Journal of Convex Analysis
Volume 13 (2006), No. 3+4, 773–784

A Qualification Free Sequential Pshenichnyi-Rockafellar Lemma and Convex Semidefinite Programming*

V. Jeyakumar
School of Mathematics, University of New South Wales, Sydney 2052, Australia
jeya@maths.unsw.edu.au

Z. Y. Wu
School of Mathematics, University of New South Wales, Sydney 2052, Australia; and: Department of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China
zhiyouwu@maths.unsw.edu.au

Dedicated to the memory of Simon Fitzpatrick.

Received: February 4, 2005

Pshenichnyi-Rockafellar Lemma provides a necessary and sufficient condition for a minimizer of a lower semi-continuous convex function over a convex set whenever a qualification condition holds. In this paper we present sequential generalizations of the lemma without any qualification condition. As an application we derive qualification free optimality conditions for a more general convex semidefinite programming model problem.

1. Introduction

The celebrated Pshenichnyi-Rockafellar Lemma [23, 16, 17] for a proper lower semi-continuous convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a convex set C, states that the inclusion

$$0 \in \partial f(a) + N_C(a) \tag{1}$$

is necessary and sufficient for the point $a \in C$ to be a minimizer of f over C, whenever the interior-point qualification, $(\operatorname{int} C) \cap dom\ f \neq \emptyset$, (or continuity of f at some $x_0 \in C \cap\ dom\ f$) holds, where $N_C(a)$ is the normal cone of C at a and $\partial f(a)$ is the subdifferential of f at a . This lemma has proved to be fundamental for the solution point characterizations of constrained optimization and approximation problems in convex optimization [5, 7, 8, 23]. However, (1) is not always satisfied at a minimizer of f over C. For instance, let $f(x) = \begin{cases} -\sqrt{x}, & \text{if } x \in [0, +\infty) \\ +\infty, & \text{if } x \in (-\infty, 0), \end{cases}$ and $C = [-1, 0]$. Then, $a = 0$ is a minimizer of f over C and $0 \notin \partial f(a) + N_C(a)$, as $\partial f(a) = \emptyset$. Moreover, the required qualification frequently fails for many important constrained convex optimization problems arising in applications. For relaxations of the qualification condition, see [1, 5, 23].

The purpose of this paper is to present new extensions of the lemma without any qualification condition. This is achieved, for instance, by relaxing the condition (1) in terms of

*The authors are grateful to Professor Gue Myung Lee, Pukyong National University, for his comments on the preliminary version of the paper.

ISSN 0944-6532 / $ 2.50

a sequence of ε_n−subdifferentials and ε_n−normal cones, where the sequence ε_n converges to 0. Thus, we show that the point $a \in C$ is a minimizer of f over a closed convex set C if and only if there exist sequences $\{\varepsilon_n\} \subset \mathbb{R}_+$ and $\{u_n\} \subset \partial_{\varepsilon_n} f(a), \{v_n\} \subset N_C^{\varepsilon_n}(a)$ such that

$$u_n + v_n \to 0 \quad \text{and} \quad \varepsilon_n \to 0 \quad \text{as } n \to \infty, \tag{2}$$

where $\partial_{\varepsilon_n} f(a)$ is the ε_n−subdifferential of f at a and $N_C^{\varepsilon_n}(a)$ is the ε_n−normal cone of C at a. As it will be shown later in the paper that, for the example above, $u_n + v_n = -n + (n + \frac{1}{n}) \to 0$, as $n \to \infty$, where $-n \in \partial_{\varepsilon_n} f(a)$, $n + \frac{1}{n} \in N_C^{\varepsilon_n}(a)$, and $\{\varepsilon_n\} = \{\frac{1}{\sqrt{n}}\}$. We also present sequential necessary and sufficient optimality conditions involving subdifferentials and normal cones at nearby points to the minimizer. The sequential conditions are shown to be equivalent to (1) under a more general qualification condition, which is much weaker than the classical interior point condition. We have chosen to avoid the use of nets and hence the results are given for reflexive Banach spaces.

As an application, we derive a qualification free necessary and sufficient optimality conditions for a general convex semidefinite programming model problem, which is increasingly becoming a basic modelling tool for many important applications in control and signal processing, eigenvalue optimization, and combinatorial optimization (see [2, 22, 21]). The results for convex semidefinite programming problems extend the corresponding recent results, given in [14, 15] (see also [19]), where the objective function is a continuous real-valued convex function.

2. Preliminaries

We begin by fixing some notation and definitions. We assume throughout that X is a (reflexive) Banach space. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset$ X, the **closure** of D will be denoted $\mathrm{cl}\,(D)$. If a set $A \subset X'$, the expression $\mathrm{cl}\,(A)$ will stand for the weak* closure. The **indicator function** δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The **support function** σ_D is defined by $\sigma_D(u) = \sup_{x \in D} u(x)$. The **normal cone** of D is given by $N_D(x) = \{v \in X' : \sigma_D(v) = v(x)\} = \{v \in X' : v(y - x) \leq 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) = \emptyset$ when $x \notin D$. Given $\varepsilon \geq 0$, the ε-**normal cone** of D is given by $N_D^\varepsilon(x) := \{v \in X' : \sigma_D(v) \leq v(x) + \varepsilon\} = \{v \in X' : v(y - x) \leq \varepsilon, \forall y \in D\}$ when $x \in D$, and $N_D^\varepsilon(x) = \emptyset$ when $x \notin D$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the **conjugate** function of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \ \mathrm{dom}\, f\}$$

where the *domain* of f, $\mathrm{dom}\, f$, is given by $\mathrm{dom}\, f = \{x \in X \mid f(x) < +\infty\}$. The *epigraph of* f, $\mathrm{Epi}\, f$, is defined by

$$\mathrm{Epi}\, f = \{(x, r) \in X \times \mathbb{R} \mid x \in \mathrm{dom}\, f,\ f(x) \leq r\}.$$

The *subdifferential of* f, $\partial f : X \rightrightarrows X'$ is defined as

$$\partial f(x) = \{v \in X' \mid f(y) \geq f(x) + v(y - x),\ \forall\, y \in X\},$$

and the ε-*subdifferential of* f, $\partial_\varepsilon f : X \rightrightarrows X'$ is defined as

$$\partial_\varepsilon f(x) = \{v \in X' \mid f(y) \geq f(x) + v(y - x) - \varepsilon,\ \forall\, y \in X\}.$$

For details see [13, 14]. Note also that $\partial\delta_D(x) = N_D(x)$ and $\partial_\varepsilon\delta_D(x) = N_D^\varepsilon(x)$. It follows easily from the definitions of Epi f^* of a proper convex function f and the ε-subdifferential of f that, if $a \in dom f$, then

$$\text{Epi } f^* = \bigcup_{\epsilon \geq 0} \{(v, v(a) + \epsilon - f(a)) \mid v \in \partial_\epsilon f(a)\}.$$

If $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are proper lower semi-continuous convex functions then the Fenchel-Moreau theorem gives us that $\text{Epi}\,(f+g)^* = \text{cl}\,(\text{Epi } f^* + \text{Epi } g^*)$. For details, see, for instance, [9, 18]. Moreover, it has recently been shown that for the proper lower semi-continuous convex functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$, if $\text{dom}\, f \cap \text{dom}\, g \neq \emptyset$ and $\text{Epi}\, f^* + \text{Epi}\, g^*$ is weak *closed then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad \forall x \in \text{dom}\, f \cap \text{dom}\, g.$$

In particular, if f and g are sublinear functions with $\text{dom}\, f \cap \text{dom}\, g \neq \emptyset$, then $\text{Epi}\, f^*$ $+\text{Epi}\, g^*$ is weak * closed if and only if $\partial f(0) + \partial g(0)$ is a weak* closed which is, in turn, equivalent to the formula, $\partial(f+g)(0) = \partial f(0) + \partial g(0)$ (for details see [4, 5]).

Proposition 2.1 (Brondsted-Rockafellar Theorem [3, 19]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then for any real number $\epsilon > 0$ and any $u \in \partial_\epsilon f(a)$ there exist $x_\epsilon \in X$ and $u_\epsilon \in \partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - a\| \leq \sqrt{\epsilon}, \|u_\epsilon - u\| \leq \sqrt{\epsilon} \quad \textit{and} \quad |f(x_\epsilon) - u_\epsilon(x_\epsilon - a) - f(a)| \leq 2\epsilon.$$

3. Generalizations of Pshenichnyi-Rockafellar Lemma

In this section, we derive extensions of the Pshenichnyi-Rockafellar Lemma without a qualification condition, and present a general qualification condition under which the extended lemma collapses to the classical lemma. Note that $\{\varepsilon_n\} \downarrow 0$ means that the sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ and $\lim_{n \longrightarrow \infty} \varepsilon_n = 0$. Note also that the weak* convergence of the sequence $\{w_n\}$ of X' to w will be denoted by $w_n \to_* w$.

Lemma 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Let C be a closed convex subset of X and let $a \in C \cap \text{dom}\, f$. Then the point a is a minimizer of f on C if and only if $(0, -f(a)) \in cl(\text{Epi}\, f^* + \text{Epi}\, \delta_C^*)$.*

Proof. Assume that a is a minimizer of f on C. Then, $0 \in \partial(f + \delta_C)(a)$, and by the Fenchel conjugation $(f+\delta_C)^*(0) + (f+\delta_C)(a) = 0$; thus, $(f+\delta_C)^*(0) = -(f+\delta_C)(a) = -f(a)$. This gives us that

$$(0, -f(a)) = (0, (f+\delta_C)^*(0)) \in \text{Epi } (f+\delta_C)^* = \text{cl}\,(\text{Epi } f^* + \text{Epi } \delta_C^*).$$

Conversely, if $(0, -f(a)) \in cl(\text{Epi}\, f^* + \text{Epi}\, \delta_C^*)$, then $(0, -f(a)) \in \text{Epi } (f+\delta_C)^*$. Thus $f(a) \leq (f+\delta_C)(x)$ for each $x \in X$, i.e., $f(x) \geq f(a)$ for any $x \in C$. Hence a is a minimizer of f on C. □

Theorem 3.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Let C be a closed convex subset of X and let $a \in C \cap \text{dom}\, f$. Suppose that there exists a convex cone $B \subset X' \times \mathbb{R}$ such that $cl(B) = \text{Epi}\, \delta_C^*$. Then a is a minimizer of f on C if and only if there exist sequences $\{\epsilon_n\} \downarrow 0$ and $\{u_n\}, \{v_n\} \subset X', \{\beta_n\} \subset \mathbb{R}$ with $u_n \in \partial_{\epsilon_n} f(a)$ and $(v_n, \beta_n) \in B$ such that $u_n + v_n \to_* 0$ and $\beta_n + u_n(a) \to 0$ as $n \to \infty$.*

Proof. By Lemma 3.1, the point a is a minimizer of f on C if and only if $(0,-f(a)) \in cl(\mathrm{Epi}\, f^* + \mathrm{Epi}\, \delta_C^*)$. Since $cl(B) = \mathrm{Epi}\ \delta_C^*$, $cl(\mathrm{Epi}\, f^* + \mathrm{Epi}\, \delta_C^*) = cl(\mathrm{Epi}\, f^* + cl(B)) = cl(\mathrm{Epi}\, f^* + B)$. Thus, a is a minimizer of f on C if and only if $(0,-f(a)) \in cl(\mathrm{Epi}\, f^* + B)$. So, there exist sequences $\{(u_n,\alpha_n)\} \subset \mathrm{Epi}\ f^*$ and $\{(v_n,\beta_n)\} \subset B$ such that $u_n + v_n \to_* 0$ and $\alpha_n + \beta_n \to -f(a)$. Since

$$\mathrm{Epi}\ f^* = \bigcup_{\epsilon \geq 0} \{(u, u(a) + \epsilon - f(a)) \mid u \in \partial_\epsilon f(a)\},$$

there exists a sequence $\{\epsilon_n\} \subset \mathbb{R}_+$ such that

$$u_n \in \partial_{\epsilon_n} f(a) \quad \text{and} \quad \alpha_n = u_n(a) + \epsilon_n - f(a).$$

As $(v_n,\beta_n) \in B \subset \mathrm{Epi}\ \delta_C^*$, we have that $\beta_n \geq v_n(a)$. Thus $u_n(a) + v_n(a) \leq \alpha_n + \beta_n + f(a) - \epsilon_n \leq \alpha_n + \beta_n + f(a)$. Now, passing to the limit as $n \to \infty$, we see $\lim_{n\to\infty} \epsilon_n = 0$ and $\lim_{n\to\infty} \beta_n + u_n(a) = \lim_{n\to\infty} \alpha_n + \beta_n - \epsilon_n + f(a) = 0$.

Conversely, suppose that there exist sequences $\{\epsilon_n\} \downarrow 0$ and $\{u_n\}, \{v_n\} \subset X', \{\beta_n\} \subset \mathbb{R}$ with $u_n \in \partial_{\epsilon_n} f(a)$ and $(v_n,\beta_n) \in B$ such that $u_n + v_n \to_* 0$ and $\beta_n + u_n(a) \to 0$ as $n \to \infty$. Then by the definition of the ϵ_n-subdifferential of f and by the fact that $B \subset \mathrm{Epi}\ \delta_C^*$, we have, for any $x \in C$ and, for each positive integer n,

$$f(x) - f(a) \geq u_n(x-a) - \epsilon_n \quad \text{and} \quad \beta_n \geq v_n(x).$$

Thus, for any $x \in C$,

$$\begin{aligned} f(x) - f(a) &\geq u_n(x-a) - \epsilon_n \\ &= (u_n(x) + v_n(x)) + (\beta_n - v_n(x)) - (\beta_n + u_n(a)) - \epsilon_n \\ &\geq (u_n(x) + v_n(x)) - (\beta_n + u_n(a)) - \epsilon_n. \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain that $f(x) - f(a) \geq 0$ for any $x \in C$. □

Theorem 3.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Let C be a closed convex subset of X and let $a \in C \cap \mathrm{dom}\, f$. Then the point a is a minimizer of f on C if and only if there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $\{u_n\}, \{v_n\} \subset X'$ with $u_n \in \partial_{\varepsilon_n} f(a)$ and $v_n \in N_C^{\varepsilon_n}(a)$ such that $u_n + v_n \to_* 0$ as $n \to \infty$.*

Proof. We shall apply Theorem 3.2 with $B = \mathrm{Epi}\ \delta_C^*$. Then $cl(B) = B = \mathrm{Epi}\ \delta_C^*$ and so, the point a is a minimizer of f on C if and only if there exist sequences $\{\theta_n\} \downarrow 0$ and $\{u_n\}, \{v_n\} \subset X', \{\beta_n\} \subset \mathbb{R}$ with $u_n \in \partial_{\theta_n} f(a)$ and $(v_n,\beta_n) \in \mathrm{Epi}\ \delta_C^*$ such that $u_n + v_n \to_* 0$ and $\beta_n + u_n(a) \to 0$ as $n \to \infty$. Since

$$\mathrm{Epi}\ \delta_C^* = \bigcup_{\gamma \geq 0} \{(v, v(a) + \gamma) \mid v \in \partial_\gamma \delta_C(a)\},$$

there exists a sequence $\{\gamma_n\} \subset \mathbb{R}_+$ such that

$$v_n \in \partial_{\gamma_n} \delta_C(a) \quad \text{and} \quad \beta_n = v_n(a) + \gamma_n.$$

Thus,

$$\beta_n + u_n(a) = v_n(a) + u_n(a) + \gamma_n \geq u_n(a) + v_n(a).$$

Passing to the limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} \gamma_n = 0$.

Let $\varepsilon_n = \max\{\theta_n, \gamma_n\}$. Then, $\{u_n\} \subset \partial_{\varepsilon_n} f(a)$ and $\{v_n\} \subset N_C^{\varepsilon_n}(a)$ and $u_n + v_n \to_* 0$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Conversely, suppose that there exist sequences $\{\varepsilon_n\} \downarrow 0$, $\{u_n\}, \{v_n\} \subset X'$ with $u_n \in \partial_{\varepsilon_n} f(a)$ and $v_n \in N_C^{\varepsilon_n}(a)$, such that $u_n + v_n \to_* 0$ as $n \to \infty$. Let $x \in C$ be arbitrary. Then by the definition of the ε_n-subdifferential of f at a and $N_C^{\varepsilon_n}(a)$, we have, for each positive integer n,

$$f(x) - f(a) \geq u_n(x-a) - \varepsilon_n \quad \text{and} \quad 0 \geq v_n(x-a) - \varepsilon_n.$$

Adding these two inequalities yield

$$f(x) - f(a) \geq (u_n + v_n)(x-a) - 2\varepsilon_n.$$

Passing to the limit as $n \to \infty$, we obtain that $f(x) - f(a) \geq 0$. □

The following example illustrates the case where Pshenichnyi-Rockafellar lemma fails whereas the sequential conditions in Theorem 3.3 hold.

Example 3.4. Let $X = \mathbb{R}$ and let $f(x) = \begin{cases} -\sqrt{x} & x \in [0, +\infty) \\ +\infty & x \in (-\infty, 0) \end{cases}$ and $C = [-1, 0]$. Then, f is a lower semicontinuous convex function on X and C is a closed convex set. Clearly, $a = 0$ is a minimizer of f over C. However, $\partial f(a) = \emptyset$ and so, $0 \notin \partial f(a) + N_C(a)$. On the other hand, if we take $\varepsilon_n = \frac{1}{\sqrt{n}}$, then $-n \in \partial_{\varepsilon_n} f(a)$. Indeed, for any $x \geq 1$, $-nx \leq -\sqrt{x}$; for $\frac{1}{n} \leq x < 1$, we have that $-nx \leq -1 < -\sqrt{x}$; and for $0 < x < \frac{1}{n}$, $-nx < -\sqrt{x} + \varepsilon_n$ as $-nx < 0$ and $-\sqrt{x} > -\frac{1}{\sqrt{n}}$. Thus, for each $x \in \mathbb{R}$, we have that $-nx \leq f(x) + \varepsilon_n$. Therefore, $-n \in \partial_{\varepsilon_n} f(a)$. Moreover, $N_C^{\varepsilon_n}(a) = \{v \mid v \geq -\varepsilon_n\}$. Let $u_n = -n \in \partial_{\varepsilon_n} f(a)$ and $v_n = n + \frac{1}{n} \in N_C^{\varepsilon_n}(a)$, then $u_n + v_n = \frac{1}{n} \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Using Proposition 2.1 and Theorem 3.3, we now derive sequential necessary and sufficient optimality conditions involving subdifferentials and normal cones at nearby points to the minimizer (see [20]).

Corollary 3.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Let C be a closed convex subset of X and let $a \in C \cap \operatorname{dom} f$. Then the point a is a minimizer of f on C if and only if there exist sequences $\{x_n\} \subset \operatorname{dom} f, \{y_n\} \subset C$ and $\{u_n\}, \{v_n\} \subset X'$ with $u_n \in \partial f(x_n)$ and $v_n \in N_C(y_n)$ such that*

$$u_n + v_n \to_* 0, \ ||x_n - a|| \to 0, \ ||y_n - a|| \to 0, \ \text{as } n \to \infty$$

and

$$f(x_n) - u_n(x_n - a) - f(a) \to 0, \ v_n(y_n - a) \to 0, \ \text{as } n \to \infty.$$

Proof. It follows from Theorem 3.3 that a is a minimizer of f on C if and only if there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $\{\bar{u}_n\}, \{\bar{v}_n\} \subset X'$ with $\bar{u}_n \in \partial_{\varepsilon_n} f(a)$ and $\bar{v}_n \in N_C^{\varepsilon_n}(a)$, such that $\bar{u}_n + \bar{v}_n \to_* 0$ as $n \to \infty$. By Proposition 2.1, there exist $\{x_n\}, \{y_n\} \subset X$ and $u_n \in \partial f(x_n), v_n \in N_C(y_n)$ such that

$$\|x_n - a\| \leq \sqrt{\varepsilon_n}, \|y_n - a\| \leq \sqrt{\varepsilon_n}, \|u_n - \bar{u}_n\| \leq \sqrt{\varepsilon_n}, \|v_n - \bar{v}_n\| \leq \sqrt{\varepsilon_n}$$

and

$$|f(x_n) - u_n(x_n - a) - f(a)| \leq 2\varepsilon_n, |\delta_C(y_n) - v_n(y_n - a) - \delta_C(a)| \leq 2\varepsilon_n.$$

Thus, $\{x_n\} \subset \text{dom } f$ and $\{y_n\} \subset C$. Since $\bar{u}_n + \bar{v}_n \to_* 0$ and $\varepsilon_n \to 0$ as $n \to \infty$,

$$u_n + v_n \to_* 0, ||x_n - a|| \to 0, \ ||y_n - a|| \to 0, \text{ as } n \to \infty$$

and

$$f(x_n) - u_n(x_n - a) - f(a) \to 0, v_n(y_n - a) \to 0, \text{ as } n \to \infty.$$

Conversely, assume there exist sequences $\{x_n\} \subset \text{dom } f, \{y_n\} \subset C$ and $\{u_n\}, \{v_n\} \subset X'$ with $u_n \in \partial f(x_n)$ and $v_n \in N_C(y_n)$ such that

$$u_n + v_n \to_* 0, ||x_n - a|| \to 0, \ ||y_n - a|| \to 0, \text{ as } n \to \infty$$

and

$$f(x_n) - u_n(x_n - a) - f(a) \to 0, \ v_n(y_n - a) \to 0, \text{ as } n \to \infty.$$

As $u_n \in \partial f(x_n)$ and $v_n \in N_C(y_n)$, we have that

$$f(x) - f(x_n) \geq u_n(x - x_n) \quad \text{and} \quad 0 \geq v_n(x - y_n) \quad \text{for any } x \in C.$$

So, for any $x \in C$,

$$\begin{aligned} f(x) &\geq f(x_n) + u_n(x - x_n) + v_n(x - y_n) \\ &= (f(x_n) - u_n(x_n - a) - f(a)) + u_n(x - a) + f(a) - v_n(y_n - a) + v_n(x - a) \\ &= (f(x_n) - u_n(x_n - a) - f(a)) - v_n(y_n - a) + (u_n + v_n)(x - a) + f(a). \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain that $f(x) \geq f(a)$ for any $x \in C$. Thus, a is a minimizer of f on C. $\square$

We now give a 2-dimensional example to illustrate our conditions in Corollary 3.1 in the case where C is a closed convex cone.

Example 3.6. Let $X = \mathbb{R}^2$ and let $f(x, y) = \begin{cases} -\ln(x+1) - \sqrt{y}, & \text{if } x > -1, y \geq 0 \\ +\infty, & \text{otherwise} \end{cases}$ and $C = \{(x, y) \mid x \leq 0, y \leq 0\}$. Then, f is a lower semicontinuous convex function on X and C is a closed convex cone. Let $a = (0, 0)$. Clearly, a is a minimizer of f on C. However, $0 \notin \partial f(a) + N_C(a)$. On the other hand, for each $n \geq 1$, let $z_n = (0, \frac{1}{n})$ and let $w_n = (0, 0)$. Then $z_n \in$ dom f, $\partial f(z_n) = \{(-1, -\frac{\sqrt{n}}{2})\}$ and $N_C(w_n) = \{(\alpha, \beta) \mid \alpha \geq 0, \beta \geq 0\}$. Now, take $u_n = (-1, -\frac{\sqrt{n}}{2}) \in \partial f(z_n)$, $v_n = (1, \frac{\sqrt{n}}{2} + \frac{1}{n}) \in N_C(w_n)$. Then $z_n \to 0$, $w_n \to 0$, $u_n + v_n = \frac{1}{n} \to 0$ and $f(z_n) - u_n(z_n - a) - f(a) = -\frac{1}{\sqrt{n}} + \frac{1}{2\sqrt{n}} = -\frac{1}{2\sqrt{n}} \to 0$, $v_n(w_n - a) = 0$ as $n \to \infty$.

Proposition 3.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous function and let C be a closed convex subset of X. Let $a \in C \cap dom\ f$. Assume that* $(\text{Epi} f^* + \text{Epi}\, \delta_C^*)$ *is weak* closed. Then the following statements are equivalent.*

(*i*) $f(a) = \min_{x \in C} f(x)$
(*ii*) $0 \in \partial f(a) + N_C(a)$
(*iii*) $(0, -f(a)) \in \text{Epi} f^* + \text{Epi}\, \delta_C^*$

(iv) $\exists\{\varepsilon_n\} \downarrow 0$ and $\{u_n\} \subset \partial_{\varepsilon_n} f(a), \{v_n\} \subset N_C^{\varepsilon_n}(a)$ such that $u_n + v_n \to_* 0$ as $n \to \infty$.

(v) $\exists\{x_n\} \subset \text{dom } f, \{y_n\} \subset C$ and $\{u_n\}, \{v_n\} \subset X'$ with $u_n \in \partial f(x_n)$ and $v_n \in N_C(y_n)$ such that as $n \to \infty$, $u_n + v_n \to_* 0$, $||x_n - a|| \to 0$, $||y_n - a|| \to 0$, $f(x_n) - u_n(x_n - a) - f(a) \to 0$, $v_n(y_n - a) \to 0$.

Proof. The conclusion will follow from Lemma 3.1, Theorem 3.3 and Corollary 3.5 if we show that *(i)* $\Longleftrightarrow$ *(ii)*. To see this, let $g = \delta_C$. Then, g is a proper lower semi-continuous convex function. Now, the point $a \in C \cap dom\ f$ is a minimizer of (P) if and only if a is a minimizer of $(f + \delta_C)$, which means that $0 \in \partial(f + \delta_C)(a)$. Now, the condition, that $(\text{Epi} f^* + \text{Epi}\,\delta_C^*)$ is weak* closed, ensures that

$$0 \in \partial(f + \delta_C)(a) = \partial f(a) + \partial\delta_C(a) = \partial f(a) + N_C(a).$$

Hence, *(i)* $\Longleftrightarrow$ *(ii)*. □

Note that if $int\ C \neq \emptyset$ and $(\text{int}\, C) \cap dom\ f \neq \emptyset$ (or if f is continuous at some $x_0 \in C \cap dom\ f$), then $(\text{Epi} f^* + \text{Epi}\,\delta_C^*)$ is weak* closed. The converse implication is, in general, not true (for details see [1, 4, 5, 6, 18]). Note also that if f is lower semi-continuous sublinear function then $\partial f(0) + N_C(0)$ is weak*closed if and only if $(\text{Epi} f^* + \text{Epi}\,\delta_C^*)$ is weak* closed.

4. Convex Semidefinite Programming

Consider the convex semi-definite programming model problem

$$(SDP) \qquad \text{Minimize } f(x)$$
$$\text{subject to } F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0,$$

where $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function and for $i = 0, 1, \cdots, m$, $F_i \in S_n$, the space of $(n \times n)$ symmetric matrices. The space S_n is partially ordered by the Löwner order, that is, for $M, N \in S_n$, $M \succeq N$ if and only if $M - N$ is positive semi-definite. The inner product in S_n is defined by $(M, N) = Tr[MN]$, where $Tr[\cdot]$ is the trace operation. Let $S := \{M \in S_n \mid M \succeq 0\}$. Then the dual cone of S, denoted by

$$S^+ := \{\theta \in S_n \mid (\theta, Z) \geq 0,\ \forall Z \in S\} = S.$$

Let $F(x) := F_0 + \sum_{i=1}^m x_i F_i$, $\hat{F}(x) = \sum_{i=1}^m x_i F_i$, $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$. Then $\hat{F}$ is a linear operator from $\mathbb{R}^m$ to S_n and its dual is defined by $\hat{F}^*(Z) = (Tr[F_1 Z], \cdots, Tr[F_m Z])$ for any $Z \in S_n$. Let $A := \{x \in \mathbb{R}^m \mid F(x) \in S\}$.

In passing, observe that the model problem (SDP) covers a wide class of constrained semidefinite programming problems. For instance, consider the following convex semidefinite programming problem

$$\text{Minimize } g(x)$$
$$\text{subject to } x \in C,\ F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0,$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is a continuous convex function and $C \subset \mathbb{R}^m$ is a closed convex set. Defining a convex function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ by $f(x) = g(x) + \delta_C(x)$, the convex semidefinite program can be reformulated in the form of (SDP).

We now obtain a sequential characterization of optimality for (SDP). A complete characterization of optimality in the case, where f is a *real-valued* convex function, can be found in [14]. Let

$$B := \bigcup_{(Z,\varepsilon)\in S\times\mathbb{R}_+} \begin{pmatrix} -\hat{F}^*(Z) \\ Tr[ZF_0] + \varepsilon \end{pmatrix}.$$

Then, B is a convex cone. We show that Epi $\delta_A^* =$cl(B).

Lemma 4.1. *Let $F_i \in S_n$, for $i = 0, 1, \cdots, m$. Suppose that $A := \{x \in \mathbb{R}^m \mid F(x) \in S\}$ is non-empty. Then* Epi $\delta_A^* = cl(B)$.

Proof. Suppose that $(v, \delta) \in$cl(B). Then there exist sequences $\{Z_n\} \subset S$ and $\{\varepsilon_n\} \subset \mathbb{R}_+$ such that:

$$v = \lim_{n\to\infty} -\hat{F}^*(Z_n) \quad \text{and} \quad \delta = \lim_{n\to\infty} Tr[Z_n F_0] + \varepsilon_n.$$

Let $x \in A$. Then

$$\begin{aligned} v^T x - \delta &= \lim_{n\to\infty} -\hat{F}^*(Z_n)^T x - Tr[Z_n F_0] - \varepsilon_n \\ &= \lim_{n\to\infty} -Tr[Z_n F(x)] - \varepsilon_n \\ &\leq 0, \end{aligned}$$

thus, $\delta_A^*(v) = \sup_{x\in A} v^T(x) \leq \delta$, and so, $(v, \delta) \in$ Epi δ_A^*.

Conversely, let $(v, \delta) \in$ Epi δ_A^*. We will show that $(v, \delta) \in$cl(B). Suppose, to the contrary, that

$$(v, \delta) \notin \text{cl}\left(\bigcup_{(Z,\varepsilon)\in S\times\mathbb{R}_+} \begin{pmatrix} -\hat{F}^*(Z) \\ Tr[ZF_0] + \varepsilon \end{pmatrix} \right).$$

Since A is non-empty, it follows that $(0, -1) \notin$ cl(B) (see [15]). So, for each $\alpha \in [0, 1]$, $\alpha(v, \delta) + (1-\alpha)(0, -1) \notin$ cl(B). Otherwise, there exist $\alpha_0 \in (0, 1)$ and sequences $\{Z_n\} \subset S$ and $\varepsilon_n \subset \mathbb{R}_+$ such that:

$$\alpha_0 v = \lim_{n\to\infty} -\hat{F}^*(Z_n) \quad \text{and} \quad \alpha_0\delta - (1-\alpha_0) = \lim_{n\to\infty} Tr[Z_n F_0] + \varepsilon_n.$$

So,

$$v = \lim_{n\to\infty} -\hat{F}^*(\frac{Z_n}{\alpha_0}) \quad \text{and} \quad \delta = \lim_{n\to\infty} Tr[\frac{Z_n}{\alpha_0} F_0] + \frac{\varepsilon_n}{\alpha_0} + \frac{(1-\alpha_0)}{\alpha_0}.$$

This gives us that $(v, \delta) \in$ cl(B) which is a contradiction. Hence, $L \cap$ cl$(B) = \emptyset$, where $L := \{\alpha(v, \delta) + (1-\alpha)(0, 1) : \alpha \in [0, 1]\}$ is the convex and compact line segment in $\mathbb{R}^m \times \mathbb{R}$. Then, by a separation theorem [12], there exists $(\bar{x}, \beta) \in \mathbb{R}^m \times \mathbb{R}$ such that for each $\alpha \in [0, 1]$,

$$[\alpha(v, \delta) + (1-\alpha)(0, -1)]^T (\bar{x}, \beta) < 0$$

and

$$u^T \bar{x} + \gamma\beta \geq 0, \;\; \forall (u, \gamma) \in \text{cl}(B).$$

By letting $\alpha = 0$, we get $\beta > 0$. By letting $\alpha = 1$ we obtain the inequalities

$$v^T \bar{x} + (\delta)\beta < 0$$

and

$$u^T \bar{x} + \gamma\beta \geq 0, \ \ \forall (u, \gamma) \in \mathrm{cl}(B).$$

Let $x' = \frac{-\bar{x}}{\beta}$. Then for each $(u, \gamma) \in \mathrm{cl}(B)$, $u^T x' - \gamma \leq 0$. So for each $Z \in S$,

$$Tr[Z(\sum_{i=1}^{m} F_i x'_i + F_0)] = \hat{F}^*(Z)^T x' + Tr[ZF_0] \geq 0.$$

This gives us $F(x') \succeq 0$. But $v^T \bar{x} + (\delta)\beta = v^T(-\beta x') + (\delta)\beta < 0$. So, $-v^T x' + \delta < 0$, since $\beta > 0$. This is a contradiction. □

We now derive a complete characterization of optimality of (SDP) as a consequence of Theorem 3.1 and Lemma 4.1.

Theorem 4.2. *For the problem (SDP), let $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function and, for $i = 0, 1, \cdots, m$, $F_i \in S_n$. Let $a \in A \cap \mathrm{dom}\, f$. Then the point a is a minimizer of (SDP) if and only if there exist sequences $\{\varepsilon_n\} \downarrow 0$, $\{Z_n\} \subset S$ and $\{u_n\} \subset \partial_{\varepsilon_n} f(a)$ such that*

$$u_n - \hat{F}^*(Z_n) \to 0 \quad \textit{and} \quad Tr[Z_n F(a)] \to 0 \quad \textit{as } n \to \infty.$$

Proof. $[\Longrightarrow]$. Assume that a is a minimizer of (SDP). Then, by Theorem 3.2 and Lemma 4.1, there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $\{u_n\}, \{v_n\} \subset X'$, $\{\beta_n\} \subset \mathbb{R}$ with $u_n \in \partial_{\varepsilon_n} f(a)$ and $(v_n, \beta_n) \in B$, such that $u_n + v_n \to 0$ and $\beta_n + u_n(a) \to 0$ as $n \to \infty$, where $B = \bigcup_{(Z,\varepsilon) \in S \times \mathbb{R}_+} \begin{pmatrix} -\hat{F}^*(Z) \\ Tr[ZF_0] + \varepsilon \end{pmatrix}$. So, there exist sequences $\{Z^n\} \subset S$ and $\{\gamma_n\} \subset \mathbb{R}_+$ such that $v_n = -\hat{F}^*(Z_n)$ and $\beta_n = Tr[Z_n F_0] + \gamma_n$. Thus,

$$\begin{aligned} u_n(a) + \beta_n &= u_n(a) + v_n(a) + \beta_n - v_n(a) \\ &= u_n(a) + v_n(a) + Tr[Z_n F_0] + \hat{F}^*(Z_n)(a) + \gamma_n \\ &= u_n(a) + v_n(a) + Tr[Z_n F(a)] + \gamma_n. \end{aligned}$$

Now, passing to the limit as $n \to \infty$, we see that $\lim_{n\to\infty}(Tr[Z_n F(a)) + \gamma_n) = 0$. This gives us that $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} Tr[Z_n F(a) = 0$ as $\{\gamma_n\}$ and $\{Tr[Z_n F(a)\} \subset \mathbb{R}_+$.

Conversely, suppose that there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $\{Z_n\} \subset S, \{u_n\} \subset \partial_{\varepsilon_n} f(a)$ such that

$$u_n - \hat{F}^*(Z_n) \to 0 \quad \text{and} \quad Tr[Z_n F](a)] \to 0 \quad \text{as } n \to \infty.$$

Since $u_n \in \partial_{\varepsilon_n} f(a)$ and $(-\hat{F}^*(Z_n), Tr[Z_n F_0]) \in B \subset \mathrm{Epi}\ \delta_A^*$, for each $x \in A$,

$$u_n(x) - u_n(a) \leq f(x) - f(a) + \varepsilon_n$$

and

$$-\hat{F}^*(Z_n)(x) - Tr[Z_n F_0] \leq 0.$$

Thus, for each $x \in A$,

$$u_n(x) - \hat{F}^*(Z_n)(x) - u_n(a) + \hat{F}^*(Z_n)(a) - Tr[Z_n F(a)] \leq f(x) - f(a) + \varepsilon_n.$$

Passing to the limit as $n \to \infty$, we obtain that $f(x) - f(a) \geq 0$ for each $x \in A$. □

Example 4.3. Consider the problem

$$(P) \qquad \min f(x)$$
$$\text{subject to} \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succcurlyeq 0,$$

where $f(x) = \begin{cases} -\sqrt{x_1} + |x_2| & \text{if } x_1 \geq 0, x_2 \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$

Let $X = \mathbb{R}^2$ and let

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $F(x) = F_0 + x_1F_1 + x_2F_2$, $x = (x_1, x_2) \in \mathbb{R}^2$. The feasible set of (P) is

$$A = \{x \in \mathbb{R}^2 \mid F(x) \succcurlyeq 0\} = \{(0, x_2) \mid x_2 \geq -1\}.$$

Clearly, $a = (0, 0)$ is the minimizer of (P).

Let $Z_n = \begin{pmatrix} n^3 & -\frac{n}{2} & 0 \\ -\frac{n}{2} & \frac{1}{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\epsilon_n = \frac{1}{\sqrt{n}}$. Clearly, $Z_n \in S$ for any $n \geq 1$ and $Tr[Z_nF_0] = \frac{1}{n}, Tr[Z_nF_1] = -n, Tr[Z_nF_2] = \frac{1}{n}$. Thus, $\widehat{F}^*(Z_n) = (Tr[Z_nF_1], Tr[Z_nF_2]) = (-n, \frac{1}{n})$. Moreover, $(-n, 0) \in \partial_{\epsilon_n} f(a)$,

$$\left(\text{since} \begin{array}{ll} -nx_1 \leq -x_1 \leq -\sqrt{x_1} + \epsilon_n, & \text{if } x_1 > 1; \\ -nx_1 \leq -1 \leq -\sqrt{x_1} < -\sqrt{x_1} + \epsilon_n, & \text{if } \frac{1}{n} \leq x_1 \leq 1; \\ -nx_1 \leq 0 < -\sqrt{x_1} + \epsilon_n, & \text{if } 0 \leq x_1 < \frac{1}{n}; \end{array} \text{ and } 0 \leq |x_2| \right).$$

Let $u_n = (-n, 0) \in \partial_{\epsilon_n} f(a)$. Then $u_n - \widehat{F}^*(Z_n) = (0, -\frac{1}{n}) \to (0, 0)$ and $Tr[Z_nF(a)] = Tr[Z_nF_0] = \frac{1}{n} \to 0$ as $n \to \infty$. Since $\partial f(a) = \emptyset$, for any $Z_k \subset S$, $\lim_{n \longrightarrow \infty} \hat{F}^*(Z_n) \notin \partial f(a)$. Note that there is no $x_0 \in \mathbb{R}^2$ such that $F(x_0)$ is positive definite.

The following Corollary extends the corresponding result, given recently in [14], where f is a real-valued continuous convex function.

Corollary 4.4. *For the problem (SDP), let $a \in A \cap \operatorname{dom} f$. Suppose that* $\operatorname{Epi} f^* + \operatorname{Epi} \delta_A^*$ *is closed. Then the point a is a minimizer of (SDP) if and only if there exists a sequence $\{Z_n\} \subset S$ such that*

$$\lim_{n \longrightarrow \infty} \hat{F}^*(Z_n) \in \partial f(a) \quad \text{and} \quad \lim_{n \longrightarrow \infty} Tr[Z_nF(a)] = 0.$$

Proof. By Proposition 3.7, we know that a is a minimizer of (SDP) if and only if $0 \in \partial f(a) + N_A(a)$; thus, a is a minimizer of (SDP) if and only if exists a $u \in \partial f(a)$ such that $-u \in N_A(a)$, which means that $F(x) \succeq 0 \Rightarrow u(x) \geq u(a)$. Now, from Lemma 4.1, $-u \in N_A(a)$ is equivalent to the inclusion

$$\begin{pmatrix} -u \\ -u(a) \end{pmatrix} \in \operatorname{cl} \left(\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \begin{pmatrix} -\hat{F}^*(Z) \\ Tr[ZF_0] + \delta \end{pmatrix} \right).$$

This gives us that there exist $\{Z_n\} \subset S$ and $\{\delta_n\} \subset \mathbb{R}_+$ such that

$$u = \lim_{n \longrightarrow \infty} \hat{F}^*(Z_n)$$

$$u(a) = \lim_{n \longrightarrow \infty} (-Tr[Z_n F_0] - \delta_n).$$

These two equations yield that $\lim_{n \longrightarrow \infty} \hat{F}^*(Z_n) \in \partial f(a)$ and $\lim_{n \longrightarrow \infty}(Tr[Z_n F(a)] + \delta_{n.}) = 0$. Since for each n, $Tr[Z_n F(a)] \geq 0$ and $\delta_{n.} \geq 0$, we get $\lim_{n \longrightarrow \infty} Tr[Z_n F(a)] = 0$. $\square$

Note that B is closed if there exists $x_0 \in \mathbb{R}^m$ such that $F(x_0)$ is positive definite. For details, see [15].

Corollary 4.5. *For the problem (SDP), let $a \in A \cap \operatorname{dom} f$. Suppose that there exists $x_0 \in \operatorname{dom} f$ such that $F(x_0)$ is positive definite. Then the point a is a minimizer of (SDP) if and only if there exists $Z \subset S$ such that*

$$\hat{F}^*(Z) \in \partial f(a) \quad \text{and} \quad Tr[ZF(a)] = 0.$$

Proof. [$\Longrightarrow$]. Assume that a is a minimizer of (SDP). By the assumption B is closed and $(\operatorname{Epi} f^* + \operatorname{Epi} \delta_A^*)$ is closed. So, by Proposition 3.7, there exists $u \in \partial f(a)$ such that $-u \in N_A(a)$. Now, from Lemma 4.1, $-u \in N_A(a)$ is equivalent to

$$\begin{pmatrix} -u \\ -u(a) \end{pmatrix} \in \bigcup_{(Z,\gamma) \in S \times \mathbb{R}_+} \begin{pmatrix} -\hat{F}^*(Z) \\ Tr[ZF_0] + \gamma \end{pmatrix}.$$

So, there exist $Z \subset S$ and $\gamma \subset \mathbb{R}_+$ such that

$$u = \hat{F}^*(Z);$$

$$u(a) = -Tr[ZF_0] - \gamma.$$

This gives us that $\hat{F}^*(Z) \in \partial f(a)$ and $Tr[ZF(a)] = 0$.

Conversely, if $\hat{F}^*(Z) \in \partial f(a)$ and $Tr[ZF(a)] = 0$, for some $Z \in S$, then from the definition of $\partial f(a)$, we obtain that

$$f(x) \geq f(a) + \hat{F}^*(Z)(x - a) = f(a) + \hat{F}^*(Z)(x) + Tr[ZF_0] - Tr[ZF_0] - \hat{F}^*(Z)(a).$$

This gives us that for each $x \in A$,

$$f(x) \geq f(a) + Tr[ZF(x)] - Tr[ZF(a)] \geq f(a) - Tr[ZF(a)] = f(a).$$

$\square$

References

[1] H. Attouch, H. Brézis: Duality for the sum of convex functions in general Banach spaces, in: Aspects of Mathematics and its Applications, J. A. Barroso (ed.), North-Holland, Amsterdam (1986) 125–133.

[2] S. Boyd, L. Vandenberghe: Convex Optimization, Cambridge University Press, Cambridge (2004).

[3] A. Brondsted, R. T. Rockafellar: On the subdifferential of convex functions, Proc. Amer. Math. Soc. 16 (1965) 605–611.

[4] R. S. Burachik, V. Jeyakumar: A simple closure condition for the normal cone intersection formula, Proc. Amer. Math. Soc. 133 (2005) 1741–1748.

[5] R. S. Burachik, V. Jeyakumar: A dual condition for the subdifferential sum formula and applications, J. Convex Analysis 12 (2005) 279–290.

[6] R. S. Burachik, V. Jeyakumar: A new geometric condition for Fenchel's duality in infinite dimensional spaces, Math. Program. B 104 (2005) 229–233.

[7] F. Deutsch, W. Li, J. Ward: A dual approach to constrained interpolation from a convex subset of Hilbert space, J. Approx. Theory 90 (1997) 385–444.

[8] F. Deutsch, W. Li, J. D. Ward: Best approximation from the intersection of a closed convex set and a polyhedron in Hilbert space, weak Slater conditions, and the strong conical hull intersection property, SIAM J. Optim. 10 (1999) 252–268.

[9] S. P. Fitzpatrick, S. Simons: The conjugates, compositions and marginals of convex functions, J. Convex Analysis 8 (2001) 423–446.

[10] J. B. Hiriart-Urruty, C. Lemarechal: Convex Analysis and Minimization Algorithms, Volumes I and II, Springer, Berlin (1993).

[11] J.-B. Hiriart-Urruty, R. R. Phelps: Subdifferential calculus using ε-subdifferentials, J. Funct. Anal. 118 (1993) 154–166.

[12] R. B. Holmes: Geometric Functional Analysis, Springer, Berlin (1975).

[13] V. Jeyakumar: Characterizing set containments involving infinite convex constraints and reverse-convex constraints, SIAM J. Optim. 13 (2003) 947–959.

[14] V. Jeyakumar, G. M. Lee, N. Dinh: New sequential Lagrange multiplier conditions characterizing optimality without constraint qualifications for convex programs, SIAM J. Optim. 14 (2003) 534–547.

[15] V. Jeyakumar, M. Nealon: Complete characterizations of optimality for convex semidefinite programming, in: Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999), M. Théra (ed.), CMS Conf. Proc. 27, Amer. Math. Soc., Providence (2000) 165–173.

[16] B. N. Pshenichnyi: Necessary Conditions for an Extremum, Marcel Dekker, New York (1971).

[17] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[18] T. Strömberg: The operation of infimal convolution, Diss. Math. 352 (1996) 1–61.

[19] L. Thibault: Sequential convex subdifferential calculus and sequential Lagrange multipliers, SIAM J. Control Optim. 35 (1997) 1434–1444.

[20] L. Thibault: Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential, in: Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999), M. Théra (ed.), CMS Conf. Proc. 27, Amer. Math. Soc., Providence (2000) 279–289.

[21] M. J. Todd: Semidefinite optimization, Acta Numerica 10 (2001) 515–560.

[22] L. Vandenberghe, S. Boyd: Semidefinite programming, SIAM Rev. 38 (1996) 49–95.

[23] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).