda > 0$ such that $\lambda x \in D(T)$. Define $\tilde{T} : X \to X^*$ in the following way. For each $x \in X$, pick up any $\lambda > 0$ such that $\lambda x \in D(T)$ and then take:

$$\tilde{T}(x) = \frac{1}{\lambda} T(\lambda x) + \left(1 - \frac{1}{\lambda}\right) T(0). \tag{21}$$

It happens to be the case that the right hand side of (21) does not depend on the specific choice of λ. In fact, we claim that

a) $\tilde{T}(x)$ is well defined and nonempty for every $x \in X$,
b) $\tilde{T} = T$ in $D(T)$,
c) $\tilde{T}$ is monotone.

We proceed to prove the claim. For proving a), we start by showing that the right hand side of (21), does not depend on λ. Suppose that $\lambda, \lambda' > 0$ are both such that $\lambda x, \lambda' x \in D(T)$. Without loss of generality, assume that $0 < \lambda' < \lambda$. Then $\lambda' x = \frac{\lambda'}{\lambda}\lambda x + (1 - \frac{\lambda'}{\lambda})0$. Since $\frac{\lambda'}{\lambda} \in (0,1)$ we can apply the hypotheses on T to get

$$T(\lambda' x) = \frac{\lambda'}{\lambda} T(\lambda x) + \left(1 - \frac{\lambda'}{\lambda}\right) T(0),$$

which implies that

$$\frac{1}{\lambda} T(\lambda x) + \left(1 - \frac{1}{\lambda}\right) T(0) = \frac{1}{\lambda'} T(\lambda' x) + \left(1 - \frac{1}{\lambda'}\right) T(0),$$

and hence $\tilde{T}(x)$ is well defined. Nonemptiness of $\tilde{T}(x)$ follows easily from the definition, since $\lambda x, 0 \in D(T)$. This gives $D(\tilde{T}) = X$. Assertion b) follows by taking $\lambda = 1$ in (21). By a), this choice of λ does not change the value of $\tilde{T}$. For proving c), we claim first that $\tilde{T}$ is affine. Indeed, since $D(\tilde{T}) = X$, this fact will follow from item i) if we show that for all $\lambda \in [0,1]$, it holds that

$$\tilde{T}(\lambda x + (1-\lambda) y) = \lambda \tilde{T}(x) + (1-\lambda)\tilde{T}(y)$$

for all $x, y \in X$. Take $\eta > 0$ small enough to ensure that $\eta x, \eta y$ and $\eta(\lambda x + (1-\lambda)y) \in D(T)$. Then,

$$\begin{aligned}
\tilde{T}(\lambda x + (1-\lambda) y) &= \tilde{T}\left(\frac{1}{\eta}(\eta\lambda x + \eta(1-\lambda)y)\right) \\
&= \frac{1}{\eta} T(\eta\lambda x + \eta(1-\lambda)y) + \left(1 - \frac{1}{\eta}\right) T(0) \\
&= \frac{\lambda}{\eta} T(\eta x) + \frac{1-\lambda}{\eta} T(\eta y) + \left(1 - \frac{1}{\eta}\right) T(0) \\
&= \lambda \left[\frac{1}{\eta} T(\eta x) + \left(1 - \frac{1}{\eta}\right) T(0)\right] + (1-\lambda)\left[\frac{1}{\eta} T(\eta y) + \left(1 - \frac{1}{\eta}\right) T(0)\right] \\
&= \lambda \tilde{T}(x) + (1-\lambda)\tilde{T}(y),
\end{aligned}$$

using the definition of $\tilde{T}$ in the second and the last equality, and the hypothesis on T in the third one. Thus, $\tilde{T}$ is affine by item ii). In this case, monotonicity of $\tilde{T}$ is equivalent to positive semidefiniteness of the linear part $\tilde{L} := \tilde{T}(\cdot) - T(0)$ of $\tilde{T}$. In other words, we must prove that $\langle \tilde{L}x, x\rangle \geq 0$ for all $x \in X$. Take $x \in X$ and $\lambda > 0$ such that $\lambda x \in D(T)$. Then

$$\begin{aligned}
\langle \tilde{L}x, x\rangle &= (1/\lambda^2)\langle \tilde{L}(\lambda x), \lambda x\rangle = (1/\lambda^2)\langle \tilde{T}(\lambda x) - T(0), \lambda x\rangle \\
&= (1/\lambda^2)\langle T(\lambda x) - T(0), \lambda x\rangle \geq 0,
\end{aligned}$$

using b) in the third equality and monotonicity of T in the inequality. Hence, $\tilde{T}$ is monotone and therefore $T = \tilde{T}$. This implies that T is affine and $D(T) = D(\tilde{T}) = X$. □

Now we are able to prove that if T cannot be enlarged, then it must be affine.

Theorem 2.15. *Let X be a reflexive Banach space and $T : X \rightrightarrows X^*$ a maximal monotone operator such that* $\operatorname{int}(D(T)) \neq \emptyset$. *If there exist $E \in \mathbb{E}(T)$ and $\bar{\varepsilon} > 0$ such that*

$$E(\bar{\varepsilon}, \cdot) = T(\cdot), \tag{22}$$

then

a) *$D(T) = X$, and*
b) *there exists a continuous linear function $L : X \to X^*$ and an element $z \in X^*$ such that $T(\cdot) = L(\cdot) + z$.*

Proof. We claim that (22) implies that $D(T)$ is open. Indeed, by Proposition 2.12 *ii)* and (22), $T(x)$ is a singleton for all x, and hence Proposition 2.13 yields the claim.

For proving *a)*, assume for the sake of contradiction that $D(T) \subsetneq X$. If $\overline{(D(T))} = X$, then by Theorem 2.6 *ii)*, it holds that $D(T) = X$. Hence we can assume that $\overline{(D(T))} \subsetneq X$. Take $x, y \in D(T)$ and $\alpha \in [0,1]$. By Definition 1.1 iii), we have

$$\alpha T(x) + (1-\alpha)T(y) \in E(\hat{\varepsilon}, \hat{x}),$$

where $\hat{\varepsilon} := \alpha(1-\alpha)\langle T(x) - T(y), x - y\rangle$ and $\hat{x} := \alpha x + (1-\alpha)y$. By Lemma 2.5, we get that $E(\hat{\varepsilon}, \hat{x}) = T(\hat{x})$, and since T is point-to-point, it holds that

$$\alpha T(x) + (1-\alpha)T(y) = T(\hat{x}).$$

We conclude from Lemma 2.14 *ii)* that T is affine and $D(T) = X$. Continuity of L follows from maximal monotonicity of T. This proves *a)* and *b)*. □

Finally, we combine the results of Theorem 2.15 with Corollary 2.3 and Proposition 2.4.

Corollary 2.16. *Let $T : X \rightrightarrows X^*$ be maximal monotone.*

i) *If T is affine with skew-symmetric linear part (i.e. $T(x) = Lx + z$ for all x, where $L : X \to X^*$ is linear and skew-symmetric, and z belongs to X^*), then T is non-enlargeable by T^e.*
ii) *If T is affine, then T is non-enlargeable by T^{se}.*
iii) *If* $\operatorname{int}(D(T)) \neq \emptyset$ *and T is non-enlargeable by T^e, then $D(T) = X$, and T is point-to-point and affine with skew-symmetric linear part.*
iv) *If* $\operatorname{int}(D(T)) \neq \emptyset$ *and T is non-enlargeable by T^{se} then $D(T) = X$, and T is point-to-point and affine.*

Proof.

i) Follows from Corollary 2.3.
ii) Follows from Proposition 2.4.
iii) Since then enlargement T^e belongs to $\mathbb{E}(T)$, we get from Theorem 2.15 that $D(T) = X$ and that T is point-to-point and affine. Using now Corollary 2.3, we conclude that the linear part of T is skew-symmetric.
iv) Again, since $T^{se} \in \mathbb{E}(T)$, we get from Theorem 2.15 that $D(T) = X$ and that T is point-to-point and affine. □

Remark. The condition $\operatorname{int}(D(T)) \neq \emptyset$ is necessary for Theorem 2.15 to hold. Consider the following example: $X = \mathbb{R}^n$, $0 \neq a \in \mathbb{R}^n$ and $T = N_V$, where $V := \{y \in \mathbb{R}^n | \langle a, y\rangle = 0\}$ and N_V is the normalizing operator of V, as defined by (9). It is clear that T is not affine, since its domain is not the whole space and it is nowhere point-to-point. Namely,

$$T(x) = \begin{cases} V^\perp = \{\lambda a \mid \lambda \in \mathbb{R}\} & \text{if } x \in V, \\ \emptyset & \text{otherwise.} \end{cases}$$

We claim that $T^e(\varepsilon, \cdot) = T(\cdot)$ for all $\varepsilon > 0$. For $x \notin V$ this equality follows from the fact that $V = D(T) \subset D(T^e(\varepsilon, \cdot)) \subset \overline{(D(T))} = V$. In this case both mappings have empty value at x. Suppose now that there exists $x \in V$ and $\varepsilon > 0$ such that $T^e(\varepsilon, \cdot) \not\subset T(\cdot)$. Then there exists $b \in T^e(\varepsilon, x)$ such that $b \notin V^\perp$. In this case, $b = ta + w$ for some $t \in \mathbb{R}$ and $0 \neq w \in V$. Since $b \in T^e(\varepsilon, x)$, for $y = x + 2\varepsilon w/|w|^2 \in V$ we have

$$-\varepsilon \leq \langle b - \lambda a, x - y\rangle = \ 2\langle (t-\lambda)a + w, -\varepsilon w/|w|^2\rangle = -2\varepsilon,$$

which is a contradiction, establishing the claim.

3. On fully enlargeable operators

We start with the formal definition of full enlargeability.

Definition 3.1. Let $T : X \rightrightarrows X^*$ be a maximal monotone operator and consider an element $E \in \mathbb{E}(T)$.

i) The enlargement E *fully enlarges* T *at the point* $x \in D(T)$ if and only if for all $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon)$ such that $T(x) + B(0, \delta) \subset E(\varepsilon, x)$.

ii) E is a *full enlargement of* T when property i) holds for all $x \in D(T)$.

iii) Given a full enlargement E of T, E is a *uniformly full enlargement of* T when the radius δ of the ball in i) does not depend on x, i.e., when for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $T(x) + B(0, \delta) \subset E(\varepsilon, x)$ for every $x \in D(T)$.

We present next a characterization of those operators T which are fully enlarged by T^e. Define $\beta_T : X \times X^* \to \mathbb{R} \cup \{\infty\}$ as

$$\beta_T(x, v) = \phi_T(x, v) - \langle x, v\rangle, \tag{23}$$

with ϕ_T as defined in (3). Also, given $w \in X^*$ and $U \subset X^*$, we will denote as $d(w, U)$ the metric distance from w to U, i.e. $d(w, U) = \inf_{u \in U} \|u - w\|$.

Theorem 3.2. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then the following statements are equivalent.*

a) *T^e is a full enlargement of T.*

b) *The Fitzpatrick function $\phi_T(x, \cdot)$ is continuous at every $v \in T(x)$, uniformly in $T(x)$.*

c) *For every $\varepsilon > 0$ and every $x \in D(T)$ there exist $\rho = \rho(\varepsilon, x) > 0$ and $\alpha = \alpha(\varepsilon, x) > 0$ such that*

$$\langle v - u, x - y\rangle \geq \alpha \|x - y\| \tag{24}$$

for all $v \in T(x)$ and all $(y, u) \in Gph(T)$ such that $\|x - y\| \geq \rho$.

Proof. Assume that a) holds. It follows easily from the definitions of β_T and T^e that, given $x \in D(T)$, w belongs to $T^e(\varepsilon, x)$ if and only if $0 \leq \beta_T(x, w) \leq \varepsilon$. In particular, given $x \in D(T)$, it holds that $\beta_T(x, w') = 0$ if and only if $w' \in T^e(0, x) = T(x)$. In view of (23), continuity of $\phi_T(x, \cdot)$ is equivalent to continuity of $\beta_T(x, \cdot)$. Thus, we proceed to prove continuity of $\beta_T(x, \cdot)$ on the set $T(x)$. Fix $\varepsilon > 0$ and consider $\delta > 0$ as given by Definition 3.1 i). Take $w \in X^*$ such that $d(w, T(x)) < \delta$. The latter inequality yields the existence of some $u \in T(x)$ such that

$$w = u + (w - u) \in T(x) + B(0, \delta) \subset T^e(\varepsilon, x),$$

using Definition 3.1 i) in the inclusion. Therefore, $|\beta_T(x, w) - \beta_T(x, v)| = \beta_T(x, w) - 0 \leq \varepsilon$ for all $v \in T(x)$. Since δ only depends on x and ε, b) holds.

Assume now that b) holds, and suppose that c) is not true for some $\bar{\varepsilon} > 0$ and some $\bar{x} \in D(T)$. By b), $\beta_T(\bar{x}, \cdot)$ is uniformly continuous on the set $T(\bar{x})$, and hence for every $\bar{v} \in T(\bar{x})$ there exists $\bar{\delta} > 0$ such that

$$\beta_T(\bar{x}, w) < \bar{\varepsilon}, \tag{25}$$

for all $w \in X^*$ such that $d(w, T(\bar{x})) < \bar{\delta}$. The assumption on $\bar{\varepsilon}$ and $\bar{x}$ entails that given any pair (α, ρ) it is posible to find $v \in T(\bar{x})$ and (y, u) in $\mathrm{Gph}(T)$ such that $\|\bar{x} - y\| \geq \rho$ but the inequality in (24) does not hold. Taking $\alpha := \bar{\delta}/2$ and $\rho := n \in \mathbb{N}$, we conclude that there exist $v_n \in T(\bar{x})$ and $(y_n, u_n) \in \mathrm{Gph}(T)$ such that $\|y_n - \bar{x}\| \geq n$ and

$$\langle v_n - u_n, \bar{x} - y_n \rangle < \frac{\bar{\delta}}{2} \|\bar{x} - y_n\| .$$

Take now $z_n \in X^*$ such that $\|z_n\| = 1$ and $\|\bar{x} - y_n\|^{-1} \langle z_n, \bar{x} - y_n \rangle = 1$, and define $w_n := v_n - \bar{\delta} z_n$. Then $d(w_n, T(\bar{x})) < \bar{\delta}$ and, by uniform continuity of $\beta_T(\bar{x}, \cdot)$, we have that $\beta_T(\bar{x}, w_n) \leq \bar{\varepsilon}$. In other words, w_n belongs to $T^e(\bar{\varepsilon}, \bar{x})$, which gives

$$\begin{aligned} -\bar{\varepsilon} \leq \langle w_n - u_n, \bar{x} - y_n \rangle &= \langle w_n - v_n, \bar{x} - y_n \rangle + \langle v_n - u_n, \bar{x} - y_n \rangle \\ &= -\bar{\delta} \|\bar{x} - y_n\| + \langle v_n - u_n, \bar{x} - y_n \rangle < -\bar{\delta} \|\bar{x} - y_n\| + \frac{\bar{\delta}}{2} \|\bar{x} - y_n\| \\ &= -\frac{\bar{\delta}}{2} \|\bar{x} - y_n\| \leq -\frac{\bar{\delta}}{2} n. \end{aligned} \tag{26}$$

Since $n \in \mathbb{N}$ can be made arbitrary large, (26) leads to contradiction, which proves c).

Assume now that c) holds. Fix $\varepsilon > 0$ and $x \in D(T)$, and consider $\rho, \alpha > 0$ as given by (24). We claim that $T(x) + B(0, \delta) \subset T^e(\varepsilon, x)$ for $\delta := \min\{\varepsilon/\rho, \alpha\}$. Indeed, take $w = v + z$, with $v \in T(x)$ and $z \in B(0, \delta)$. Then, for all $(y, u) \in \mathrm{Gph}(T)$, we have

$$\langle w - u, x - y \rangle = \langle z, x - y \rangle + \langle v - u, x - y \rangle \geq -\|z\| \|x - y\| + \langle v - u, x - y \rangle. \tag{27}$$

We consider now two cases.

Case 1. $y \in B(x, \rho)$. We get from (27)

$$\langle w - u, x - y \rangle \geq -\|z\| \rho + \langle v - u, x - y \rangle \geq -\|z\| \rho \geq -\delta \rho \geq -\varepsilon,$$

which implies that $w \in T^e(\varepsilon, x)$ in this case.

Case 2. $y \notin B(x,\rho)$. From (27) and assumption (c) we get

$$\langle w-u, x-y\rangle \geq -\|z\|\,\|x-y\| + \alpha\|x-y\| \geq (\alpha-\delta)\|x-y\| \geq 0 \geq -\varepsilon,$$

which also yields $w \in T^e(\varepsilon, x)$. Therefore T^e is a full enlargement of T at x. □

Next we characterize operators T for which T^e is a uniformly full enlargement.

Theorem 3.3. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then the following statements are equivalent.*

a) *T is strictly $^\infty$monotone, as in Definition* 1.2.
b) *The function $\beta(x,\cdot)$, as defined in* (23)*, is locally Lipschitz around $T(x)$, with a local Lipschitz constant which does not depend on x.*
c) *T^e is a uniformly full enlargement of T.*

Proof. Assume that *a)* holds. Since T is strictly $^\infty$monotone, there exists a function φ as in Definition 1.2, and hence there exist constants $M > 0$ and $r > 0$ such that $\varphi(t)/t > r$ for all $t > M$. Fix $x \in D(T)$, and take $w \in X^*$ such that $d(w, T(x)) < \delta := r/2$. For each $v \in T(x)$ we have

$$\begin{aligned} |\beta_T(x,w) - \beta_T(x,v)| &= \beta_T(x,w) = \sup_{(z,u)\in \mathrm{Gph}(T)} \langle u-w, x-z\rangle \\ &= \sup_{(z,u)\in \mathrm{Gph}(T)} \{\langle u-v, x-z\rangle + \langle v-w, x-z\rangle\} \\ &\leq \sup_{(z,u)\in \mathrm{Gph}(T)} \{-\varphi(\|x-z\|) + \langle v-w, x-z\rangle\} \\ &\leq \sup_{(z,u)\in \mathrm{Gph}(T)} \|x-z\| \left(\left\langle v-w, \frac{x-z}{\|x-z\|}\right\rangle - \frac{\varphi(\|x-z\|)}{\|x-z\|}\right) \\ &\leq \sup_{(z,u)\in \mathrm{Gph}(T)} \|x-z\| \left(\|v-w\| - \frac{\varphi(\|x-z\|)}{\|x-z\|}\right) \end{aligned} \tag{28}$$

Using the facts that $\|v-w\| < \delta = r/2$ and $\mathrm{Im}(\varphi) \subset \mathbb{R}_+$, and the assumption on M, r, we get that $\|v-w\| - \varphi(\|x-z\|)/\|x-z\| < 0$ when $\|x-z\| > M$. Since the supremum in the righmost expression of (28) is nonnegative, because we can take $z := x \in D(T)$, such supremum is attained at some z such that $\|x-z\| \leq M$. Therefore, since $\mathrm{Im}(\varphi) \subset \mathbb{R}_+$,

$$|\beta_T(x,w) - \beta_T(x,v)| \leq \sup_{(z,u)\in \mathrm{Gph}(T), \|x-z\|\leq M} \|x-z\| \left(\|v-w\| - \frac{\varphi(\|x-z\|)}{\|x-z\|}\right) \leq M\|v-w\|,$$

and hence *b)* holds.

Assume now that *b)* holds. Fix $x \in D(T)$ and $\varepsilon > 0$. We know that there exists $L, \delta > 0$, with δ independent of x, such that, whenever $w \in X^*$ is such that $d(w, T(x)) < \delta$, we have

$$|\beta_T(x,w) - \beta_T(x,v)| = \beta_T(x,w) \leq L\|v-w\|, \tag{29}$$

for all $v \in T(x)$. Define $\bar{\delta} := \min\{(1/2)\delta, \varepsilon/L\}$. We claim that $Tx + B(0,\bar{\delta}) \subset T^e(\varepsilon, x)$. Indeed, take $w = u + z$, with $u \in Tx$ and $z \in B(0,\bar{\delta})$. Since $\bar{\delta} < \delta$, we can use (29) for w and $u \in Tx$, yielding

$$\beta_T(x,w) \leq L\|u-w\| \leq L\bar{\delta} \leq \varepsilon,$$

which gives $w \in T^e(\varepsilon, x)$. Hence T^e is a uniformly full enlargement of T and $c)$ holds.

Assume now that T^e is a uniformly full enlargement of T. Take $\varepsilon = 1$ and $x \in D(T)$. Then there exists $\delta > 0$, not depending on x, such that $T(x) + B(0, \delta) \subset T^e(1, x)$. Take $(y, u) \in \mathrm{Gph}(T)$ with $z \neq x$ and let $\bar{z} \in X^*$ be such that $\|\bar{z}\| = 1$ and $\|x - y\|^{-1} \langle \bar{z}, x - y\rangle = 1$. Define also $z := -\delta \bar{z}$. Then $z \in B(0, \delta)$ and hence, for all $v \in T(x)$ we have

$$-1 \leq \langle (v + z) - u, x - y\rangle = \langle v - u, x - y\rangle + \langle z, x - y\rangle = \langle v - u, x - y\rangle - \delta\|x - y\|,$$

which can be rewritten as

$$\langle v - u, x - y\rangle \geq \delta\|x - y\| - 1, \tag{30}$$

for all $(y, u), (x, v) \in \mathrm{Gph}(T)$. Define now the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ as $\varphi(t) := \max\{0, \delta\, t - 1\}$. From (30) and monotonicity of T, we get

$$\langle v - u, x - y\rangle \geq \max\{0, \delta\|x - y\| - 1\} = \varphi(\|x - y\|).$$

Note that $\varphi(t) \geq 0$ for all t and that $\liminf_{t\to\infty} \varphi(t)/t = \delta > 0$. We conclude that T is strictly $^\infty$monotone. □

Corollary 3.4. *If T^e is a uniformly full enlargement of T, then the Fitzpatrick function $\phi_T(x, \cdot)$ is locally Lispchitz on the set $T(x)$ (but for ϕ_T the Lipschitz constant depends on x, at variance with the case of β_T).*

Proof. Since $\beta_T(x, \cdot)$ is locally Lipschitz on $T(x)$ by Theorem 3.3 and $\phi_T(x, \cdot) = \beta_T(x, \cdot) + \langle x, \cdot\rangle$, we get the result, because $\langle x, \cdot\rangle$ is obviously Lipschitz continuous on the whole space with constant $L = \|x\|$. □

When T is strictly $^\infty$monotone, and the Fenchel conjugate φ^* of φ is available, it is possible to obtain a more explicit expression for the radius δ in Definition 1.2, in which case we can effectively construct points in $T^e(\varepsilon, x) \setminus T(x)$. We need a preliminary lemma. We recall that, for a general $f : \mathbb{R} \to \mathbb{R}$, f^* is defined as $f^*(s) = \sup_{t\in\mathbb{R}}\{st - f(t)\}$. Since we are dealing with a function φ whose domain is $\mathbb{R}_+$, it is natural to consider $\varphi^*(s) = \sup_{t\in\mathbb{R}_+}\{st - \varphi(t)\}$.

Lemma 3.5. *Take $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$. Then*

i) $\limsup_{s\to 0^+} \varphi^*(s) \leq 0$.

ii) *If additionally $\varphi(0) = 0$ then $\lim_{s\to 0^+} \varphi^*(s) = \varphi^*(0) = 0$.*

Proof. *i)* Otherwise, there exists a sequence $\{s_k\} \subset \mathbb{R}_{++}$ and a constant $\theta > 0$ such that $s_k \searrow 0$ and $\varphi^*(s_k) > \theta > 0$ for all k. By definition of φ^*, the latter expression yields the existence of a sequence $\{t_k\} \subset \mathbb{R}_+$ such that $s_k t_k - \varphi(t_k) > \theta/2$ for all k. We claim that $\{t_k\}$ is unbounded. Otherwise, there exists $\bar{t}$ such that $t_k \leq \bar{t}$ for all k. Then

$$0 < \theta/2 \leq s_k t_k - \varphi(t_k) \leq s_k t_k \leq s_k \bar{t}. \tag{31}$$

Since the rightmost expression in (31) tends to zero as $k \to \infty$, (31) entails a contradiction, and hence $\{t_k\}$ is unbounded, i.e. there exists a subsequence $\{t_{i_k}\} \subset \{t_k\}$ such that $\lim_{k\to\infty} t_{i_k} = +\infty$. In view of the first two inequalities in (31),

$$0 < \theta/2 \leq t_{i_k}\left(s_{i_k} - \frac{\varphi(t_{i_k})}{t_{i_k}}\right),$$

implying that $s_{i_k} \geq \varphi(t_{i_k})/t_{i_k} \geq 0$ for all k. Combining this inequality with the fact that $\lim_{k\to\infty} s_{i_k} = 0$, we get $\lim_{k\to\infty} \varphi(t_{i_k})/t_{i_k} = 0$, contradicting the fact that $\limsup_{t\to\infty}\varphi(t)/t > 0$. Hence $\limsup_{s\to 0^+}\varphi^*(s) \leq 0$.

ii) If $\varphi(0) = 0$, then

$$\varphi^*(s) = \sup_{t\geq 0}\{st - \varphi(t)\} \geq s\,0 - \varphi(0) = -\varphi(0) = 0, \tag{32}$$

for all $s \geq 0$. Using (32) and *i*), we get $0 \geq \limsup_{s\to 0^+}\varphi^*(s) \geq 0$, so that $\lim_{s\to 0^+} \varphi^*(s) = 0$. On the other hand, $0 = -\varphi(0) \leq \sup_{t\geq 0}\{-\varphi(t)\} = \varphi^*(0) = \sup_{t\geq 0}\{-\varphi(t)\} \leq 0$ because $\varphi(t) \geq 0$ for all $t \geq 0$. We conclude that $\varphi^*(0) = 0$. □

Now we obtain a specific expression for δ, in terms of φ^*.

Proposition 3.6. *If T is strictly $^\infty$monotone then $T(x) + B(0,\delta) \subset T^e(\varepsilon, x)$ for any $\delta > 0$ such that $\varphi^*(\delta) \leq \varepsilon$.*

Proof. Note that the function φ in the definition of strictly $^\infty$monotone satisfies $\varphi(0) = 0$. Indeed, for all $(v, x) \in \mathrm{Gph}(T)$ we have

$$0 = \langle v - v, x - x\rangle \geq \varphi(\|x - x\|) = \varphi(0) \geq 0.$$

Thus Lemma 3.5 *ii*) applies, and hence $\lim_{s\to 0^+} \varphi^*(s) = 0$. Therefore, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi^*(\delta) < \varepsilon$. Take $w \in T(x) + B(0,\delta)$, so that $w = v + z$ with $v \in T(x)$ and $\|z\| \leq \delta$. For all $(y, u) \in \mathrm{Gph}(T)$ we have

$$\begin{aligned}\langle w - u, x - y\rangle &= \langle v - u, x - y\rangle + \langle z, x - y\rangle \geq \varphi(\|x - y\|) - \|z\|\,\|x - y\| \\ &\geq \varphi(\|x - y\|) - \delta\,\|x - y\| = -\left(\delta\,\|x - y\| - \varphi(\|x - y\|)\right) \\ &\geq -\sup_{t\geq 0}\left(\delta\, t - \varphi(t)\right) = -\varphi^*(\delta) \geq -\varepsilon,\end{aligned}$$

which yields $w \in T^e(\varepsilon, x)$. □

We present next some examples in which δ can be explicitly computed.

Example 3.7. Let T be strictly $^\infty$monotone with $\varphi(t) = \alpha t^\gamma$, with $\alpha > 0$ and $\gamma > 1$. Then $\varphi^*(s) = (\gamma - 1)(\frac{s}{\alpha\gamma})^{\frac{\gamma}{\gamma-1}}$, so that we can take $\delta = \alpha\,\gamma(\frac{\varepsilon}{\gamma-1})^{1-\frac{1}{\gamma}}$.

Example 3.8. Let T be strictly $^\infty$monotone with $\varphi(t) = \alpha t$, with $\alpha > 0$. Then

$$\varphi^*(s) = \begin{cases} 0 & \text{if } s \leq \alpha, \\ \infty & \text{if } s > \alpha, \end{cases}$$

so that we can take $\delta = \alpha$

Next, we study conditions upon which some relevant maximal monotone operators are strictly $^\infty$monotone, and hence uniformly fully enlargeable.

Proposition 3.9. *Let X be a Hilbert space and $C \subset X$ a closed and convex set. Let $P_C : X \to X$ be the metric projection onto C, and take $T : X \to X$ defined as $T(x) =$*

$x - P_C(x)$. T is strictly $^\infty$monotone if and only if C is bounded. In this situation, the function φ in Definition 1.2 can be taken as

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 4\rho, \\ (1/2)t^2 & \text{if } t > 4\rho, \end{cases} \tag{33}$$

where ρ is any positive constant such that $C \subset B(0,\rho)$.

Proof. If $x, y \in D(T)$ are such that $\|x - y\| \le 4\rho$, then

$$\langle T(x) - T(y), x - y\rangle \ge 0 = \varphi(\|x - y\|),$$

by monotonicity. Assume now that $\|x - y\| > 4\rho$. Note that this inequality can be equivalently written as $\|x - y\|^2 - 2\rho\,\|x - y\| > (1/2)\|x - y\|^2$. Using the fact that $\|P_C(x) - P_C(y)\| \le 2\rho$ we get

$$\begin{aligned} & \langle Tx - Ty, x - y\rangle \\ = \;& \|x - y\|^2 - \langle P_C(x) - P_C(y), x - y\rangle \\ \ge \;& \|x - y\|^2 - \|P_C(x) - P_C(y)\|\,\|x - y\| \ge \|x - y\|^2 - 2\rho\,\|x - y\| > (1/2)\|x - y\|^2, \end{aligned}$$

and therefore we can take φ as in (33). It can be checked that

$$\varphi^*(s) = \begin{cases} 4\rho s & \text{if } s \le 8\rho, \\ (1/2)s^2 & \text{if } s > 8\rho, \end{cases}$$

so that δ can be taken as

$$\delta = \begin{cases} \frac{\varepsilon}{4\rho} & \text{if } \varepsilon \le 32\rho^2, \\ \sqrt{2\varepsilon} & \text{otherwise.} \end{cases}$$

In order to prove the converse statement, assume that C is unbounded. Fix $\bar{x} \in C$ and an unbounded sequence $\{z_n\} \subset C$. If T were strictly $^\infty$monotone, we would have

$$\begin{aligned} \varphi(\|\bar{x} - z_n\|) &\le \langle T(\bar{x}) - T(z_n), \bar{x} - z_n\rangle = \|\bar{x} - z_n\|^2 - \langle P_C(\bar{x}) - P_C(z_n), \bar{x} - z_n\rangle \\ &= \|\bar{x} - z_n\|^2 - \|\bar{x} - z_n\|^2 = 0, \end{aligned}$$

implying that $\liminf_{n\to\infty}\varphi(\|\bar{x} - z_n\|)/\|\bar{x} - z_n\| \le 0$ in contradiction with $\liminf_{t\to\infty}\varphi(t)/t > 0$. □

Proposition 3.10. *Let X be a Banach space, $C \subset X$ a closed and convex set, and $N_C : X \rightrightarrows X^*$ the normalizing operator of C, as in (9). N_C is strictly $^\infty$monotone if and only if C is bounded.*

Proof. Assume first that C is a bounded set. In view of Theorem 3.3, for proving that N_C is strictly $^\infty$monotone it suffices to show that T^e is a uniformly full enlargement of T. Let $\rho > 0$ be such that $C \subset B(0,\rho)$ and take $\varepsilon > 0$, $x \in D(T)$. We claim that $T(x) + B(0,\delta) \subset T^e(\varepsilon, x)$ with $\delta = \varepsilon/(2\rho)$. Take $w = v + z$, with $v \in T(x)$ and $\|z\| \le \delta$. Then, for all $y \in C$ and $u \in N_C(y)$ we have

$$\langle w - u, x - y\rangle = \langle v - u, x - y\rangle + \langle z, x - y\rangle \ge -\|z\|\,\|x - y\| \ge -2\delta\rho \ge -\varepsilon,$$

using the fact that $\|x - y\| \leq 2\rho$.

Suppose now that C is unbounded. Fix $\bar{x} \in C$ and an unbounded sequence $\{z_n\} \subset C$. If T were strictly $^{\infty}$monotone, we would have, since $0 \in N_C(x)$ for all $x \in C$, $\varphi(\|\bar{x} - z_n\|) \leq \langle 0 - 0, \bar{x} - z_n \rangle = 0$, implying that $\liminf_{n\to\infty} \varphi(\|\bar{x} - z_n\|)/\|\bar{x} - z_n\| \leq 0$, in contradiction with $\liminf_{t\to\infty} \varphi(t)/t > 0$. □

Remark. Consider T as in the previous example with $C = B(0,\rho)$ and the operator $\hat{T}$ defined as $\hat{T}(x) = 0$ for all $x \in C$. Note that T and $\hat{T}$ conicide in the interior of $B(0,\rho)$. However, for $x \in B(0,\rho)$ we have $B(0,\delta) \subset T(x) + B(0,\delta) \subset T^e(\varepsilon, x)$ whenever $\delta = \varepsilon/(2\rho)$, while $\hat{T}^e(\varepsilon, x) = \{0\}$ for all $x \in X$. Hence, knowledge of the values of T in an arbitrarily large neighborhood of a point x is not enough for computing $T^e(\varepsilon, x)$ for any $\varepsilon > 0$. In other words, the determination of T^e requires a "global" knowledge of T.

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