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The Fitzpatrick Function and Nonreflexive Spaces

S. Simons
*Department of Mathematics, University of California,
Santa Barbara, CA 93106-3080, USA
simons@math.ucsb.edu*

Dedicated to the memory of Simon Fitzpatrick, who was a scholar and a gentleman.

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In this paper, we show how Fitzpatrick functions can be used to obtain various results on the local boundedness, domain and surjectivity of monotone and maximal monotone multifunctions on a Banach space, and also to clarify the relationships between different subclasses of the set of maximal monotone multifunctions.

Introduction

In this paper, we show how Fitzpatrick functions can be used to obtain a number of results on maximal monotone multifunctions on (possibly nonreflexive) Banach spaces. We give the definition and the basic properties of Fitzpatrick functions in Section 1. That section also introduces Gossez's extension, $\overline{S}$, of a monotone multifunction S, and the concept of "type (NI)".

Several of the results in this paper were already established in [22] and [24], but using a minimax theorem and the "free convexification" of a multifunction. The use of the Fitzpatrick function and convex analysis provides much shorter and simpler proofs, under the assumption that one has sufficiently refined convex analysis tools. One such tool is the rather bizarre result on norm $\times$ weak* lower semicontinuous functions proved in Lemma 2.2, and another is the extension of the Attouch–Brezis version of the Fenchel duality theorem given in Theorem 4.1.

In fact, the mathematics in this paper follows two totally independent lines of development. The first goes by way of Lemma 2.2, Theorem 3.1, Theorem 3.6 and Theorem 3.8, while the second goes by way of Theorem 4.1, Theorem 4.5, Theorem 5.1, Theorem 6.6, Theorem 7.2, Theorem 8.2 and Theorem 9.3.

Section 2 is devoted to Lemma 2.2 referred to above and, in Section 3, we give applications of Lemma 2.2 to the local boundedness of monotone multifunctions on a Banach space and also give new proofs of the "six set theorem" and the "nine set theorem" originally established in [22, Section 18] for maximal monotone multifunctions on a Banach space.

Section 4 starts with the extension of the Attouch–Brezis theorem referred to above, and gives a number of consequences. Theorem 4.5 is the form which is most convenient for our applications to maximal monotone multifunctions of type (NI) in Section 5. Theorem 4.5 follows by bootstrapping Lemma 4.4 using some simple changes of variable.

Section 5 is devoted to the single result, Theorem 5.1. Here we suppose that S is maximal

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monotone of type (NI) and obtain sufficient conditions for $(w, w^*) \in G(S + \partial g)$ and $(w^*, w) \in G(S^{-1} + \partial g^{-1})$, where g is an appropriate proper, convex lower semicontinuous function. These conditions involve the extension $\overline{S}$ of S introduced in Section 1. For the record, we note that the multifunction $\left(S^{-1} + \partial g^{-1}\right)^{-1}$ is sometimes known as the "parallel sum" of S and ∂g.

Various subclasses of the set of maximal monotone multifunctions are introduced in Sections 6–8. There is a full discussion of their properties in the relevant sections; however, here is an overview. The smallest general class of maximal monotone multifunctions that we consider will be those of type (ED). However, this class is large enough to include the subdifferentials of all proper convex lower semicontinuous functions on a general Banach space, and all maximal monotone multifunctions on a reflexive space. Many of the known results about these functions are summed up in the following two charts relating to the classes in general:

$$\begin{tikzcd}
 & \text{(ED)} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{(FPV)} & \text{(D)} \arrow[dl] \arrow[dr] & \text{Strongly maximal} \\
\text{(FP)} & & \text{(NI)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{``Tail'' operator} \arrow[d] \arrow[dr, "/" marking] & \\
\text{Linear and maximal monotone} \arrow[d] & \text{(D)} \arrow[d] \\
\text{(FPV)} & \text{Strongly maximal;}
\end{tikzcd}$$

and the following two charts relating to their domains and ranges:

$$\begin{tikzcd}
\text{Surjective} \arrow[d] \\
\text{(FP)} \arrow[d] \\
\text{Range has convex closure}
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Full domain} \arrow[d] \\
\text{(FPV)} \arrow[d] \\
\text{Domain has convex closure.}
\end{tikzcd}$$

In Section 6, we introduce maximal monotone multifunctions of type (D) and (ED) and give more sufficient conditions in Theorem 6.6 and Corollary 6.7 for $(w, w^*) \in G(S + \partial g)$ and $(w^*, w) \in G(S^{-1} + \partial g^{-1})$, where g is an appropriate proper, convex lower semicontinuous function. Unlike the results of Section 5, these conditions involve S directly rather than $\overline{S}$.

In Section 7, we introduce maximal monotone multifunctions of type (FP) (locally maximal monotone) and (FPV) (maximal monotone locally) and use Theorem 6.6 to prove that maximal monotone multifunctions of type (ED) are of type (FPV), and maximal monotone multifunctions of type (D) are of type (FP). Section 7 concludes with a number of open problems.

In Section 8, we introduce strong maximality, and use Corollary 6.7 to prove that every maximal monotone multifunction of type (ED) is strongly maximal. In Section 9, we first give in Theorem 9.2 a proof using Fitzpatrick functions that a maximal monotone multifunction with bounded range has full domain. Using Corollary 6.7 again, we then prove in Corollary 9.4 that any maximal monotone multifunction of type (ED) with coercive inverse has full domain. This implies the known result that any coercive maximal monotone multifunction on a reflexive space is surjective.

1. The Fitzpatrick function and type (NI)

We start off by recalling some standard notation from convex analysis. If F is a normed space and $f\colon\ F \mapsto]-\infty,\infty]$ then we write $\operatorname{dom} f := \{x \in F\colon\ f(x) \in \mathbb{R}\}$. f is said to be *proper* if $\operatorname{dom} f \neq \emptyset$. If f is proper and convex then the *conjugate function* of f is the convex function $f^*\colon\ F^* \mapsto]-\infty,\infty]$ defined by $f^*(x^*) := \sup_F[x^* - f]\quad (x^* \in F^*)$. If f is lower semicontinuous then f^* is proper, and we then write f^{**} for the conjugate function of f^*, so $f^{**}\colon\ F^{**} \mapsto]-\infty,\infty]$ and $f^{**}(x^{**}) := \sup_{F^*}[x^{**} - f^*]\quad (x^{**} \in F^{**})$. We should warn the reader that our notation is not the same as that used in [28], to which we will refer later. There, the conjugate function is defined with reference to a pair of spaces. This does not affect the definition of f^*, but f^{**} is then defined on F rather than on F^{**}.

We now assume that E is a nonzero real (not necessarily reflexive) normed space and E^* is its topological dual space. We norm $E \times E^*$ by $\|(x,x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$. Then the topological dual of $E \times E^*$ is $E^* \times E^{**}$, under the pairing $\langle (x,x^*),(u^*,u^{**})\rangle := \langle x,u^*\rangle + \langle x^*,u^{**}\rangle$. Further, $\|(u^*,u^{**})\| = \sqrt{\|u^*\|^2 + \|u^{**}\|^2}$. We always use $\widehat{\ }$ for the canonical map from a normed space into its bidual. Let $S\colon\ E \rightrightarrows E^*$ be monotone with graph $G(S) := \{(x,x^*) \in E \times E^*\colon\ x^* \in Sx\} \neq \emptyset$. We define $\psi_S\colon\ E \times E^* \mapsto]-\infty,\infty]$ by

$$\psi_S(x,x^*) := \sup_{(s,s^*)\in G(S)} \langle x - s, s^* - x^*\rangle,$$

and the *Fitzpatrick function* $\varphi_S\colon\ E \times E^* \mapsto]-\infty,\infty]$ associated with S by

$$\varphi_S(x,x^*) := \sup_{(s,s^*)\in G(S)} [\langle s,x^*\rangle + \langle x,s^*\rangle - \langle s,s^*\rangle] = \psi_S(x,x^*) + \langle x,x^*\rangle. \tag{1}$$

(The function φ_S was introduced by Fitzpatrick in [5, Definition 3.1, p. 61] under the notation L_S.) The monotonicity of S and (1) imply that

$$(x,x^*) \in G(S) \quad\Longrightarrow\quad \psi_S(x,x^*) = 0 \iff \varphi_S(x,x^*) = \langle x,x^*\rangle, \tag{2}$$

and so φ_S is proper, convex and lower semicontinuous.

We use the notation pr_E and pr_{E^*} to stand for the projections of $E \times E^*$ onto E and E^*, respectively. We also use the standard notation $D(S) := \{x \in E\colon\ Sx \neq \emptyset\} = \mathrm{pr}_E G(S)$ and $R(S) := \bigcup_{x\in E} Sx = \mathrm{pr}_{E^*} G(S)$. (2) implies that that $G(S) \subset \operatorname{dom}\varphi_S$, from which $D(S) \subset \mathrm{pr}_E \operatorname{dom}\varphi_S$ and $R(S) \subset \mathrm{pr}_{E^*} \operatorname{dom}\varphi_S$. Since $\operatorname{dom}\varphi_S$ is convex, it follows that

$$D(S) \subset \operatorname{co} D(S) \subset \mathrm{pr}_E \operatorname{dom}\varphi_S \quad\text{and}\quad R(S) \subset \operatorname{co} R(S) \subset \mathrm{pr}_{E^*} \operatorname{dom}\varphi_S, \tag{3}$$

where "co" stands for "convex hull".

Define $\widetilde{S}\colon\ E^* \rightrightarrows E^{**}$ by $G(\widetilde{S}) := \{(s^*,\widehat{s})\colon\ (s,s^*) \in G(S)\}$. $\widetilde{S}$ is also monotone. Clearly, for all $(x,x^*) \in E \times E^*$,

$$\psi_{\widetilde{S}}(x^*,\widehat{x}) = \psi_S(x,x^*) \quad\text{and}\quad \varphi_{\widetilde{S}}(x^*,\widehat{x}) = \varphi_S(x,x^*). \tag{4}$$

Let $(z^*, z^{**}) \in E^* \times E^{**}$. Then we see from (2) that

$$\left.\begin{aligned} \varphi_{\widetilde{S}}(z^*, z^{**}) &= \sup_{(s,s^*)\in G(S)} [\langle s^*, z^{**}\rangle + \langle z^*, \widehat{s}\rangle - \langle s^*, \widehat{s}\rangle] \\ &= \sup_{(s,s^*)\in G(S)} [\langle s, z^*\rangle + \langle s^*, z^{**}\rangle - \varphi_S(s, s^*)] \\ &\le \sup_{(y,y^*)\in E\times E^*} [\langle y, z^*\rangle + \langle y^*, z^{**}\rangle - \varphi_S(y, y^*)] \\ &= \sup_{(y,y^*)\in E\times E^*} [\langle (y, y^*), (z^*, z^{**})\rangle - \varphi_S(y, y^*)] = {\varphi_S}^*(z^*, z^{**}). \end{aligned}\right\} \tag{5}$$

Under certain circumstances, we have an inequality in the reverse direction for ${\varphi_S}^*$. More precisely,

$$(s, s^*) \in G(S) \quad\Longrightarrow\quad {\varphi_S}^*(s^*, \widehat{s}) \le \langle s, s^*\rangle. \tag{6}$$

To see this, let $(s, s^*) \in G(S)$. Then, for all $(y, y^*) \in E \times E^*$, the definition of $\varphi_S(y, y^*)$ yields $\varphi_S(y, y^*) \ge \langle y, s^*\rangle + \langle s, y^*\rangle - \langle s, s^*\rangle = \langle (y, y^*), (s^*, \widehat{s})\rangle - \langle s, s^*\rangle$, from which

$$\langle (y, y^*), (s^*, \widehat{s})\rangle - \varphi_S(y, y^*) \le \langle s, s^*\rangle,$$

and (6) follows by taking the supremum over $(y, y^*) \in E \times E^*$.

Monotone multifunctions of "type (NI)" ("NI" stands for "negative infimum") were introduced in [21, Definition 10, p. 183], motivated by some questions about the range of maximal monotone operators in nonreflexive spaces. Formally, S is *of type (NI)* if

$$(z^*, z^{**}) \in E^* \times E^{**} \quad\Longrightarrow\quad \inf_{(s,s^*)\in G(S)} \langle s^* - z^*, \widehat{s} - z^{**}\rangle \le 0.$$

This is clearly equivalent to the statement that $\psi_{\widetilde{S}} \ge 0$ on $E^* \times E^{**}$.

For the remainder of this section, we suppose that S is maximal monotone. In this case, (2) can be strengthened to the two statements

$$(x, x^*) \in E \times E^* \quad\Longrightarrow\quad \psi_S(x, x^*) \ge 0 \iff \varphi_S(x, x^*) \ge \langle x, x^*\rangle \tag{7}$$

and

$$\psi_S(x, x^*) = 0 \iff \varphi_S(x, x^*) = \langle x, x^*\rangle \iff (x, x^*) \in G(S). \tag{8}$$

(See [5, Corollary 3.9, p. 62].) Combining (4), (5) and (7) yields

$$(x, x^*) \in E \times E^* \quad\Longrightarrow\quad {\varphi_S}^*(x^*, \widehat{x}) \ge \langle x, x^*\rangle, \tag{9}$$

and combining (4), (5), (7) and (8) yields

$${\varphi_S}^*(x^*, \widehat{x}) \le \langle x, x^*\rangle \quad\Longrightarrow\quad (x, x^*) \in G(S). \tag{10}$$

$\widetilde{S}$ is also closely related to the multifunction $\overline{S}$ introduced by Gossez in [8]. Precisely, $\overline{S}$ is the multifunction from E^{**} into E^* defined by

$$G(\overline{S}) = \{(z^{**}, z^*) \in E^{**} \times E^*\colon\ \psi_{\widetilde{S}}(z^*, z^{**}) \le 0\}.$$

Combining this with (4), (7) and (8) gives

$$w \in E \quad\Longrightarrow\quad \overline{S}(\widehat{w}) = S(w). \tag{11}$$

It is this implication that allows us to describe $\overline{S}$ as an *extension* of S. It is also clear from the above considerations that if E is reflexive then S is of type (NI). We shall give other less trivial examples of maximal monotone multifunctions of type (NI) in Section 6.

The reader may ask why we have introduced both the functions φ_S and ψ_S, which are so closely related. As observed in [26], the reason for this in the reflexive case is that φ_S is convex and lower semicontinuous (while ψ_S is generally neither). On the other hand, as noted in (7), ψ_S is positive (while φ_S is generally not). Unfortunately, the positivity of ψ_S does not seem to be adequate in the nonreflexive case: what seems to be needed is the positivity of $\psi_{\widetilde{S}}$. This is the reason for the introduction of "type (NI)". This point is illustrated quite well by Theorem 5.1. The positivity of $\psi_{\widetilde{S}}$ is critical for the manipulations in Theorem 5.1. On the other hand, it is the convexity and lower semicontinuity of φ_S that enable us (in Theorem 4.5) to establish (24) and (25).

It is worth pointing out that the interest of the class of multifunctions of type (NI) is just as much the insight that it gives on other classes of multifunctions as its interest as a class of multifunctions in its own right.

2. The $\mathcal{T}_{\| \ \|} \times\ w(E^*, E)$ topology on $E \times E^*$

We start off by stating Rockafellar's version of the Fenchel duality theorem for locally convex spaces.

Theorem 2.1. *Let F be a nonzero Hausdorff real locally convex space with topological dual F^*, $f\colon\ F \mapsto]-\infty, \infty]$ be proper and convex, $g\colon\ F \mapsto \mathbb{R}$ be convex and continuous and $f + g \geq 0$ on F. Then there exists $z^* \in F^*$ such that $f^*(z^*) + g^*(-z^*) \leq 0$.*

Proof. See Rockafellar, [16, Theorem 1, pp. 82–83] or Zălinescu, [28, Theorem 2.8.3(iii), pp. 123–124]. □

Now let E be a nonzero Banach space and $\mathcal{M}$ be the $\mathcal{T}_{\| \ \|} \times\ w(E^*, E)$ topology on $E \times E^*$. ("$\mathcal{M}$" stands for "mixed".) Then $(E \times E^*, \mathcal{M})$ is a Hausdorff locally convex space with topological dual $E \times E^*$ under the pairing $\lfloor (x, x^*), (y, y^*) \rfloor := \langle x, y^* \rangle + \langle y, x^* \rangle$. Lemma 2.2 below will be applied in Theorem 3.1 and Lemma 3.5, and part of the argument of Lemma 2.2(e) will be used again in Lemma 9.1($\Longleftarrow$).

Lemma 2.2. *Let E be a nonzero Banach space and $\varphi\colon\ E \times E^* \mapsto]-\infty, \infty]$ be $\mathcal{M}$–lower semicontinuous. We write $^{@}$ for the operation of $\mathcal{M}$–conjugacy, so that $\varphi^{@}\colon\ E \times E^* \mapsto]-\infty, \infty]$ is defined by $\varphi^{@}(y, y^*) := \sup_{(x,x^*) \in E \times E^*}[\langle x, y^* \rangle + \langle y, x^* \rangle - \varphi(x, x^*)] = \varphi^*(y^*, \widehat{y})$.*

(a) *$\varphi^{@}$ is proper and convex.*

(b) *$\varphi = \varphi^{@@}$ on $E \times E^*$.*

(c) *Suppose that $s_0 \in E$ and $\mathrm{pr}_E \mathrm{dom}\, \varphi - s_0$ is absorbing. Then there exist $K > 0$ and $\eta \in]0, 1]$ such that*

$$\|s - s_0\| \leq \eta \text{ and } (y, y^*) \in E \times E^* \quad \Longrightarrow \quad \varphi^{@}(y, y^*) + K\|y - s\| - \langle s, y^* \rangle \geq \eta(\|y^*\| - K).$$

(d) *In the notation of (c),*

$$(s, s^*) \in E \times E^*, \|s - s_0\| \leq \eta \text{ and } \varphi^{@}(s, s^*) \leq \langle s, s^* \rangle \quad \Longrightarrow \quad \|s^*\| \leq K.$$

(e) *Suppose that* $s_0 \in E$, $\mathrm{pr}_E\mathrm{dom}\,\varphi - s_0$ *is absorbing and, in addition,*

$$(y, y^*) \in E \times E^* \quad\Longrightarrow\quad \varphi^{@}(y, y^*) \geq \langle y, y^* \rangle. \tag{12}$$

Then $s_0 \in \mathrm{int}\,\mathrm{pr}_E\{(x, x^*) \in E \times E^*\colon\ \varphi(x, x^*) \leq \langle x, x^* \rangle\}$.

Proof. (a) and (b) follow from the $\mathcal{M}$–lower semicontinuity of φ and the Fenchel–Moreau theorem for locally convex spaces (see Zălinescu, [28, Theorem 2.3.3, pp. 77–78]).

(c) We define the function $\varphi^{\div}\colon\ E \to]-\infty, \infty]$ by

$$\varphi^{\div}(x) := \sup\nolimits_{(y,y^*)\in E\times E^*} \frac{\langle x, y^* \rangle - \varphi^{@}(y, y^*)}{1 + \|y\|}.$$

Since $\varphi^{\div}$ is the supremum of a family of continuous affine functions and $\varphi^{@}$ is proper, $\varphi^{\div}$ is proper, convex and lower semicontinuous. We now prove that

$$\mathrm{pr}_E\mathrm{dom}\,\varphi \subset \mathrm{dom}\,\varphi^{\div}. \tag{13}$$

To this end, let x be an arbitrary element of $\mathrm{pr}_E\mathrm{dom}\,\varphi$. Then there exists $x^* \in E^*$ such that $\varphi(x, x^*) < \infty$. The Fenchel–Young inequality now implies that, for all $(y, y^*) \in E \times E^*$, $\langle x, y^* \rangle + \langle y, x^* \rangle \leq \varphi(x, x^*) + \varphi^{@}(y, y^*)$, from which $\langle x, y^* \rangle - \varphi^{@}(y, y^*) \leq \varphi(x, x^*) - \langle y, x^* \rangle \leq \varphi(x, x^*) + \|y\|\|x^*\| \leq (\varphi(x, x^*) \vee \|x^*\|)(1 + \|y\|)$. Thus $\varphi^{\div}(x) \leq \varphi(x, x^*) \vee \|x^*\| < \infty$, which completes the proof of (13). Our assumptions now imply that $\mathrm{dom}\,\varphi^{\div} - s_0$ is absorbing. Since $\varphi^{\div}$ is proper, convex and lower semicontinuous, it follows from Rockafellar, [17, Corollary 7C, p. 61] (see Moreau, [10, Proposition 5.f, p. 30] for a simpler proof of Rockafellar's result) that $\varphi^{\div}$ is continuous at s_0. Let $N := \varphi^{\div}(s_0) \vee 0 + 1$. Then there exists $\eta \in]0, 1]$ such that

$$x \in E \text{ and } \|x\| \leq 2\eta \quad\Longrightarrow\quad \varphi^{\div}(x + s_0) \leq N.$$

Let (y, y^*) be an arbitrary element of $E \times E^*$. Then the definition of $\varphi^{\div}$ gives

$$x \in E \text{ and } \|x\| \leq 2\eta \quad\Longrightarrow\quad \langle x + s_0, y^* \rangle - {\varphi_S}^{@}(y, y^*) \leq N(1 + \|y\|).$$

Taking the supremum over x: ${\varphi_S}^{@}(y, y^*) - \langle s_0, y^* \rangle + N(1 + \|y\|) \geq 2\eta\|y^*\|$, from which

$$\|s - s_0\| \leq \eta \quad\Longrightarrow\quad {\varphi_S}^{@}(y, y^*) - \langle s, y^* \rangle + N(2 + \|s_0\| + \|y - s\|) \geq \eta\|y^*\|.$$

We now obtain (c) with $K := (2 + \|s_0\|)N/\eta > N$.

(d) follows from (c) by setting $(y, y^*) = (s, s^*)$.

(e) Let $K > 0$ and $\eta \in (0, 1]$ be as in (c). We will show that

$$s \in E \text{ and } \|s - s_0\| \leq \eta \quad\Longrightarrow\quad s \in \mathrm{pr}_E\{(x, x^*) \in E \times E^*\colon\ \varphi(x, x^*) \leq \langle x, x^* \rangle\}, \tag{14}$$

which implies the desired result. So let $s \in E$ and $\|s - s_0\| \leq \eta$. Then, from (c),

$$(y, y^*) \in E \times E^* \quad\Longrightarrow\quad \varphi^{@}(y, y^*) + K\|y - s\| - \langle s, y^* \rangle \geq \eta(\|y^*\| - K). \tag{15}$$

We next show that

$$(y, y^*) \in E \times E^* \quad\Longrightarrow\quad \varphi^{@}(y, y^*) + K\|y - s\| - \langle s, y^* \rangle \geq 0. \tag{16}$$

To this end, let $(y, y^*) \in E \times E^*$. (16) is immediate from (15) if $\|y^*\| \geq K$. If, on the other hand, $\|y^*\| < K$ then, from (12),

$$\varphi^@(y, y^*) + K\|y - s\| - \langle s, y^* \rangle \geq K\|y - s\| + \langle y - s, y^* \rangle \geq K\|y - s\| - \|y - s\|\|y^*\| \geq 0,$$

which completes the proof of (16). (16) implies that $\varphi^@ + g \geq 0$ on $E \times E^*$, where $g\colon\ E \times E^* \mapsto \mathbb{R}$ is defined by $g(y, y^*) := K\|y - s\| - \langle s, y^* \rangle$. From Theorem 2.1, there exists $(z, z^*) \in E \times E^*$ such that $\varphi^{@@}(z, z^*) + g^@(-z, -z^*) \leq 0$. By direct computation,

$$g^@(-z, -z^*) = \begin{cases} -\langle s, z^* \rangle & \|z^*\| \leq K \text{ and } z = s; \\ \infty & \text{otherwise.} \end{cases}$$

Thus $\|z^*\| \leq K$, $z = s$ and, using (b),

$$\varphi(s, z^*) - \langle s, z^* \rangle = \varphi^{@@}(s, z^*) - \langle s, z^* \rangle = \varphi^{@@}(z, z^*) + g^@(-z, -z^*) \leq 0,$$

which gives (14) and completes the proof of (e). □

3. Local boundedness, and the relationship between $\mathrm{pr}_E \mathrm{dom}\, \varphi_S$ and $D(S)$

We recall that if E is a nonzero Banach space, $S\colon E \rightrightarrows E^*$ and $s_0 \in E$ then S is *locally bounded at* s_0 if there exist $\eta, K > 0$ such that

$$(s, s^*) \in G(S) \text{ and } \|s - s_0\| \leq \eta \quad \Longrightarrow \quad \|s^*\| \leq K.$$

Theorem 3.1. *Let E be a nonzero Banach space, $S\colon E \rightrightarrows E^*$ be monotone, $s_0 \in E$ and $\mathrm{pr}_E \mathrm{dom}\, \varphi_S - s_0$ be absorbing. Then S is locally bounded at s_0.*

Proof. This is immediate from (6) and Lemma 2.2(d) with $\varphi = \varphi_S$. □

In order to proceed further with this analysis, we need to introduce some additional notation.

Definition 3.2. We write "$x \in \mathrm{sur}\, A$" and say that "A surrounds x" if, for each $w \in E \setminus \{0\}$, there exists $\delta > 0$ such that $x + \delta w \in A$. The statement "$x \in \mathrm{sur}\, A$" is related to x being an "absorbing point" of A (see Phelps, [12, Definition 2.27(b), p. 28]), but differs in that we do not require that $x \in A$. We also note that, if A is convex then $\mathrm{sur}\, A \subset A$ and so $\mathrm{sur}\, A$ is identical with the "core" or algebraic interior of A. In particular:

$$\text{if } A \text{ is convex then} \qquad (0 \in \mathrm{sur}\, A \iff A \text{ is absorbing}). \tag{17}$$

Corollary 3.3. *Let E be a nonzero Banach space, $S\colon E \rightrightarrows E^*$ be monotone, $s_0 \in E$ and $\mathrm{co}\, D(S)$ surround s_0. Then S is locally bounded at s_0.*

Proof. (3) and (17) imply that $\mathrm{pr}_E \mathrm{dom}\, \varphi_S$ surrounds s_0 and then that the set $\mathrm{pr}_E \mathrm{dom}\, \varphi_S - s_0$ is absorbing. The result follows from Theorem 3.1. □

Remark 3.4. Corollary 3.3 extends the result proved by Borwein–Fitzpatrick in [4]. As an example, if $S\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is monotone and the four points $(\pm 1, \pm 1)$ are in $D(S)$ then Corollary 3.3 implies that S is locally bounded at 0 (even if $0 \notin D(S)$).

We observed in (3) that $D(S) \subset \mathrm{pr}_E \mathrm{dom}\, \varphi_S$. We now investigate the subtler relationship between these two sets when S is maximal monotone.

Lemma 3.5. *Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be maximal monotone. Then* $\mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S) \subset \mathrm{int}\, D(S)$.

Proof. Let s_0 be an arbitrary element of $\mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S)$. From (17), $\mathrm{pr}_E \mathrm{dom}\, \varphi_S - s_0$ is absorbing. We apply Lemma 2.2(e) with $\varphi = \varphi_S$, noting that (12) follows from (9), and deduce that $s_0 \in \mathrm{int}\, \mathrm{pr}_E\{(x, x^*) \in E \times E^*\colon\ \varphi_S(x, x^*) \le \langle x, x^*\rangle\}$. (7) and (8) now imply that $s_0 \in \mathrm{int}\, D(S)$. □

Theorem 3.6. *Let E be a nonzero Banach space and $S\colon E \rightrightarrows E^*$ be maximal monotone. Then*

$$\begin{aligned} &\mathrm{int}\, D(S) = \mathrm{int}\,(\mathrm{co}\, D(S)) = \mathrm{int}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S) \\ &= \mathrm{sur}\, D(S) = \mathrm{sur}\,(\mathrm{co}\, D(S)) = \mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S). \end{aligned}$$

Proof. From (3),

$$\mathrm{int}\, D(S) \subset \mathrm{int}\,(\mathrm{co}\, D(S)) \subset \mathrm{int}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S)$$

and

$$\mathrm{sur}\, D(S) \subset \mathrm{sur}\,(\mathrm{co}\, D(S)) \subset \mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S).$$

Obviously $\mathrm{int}\,(\dots) \subset \mathrm{sur}\,(\dots)$, and the result follows from Lemma 3.5. □

Remark 3.7. Following on from the comments of Remark 3.4, if $S\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is maximal monotone and the four points $(\pm 1, \pm 1)$ are in $D(S)$ then Theorem 3.6 implies that $]-1, 1[\, \times\,]-1, 1[\, \subset D(S)$ (even if we do not assume that $0 \in D(S)$).

Theorem 3.8. *Let E be a nonzero Banach space, $S\colon E \rightrightarrows E^*$ be maximal monotone and* $\mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S) \ne \emptyset$. *Then*

$$\begin{aligned} &\overline{D(S)} = \overline{\mathrm{co} D(S)} = \overline{\mathrm{pr}_E \mathrm{dom}\, \varphi_S} \\ &= \overline{\mathrm{int}\, D(S)} = \overline{\mathrm{int}\,(\mathrm{co}\, D(S))} = \overline{\mathrm{int}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S)} \\ &= \overline{\mathrm{sur}\, D(S)} = \overline{\mathrm{sur}\,(\mathrm{co}\, D(S))} = \overline{\mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S)}. \end{aligned}$$

Proof. Obviously, $\overline{\mathrm{int}\, D(S)} \subset \overline{D(S)}$ and, from (3), $\overline{D(S)} \subset \overline{\mathrm{co} D(S)} \subset \overline{\mathrm{pr}_E \mathrm{dom}\, \varphi_S}$. From Theorem 3.6, $\mathrm{int}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S) = \mathrm{sur}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S) \ne \emptyset$, hence (see, for instance, Kelly–Namioka, [9, 13.1(i), pp. 110–111]),

$$\overline{\mathrm{pr}_E \mathrm{dom}\, \varphi_S} = \overline{\mathrm{int}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S)}.$$

Thus we have

$$\overline{\mathrm{int}\, D(S)} \subset \overline{D(S)} \subset \overline{\mathrm{co} D(S)} \subset \overline{\mathrm{pr}_E \mathrm{dom}\, \varphi_S} = \overline{\mathrm{int}\,(\mathrm{pr}_E \mathrm{dom}\, \varphi_S)}.$$

The result now follows by combining this with Theorem 3.6. □

Remark 3.9. It was observed in Simons–Zălinescu, [26, Remark 5.6] that, in the notation of Simons, [22, Definition 15.1], $\mathrm{pr}_E \mathrm{dom}\, \varphi_S = \mathrm{dom}\, \chi_S$, so Theorems 3.6 and 3.8 are, in fact, identical with [22, Theorems 18.3 and 18.4].

4. Fenchel duality for functions of two variables

We start off by stating a generalization of the Attouch–Brezis version of the Fenchel duality theorem ([1]). We use the notation pr_E and pr_F to stand for the projections of $E \times F$ onto E and F, respectively.

Theorem 4.1. *Let E and F be Banach spaces, $\sigma,\ \tau\colon\ E \times F \mapsto]-\infty,\infty]$ be proper, convex and lower semicontinuous and*

$$\bigcup\nolimits_{\lambda>0} \lambda[\mathrm{pr}_E \operatorname{dom}\sigma - \mathrm{pr}_E \operatorname{dom}\tau] \text{ be a closed subspace of } E.$$

For all $(x,y) \in E \times F$, let

$$\rho(x,y) := \inf_{u\in F}[\sigma(x,u) + \tau(x,y-u)] > -\infty.$$

ρ is a proper convex function from $E \times F$ into $]-\infty,\infty]$. Then

$$(z^*,y^*) \in E^* \times F^* \quad \Longrightarrow \quad \rho^*(z^*,y^*) = \min_{t^*\in E^*}[\sigma^*(z^*-t^*,y^*) + \tau^*(t^*,y^*)].$$

Proof. See Simons–Zălinescu [26, Theorem 4.2]. □

Corollary 4.2(a) below is immediate from Theorem 4.1, while Corollary 4.2(b) follows from Corollary 4.2(a) by reversing the roles of E and F.

Corollary 4.2. *Let E and F be Banach spaces and $\sigma,\ \tau\colon\ E \times F \mapsto]-\infty,\infty]$ be proper, convex and lower semicontinuous.*

(a) *Let $\mathrm{pr}_E \operatorname{dom}\sigma \cap \operatorname{int}\mathrm{pr}_E \operatorname{dom}\tau \neq \emptyset$ and, for all $(x,y) \in E \times F$,*

$$\rho(x,y) := \inf_{u\in F}[\sigma(x,u) + \tau(x,y-u)] > -\infty. \tag{18}$$

ρ is a proper convex function from $E \times F$ into $]-\infty,\infty]$. Then

$$(z^*,y^*) \in E^* \times F^* \quad \Longrightarrow \quad \rho^*(z^*,y^*) = \min_{v^*\in E^*}[\sigma^*(z^*+v^*,y^*) + \tau^*(-v^*,y^*)].$$

(b) *Let $\mathrm{pr}_F \operatorname{dom}\sigma \cap \operatorname{int}\mathrm{pr}_F \operatorname{dom}\tau \neq \emptyset$ and, for all $(x,y) \in E \times F$,*

$$\rho(x,y) := \inf_{u\in E}[\sigma(u,y) + \tau(x-u,y)] > -\infty. \tag{19}$$

ρ is a proper convex function from $E \times F$ into $]-\infty,\infty]$. Then

$$(z^*,y^*) \in E^* \times F^* \quad \Longrightarrow \quad \rho^*(z^*,y^*) = \min_{v^*\in F^*}[\sigma^*(z^*,y^*+v^*) + \tau^*(z^*,-v^*)].$$

Notation 4.3. In order to simplify some rather cumbersome algebraic expressions, we will define $\Delta_E\colon\ E \times E^* \mapsto \mathbb{R}$ by $\Delta_E(y,y^*) := \langle y,y^*\rangle + \frac{1}{2}\|(y,y^*)\|^2$. "$\Delta$" stands for "discriminant". We note then that, for all $(y,y^*) \in E \times E^*$,

$$\Delta_E(y,y^*) = \tfrac{1}{2}\|y\|^2 + \langle y,y^*\rangle + \tfrac{1}{2}\|y^*\|^2 \geq \tfrac{1}{2}\|y\|^2 - \|y\|\|y^*\| + \tfrac{1}{2}\|y^*\|^2 \geq 0. \tag{20}$$

Clearly $\Delta_E(y,y^*) = 0 \Longrightarrow \|y^*\| = \|y\|$. Plugging this back into (20), we have

$$\Delta_E(y,y^*) = 0 \quad \Longrightarrow \quad \langle y,y^*\rangle = -\|y\|^2 = -\|y^*\|^2 = -\|y\|\|y^*\|.$$

The significance of this is that, if $J_E\colon E \rightrightarrows E^*$ is the duality map, then

$$\Delta_E(y,y^*) = 0 \iff y^* \in -J_E y.$$

Lemma 4.4. *Let E be a Banach space, $\sigma\colon\ E \times E^* \mapsto]-\infty,\infty]$ and $\theta\colon\ E \mapsto]-\infty,\infty]$ be proper, convex and lower semicontinuous, and*

$$(x,x^*) \in E \times E^* \quad \Longrightarrow \quad \sigma(x,x^*) \geq \langle x,x^*\rangle. \tag{21}$$

(a) *If* $\mathrm{pr}_E \operatorname{dom}\sigma \cap \operatorname{int}\operatorname{dom}\theta \neq \emptyset$ *then there exist* $z^*, v^* \in E^*$ *and* $z^{**} \in E^{**}$ *such that*

$$\sigma^*(z^*+v^*, z^{**}) + \theta^*(-v^*) + \theta^{**}(z^{**}) + \tfrac{1}{2}\|(-z^*,-z^{**})\|^2 \leq 0.$$

(b) *If* $\mathrm{pr}_{E^*} \operatorname{dom}\sigma \cap \operatorname{int}\operatorname{dom}\theta^* \neq \emptyset$ *then there exist* $z^* \in E^*$ *and* $z^{**}, v^{**} \in E^{**}$ *such that*

$$\sigma^*(z^*, z^{**}+v^{**}) + \theta^*(z^*) + \theta^{**}(-v^{**}) + \tfrac{1}{2}\|(-z^*,-z^{**})\|^2 \leq 0.$$

Proof. We define $\tau\colon\ E \times E^* \mapsto]-\infty,\infty]$ by $\tau(x,x^*) := \theta(x) + \theta^*(x^*) \geq \langle x,x^*\rangle$. Writing $F := E^*$, if ρ is defined either by (18) or (19) then, from (21), for all $(x,x^*) \in E \times E^*$,

$$\rho(x,x^*) \geq \langle x,x^*\rangle > -\infty.$$

Consequently, it follows from (20) that, for all $(x,x^*) \in E \times E^*$,

$$\rho(x,x^*) + \tfrac{1}{2}\|(x,x^*)\|^2 = \rho(x,x^*) - \langle x,x^*\rangle + \Delta_E(x,x^*) \geq 0.$$

We now apply Theorem 2.1 and obtain an element (z^*,z^{**}) of $E^* \times E^{**}$ such that

$$\rho^*(z^*,z^{**}) + \tfrac{1}{2}\|(-z^*,-z^{**})\|^2 \leq 0. \tag{22}$$

Now $\operatorname{dom}\tau = \operatorname{dom}\theta \times \operatorname{dom}\theta^*$ and, since $\operatorname{dom}\theta^* \neq \emptyset$ and $\operatorname{dom}\theta \neq \emptyset$, we have

$$\mathrm{pr}_E\operatorname{dom}\tau = \operatorname{dom}\theta \quad \text{and} \quad \mathrm{pr}_{E^*}\operatorname{dom}\tau = \operatorname{dom}\theta^*.$$

By direct computation, for all $(x^*,x^{**}) \in E^* \times E^{**}$,

$$\tau^*(x^*,x^{**}) = \theta^*(x^*) + \theta^{**}(x^{**}),$$

and so the results follow by substituting into (22) the formulae for ρ^* that we obtained in Corollary 4.2. $\square$

In Theorem 4.5, we modify Lemma 4.4 with some simple changes of variable and the introduction of the function Δ_{E^*} to obtain results that will be useful to us in Theorem 5.1.

Theorem 4.5. *Let E be a Banach space, $(w,w^*) \in E \times E^*$, $f\colon\ E \times E^* \mapsto]-\infty,\infty]$ and $g\colon\ E \mapsto]-\infty,\infty]$ be proper, convex and lower semicontinuous, and*

$$(x,x^*) \in E \times E^* \quad \Longrightarrow \quad f(x,x^*) \geq \langle x,x^*\rangle.$$

(a) *If* $\mathrm{pr}_E \operatorname{dom} f \cap \operatorname{int}\operatorname{dom} g \neq \emptyset$ *then there exist* $\xi^*, v^* \in E^*$ *and* $\xi^{**} \in E^{**}$ *such that*

$$f^*(\xi^*,\xi^{**}) - \langle \xi^*,\xi^{**}\rangle + g^*(w^*-v^*) + g^{**}(\xi^{**}) + \langle v^*-w^*,\xi^{**}\rangle + \Delta_{E^*}(v^*-\xi^*, \widehat{w}-\xi^{**}) \leq 0.$$

(b) *If* $\mathrm{pr}_{E^*} \operatorname{dom} f \cap \operatorname{int}\operatorname{dom} g^* \neq \emptyset$ *then there exist* $\xi^* \in E^*$ *and* $\xi^{**}, v^{**} \in E^{**}$ *such that*

$$f^*(\xi^*,\xi^{**}) - \langle \xi^*,\xi^{**}\rangle + g^*(\xi^*) + g^{**}(\widehat{w}-v^{**}) + \langle \xi^*, v^{**}-\widehat{w}\rangle + \Delta_{E^*}(w^*-\xi^*, v^{**}-\xi^{**}) \leq 0.$$

Proof. We define $\theta := g(\cdot + w) - w^*$. By direct computation, $\theta^* = g^*(\cdot + w^*) - \widehat{w} - \langle w, w^* \rangle$ and $\theta^{**} := g^{**}(\cdot + \widehat{w}) - \widehat{w^*}$. It follows, in particular, that

$$\operatorname{dom}\theta = \operatorname{dom} g - w \quad \text{and} \quad \operatorname{dom}\theta^* = \operatorname{dom} g^* - w^*. \tag{23}$$

(a) For all $(x, x^*) \in E \times E^*$, let $\sigma(x, x^*) := f(x + w, x^*) - \langle w, x^* \rangle \geq \langle x, x^* \rangle$. It follows that $\operatorname{pr}_E \operatorname{dom}\sigma = \operatorname{pr}_E \operatorname{dom} f - w$, and so (23) gives $\operatorname{pr}_E \operatorname{dom}\sigma \cap \operatorname{int}\operatorname{dom}\theta \neq \emptyset$. Now, for all $(y^*, y^{**}) \in E^* \times E^{**}$, $\sigma^*(y^*, y^{**}) = f^*(y^*, y^{**} + \widehat{w}) - \langle w, y^* \rangle$, and the result follows from Lemma 4.4(a) and some elementary algebra, using the substitutions $\xi^* := z^* + v^*$ and $\xi^{**} := z^{**} + \widehat{w}$.

(b) For all $(x, x^*) \in E \times E^*$, let $\sigma(x, x^*) := f(x, x^* + w^*) - \langle x, w^* \rangle \geq \langle x, x^* \rangle$. It follows that $\operatorname{pr}_{E^*} \operatorname{dom}\sigma = \operatorname{pr}_{E^*} \operatorname{dom} f - w^*$, and so (23) gives $\operatorname{pr}_{E^*} \operatorname{dom}\sigma \cap \operatorname{int}\operatorname{dom}\theta^* \neq \emptyset$. Now, for all $(y^*, y^{**}) \in E^* \times E^{**}$, $\sigma^*(y^*, y^{**}) = f^*(y^* + w^*, y^{**}) - \langle w^*, y^{**} \rangle$, and the result follows from Lemma 4.4(b) and some elementary algebra, using the substitutions $\xi^* := z^* + w^*$ and $\xi^{**} := z^{**} + v^{**}$. □

5. Maximal monotone multifunctions of type (NI)

In this section, we combine the results of Sections 1 and 4 to obtain a two–part result on maximal monotone multifunctions of type (NI), which we will apply in Theorem 6.6.

Theorem 5.1. *Let E be a Banach space, $S\colon\ E \rightrightarrows E^*$ be maximal monotone of type (NI), $g\colon\ E \mapsto]-\infty, \infty]$ be proper, convex and lower semicontinuous and $(w, w^*) \in E \times E^*$.*

(a) *Let $\operatorname{co} D(S) \cap \operatorname{int}\operatorname{dom} g \neq \emptyset$. Then*

$$\inf_{(\xi^{**}, \xi^*) \in G(\overline{S})} [\langle \xi^* - w^*, \xi^{**} - \widehat{w} \rangle + g^{**}(\xi^{**})] \geq g(w) \quad \Longrightarrow \quad w^* \in (S + \partial g)(w).$$

(b) *Let $\operatorname{co} R(S) \cap \operatorname{int}\operatorname{dom} g^* \neq \emptyset$ and $\partial g^*(w^*) \subset \widehat{E}$. Then*

$$\inf_{(\xi^{**}, \xi^*) \in G(\overline{S})} [\langle \xi^* - w^*, \xi^{**} - \widehat{w} \rangle + g^*(\xi^*)] \geq g^*(w^*) \quad \Longrightarrow \quad w \in (S^{-1} + \partial g^{-1})(w^*).$$

Proof. Taking (3) into account, we can apply the appropriate part of Theorem 4.5 with $f := \varphi_S$. Using the observation made in (5) that $\varphi_S{}^* \geq \varphi_{\widetilde{S}}$ on $E^* \times E^{**}$, in (a) we obtain $\xi^*, v^* \in E^*$ and $\xi^{**} \in E^{**}$ such that

$$\varphi_{\widetilde{S}}(\xi^*, \xi^{**}) - \langle \xi^*, \xi^{**} \rangle + g^*(w^* - v^*) + g^{**}(\xi^{**}) + \langle v^* - w^*, \xi^{**} \rangle + \Delta_{E^*}(v^* - \xi^*, \widehat{w} - \xi^{**}) \leq 0,$$

that is to say

$$\psi_{\widetilde{S}}(\xi^*, \xi^{**}) + [g^*(w^* - v^*) + g^{**}(\xi^{**}) + \langle v^* - w^*, \xi^{**} \rangle] + \Delta_{E^*}(v^* - \xi^*, \widehat{w} - \xi^{**}) \leq 0, \tag{24}$$

while in (b) we obtain $\xi^* \in E^*$ and $\xi^{**}, v^{**} \in E^{**}$ such that

$$\varphi_{\widetilde{S}}(\xi^*, \xi^{**}) - \langle \xi^*, \xi^{**} \rangle + g^*(\xi^*) + g^{**}(\widehat{w} - v^{**}) + \langle \xi^*, v^{**} - \widehat{w} \rangle + \Delta_{E^*}(w^* - \xi^*, v^{**} - \xi^{**}) \leq 0,$$

that is to say

$$\psi_{\widetilde{S}}(\xi^*, \xi^{**}) + [g^*(\xi^*) + g^{**}(\widehat{w} - v^{**}) + \langle \xi^*, v^{**} - \widehat{w} \rangle] + \Delta_{E^*}(w^* - \xi^*, v^{**} - \xi^{**}) \leq 0. \tag{25}$$

(a) (20) and the fact that S is of type (NI) imply that each of the three terms in (24) is non–negative, hence they are all zero. Using the definition of $G(\overline{S})$ in terms of $\psi_{\widetilde{S}}$, (24) thus implies the three assertions $\xi^* \in \overline{S}(\xi^{**})$, $\xi^{**} \in \partial g^*(w^* - v^*)$, or equivalently $g^{**}(\xi^{**}) = \langle w^* - v^*, \xi^{**}\rangle - g^*(w^* - v^*)$, and $\widehat{w} - \xi^{**} \in -J_{E^*}(v^* - \xi^*)$. Combining this with our initial assumption that $\langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle + g^{**}(\xi^{**}) \geq g(w)$, we obtain from the Fenchel–Young inequality that

$$\begin{aligned}\langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle &\geq g(w) - g^{**}(\xi^{**}) = g(w) + g^*(w^* - v^*) - \langle w^* - v^*, \xi^{**}\rangle \\ &\geq \langle w, w^* - v^*\rangle - \langle w^* - v^*, \xi^{**}\rangle = \langle v^* - w^*, \xi^{**} - \widehat{w}\rangle,\end{aligned}$$

from which $\langle \xi^* - v^*, \xi^{**} - \widehat{w}\rangle \geq 0$. Since $\widehat{w} - \xi^{**} \in -J_{E^*}(v^* - \xi^*)$ it follows that $\xi^* = v^*$ and $\xi^{**} = \widehat{w}$. (11) now implies that $v^* = \xi^* \in \overline{S}(\xi^{**}) = \overline{S}(\widehat{w}) = Sw$. Furthermore, $\widehat{w} = \xi^{**} \in \partial g^*(w^* - v^*)$, and so the Fenchel–Moreau theorem gives us that $w^* - v^* \in \partial g(w)$. Thus $w^* = v^* + (w^* - v^*) \in (S + \partial g)(w)$, which completes the proof of (a).

(b) (20) and the fact that S is of type (NI) imply that each of the three terms in (25) is positive, hence they are all zero. Using the definition of $G(\overline{S})$ in terms of $\psi_{\widetilde{S}}$, (25) thus implies the three assertions $\xi^* \in \overline{S}(\xi^{**})$, $\widehat{w} - v^{**} \in \partial g^*(\xi^*)$ and $v^{**} - \xi^{**} \in -J_{E^*}(w^* - \xi^*)$. Combining the second of these with our assumption that $\langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle + g^*(\xi^*) \geq g^*(w^*)$, we obtain

$$\langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle \geq g^*(w^*) - g^*(\xi^*) \geq \langle w^* - \xi^*, \widehat{w} - v^{**}\rangle = \langle \xi^* - w^*, v^{**} - \widehat{w}\rangle,$$

from which $\langle \xi^* - w^*, \xi^{**} - v^{**}\rangle \geq 0$. Since $v^{**} - \xi^{**} \in -J_{E^*}(w^* - \xi^*)$ it follows that $\xi^* = w^*$ and $\xi^{**} = v^{**}$. Thus $w^* = \xi^* \in \overline{S}(\xi^{**}) = \overline{S}(v^{**})$ and $\widehat{w} - v^{**} \in \partial g^*(\xi^*) = \partial g^*(w^*) \subset \widehat{E}$. Consequently, there exists $v \in E$ such that $\widehat{v} = v^{**}$. (11) now implies that $w^* \in \overline{S}(\widehat{v}) = S(v)$. Further, $\widehat{w} - \widehat{v} \in \partial g^*(w^*)$, and so the Fenchel–Moreau theorem gives us that $w^* \in \partial g(w - v)$. Thus $w = v + (w - v) \in (S^{-1} + \partial g^{-1})(w^*)$, which completes the proof of (b). □

Remark 5.2. The converse of Theorem 5.1(a) is true even if S is not of type (NI) and there is no constraint qualification. To see this, suppose that there exists $v^* \in S(w)$ such that $w^* - v^* \in \partial g(w)$, which implies that $\widehat{w^*} - \widehat{v^*} \in \partial g^{**}(\widehat{w})$. Let $(\xi^{**}, \xi^*) \in G(\overline{S})$. Then, since the definition of $\overline{S}$ implies that $\langle \xi^* - v^*, \xi^{**} - \widehat{w}\rangle \geq 0$,

$$\begin{aligned}\langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle + g^{**}(\xi^{**}) &\geq \langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle + \langle \xi^{**} - \widehat{w}, \widehat{w^*} - \widehat{v^*}\rangle + g^{**}(\widehat{w}) \\ &= \langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle + \langle w^* - v^*, \xi^{**} - \widehat{w}\rangle + g(w) \\ &= \langle \xi^* - v^*, \xi^{**} - \widehat{w}\rangle + g(w) \geq g(w).\end{aligned}$$

6. Type (D), the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, and type (ED)

Maximal monotone multifunctions of type (D) were introduced by Gossez in order to generalize to nonreflexive spaces some results that were known in reflexive spaces — see Phelps, [13, Section 3] for an exposition. It is clear that every maximal monotone multifunction of type (D) is of type (NI). In what follows, we write "$\mathcal{T}_{\|\ \|}$" for "norm topology of". Theorem 6.6 will be used in Theorem 7.2, and Corollary 6.7 will be used in Theorem 8.2 and Theorem 9.3.

Definition 6.1. $S\colon\ E \rightrightarrows E^*$ is *maximal monotone of type* (D) if S is maximal monotone and, for all $(\xi^{**}, \xi^*) \in G(\overline{S})$, there exists a bounded net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(S)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (\xi^{**}, \xi^*)$ in $w(E^{**}, E^*) \times \mathcal{T}_{\|\ \|}(E^*)$.

Lemma 6.2. *Let $S\colon\ E \rightrightarrows E^*$ be maximal monotone of type (D) and $(\xi^{**},\xi^*) \in G(\overline{S})$. Then there exists a net $\{(s_\gamma,s_\gamma^*)\}$ of elements of $G(S)$ such that, for all $(w,w^*) \in E \times E^*$, $\langle s_\gamma - w, s_\gamma^* - w^*\rangle \to \langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle$ and $s_\gamma^* \to \xi^*$ in $\mathcal{T}_{\|\ \|}(E^*)$.*

Proof. Immediate. $\square$

Maximal monotone multifunctions of type (ED) were first introduced by Simons in [22] and then studied systematically in [23] and [24]. In particular, it was shown in [23] that these multifunctions have a property similar to the Brøndsted–Rockafellar property of subdifferentials. We must first define the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ on E^{**}.

Definition 6.3. We write $\mathcal{CLB}(E)$ for the set of all convex functions $f\colon\ E \mapsto \mathbb{R}$ that are Lipschitz on the bounded subsets of E, or equivalently bounded on the bounded subsets of E. We define the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ on E^{**} to be the coarsest topology on E^{**} making all the functions $h^{**}\colon\ E^{**} \mapsto \mathbb{R}\quad (h \in \mathcal{CLB}(E))$ continuous. The properties of the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ are discussed fully in [23, Section 3].

Definition 6.4. $S\colon\ E \rightrightarrows E^*$ is *maximal monotone of type (ED)* if S is maximal monotone and, for all $(\xi^{**},\xi^*) \in G(\overline{S})$, there exists a net $\{(s_\gamma,s_\gamma^*)\}$ of elements of $G(S)$ such that $(\widehat{s_\gamma},s_\gamma^*) \to (\xi^{**},\xi^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$.

Since $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ is stronger than $w(E^{**},E^*)$, it is clear that every maximal monotone multifunction of type (ED) is of type (D). On the other hand, as is pointed out in the introduction to [23], *in every case where it has been proved that a multifunction is maximal monotone of type (D) then it is also of type (ED).*

Lemma 6.5. *Let $S\colon\ E \rightrightarrows E^*$ be maximal monotone of type (ED) and $(\xi^{**},\xi^*) \in G(\overline{S})$. Then there exists a net $\{(s_\gamma,s_\gamma^*)\}$ of elements of $G(S)$ such that, for all $(w,w^*) \in E \times E^*$, $\langle s_\gamma - w, s_\gamma^* - w^*\rangle \to \langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle$ and $\widehat{s}_\gamma \to \xi^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$.*

Proof. Let $\{(s_\gamma,s_\gamma^*)\}$ be as in Definition 6.4. It follows from [23, Lemma 3.1(c), p. 263] that $(\widehat{s}_\gamma - \widehat{w}, s_\gamma^* - w^*) \to (\xi^{**} - \widehat{w}, \xi^* - w^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$, and so [23, Lemma 3.1(e), p. 263] implies that $\langle s_\gamma - w, s_\gamma^* - w^*\rangle \to \langle \xi^* - w^*, \xi^{**} - \widehat{w}\rangle$. $\square$

Theorem 6.6. *Let E be a nonzero Banach space, $S\colon\ E \rightrightarrows E^*$ be maximal monotone, $g\colon\ E \mapsto]-\infty,\infty]$ be proper, convex and lower semicontinuous and $(w,w^*) \in E \times E^*$.*

(*a*) *Let S be of type (ED), $\operatorname{co} D(S) \cap \operatorname{int}\operatorname{dom} g \neq \emptyset$ and*

$$\limsup\nolimits_\gamma \langle s_\gamma - w, s_\gamma^* - w^*\rangle + g^{**}(\xi^{**}) \geq g(w)$$

whenever $\{(s_\gamma,s_\gamma^*)\}$ is a net of elements of $G(S)$ such that $\widehat{s}_\gamma \to \xi^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Then $w^* \in (S + \partial g)(w)$.

(*b*) *Let S be of type (D), $\operatorname{co} R(S) \cap \operatorname{int}\operatorname{dom} g^* \neq \emptyset$, $\partial g^*(w^*) \subset \widehat{E}$ and*

$$\limsup\nolimits_\gamma \langle s_\gamma - w, s_\gamma^* - w^*\rangle + g^*(\xi^*) \geq g^*(w^*)$$

whenever $\{(s_\gamma,s_\gamma^*)\}$ is a net of elements of $G(S)$ such that $s_\gamma^* \to \xi^*$ in $\mathcal{T}_{\|\ \|}(E^*)$. Then $w \in (S^{-1} + \partial g^{-1})(w^*)$.

Proof. (a) Let $(\xi^{**},\xi^*)\in G(\overline{S})$. Let $\{(s_\gamma,s_\gamma^*)\}$ be a net of elements of $G(S)$ as in Lemma 6.5. By assumption, $\langle\xi^*-w^*,\xi^{**}-\widehat{w}\rangle+g^{**}(\xi^{**})=\lim_\gamma\langle s_\gamma-w,s_\gamma^*-w^*\rangle+g^{**}(\xi^{**})\ge g(w)$, and (a) follows from Theorem 5.1(a).

(b) Let $(\xi^{**},\xi^*)\in G(\overline{S})$. Let $\{(s_\gamma,s_\gamma^*)\}$ be a net of elements of $G(S)$ as in Lemma 6.2. By assumption, $\langle\xi^*-w^*,\xi^{**}-\widehat{w}\rangle+g^*(\xi^*)=\lim_\gamma\langle s_\gamma-w,s_\gamma^*-w^*\rangle+g^*(\xi^*)\ge g^*(w^*)$, and (b) follows from Theorem 5.1(b). □

Corollary 6.7. *Let E be a nonzero Banach space, $S\colon\ E\rightrightarrows E^*$ be maximal monotone, $g\colon\ E\mapsto\,]-\infty,\infty]$ be proper, convex and lower semicontinuous and $(w,w^*)\in E\times E^*$.*

(a) *Let S be of type (ED), $g\in\mathcal{CLB}(E)$ and*

$$(s,s^*)\in G(S)\quad\Longrightarrow\quad\langle s-w,s^*-w^*\rangle+g(s)\ge g(w).$$

Then $w^*\in(S+\partial g)(w)$.

(b) *Let S be of type (D), g^* be finite and norm–continuous on E^*, $\partial g^*(w^*)\subset\widehat{E}$, and*

$$(s,s^*)\in G(S)\quad\Longrightarrow\quad\langle s-w,s^*-w^*\rangle+g^*(s^*)\ge g^*(w^*).$$

Then $w\in(S^{-1}+\partial g^{-1})(w^*)$.

Proof. (a) This is immediate from Theorem 6.6(a) since $\operatorname{dom}g=E$ and the $\mathcal{T}_{\mathcal{CLB}}(E^{**})$–continuity of g^{**} implies that, with the notation of Theorem 6.6(a),

$$\limsup_\gamma\langle s_\gamma-w,s_\gamma^*-w^*\rangle+g^{**}(\xi^{**})=\limsup_\gamma[\langle s_\gamma-w,s_\gamma^*-w^*\rangle+g(s_\gamma)]\ge g(w).$$

(b) This is immediate from Theorem 6.6(b) since $\operatorname{dom}g^*=E^*$ and the norm continuity of g^* implies that, with the notation of Theorem 6.6(b),

$$\limsup_\gamma\langle s_\gamma-w,s_\gamma^*-w^*\rangle+g^*(\xi^*)=\limsup_\gamma[\langle s_\gamma-w,s_\gamma^*-w^*\rangle+g^*(s_\gamma^*)\ge g^*(w^*)].$$

□

7. Type (FP) and type (FPV)

Maximal monotone multifunctions of "type (FP)" were introduced by Fitzpatrick–Phelps in [6, Section 3] under the name of "locally maximal monotone" multifunctions. Their introduction was motivated by the problem of approximating maximal monotone multifunctions by simpler ones. Maximal monotone multifunctions of "type (FPV)" were introduced independently by Fitzpatrick–Phelps and Verona–Verona in [7, p. 65] and [27, p. 268] by dualizing the definition of "type (FP)".

Definition 7.1. Let E be a nonzero Banach space and $S\colon\ E\rightrightarrows E^*$ be maximal monotone. We say that S is *of type (FPV)* or *maximal monotone locally* provided that

$$(w,w^*)\in G(S)\tag{26}$$

whenever U is a convex open subset of E with $U\cap D(S)\ne\emptyset$, $(w,w^*)\in U\times E^*$ and

$$(s,s^*)\in G(S)\text{ and }s\in U\quad\Longrightarrow\quad\langle s-w,s^*-w^*\rangle\ge 0.\tag{27}$$

We say that S is *of type (FP)* or *locally maximal monotone* provided that (26) holds whenever U is a convex open subset of E^* with $U\cap R(S)\ne\emptyset$, $(w,w^*)\in E\times U$ and

$$(s,s^*)\in G(S)\text{ and }s^*\in U\quad\Longrightarrow\quad\langle s-w,s^*-w^*\rangle\ge 0.\tag{28}$$

The results of Theorem 7.2 were first established by Simons in [24, Theorem 20, pp. 407–409 and Theorem 17, pp. 405–406], using the "free convexification" of a multifunction and a minimax theorem. The proofs given here using the Fitzpatrick function provide an enormous simplification, and avoid the excursion to E^{***} made in the proof of [24, Theorem 20].

Theorem 7.2. *Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be maximal monotone.*

(a) *Let S be of type (ED). Then S is of type (FPV).*
(b) *Let S be of type (D). Then S is of type (FP).*

Proof. (a) Let U be an open convex subset of E such that $U \cap D(S) \neq \emptyset$ and $(w, w^*) \in U \times E^*$ satisfy (27). We want to prove that (26) holds. We first find $\tau \in U \cap D(S)$, choose $\varepsilon > 0$ so that $[w, \tau] + \{x \in E\colon\ \|x\| \le 2\varepsilon\} \subset U$, and let

$$C := [w, \tau] + \{x \in E\colon\ \|x\| \le \varepsilon\} \quad \text{and} \quad B := [\widehat{w}, \widehat{\tau}] + \{x^{**} \in E^{**}\colon\ \|x^{**}\| \le \varepsilon\}.$$

We will apply Theorem 6.6(a) with $g := \mathbb{I}_C$, where $\mathbb{I}_C$ is the indicator function of C. Let $\{(s_\gamma, s_\gamma^*)\}$ be a net of elements of $G(S)$ such that $\widehat{s}_\gamma \to \xi^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Suppose first that $\mathbb{I}_B(\xi^{**}) < \infty$. Since $\xi^{**} \in B$, there exists $u \in [w, \tau]$ such that $\|\xi^{**} - \widehat{u}\| \le \varepsilon$. Now Simons, [23, Lemma 3.1(b), p. 263] implies that $\|s_\gamma - u\| \to \|\xi^{**} - \widehat{u}\|$, and so eventually $\|s_\gamma - u\| \le 2\varepsilon$, from which eventually $s_\gamma \in U$. From (27), eventually $\langle s_\gamma - w, s_\gamma^* - w^*\rangle \ge 0$. Since $\mathbb{I}_B(\xi^{**}) = 0$ and $\mathbb{I}_C(w) = 0$, in fact eventually

$$\langle s_\gamma - w, s_\gamma^* - w^*\rangle + \mathbb{I}_B(\xi^{**}) \ge \mathbb{I}_C(w).$$

Of course, this inequality is trivially true if $\mathbb{I}_B(\xi^{**}) = \infty$. It follows from the Goldstine–Weston theorem (see, for instance, Schechter, [18, §28.40, p. 777]) that B is the closure of $\widehat{C}$ in $w(E^{**}, E^*)$, which implies that $\mathbb{I}_B = {\mathbb{I}_C}^{**}$ and so, since $D(S) \cap \operatorname{int}\operatorname{dom}\mathbb{I}_C = D(S) \cap \operatorname{int} C \ni \tau$, Theorem 6.6($a$) gives $w^* \in (S + \partial\mathbb{I}_C)(w)$. However, $w \in \operatorname{int} C$, and so $\partial\mathbb{I}_C(w) = \{0\}$. This establishes (26), completing the proof of (a).

(b) Let U be an open convex subset of E^* such that $R(S) \cap U \neq \emptyset$ and $(w, w^*) \in E \times U$ satisfy (28). Again, we want to prove that (26) holds. We first find $\tau^* \in U \cap R(S)$ and choose $\varepsilon > 0$ so that

$$B := [w^*, \tau^*] + \{x^* \in E^*\colon\ \|x^*\| \le \varepsilon\} \subset U.$$

Define $g\colon\ E \mapsto \mathbb{R}$ by $g(x) := \sup\langle x, B\rangle$. Since B is $w(E^*, E)$–closed, $g^* = \mathbb{I}_B$ and so $R(S) \cap \operatorname{int}\operatorname{dom} g^* = R(S) \cap \operatorname{int} B \ni \tau^*$. Now $w^* \in \operatorname{int} B = \operatorname{int}\operatorname{dom} g^*$, from which it follows that $\partial g^*(w^*) = \{0\} \subset \widehat{E}$. We will apply Theorem 6.6(b). Let $\{(s_\gamma, s_\gamma^*)\}$ be a net of elements of $G(S)$ such that $s_\gamma^* \to \xi^*$ in $\mathcal{T}_{\|\ \|}(E^*)$. Suppose first that $\mathbb{I}_B(\xi^*) < \infty$. Then $\xi^* \in B \subset U$, and so eventually $s_\gamma^* \in U$. From (28), eventually $\langle s_\gamma - w, s_\gamma^* - w^*\rangle \ge 0$. Since $\mathbb{I}_B(\xi^*) = 0$ and $\mathbb{I}_B(w^*) = 0$, eventually

$$\langle s_\gamma - w, s_\gamma^* - w^*\rangle + \mathbb{I}_B(\xi^*) \ge \mathbb{I}_B(w^*).$$

Of course, this inequality remains true if $\mathbb{I}_B(\xi^*) = \infty$. Thus Theorem 6.6($b$) implies that $w \in (S^{-1} + \partial g^{-1})(w^*)$. However, $\widehat{\partial g^{-1}(w^*)} \subset \partial g^*(w^*) = \{0\}$, thus $\partial g^{-1}(w^*) = \{0\}$, and so $w \in S^{-1}w^*$. This gives (26), completing the proof of (b). $\square$

Let E be a nonzero Banach space. It was proved by Simons in [22, Theorem 26.3, p. 103] that *if* $S\colon\ E \rightrightarrows E^*$ *is maximal monotone of type (FPV) then* $\overline{D(S)}$ *is convex*, consequently Theorem 7.2(a) implies that if S is maximal monotone of type (ED) then $\overline{D(S)}$ is convex. It was proved by Fitzpatrick–Phelps in [6, Theorem 3.5, p. 585] that *if* S *is maximal monotone of type (FP) then* $\overline{R(S)}$ *is convex*, consequently Theorem 7.2(b) implies the result proved essentially by Gossez in [8] (see Phelps, [13, Theorem 3.8, p. 221] for an exposition) that *if* S *is maximal monotone of type (D) then* $\overline{R(S)}$ *is convex.* Taken together, these two observations lead to the following result:

Corollary 7.3. *Let* E *be a nonzero Banach space and* $S\colon\ E \rightrightarrows E^*$ *be maximal monotone of type (ED). Then both* $\overline{D(S)}$ *and* $\overline{R(S)}$ *are convex.*

Let E be a Banach space and $f\colon\ E \mapsto\,]-\infty, \infty]$ be proper, convex and lower semicontinuous. It was proved by Simons in [22, Theorem 35.3, p. 139] and [23, Theorem 12.6(b), p. 287] that $\partial f\colon E \rightrightarrows E^*$ *is maximal monotone of type (ED).* Consequently, Theorem 7.2 implies the results proved by Fitzpatrick–Phelps in [7, Corollary 3.4, p. 66] and by Verona–Verona in [27, Theorem 3, p. 269] that ∂f *is maximal monotone of type (FPV)*, and by Simons in [19, Main theorem, p. 470] that ∂f *is maximal monotone of type (FP).*

Let E be a nonzero reflexive Banach space and $S\colon\ E \rightrightarrows E^*$ be maximal monotone. It is trivial to see that S is of type (ED). Consequently, Theorem 7.2(b) implies the result proved by Fitzpatrick–Phelps in [6, Proposition 3.3, p. 585] that S *is of type (FP)* (from which it follows easily that S *is also of type (FPV)*).

Since (see the introduction to [24]) the "tail" operator from ℓ^1 into ℓ^∞ defined by $(Tx)_n := \sum_{k\ge n} x_k$ is everywhere defined, maximal monotone and linear but not of type (ED), Theorem 7.2(a) does not imply the result proved by Simons in [22, Theorem 38.2, p. 146] that *if* $D(T)$ *is a subspace of a nonzero Banach space* E *and* $T\colon\ D(T) \mapsto E^*$ *is* any *maximal monotone linear operator then* T *is of type (FPV)*, and also does not imply the result proved by Fitzpatrick–Phelps in [7, Theorem 3.10, p. 68] that *if* S *is maximal monotone and* $D(S) = E$ *then* S *is of type (FPV).* On the other hand, Theorem 7.2(b) does imply the result proved by Phelps–Simons in [14, Theorem 6.7, p. 320] that *if* $D(T)$ *is a subspace of a nonzero Banach space* E *and* $T\colon\ D(T) \mapsto E^*$ *is a maximal monotone linear operator of type (D) then* T *is of type (FP).* We do not know if Theorem 7.2(b) implies the result proved by Fitzpatrick–Phelps in [7, Theorem 3.7, p. 67] that *if* S *is maximal monotone and* $R(S) = E^*$ *then* S *is of type (FP).* In other words, we have the following problem:

Problem 7.4. If E is a nonzero Banach space, $S\colon\ E \rightrightarrows E^*$ is maximal monotone and $R(S) = E^*$ then is S of type (D)? (For linear maps, the solution to this problem is in the affirmative — see Phelps–Simons, [14, Theorem 6.7, pp. 320–323].)

We note that it was proved by Bauschke–Borwein in [2, Theorem 4.1] (see also [14, Theorem 8.1, p. 327]) that every *continuous single–valued linear* maximal monotone multifunction of type (FP) is necessarily of type (D). However, we do not know the solution to the following problem:

Problem 7.5. Is every maximal monotone multifunction of type (FP) necessarily of type (D)?

While the tail operator is not of type (FP), we do not know the solution to the following problem either:

Problem 7.6. Is every maximal monotone multifunction of type (FPV)?

Remark 7.7. It is clear from the proof of Theorem 7.2 that if S is of type (ED) then in order to prove that (27) implies (26) it suffices to assume that $U \cap \operatorname{co} D(S) \neq \emptyset$ instead of $U \cap D(S) \neq \emptyset$, and that if S is of type (D) then in order to prove that (28) implies (26) it suffices to assume that $U \cap \operatorname{co} R(S) \neq \emptyset$ instead of $U \cap R(S) \neq \emptyset$.

8. Strong maximality

We now discuss "strong maximality", which is actually defined in terms of two simpler concepts.

Definition 8.1. Let $S\colon\ E \rightrightarrows E^*$ be monotone. We say that S is *$w(E^*,E)$–cc maximal* if, whenever C is a nonempty $w(E^*,E)$–compact convex subset of E^*, $w \in E$ and

$$(s,s^*) \in G(S) \quad\Longrightarrow\quad \text{there exists } w^* \in C \text{ such that } \quad \langle s-w, s^*-w^*\rangle \geq 0 \tag{29}$$

then

$$Sw \cap C \neq \emptyset. \tag{30}$$

We say that S is *$w(E,E^*)$–cc maximal* if, whenever C is a nonempty $w(E,E^*)$–compact convex subset of E, $w^* \in E^*$ and

$$(s,s^*) \in G(S) \quad\Longrightarrow\quad \text{there exists } w \in C \text{ such that } \quad \langle s-w, s^*-w^*\rangle \geq 0 \tag{31}$$

then

$$S^{-1}w^* \cap C \neq \emptyset. \tag{32}$$

We say that S is *strongly maximal* if S is both $w(E^*,E)$–cc maximal and $w(E,E^*)$–cc maximal. It is clear (by taking C to be a singleton) that every $w(E^*,E)$–cc maximal or $w(E,E^*)$–cc maximal or strongly maximal monotone multifunction is maximal monotone.

The result of Theorem 8.2(c) was first established by Simons in [24, Theorem 15, pp. 400–402] using the "free convexification" of a multifunction and a minimax theorem. Again, the proof given here using the Fitzpatrick function provide an enormous simplification.

Theorem 8.2. *Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be maximal monotone.*

(a) *Let S be of type (ED). Then S is $w(E^*,E)$–cc maximal.*
(b) *Let S be of type (D). Then S is $w(E,E^*)$–cc maximal.*
(c) *Let S be of type (ED). Then S is strongly maximal.*

Proof. (a) We suppose first that C is a nonempty $w(E^*,E)$–compact convex subset of E^*, $w \in E$ and (29) is satisfied and, we will prove that (30) is satisfied. Define $g\colon\ E \mapsto \mathbb{R}$ by $g(x) := \sup\langle w-x, C\rangle$. It follows from (29) that

$$(s,s^*) \in G(S) \quad\Longrightarrow\quad \langle s-w, s^*\rangle + g(s) \geq 0 = g(w).$$

Since $g \in \mathcal{CLB}(E)$, Corollary 6.7(a) implies that $0 \in (S+\partial g)(w)$, so there exists $v^* \in S(w)$ such that $-v^* \in \partial g(w)$, from which $g^*(-v^*) \leq \infty$. Since $g^*(-v^*) = \mathbb{I}_C(v^*) - \langle w, v^*\rangle$, this implies in turn that $v^* \in C$, and so $Sw \cap C \ni v^*$, giving (30) and completing the proof of (a).

(b) Now we suppose that C is a nonempty $w(E,E^*)$–compact convex subset of E, $w^* \in E^*$ and (31) is satisfied and we will prove that (32) is satisfied. Here, define $g\colon E \mapsto \,]-\infty,\infty]$ by $g := w^* + \mathbb{I}_{-C}$. g is proper, convex and lower semicontinuous and, for all $s^* \in E^*$, $g^*(s^*) = \sup\langle C, w^* - s^*\rangle$. It follows from (31) that

$$(s,s^*) \in G(S) \quad \Longrightarrow \quad \langle s, s^* - w^*\rangle + g^*(s^*) \geq 0 = g^*(w^*).$$

Now g^* is finite and continuous on E^*. Further, $g^{**} = \mathbb{I}_{\widehat{-C}} + \widehat{w^*}$, which implies that $\partial g^*(w^*) \subset \operatorname{dom} g^{**} \subset \widehat{-C} \subset \widehat{E}$. Thus Corollary 6.7($b$) gives us that $0 \in (S^{-1} + \partial g^{-1})(w^*)$, so there exists $v \in S^{-1}w^*$ such that $w^* \in \partial g(-v)$, from which $-v \in \operatorname{dom} g = -C$. Thus $v \in C$, and so $S^{-1}w^* \cap C \ni v$, giving (32) and completing the proof of (b).

(c) This is immediate from (a) and (b). □

Let E be a Banach space and $f\colon E \mapsto \,]-\infty,\infty]$ be proper, convex and lower semicontinuous. As we have already observed, it was proved in [22] and [23] that *$\partial f\colon E \rightrightarrows E^*$ is maximal monotone of type (ED).* Consequently, Theorem 8.2 implies the results proved by Simons in different ways in [20, Theorems 6.1 and 6.2, p. 1386] and [22, Theorem 32.5, p. 128] that *∂f is strongly maximal.*

Let E be a nonzero reflexive Banach space and $S\colon E \rightrightarrows E^*$ be maximal monotone. It is trivial to see that S is of type (ED). Consequently, Theorem 8.2 implies the result proved by Przeworski–Zagrodny in [15, Theorem 3.2, p. 151] that *if E is reflexive then every maximal monotone multifunction $E \rightrightarrows E^*$ is strongly maximal.*

Since, as we have already observed, the "tail" operator from ℓ^1 into ℓ^∞ is everywhere defined, maximal monotone and and linear but not of type (ED), Theorem 8.2 does not imply the result proved by Bauschke–Simons in [3, Theorem 1.1, p. 166] that *if $D(T)$ is a subspace of E and $T\colon D(T) \mapsto E^*$ is linear and maximal monotone then T is strongly maximal.*

These observations lead to the following problem:

Problem 8.3. Is every maximal monotone multifunction strongly maximal?

9. Full domain and surjectivity

In Theorem 9.2(b) and Theorem 9.3, we give sufficient conditions for a maximal monotone multifunction to have full domain. The first of these is established using Lemma 9.1, a result of independent interest which depends on Theorem 2.1. Theorem 9.2(b) has been established in many other ways — see the remarks preceding Theorem 9.2 for a discussion of this. Theorem 9.3 is a subtler result for maximal monotone multifunctions of type (ED), from which we deduce in Corollary 9.5, the classical result that a coercive maximal monotone multifunction on a reflexive Banach space is surjective

Lemma 9.1. *Let E be a nonzero Banach space, $S\colon E \rightrightarrows E^*$ be maximal monotone, $s \in E$ and $K \geq 0$. Define $g\colon E \times E^* \mapsto \mathbb{R}$ by $g(x,x^*) := K\|s - x\| - \langle s, x^*\rangle$. Then*

$$\text{there exists } s^* \in Ss \text{ such that } \|s^*\| \leq K \iff \varphi_S + g \geq 0 \text{ on } E \times E^*.$$

Proof. ($\Longrightarrow$) Let $s^* \in Ss$ and $\|s^*\| \le K$. Then, for all $(x, x^*) \in E \times E^*$,

$$\begin{aligned}\varphi_S(x, x^*) + g(x, x^*) &\ge \langle x, s^*\rangle + \langle s, x^*\rangle - \langle s, s^*\rangle + g(x, x^*)\\ &= \langle x, s^*\rangle - \langle s, s^*\rangle + K\|s - x\| = K\|s - x\| - \langle s - x, s^*\rangle \ge 0.\end{aligned}$$

($\Longleftarrow$) The argument here follows the same lines as in Lemma 2.2(e). From Theorem 2.1, there exists $(z, z^*) \in E \times E^*$ such that ${\varphi_S}^@(z, z^*) + g^@(-z, -z^*) \le 0$. By direct computation,

$$g^@(-z, -z^*) = \begin{cases} -\langle s, z^*\rangle & \|z^*\| \le K \text{ and } z = s;\\ \infty & \text{otherwise.}\end{cases}$$

Thus $\|z^*\| \le K$, $z = s$ and ${\varphi_S}^*(z^*, \widehat{s}) - \langle s, z^*\rangle = {\varphi_S}^@(z, z^*) + g^@(-z, -z^*) \le 0$. (10) now implies that $(s, z^*) \in G(S)$. □

We now show how Lemma 9.1 can be used to give a proof of the result that if a maximal monotone multifunction has bounded range then it has full domain. Theorem 9.2 can also be established using the Debrunner–Flor extension theorem (which depends on Brouwer's fixed–point theorem, see Phelps, [13, Lemma 1.7, p. 4] and the comments preceding), or the Farkas Lemma (see Fitzpatrick–Phelps, [6, Lemma 2.4, pp. 580–581]), or a minimax theorem (see Simons, [22, Lemma 11.1, p. 41]), or a new version of the Hahn–Banach theorem (see Simons, [25, Theorem 4.1, p. 639]).

Theorem 9.2. *Let E be a nonzero Banach space, $S\colon\ E \rightrightarrows E^*$ be maximal monotone, and suppose that there exists K such that $(s, s^*) \in G(S) \Longrightarrow \|s^*\| \le K$. Then:*

(a) *For all $(x, w, x^*) \in E \times E \times E^*$, $\varphi_S(x, x^*) + K\|w - x\| \ge \varphi_S(w, x^*)$. (It follows from this that φ_S is K–Lipschitz in its first variable.)*

(b) *$D(S) = E$.*

Proof. (a) Using the hypothesis $(s, s^*) \in G(S) \Longrightarrow \|s^*\| \le K$, we have

$$\begin{aligned}\varphi_S(x, x^*) + K\|w - x\| &= \sup_{(s,s^*)\in G(S)} [\langle s, x^*\rangle + \langle x, s^*\rangle - \langle s, s^*\rangle + K\|w - x\|]\\ &\ge \sup_{(s,s^*)\in G(S)} [\langle s, x^*\rangle + \langle w, s^*\rangle - \langle s, s^*\rangle] = \varphi_S(w, x^*).\end{aligned}$$

(b) Let s be an arbitrary element of E. Then, using (a) and the notation of Lemma 9.1, for all $(x, x^*) \in E \times E^*$,

$$\varphi_S(x, x^*) + g(x, x^*) = \varphi_S(x, x^*) + K\|s - x\| - \langle s, x^*\rangle \ge \varphi_S(s, x^*) - \langle s, x^*\rangle \ge 0.$$

Thus it follows from Lemma 9.1 that $s \in D(S)$. This completes the proof of (b). □

Theorem 9.3. *Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be maximal monotone of type (ED). Suppose that, for all $w \in E$, there exists $K \ge 0$ such that*

$$(s, s^*) \in G(S) \text{ and } \|s^*\| > K \quad\Longrightarrow\quad \langle s - w, s^*\rangle \ge 0.$$

Then $D(S) = E$.

Proof. Let w be an arbitrary element of E. Define $g\colon\ E \mapsto \mathbb{R}$ by $g(x) := K\|w-x\| \geq 0$. Let $(s,s^*) \in G(S)$. If $\|s^*\| > K$ then $\langle s-w, s^*\rangle + g(s) \geq \langle s-w,s^*\rangle \geq 0 = g(w)$, while if $\|s^*\| \leq K$ then $\langle s-w,s^*\rangle + g(s) = K\|w-s\| - \langle w-s, s^*\rangle \geq 0 = g(w)$: so Corollary 6.7($a$) gives $0 \in (S+\partial g)(w)$, from which $w \in D(S)$. □

Corollary 9.4. *Let E be a nonzero Banach space, $S\colon\ E \rightrightarrows E^*$ be maximal monotone of type (ED), and $S^{-1}\colon\ E^* \rightrightarrows E$ be coercive, that is to say* $\inf\langle S^{-1}x^*, x^*\rangle/\|x^*\| \to \infty$ *as* $\|x^*\| \to \infty$*. Then* $D(S) = E$.

Proof. Let w be an arbitrary element of E. Chose $K \geq 0$ so that

$$x^* \in E^* \text{ and } \|x^*\| > K \quad\Longrightarrow\quad \inf\langle S^{-1}x^*, x^*\rangle/\|x^*\| \geq \|w\|.$$

If (s,s^*) is an arbitrary element of $G(S)$ and $\|s^*\| > K$ then $\langle s, s^*\rangle/\|s^*\| \geq \|w\|$, from which $\langle s-w, s^*\rangle \geq 0$. The result now follows from Theorem 9.3. □

Corollary 9.5. *Let F be a nonzero reflexive Banach space and $T\colon\ F \rightrightarrows F^*$ be maximal monotone and coercive. Then* $R(T) = F^*$.

Proof. This immediate from Theorem 9.3 with $E := F^*$ and $S := T^{-1}$. □

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