Journal of Convex Analysis
Volume 13 (2006), No. 2, 207–224

Inertia and Reactivity in Decision Making as Cognitive Variational Inequalities

Hedy Attouch
I3M UMR CNRS 5149, Université Montpellier II,
Place Eugène Bataillon, 34095 Montpellier, France
attouch@math.univ-montp2.fr

Antoine Soubeyran
GREQAM, Université Aix-Marseille,
Centre de la Vieille Charité, 13236 Marseille Cedex, France
soubey@romarin.univ-aix.fr

Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

Received: March 1, 2005

In this paper, we modelize a Decision Process of an agent where Inertia and Reactivity Aspects help to converge to Stable Routines. A decision is a move. We consider an agent who can only take incremental decisions, moving step by step on an unknown landscape, due to limited knowledge of his environment, resources, efforts, energy, money or time constraints. The agent explores around to be able to compare incremental advantages and costs of moving (changing). The agent reaches a stable routine, stops moving, and prefers to stay there, when advantages to move are lower than costs to move. Inertia is modelized by costs to move quickly (Reactivity is the learned ability to move quickly in a cheap way). We apply our "Worthwhile to Move Approach" (Attouch/Soubeyran, 2005, [6]) to build "Cognitive Versions of Variational Inequalities" for "Second Order Dynamical Gradients Systems with Inertia" ("HBF" differential equations). In this model, we have mainly insisted on costs to move. Advantages to move (the other side of the balance which drives a "worthwhile to move decision") are detailed in Attouch/Soubeyran (2005), [6]. To save space the annex concentrates only on costs to move.

Keywords: Inertia and Reactivity, worthwhile to move process, costs to move, improving with cost of improving quickly, second order dynamical gradient process, differential inclusions, viscous and dry friction.

1. Introduction

In daily life, every agent has a lot of problems to solve. A problem is defined by a list of criteria or aspects which are not fully satisfied. In this case the problem will be said to remain partially solved or partially unsolved (this view depends of how the agent sees the bottle, half full or half empty, depending of his character, resilience, ability to deal with unsatisfaction, "aptitude au bonheur", neighbours opinions...). The degree of resolution of a problem is the vector of the degree of resolution of each criteria which matter to define the problem. The agent feels some unsatisfaction (deception) when at least a problem remains partially solved or unsolved. This means that the degree of resolution of some criteria which matter to define this problem are not at their maximum, if a maximum can be defined precisely.

Problems to solve are of various kinds: in Economics a consumer can be hungry and

ISSN 0944-6532 / $ 2.50

feels some unsatisfaction not to eat enough in terms of quantity and quality or can be unsatisfied to buy a good at a too high price, a producer can search for a new technology (a new way to produce a final good), in Psychology an agent can search for a higher social status...

The agent feels some satisfaction (contentment) when he has succeeded in improving the degree of resolution of some criteria which characterizes a partially solved problem. We define the position or performance of an agent by the vector of degree of resolution of all the criteria which matter to define his unsolved problems. A choice is defined as a move from one position (performance) to another one. In real life, an agent does not optimize. He only tries to improve his position, more or less, moving from one position to a better one, where he succeeds in improving the degree of resolution of some criteria which characterizes some partially unsolved problems.

But to improve his position, an agent will have to bear some (several) costs of improving. The simplest decision rule he can use is to compare “Advantages and Costs to Move" and take care that “Improving implies some Costs of Improving".

Costs of Moving are central to our model, as well as Advantages to Move, using some Exploration Process to evaluate this balance or comparison at each step.

We have developed in Attouch/Soubeyran [5] a basic model which modelizes this balance between advantages and costs to move, ”Improving and Satisficing Enough by Exploration and Exploitation on an Unknown Landscape”. In that paper [5] and the revised version ”From Procedural Rationality to Routines: A Worthwhile to Move Approach of Satisficing with Not Too Much Sacrificing” (2005), we proposed a Cognitive proof of the Ekeland Theorem as a by product (see also Martinez-Legaz/Soubeyran, 2003, [34] for a model of Choice by Rejection of Alternatives which do not Improve Enough, but without explicit comparisons between advantages and costs of improving).

In the present paper, our goal is to examine such incremental dynamic decision process which balances, step by step, between advantages and costs to move when the speed of decision making matters much, especially when moving quickly is costly. In these cases, Costs to move present Inertia aspects. At the opposite, Reactivity is the ability to move quickly, at a low cost. ”Learning to Learn” is the best way to become more reactive. This is the major goal of all modern organization, after a first phase of mass production during the industrialization period, and a second phase of product differentiation. Agents and modern organizations enter now in a new phase where reactivity represents a major ability to be able to be the first to capture new opportunities and to enter new markets.

In these two cases Second Order dynamics are at work. The second order gradient process with Inertia of Attouch et al. [7], [4] (2000, 2002) presents such a cognitive aspect. The HBF and DIN equations (Attouch et al.) are important special cases (as an equality) of these kinds of ”Cognitive Variational Inequalities” described in Attouch/Soubeyran (2004, 2005) [5], [6].

2. Solving a Problem as Moving on an Unknown Landscape

Consider an agent who has a problem to solve, some action to do, a choice to make. Choosing is an action, a move on an unknown landscape.

The agent tries to solve this problem step by step, i.e., incrementally, and at the same time tries to make more explicit an initial rather vague and unknown objective, i.e., the unknown utility of the partial degree of resolution of the problem. The utility of the full resolution of the problem is unknown (the objective is not necessarily bounded). Then, at each step, the unsatisfaction of the agent is not precisely defined, while the contentment or satisfaction is revealed after exploration around.

Let $x = (x_i)_{i \in I} \in X$ be a performance, i.e., the vector of all the degrees of resolution $x_i \in [0, \overline{x_i}] \subset \mathbb{R}$, $\overline{x_i} \leq +\infty$, of all criteria $i \in I$ that the agent considers concerning this problem. The set I is the set of all criteria which matter for this problem. Let $g(x)$ the per unit of time gross utility (or valence) of the degree of resolution x of this problem. For example $g(x) = \sum_{i \in I} \mu_i g_i(x_i)$, where each real number $g_i(x_i) \in \mathbb{R}$, is the per unit of time gross utility of the degree of resolution $x_i \in [0, \overline{x_i}]$ of each criterion $i \in I$, and $\mu_i > 0$ are weights that the agent attaches to each criterion, relative to his problem. Let $\overline{g} = \sup \{g(x), x \in X\} \leq +\infty$.

If $g(.) : x \in X \longmapsto g(x) \in \mathbb{R}$, is bounded, (i.e., $\overline{g} < +\infty$, and more precisely $\overline{g} = 1$ when the problem is completely solved) and if the agent starts from some given performance $x \in X$ where he feels some unsatisfaction, i.e., $[\overline{g} - g(x)] > 0$, and is able to move to a new performance y such that $g(y) > g(x)$, his contentment will be $\delta(x, y) = [g(y) - g(x)] \,/\, [\overline{g} - g(x)] > 0$, while his deception will be $\omega(x, y) = [\overline{g} - g(y)] \,/\, [[\overline{g} - g(x)]]$, for $[\overline{g} - g(x)] > 0$. This was the case for our Cognitive approach of the Ekeland Theorem where $\overline{g} < +\infty$ (see Attouch/Soubeyran, 2005, [6]).

3. The Present Exploration-Exploitation Phase $[x]$

We suppose, as in Attouch/Soubeyran (2005), [6], that the agent does not know globally his landscape, made both of the per unit of time gross utility $g(.)$ and the local physical and physiological moving costs $C(x, .)$ to be defined later, but can discover them only locally, step by step, by exploring around a given known present performance $x \in X$. At each step, the agent builds or tries to discover his unknown landscape around some performance x, namely $(g(.), C(x, .))$.

In our model, the agent wants to solve a problem quickly enough, without suffering of too much temporary sacrifices. The goal of the agent is to improve his per unit of time gross utility step by step, by exploring around, to try to reach his ambition level or final goal $\widehat{g}$, which is an estimation of the unknown supremum $\overline{g}$ of $g(.)$. Each step, the agent has unknown costs to move and hence intermediary payoffs. He wants to satisfy step by step endogenous constraints, to be discovered each step by exploration. These constraints impose not too low (for example non negative) intermediary payoffs, which are the difference between local advantages and physical and physiological costs to move. They represent the desire to make, each step, not too high temporary sacrifices, because the agent has some preference for the present. Exploration costs can be chosen step by step and are lost once they have been spent (like fixed costs). They are very different from physical and physiological costs to move. They can be included in a lot of different ways in this balance between step by step local advantages and costs to move: from time to time, at the end, or at each period, in an adaptative way or not... The way the agent includes these costs of exploration in the balance modifies the speed of improvement of his gross utility. The case of a group of agents is even more complex, because the group can escape to time constraints by hiring engineers to explore around. To save space, we will present here one of the simplest way to include costs of exploration in the balance, restricting our model to the case of one agent.

For us exploration is not a move. It is a static phase of preparation to action, before moving. It represents the preliminary phase necessary to estimate advantages and costs to move, by mental simulation, gathering information,..., estimation...

Consider a given period of length $t(x)$ where, starting from a known given performance $x \in X$ and a known valence $g(x)$, the agent both

i) first tries to enjoy for the per unit of time gross utility $g(x)$ of his present performance $x \in X$, (note that the per unit of time net utility will include some per unit of time maintenance costs, see below) and

ii) tries also to explore around to try to improve enough from performance x to a new performance y which improves with respect to $x : g(y) - g(x) > 0$.

Let $\lambda(x)t(x)$, $0 \leq \lambda(x) \leq 1$ be the portion of time spent to exploit (to enjoy the per unit of time gross utility $g(x)$) during the phase of length $t(x)$, which offers to the agent the total gross utility $\lambda(x)t(x)g(x)$, spending some costs of maintenance $\lambda(x)m(x)t(x) > 0$, which are costs to be able to repeat performance x during this portion of time. The per unit of time net utility $u(x) = g(x) - m(x)$ is in fact the difference between a per unit of time revenue or gross utility $g(x)$ and a per unit of time cost of maintenance $m(x)$.

The total net utility to repeat performance x over a portion of time of this period is $U(x) = \lambda(x)t(x)u(x)$.

Let $[1 - \lambda(x)]\, t(x)$ be the portion of time spent to explore around x, spending a per unit of time effort of exploration $k(x) \geq 0$. The total costs of exploration are, over the period

$$K(x) = [1 - \lambda(x)]\, t(x)k(x).$$

During this exploration-exploitation period the total intermediary payoff is $F(x) = U(x) - K(x)$, i.e., $F(x) = t(x)f(x)$, where

$$f(x) = \lambda(x)g(x) - [\lambda(x)m(x) + [1 - \lambda(x)]\, k(x)]$$

is the per unit of time intermediary payoff (net gain).

The lenght $t(x)$ of the exploration-exploitation period is a choice variable which partially determines the strenght of the exploration process. This per unit of time intermediary payoff $f(x)$ can be troublesome because it includes both exogenous and endogenous terms. The per unit of time gross utility and maintenance costs $(g(x), m(x))$ are exogenous (given) and must be discovered, each step. The long term goal of the agent is to improve step by step, from $g(x)$ to $g(y) > g(x)$. The per unit of time cost of exploration $k(x)$ is a choice variable which determines partially the intensity of the exploration process, together with the choice of $t(x)$ and $\lambda(x)$.

To simplify the exposition, and to better understand what is given (exogenous) and what can be chosen (endogenous) we will consider the simple polar case of identical and constant per unit of time maintenance and exploration costs: $m(x) = k(x) = k > 0$ for all $x \in X$. We will also suppose a constant portion of time devoted to exploration, each period: $0 < \lambda(x) = \lambda < 1$.

Then, the per unit of time intermediary payoff of the agent reduces to $f(x) = \lambda g(x) - k$ and $f(y) - f(x) = \lambda\, [g(y) - g(x)]$. These remarks will greatly simplify the expression of advantages to move $A(x, y)$, to be defined later. This model supposes that, each step, the agent is motivated to explore, because he wants to reduce step by step his remaining unsatisfaction $\widehat{g} - g(x) > 0$ by improving his valence $g(.)$. Animal spirits (motivation to improve, like curiosity, needs...) push him to explore again and again. Once he has discovered a new improving performance such that $g(y) > g(x)$, the agent knows that he will explore again to try to move and improve again in a near future.

In this model, we do not address the much more complicated and fascinating questions like: how much to explore, when to stop exploration, when to recover or to do not include exploration costs in the step by step balance between advantages and costs to move, in order to improve the speed of improvement? (the trade off between higher temporary sacrifices and a higher speed of improvement, the quality, the speed and the cost of decision making).

Starting from $x \in X$, the exploration costs $K(x)$ allow the agent to explore around performance x, within the exploration set $E\left[x, K(x)\right]$.

This exploration process allows the agent to discover both

i) the a priori unknown per unit of time gross utility $g(y)$, and the per unit of time maintenance costs $m(y)$ for all $y \in E\left[x, K(x)\right]$.
ii) the unknown Physical and Physiological Costs to move from x to y, namely $C(x, y)$ for all $y \in E\left[x, K(x)\right]$.

The Exploration phase can be divided into two successive steps .

First, the agent has to materialize his motivation to move. The agent will try to improve his valence from $g(x)$ to an unknown $g(y) > g(x)$, $y \in E\left[x, K(x)\right]$. Let $I\left[x, K(x)\right] \subset E\left[x, K(x)\right]$ be the subset of performances which improve his valence with respect to $x \in X$. For each $y \in I\left[x, K(x)\right]$, the incremental per unit of time advantages to move are $[g(y) - g(x)] > 0$. But, before having the ability to exploit (to benefit of) the incremental improvement $g(y) - g(x) > 0$, there is, within the exploration phase, a second step which is necessary to evaluate and estimate the costs to move from x to y, namely the physical and physiological costs to move $C(x, y)$.

Then, a new dynamic phase $[x, y]$ follows, moving from the present static phase of exploration-exploitation $[x]$ to a new static one $[y]$.

4. The Moving Phase $[x, y]$

Suppose that the agent has decided to move from x to y, leaving a temporary routine phase, or static period of exploitation-exploration $[x]$, to enter a new temporary routine phase $[y]$. He is now able to decide to move or not (to choose between moving or not) because he has estimated by exploration during this initial period $[x]$, the balance between advantages and costs to move. More precisely, he has estimated both the per unit of time advantage to move $f(y) - f(x) = \lambda\left[g(y) - g(x)\right] > 0$ (be careful, not the total advantage to move $A(x, y)$ to be defined later) and the total cost to move $C(x, y) \geq 0$ from x to y. Let $t(x, y) = h > 0$ be the estimated time spent to physically move from x to $y \neq x$, and $c(x, y) > 0$ the per unit of time cost to move spent for that purpose. Then $C(x, y) = c(x, y)t(x, y) = c(x, y)h$. Let $d(x, y) \geq 0$ be the distance between performance x and y, and $e(x, y) \geq 0$ the effort, or per unit of distance cost to move. Then the physical cost of moving is $C(x, y) = e(x, y)d(x, y)$. It depends both on the effort $e(x, y)$, and on the distance between performances x and y. The modulus of the speed of moving from the present exploitation-exploration phase $[x]$ to a future one $[y]$ is $\rho(x,y) = d(x,y)/t(x,y)$.

The relation between the per unit of time cost to move $c(x,y) = C(x,y)/t(x,y)$ and the per unit of distance cost to move $e(x,y) = C(x,y)/d(x,y)$ is

$$t(x,y)c(x,y) = d(x,y)e(x,y) \Longleftrightarrow c(x,y) = \rho(x,y)e(x,y).$$

Notice that the per unit of time advantage to move $\lambda\,[g(y) - g(x)] > 0$ and the cost to move $C(x,y)$ from x to y are known by the agent within his exploration set, i.e., for all $y \in E\,[x, K(x)]$.

The agent has a Bounded Rational behavior because he has chosen to pay (via exploration costs) for the ability to be able to compare locally advantages and costs to move. This definition of Bounded Rationality is more precise and more operational than the one pioneered by Simon (1955), [45].

5. The Future Exploitation-Exploration Phase $[y]$

If we consider the temporary routine phase $[x]$ where the agent both exploits and explores, at some time the agent has to decide when to leave this temporary routine phase $[x]$ (i.e., to choose the length $t(x)$ of this ongoing present phase) to move and enter a new temporary routine phase $[y]$.

The agent continues the temporary routine phase $[x]$, as long as he has not founded (estimated), by exploration, some new performance y whose estimated valence improves, i.e., $g(y) - g(x) > 0$. Then, the agent has to estimate the cost of moving $C(x,y)$ to be able to compare advantages and costs of moving. We turn now to the estimated total incremental advantages to move $A(x,y)$. They depend on the length $t(y)$, chosen ex ante, (it is a choice variable for the agent) of the new temporary routine phase $[y]$ and of the (per unit of time) incremental advantages to move, $f(y) - f(x) = \lambda\,[g(y) - g(x)] > 0$. Notice again that the length $t(y)$ of the new temporary routine phase $[y]$ is a choice variable for the agent, as well as the length $t(x)$ of the previous temporary routine phase $[x]$.

These estimated total incremental advantages to move are

$$A(x,y) = t(y)\,[f(y) - f(x)] = \lambda t(y)\,[g(y) - g(x)] > 0.$$

The Costs to move of the agent incorporate a hidden part, an Opportunity cost, which represents the loss of exploitation during the moving phase $([x]\,,[y])$ of length $t(x,y)$. These opportunity costs are $O(x,y) = t(x,y)\,[g(x) - k] = h\,[g(x) - k]$, if $t(x,y) = h$, because the agent stops to exploit during the moving period.

6. The Decision to Move or Not: Dynamic Gains

The decision to move. Within his exploration set, the agent decides to move from some performance x to a new performance $y \in E\,[x, K(x)]$, if the advantages to move are greater than the costs to move, i.e., if

$$A(x,y) \geq C(x,y) + \chi(x)O(x,y),$$

where $\chi(x) \in \{0,1\}$.

The agent can decide to include or not the opportunity costs to move $O(x, y)$. If he decides to do it, then $\chi(x) = 1$, while if he does not include them, $\chi(x) = 0$. The advantages to choose to do not include them is to be able to obtain more easily an y such that advantages to move are greater than costs to move. To accept some sacrifices can be rewarding because this can speed up the improvement process by saving some time necessary to find some new performance such that advantages to move be higher than costs to move (this constraint becomes easier to satisfy).

Each step, because of a persistent feeling of unsatisfaction, as long as the residual unsatisfaction $(\widehat{g} - g(x)) > 0$ persists, the agent will have the motivation to choose to pay some preliminary costs of exploration around x (time, effort, physical and psychological costs and money). Then, he will be able to decide in a limited rational way (only around x, within $E[x, K(x)]$) when to decide to move. This will be the case when, for some $y \in E[x, K(x)]$, the estimated advantages to move $A(x, y)$ are greater than the estimated costs to move.

Dynamic Gain. The dynamic gain of an agent groups together the costs of moving $C(x, y) + \chi(x)O(x, y)$ attached to the moving period $[x, y]$ and the future intermediary payoff $t(y)f(y)$, attached to the future period $[y]$.

It does not include the present period $[x]$ in the balance which helps to decide between moving or not from x to y. This dynamic gain is, when the agent decides to move,

$$G[[x, y], [y]] = t(y)f(y) - C(x, y) - \chi(x)O(x, y).$$

If the agent decides to stay at x, the dynamic gain reduces to $G[(x, x), [x]] = t(y)f(x) - C(x, x) - \chi(x)O(x, x) = t(y)f(x)$, because both $C(x, x)$ and $O(x, x)$ are equal to zero.

The difference $\Delta(x, y) = G[[x, y], [y]] - G[([x], [x]), [x]]$ is equal to

$$\Delta(x, y) = A(x, y) - C(x, y) - \chi(x)O(x, y).$$

The agent prefers to stay at x, i.e., not to move iff $\Delta(x, y) < 0$ for $y \in E[x, K(x)], y \neq x$.

7. Stable Routines

If no $y \neq x$ exists such that $A(x, y) \geq C(x, y) + \chi(x)O(x, y)$, either the agent chooses to explore more, in a set $E'[x, K'(x)]$ larger than $E[x, K(x)]$, or he will choose to stop exploration and prefer to stay at x. In this case, performance x will be said to be a stable routine.

Definition 7.1. $x^* \in X$ is a stable routine with respect to the exploration map $x \in X \longmapsto E[x, K(x)]$, iff $A(x^*, y) < C(x^*, y) + \chi(x^*)O(x^*, y)$ for all $y \in E[x^*, K(x^*)]$, $y \neq x^*$.

In our dynamic model of decision making the agent behavior is governed by Bounded Rationality in a sense similar to, but more precise, than the formulation given by H. Simon (1955), [45]. The main point to be understood is that the agent has first to decide to explore around if he wants to be able to choose between moving or not. To choose between moving or not is costly: ex ante exploration costs are necessary to be a bit rational (i.e., to give some partial justification to prefer to move from x to y than to stay at x). The agent has the obligation to pay some fixed cost to have the permission to see and to be able to compare between moving or not, taking this decision in a limited rational way. Then, it is costly for an agent to be able to rationalize a decision (to give himself some good reasons to move or not), even if, finally, the agent decides to do not move. For us this is the essence of Bounded Rationality.

8. Incremental Improving and Worthwhile Decision Process

We will say that it is worthwhile for the agent to move from x to y, after some exploration around x, iff the advantages to move are greater than the costs to move, i.e., iff $y \in W(x)$, where

$$W(x) = \big\{y \in E[x, K(x)] : A(x,y) \geq C(x,y) + \chi(x)O(x,y), \\ \text{i.e., } \Delta(x,y) \geq 0\big\}.$$

Our model presents a strong Anchoring effect (Kahneman/Tversky (see Gilovich/Griffin/Kahneman, 2002)) where the agent decision is based on a comparison with what he has just done before and not between any two positions, starting from nowhere. Our model of choice formalizes incremental improving moves around successive performances. We will consider a sequence of successive worthwhile moves $x_{n+1} \in W(x_n), n \in \mathbb{N}$, where $x = x_n$ and $y = x_{n+1}$:

$$\begin{aligned} y \in W(x) &\Longleftrightarrow A(x,y) \geq C(x,y) + \chi(x)O(x,y) \\ &\Longleftrightarrow \lambda t(y)\,[g(y) - g(x)] \geq c(x,y)t(x,y) + \chi(x)\,[g(x) - k]\,t(x,y). \end{aligned}$$

Let $t(x,y) = h \geq 0$ and $t(y) = [1 + \gamma(x,y)]\,h \geq 0$ where $1 + \gamma(x,y) > 0$. Then $y \in W(x)$ iff

$$\begin{aligned} &\lambda\,[1 + \gamma(x,y)]\,h\,[g(y) - g(x)] \geq [\chi(x)\,[g(x) - k] + c(x,y)]\,h \\ \Longleftrightarrow\ &\lambda\,[1 + \gamma(x,y)]\,[g(y) - g(x)] \geq \chi(x)\,[g(x) - k] + c(x,y). \end{aligned}$$

We have $c(x,y) = \rho(x,y)e(x,y)$. This leads us to the following definition:

Definition 8.1. It is worthwhile to move from x to y iff $y \in W(x)$, i.e.,

$$\lambda\,[1 + \gamma(x,y)]\,[g(y) - g(x)] \geq \chi(x)\,[g(x) - k] + \rho(x,y)e(x,y).$$

9. Inertia: Costs to Move Quickly

Costs to move present Inertia when it is costly to move quickly, i.e., costly to change speed in a short period of time, for example when the agent starts from a zero speed, or stops to end with a zero speed. This is the case when the cost to move per unit of distance $e(x,y)$ is an increasing function, both of the distance $d(x,y)$ from x to y, the initial speed $v = v(x,y)$ at x, and of the variation of the speed, Δv. Formally $e(x,y) = e\,[d(x,y), v, \Delta v]$.

Suppose that the set of performances X is a Hilbert space, with the distance $d(x,y) = \|y - x\|$ between performances x and y. Let $t(x,y) = h > 0$ be the estimated time spent to physically move from x to $y \neq x$, and $v(x,y) = v = (y - x)/h$ the vectorial speed to move from x to y. We have $d(x,y) = \|y - x\| = h\,\|v(x,y)\|$. The norm of the speed to move from x to y is $\rho(x,y) = \|v(x,y)\| = d(x,y)/t(x,y)$.

The total cost to move from x to y is $C(x,y) = c(x,y)t(x,y) = c(x,y)h$, where $c(x,y) > 0$ is the per unit of time cost to move from x to y. Let $e(x,y) \geq 0$ be the effort or cost to move per unit of distance. Then, the physical cost of moving is $C(x,y) = e(x,y)d(x,y) = e(x,y)\,\|v(x,y)\|\,h$. It depends both on the effort, or cost per unit of distance $e(x,y)$, and on the distance between performances x and y.

Remember that the relation between the per unit of time cost to move $c(x,y) = C(x,y)/t(x,y)$ and the per unit of distance cost to move $e(x,y) = C(x,y)/d(x,y)$ is

$$\begin{aligned} & t(x,y)c(x,y) = d(x,y)e(x,y) \\ \iff & c(x,y) = \|v(x,y)\|\,e(x,y) = \rho(x,y)e(x,y). \end{aligned}$$

Then $y = x + hv \in E\,[x, K(x)]$ is equivalent to $v = (y-x)/h \in F\,[x, K(x)]$, where $E\,[x, K(x)] = x + hF\,[x, K(x)]$.

Our basic example will be the "linear case" where

$$e(x,y) = e[d(x,y), \langle v/\|v\|, \Delta v\rangle] = \widetilde{\alpha} d(x,y) + \widetilde{\beta}\langle v/\|v\|, \Delta v\rangle,$$

i.e.,

$$e(x,y) = \widetilde{\alpha}\|v\|h + \widetilde{\beta}\langle [v/\|v\|], \Delta v\rangle, \quad \text{with } \widetilde{\alpha} > 0, \text{and } \widetilde{\beta} \geq 0.$$

This formulation describes Inertia. It means that the cost or effort to move $e(x,y)$, per unit of distance, is first an increasing function of the norm of the speed to move $\|v\|$ and of the time spent to move h. It also increases with the improvement of the speed to move Δv, more and more as this speed is higher. Inertia represents the difficulty to move quickly, which increases with the speed of movement. Consider to simplify the case where the agent moves to the right, on the real line $x \in R$. Our formulation modelizes the two opposite cases where

i) moving is more costly when the agent increases his speed of moving, from $v > 0$ to $v + \Delta v$, with $\Delta v > 0$. Then $\langle v, \Delta v\rangle$ is > 0 and the cost of moving increases.
ii) moving is less costly when the agent reduces his speed of moving. Suppose that the agent reduces his speed from $v > 0$ to $v + \Delta v$, with $\Delta v < 0$. Then $\langle v, \Delta v\rangle$ is < 0 and the cost of moving decreases.

Take the ratio between the length of an "exploration-exploitation" period and the length of a moving period, $-1 < \gamma(x,y) \leq \gamma$ lower than a given constant $\gamma > 0$.

Take also $\chi(x) = 0$, i.e., the agent does not consider opportunity costs of moving. Then, a worthwhile move from x to y is such that

$$\begin{aligned} \lambda\,[1+\gamma]\,h\,[g(y) - g(x)] &\geq c(x,y)h = \|v(x,y)\|\,e(x,y)h, \quad \text{i.e.,} \\ \lambda[1+\gamma][g(y) - g(x)] &\geq \|v(x,y)\|e(x,y), \quad \text{i.e.,} \\ [1+\gamma][g(y) - g(x)] &\geq \|v\|\alpha\|v\|h + \beta\|v\|\langle [v/\|v\|], \Delta v\rangle \end{aligned}$$

where $\alpha = \widetilde{\alpha}/\lambda > 0$ and $\beta = \widetilde{\beta}/\lambda > 0$.

We finally obtain

Proposition 9.1. *In the linear case, it is worthwhile to move from x to y iff*

$$[1+\gamma][g(y) - g(x)]/h \geq \alpha\langle v, v\rangle + \beta\langle v, \Delta v/h\rangle.$$

10. Convergence in the Discrete Time Version

The central questions for these "Dynamic Worthwhile to Move Process" are convergence, and finite time convergence.

Consider first the discrete time version. In this discrete time framework, a worthwhile to move process is defined by the variational inequality $x_{n+1} \in W(x_n)$, $n \in \mathbb{N}$, such that

$[1+\gamma]\ [g(x_{n+1}) - g(x_n)]/h_n \geq \alpha\langle v_n, v_n\rangle + \beta\langle v_n, (\Delta v_n/h_n)\rangle$, where $v_n = [x_{n+1} - x_n]/h_n$ and $\Delta v_n = v_{n+1} - v_n$.

There are two polar cases: i) for an isolated agent, or ii) for a group. An agent alone is limited by time constraints. A group can escape to them partially.

i) With Opportunity Costs for an agent alone. The worthwhile to move condition is $[1+\gamma(x,y)]\,[g(y) - g(x)] \geq \chi(x)g(x) + \rho(x,y)e(x,y)$. Let $\chi(x) = 1$.

It can be written in the discrete time case as

$$[1+\gamma_n]\,[g(x_{n+1}) - g(x_n)] \geq g(x_n) + \rho_n e_n.$$

Suppose that the ratio $\gamma(x,y) = \gamma_n$ between the length of a moving phase and an exploration-exploitation phase is bounded above $-1 < \gamma_n \leq \gamma$ for all $n \in N$. Because g is bounded above, and $g(x_n)$ is not decreasing, then $g(x_n) \longrightarrow g^*$, and $[g(x_{n+1}) - g(x_n)] \longrightarrow 0$. Then the right hand side of the inequality tends to zero, and the right hand side is greater than g^*. The process stops in finite time.

ii) Without Opportunity Costs for a Group. This is the case for a Group which can escape to time constraints by asking workers (engineers) to explore around. In this case, opportunity costs disappear. Exploration costs are external to the agent and just add to other costs. More formally, the length of the exploitation period is $t(x)$, (and not a fraction of it, $\lambda t(x)$). During this period $t(x)$ the gross utility of the group is $t(x)g(x)$, exploration costs are $K(x) = t(x)k(x)$ and maintenance costs are $t(x)m(x)$. The intermediary payoff of the group over this period is $F(x) = t(x)\,[g(x) - m(x) - k(x)]$. In the case of one agent (alone) we have $F(x) = t(x)\,[\lambda g(x) - \lambda m(x) - (1-\lambda)k(x)]$, $0 < \lambda \leq 1$. During the moving phase of length $t(x,y)$ the agent (a group, a firm) continues to fully exploit and stops to explore around. The group gets $t(x,y)\,[g(x) - m(x)]$ from exploitation during this period. If he decides to move, he pays the moving costs $C(x,y)$. If not, he pays nothing more. The difference between its net gain between moving or not during the moving period is only the moving cost $C(x,y)$. Then, Opportunity costs disappear. We are in the situation examined by Adly/Attouch/Cabot (2004), [1].

11. Continuous Time Version

Consider a path of "worthwhile moves" $t \in T \longmapsto x(t) \in X$. Let $h \longrightarrow 0$. Then, we get the cognitive variational inequality $\dot{z}(t) \in \Omega[z(t)]$ iff

$$[1+\gamma]\langle \nabla g[x(t)], \dot{x}(t)\rangle \geq \langle [\alpha[\dot{x}(t)] + \beta[\ddot{x}(t)]],\ [\dot{x}(t)]\rangle,$$

where $z(t) = [x(t), \dot{x}(t)]$. The HBF [7] and DIN [4] Equations of Attouch et al. (2000, 2002, ...) represent important special cases of our "Dynamic Worthwhile Process to Move" as equalities rather than as inequalities. The dynamic gain of the agent is

$$\Delta[z(t), \dot{z}(t)] = [1+\gamma]\nabla g[x(t)] - \alpha[\dot{x}(t)] - \beta[\ddot{x}(t)].$$

The Cognitive Variational Inequality is

$$\langle \Delta[z(t), \dot{z}(t)], \dot{x}(t) \rangle \geq 0 \Longleftrightarrow \dot{z}(t) \in \Omega[z(t)].$$

The function $\Phi[z(t)] = [1+\gamma]g[x(t)] - \frac{\beta}{2}\|\dot{x}(t)\|^2$ is the dual of an energy function.

Comments about Inertia:

Agents change only with difficulty, but why is change difficult? Organizational Ecology defines Inertia as change which is slow relative to environment (Hannan and Freeman, 1984, Rumelt, 1990). This comes from specialized investments in physical assets and social structures. Organizations have "cores" which are very difficult to change relatively to more peripheral elements. Evolutionary Economics explains Inertia via Bounded Rationality, Routines, and Tacitness (Nelson/Winter, 1982).

Routines bound skills and capabilities. They are the skill set and the memory of the agent.

In Management Sciences, the main sources of Inertia are five. Distorted Perception-Dulled Motivation-Failed Creative Response-Political Deadlocks-Action Disconnects. Distorted Perception coming from Myopia, (the unability to look into future with clarity), Denial (a defensive behavior, which is the rejection of information that is contrary to what is desired or what is believed to be true). Denial may stem from hubris or from fear. Information Filtering rejects information which is unpopular, unpleasant or contrary to doctrine. Grooved Thinking rejects informations which deviate too much from orthodoxy and mental habits. Dulled Motivation describes the lack of sufficient motivation to change, due to the abandonment of costly sunk specific investments. Failed Creative Responses concerns the difficulty to choose a direction because of the complexity of the choice, the speed of change (things happen too fast), inhibition and reactive mind-set (problems are natural and inevitable), or inadequate strategic vision. Political Deadlocks come from the three main sources of disagreement among agents: difference in personal interest, difference in beliefs, and difference in fundamental values. Action Disconnects comes from leadership inaction (lack of vision, lack of leading by example, attachment to status quo, embedding in routines in complex process with great tacit aspects, where changing one part of the process requires to change a lot of other parts.

12. Cognitive Second Order Gradient Process with Inertia

In this section, we consider some continuous time versions of our cognitive variational inequality. We show that the modelization of inertia via costs to move quickly is quite flexible and allows to cover various situations. Of particular interest are the trajectories for which equality (instead of inequality) holds, which is precisely the case of the dynamics associated to second order gradient process with inertia. This important fact provides a natural bridge between our model and dissipative dynamical systems with inertia from physics and mechanics.

A) "Viscous friction" and costs to move quickly. That's the basic example which has been introduced in the preceding section. Noticing that $d/dt[(1/2)\|\dot{x}(t)\|^2] = \langle \ddot{x}(t), \dot{x}(t) \rangle$, the continuous time version of the cognitive variational inequality can be rewritten in the following form:

$$(1+\gamma)d/dt[g[x(t)]] \geq \alpha\|\dot{x}(t)\|^2 + [\beta/2]d/dt[\|\dot{x}(t)\|^2].$$

The cognitive interpretation of this differential variational inequality is clear: trajectories which satisfy are such that the instantaneous rate of gain is greater than the instantaneous costs of changing position and velocity. This variational inequality bears close connections with mechanics and physics. To see this, let us rewrite it in the following form

$$d/dt[(1+\gamma)g[x(t)] - (\beta/2)\|\dot{x}(t)\|^2] \geq \alpha\|\dot{x}(t)\|^2.$$

If we think to g (a gain function which is to maximize) as the opposite of φ (a cost function which is to minimize), and after changing signs, we obtain,

$$d/dt[(1+\gamma)\varphi[x(t)] + (\beta/2)\|\dot{x}(t)\|^2] \leq -\alpha\|\dot{x}(t)\|^2.$$

The physical interpretation is now clear. The quantity $E[x(t), \dot{x}(t)] = (1+\gamma)\varphi[x(t)] + (\beta/2)\|\dot{x}(t)\|^2$ is the global energy of the system represented by $x(t)$ at time t, it is the sum of the potential energy $(1+\gamma)\varphi[x(t)]$ and of the kinetic energy $(\beta/2)\|\dot{x}(t)\|^2$. The above variational inequality

$$d/dt[E[x(t), \dot{x}(t)]] \leq -\alpha\|\dot{x}(t)\|^2$$

expresses that, along trajectories of the differential variational inequality, the global energy is dissipated ($\alpha \geq 0$) following a certain rule. We have a dissipative dynamical system with the physical energy as a Liapunov function. Symmetrically, when returning to the cognitive situation with $g = -\varphi$, we have an expansive dynamical system with entropy Ψ,

$$\Psi[x(t), \dot{x}(t)] = (1+\gamma)g[x(t)] - (\beta/2)\|\dot{x}(t)\|^2$$

and the cognitive variational inequality becomes

$$[d/dt]\Psi[x(t), \dot{x}(t)] \geq \alpha\|\dot{x}(t)\|^2.$$

Of particular interest is the case when equality holds, i.e.,

$$[d/dt]\Psi[x(t), \dot{x}(t)] = \alpha\|\dot{x}(t)\|^2.$$

Solutions of this differential equation can be obtained (among others) by considering the trajectories of the "heavy ball with friction" (HBF in short) dynamical system (which is expressed here with $g = -\varphi$),

$$\text{(HBF)} : \beta\ddot{x}(t) + \alpha\dot{x}(t) - (1+\gamma)\nabla g[x(t)] = 0.$$

Indeed, we can easily verify that for any trajectory $x(.)$ of (HBF) the following equality holds (take the scalar product with $\dot{x}(t)$ in (HBF)):

$$\beta\langle\ddot{x}(t), \dot{x}(t)\rangle + \alpha\|\dot{x}(t)\|^2 - (1+\gamma)\langle\nabla g[x(t)], \dot{x}(t)\rangle = 0$$

that is,

$$[d/dt]\left[(1+\gamma)g[x(t)] - (\beta/2)\|\dot{x}(t)\|^2\right]] = \alpha\|\dot{x}(t)\|^2.$$

Because of its central role in physics ((HBF) is a model example of a damped oscillator), the (HBF) equation has been intensively studied in the linear case (g quadratic). More recently, the interest of this dynamical system in optimization has been put to the fore in Attouch/Goudou/Redont (2000), [7]. Thanks to its inertial properties it allows to explore local maxima of g, see also Aubin/Lesne (2004), [9] for a related but different approach. In such a context, it is important to be able to work with a large class of gain function g (or potential φ). A central question is the study of the asymptotical behavior, as time $t \longrightarrow +\infty$, of the trajectories of (HBF). Noticing that $(1+\gamma)[d/dt]g[x(t)] \geq \alpha\|\dot{x}(t)\|^2$, one can easily obtain, after integration, and using the assumption $\sup\{g(x), x \in X\} < +\infty$ that $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$. When g is smooth, by using (HBF) equation, one easily obtains that, for any bounded trajectory of (HBF), one has $\sup\{\|\ddot{x}(t)\|,\ t \geq 0\} < +\infty$. This, together with $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$, classically implies that $\lim_{t\longrightarrow+\infty} \dot{x}(t) = 0$. Returning to (HBF) one can easily derive that $\lim_{t\longrightarrow+\infty} \nabla[g[x(t)]] = 0$.

Note that this holds for any smooth gain function g which is bounded from above and for any bounded trajectory of (HBF). As a consequence, any limit point $\overline{x}$ of such a trajectory is a critical point of g ($x(t_k) \longrightarrow \overline{x}$ for $t_k \longrightarrow +\infty$). The crucial point is that, without any further assumption on g, it may happen that the whole trajectory does not converge (counterexample of Redont [7] which holds in $X = R^2$ with a C^1 function, and further improved by Jendoubi with a C^∞ function). Thus, one needs to make some further assumptions on g in order to ensure the asymptotical convergence of the trajectories of (HBF). The main cases where positive answers to this difficult question have been obtained are:

a) The convex case (i.e., g concave) has been solved by Alvarez [3]. The convergence result has been recently extended to the case of a quasi concave function by Goudou/Munier [24], [25]. The proof relies on Opial's lemma.

b) The real analytical case has been solved by Haraux/Jendoubi [26]. It relies on the Lojasiewicz inequality, see also Bolte [15].

c) When the critical points of g are isolated, the convergence of the trajectories of (HBF) follows from an easy topological argument. As a striking result, one can prove that for such Morse functions then, generically with initial data (position and velocity), the trajectories of (HBF) converge to a local maxima of g. All this subtle analysis comes from the fact that the cost of moving for small changes is low. This corresponds to viscous friction in physics and gives rise to the global energy expenses for moving costs $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$, which is not enough information in order to ensure convergence of the trajectories. This makes a big contrast with situations where the corresponding L^1 estimate holds $\int_0^{+\infty} \|\dot{x}(t)\| dt < +\infty$, which clearly implies the convergence of such a trajectory. This makes a natural transition with the next situation where we consider dry friction instead of viscous friction, in which case this L^1 estimate on the velocity holds.

B) "Dry friction" and costs to move quickly. Let us consider the case where $e(x, y)$, which is the cost to move per unit of distance between performance x and y, is given by $e(x, y) = \alpha + \beta\langle v/\|v\|, \Delta v\rangle$. We call it, and this will be justified below, the dry friction model. Note the big difference with the previous situation $e(x, y) = \alpha d(x, y) + \beta\langle v/\|v\|, \Delta v\rangle$ where small changes of position are costless (essentially free).

Now, in this dry friction model, small changes are costly, and we shall see that this is the main explanation of the good stabilization properties (as $t \longrightarrow +\infty$) of the corresponding differential variational inequality,

$$(1 + \gamma)d/dt[g[x(t)]] \geq \alpha\|\dot{x}(t)\| + \beta\langle\ddot{x}(t), \dot{x}(t)\rangle.$$

Equivalently

$$d/dt[(1+\gamma)g[x(t)] - (\beta/2)\|\dot{x}(t)\|^2] \geq \alpha\|\dot{x}(t)\|.$$

Clearly, for any trajectory $x(.)$ which satisfies the above inequality

$$\alpha \int_0^{+\infty} \|\dot{x}(t)\| dt \leq [1+\gamma][\sup g - g(x_0)] < +\infty,$$

and the convergence of $x(t)$ as $t \longrightarrow +\infty$ to some $\overline{x} \in X$ holds. Of particular interest are the trajectories provided by the following differential inclusion from mechanics ($X = R^n$ for simplicity) ("heavy ball with dry friction")

$$\beta\ddot{x}(t) + \partial\Phi[\dot{x}(t)] - (1+\gamma)\nabla g[x(t)] \ni 0,$$

where $\Phi : R^n \longmapsto R$ is a convex function which satisfies $\Phi(0) = 0$ and $\Phi(\xi) \geq \alpha\|\xi\|$ for all $\xi \in R^n$. Here $\partial\Phi$ stands for the subdifferential of convex analysis.

Note that the assumption $\Phi(\xi) \geq \alpha\|\xi\|$ for some $\alpha > 0$ is equivalent to $0 \in int\partial\Phi(0)$. In mechanics, the term $\partial\Phi[\dot{x}(t)]$ can be interpreted as a dry friction force. The model situation is $\Phi(\xi) = \alpha\|\xi\|$ in which case $\partial\Phi(\xi) = \alpha(\xi/\|\xi\|)$ for $\xi \neq 0$, and $\partial\Phi(0) = \alpha B(0,1)$. The mathematical analysis of the above system has been developed in Adly/Attouch/Cabot (2004), [1]. Let us first make the connection with our variational inequality: the convex subdifferential inequality $0 = \Phi(0) \geq \Phi[\dot{x}(t)] + \langle z, -\dot{x}(t)\rangle$ holds for any $z \in \partial\Phi[\dot{x}(t)]$. Taking account of the fact that $x(.)$ is a solution of the heavy ball with dry friction differential inclusion one can take $z = -\beta\ddot{x}(t) + (1+\gamma)\nabla g[x(t)]$ in the above inequality. One obtains

$$\langle -\beta\ddot{x}(t) + (1+\gamma)\nabla g[x(t)], \dot{x}(t)\rangle \geq \Phi[\dot{x}(t)].$$

Using $\Phi(\xi) \geq \alpha\|\xi\|$ (that's the dry friction assumption!), we get

$$d/dt[(1+\gamma)g[x(t)] - (\beta/2)\|\dot{x}(t)\|^2] \geq \alpha\|\dot{x}(t)\|$$

that's our variational inequality.

In the model case $\Phi(\xi) = \alpha\|\xi\|$, noticing that for $\xi \neq 0$ the set $\partial\Phi(\xi)$ is reduced to a single element $\alpha\xi/\|\xi\|$, we have for all $\xi \neq 0$, $\langle\partial\Phi(\xi), \xi\rangle = \langle\alpha\xi/\|\xi\|, \xi\rangle = \alpha\|\xi\|$ and for $\xi = 0$, $\langle\partial\Phi(\xi), \xi\rangle = 0 = \alpha\|\xi\|$. Note that $\partial\Phi(0)$ is bounded. In that case

$$d/dt\left[(1+\gamma)g[x(t)] - (\beta/2)\|\dot{x}(t)\|^2\right] = \alpha\|\dot{x}(t)\|$$

and we can say that trajectories of the heavy ball with dry friction provide equality in the case $\Phi(\xi) = \alpha\|\xi\|$, this as a particular case of the general differential inequality. The asymptotic behavior of the trajectories of the heavy ball with dry friction differential inclusion exhibits some striking properties:

1. As we already stressed, any trajectory $x(.)$ of this system converges, let $x(t) \longrightarrow \overline{x}$, as $t \longrightarrow +\infty$.
2. Any limit $\overline{x}$ is an equilibrium which satisfies $(1+\gamma)\nabla g(\overline{x}) \in \partial\Phi(0)$.
   In the model situation $\Phi(\xi) = \alpha\|\xi\|$, this expresses that $\|\nabla g(\overline{x})\| \leq \alpha/(1+\gamma)$. Taking $\varepsilon = \alpha/(1+\gamma)$, we have an equilibrium corresponding to Ekeland's ε variational principle.

3. Generically with initial data, convergence holds within a finite time, i.e., for each trajectory there exists a finite $T < +\infty$ such that $x(t) = \overline{x}$ for all $t \geq T$. For a detailed presentation of these results, one can consult Adly/Attouch/Cabot [1].

C) Costs to move far from the natural steepest descent. A natural dynamic in decision sciences, physics, ... is the steepest descent dynamics, i.e., $\dot{x}(t) = -\nabla g\,[x(t)]$. Indeed, $\nabla g\,[x(t)]$ provides to the agent the best local direction in which to move in order to make increase his gain function. Accordingly, the agent pays attention not to deviate too much from this natural direction and costs to move quickly are naturally attached both to $\|\dot{x}(t)\|^2$ and $\|\dot{x}(t) - \beta\nabla g[x(t)]\|^2$. This yields as a cognitive differential variational inequality, with $b \geq 0$,

$$d/dt[(1+\gamma)g[x(t)]] \geq \alpha\|\dot{x}(t)\|^2 + (b/2)[d/dt]\|\dot{x}(t)\|^2 \\ + [1/2]d/dt\|\dot{x}(t) - \beta\nabla g[x(t)]\|^2.$$

As a model example, let us consider the case $b = 0$ and $\gamma = \alpha\beta$, and show that the trajectories of the following dynamical system (DIN)

$$\ddot{x}(t) + \alpha\dot{x}(t) - \beta\nabla^2 g[x(t)]\dot{x}(t) - \nabla g[x(t)] = 0$$

are solutions of the above differential inequality.

Set $\Psi(t) = (1+\alpha\beta)g[x(t)] - (1/2)\|\dot{x}(t) - \beta\nabla g[x(t)]\|^2$ and compute $\dot{\Psi}(t)$. After derivation, and using the (DIN) equation, we obtain

$$\dot{\Psi}(t) = \alpha\|\dot{x}(t)\|^2 + \beta\|\nabla g[x(t)]\|^2.$$

Hence

$$d/dt[(1+\gamma)g[x(t)]] \geq \alpha\|\dot{x}(t)\|^2 + [1/2]d/dt\|\dot{x}(t) - \beta\nabla g[x(t)]\|^2.$$

The above dynamical system is called the dynamical inertial Newton-like system, (DIN) in short. It has been studied by Alvarez/Attouch/Bolte/Redont (2002), [4]. It is an interesting model for decision making, the geometrical damping term $\nabla^2 g\,[x(t)]\,\dot{x}(t)$ prevents undesirable oscillations. This open perspectives towards similar developments for gradient systems in the context of Riemannian metrics, which is a largely open question for second order dynamics.

13. Conclusion

In this paper, we have modelized Inertia and Reactivity Aspects of Decision Making which help to converge to stable routines. Inertia represents the costs to move quickly and Reactivity the (learned) ability to move quickly in a cheap way. We have applied our “Worthwhile to Move Model" (Attouch/Soubeyran, 2005, [6]) to build “Cognitive Versions of Variational Inequalities" for “Second Order Dynamical Gradients Systems with Inertia" (“HBF" differential equations). The model is the following.

A decision is a move. The agent is motivated to move to fill unsatisfied needs (advantages to move). The agent cannot fill his needs in one step. The agent is forced to accept to move step by step on his landscape, due to limited knowledge of his environment, resources, efforts, energy, money or time constraints. There are costs to move. The agent explores around to be able to compare incremental advantages and costs of moving (changing). The agent reaches a stable routine, stops moving, and prefers to stay there, when advantages to move are lower than costs to move. We have classified costs of moving in two branches: low (or high) costs to move locally (viscous or dry friction). This classification helps us to revisit Second Order Dynamical Gradient Process (Attouch et al., 2002). In this model, we have mainly insisted on costs to move. Advantages to move (the other side of the balance which drives a "worthwhile to move decision") are detailed in Attouch/Soubeyran (2005). To save space the annex concentrates only on costs to move.

Our model has already succeeded in giving a Cognitive approach of the "Ekeland ε Variational Principle". We expect a lot of other applications of our "worthwhile to move model" both in Economics-Management and in Game Theory, concerning Stable Routines as rest points of "Worthwhile to Move Dynamic Games" for a population or for groups and sub-groups. In Applied Mathematics, our model can allow us to give Cognitive versions of Proximal Algorithms, Decomposition Algorithms, Direct Search Methods... (see Attouch/Soubeyran, 2005, for a list of applications like the Ekeland Theorem and other possible applications).

14. Annex

Typology of Physical and Physiological Moving Costs: Costs to Change Actions. We classify Physical and Physiological Moving Costs in four categories: Costs of Dissimilarity, Costs to Start Actions (Fixed costs and Excitation costs), Costs to Stop (and Inihibition costs), and Costs of Reactivity (costs to move more quickly).

There are several types of Costs of doing (building and choosing) new actions which are influenced by the past. There are fixed costs to start an action (training or preliminary "echauffement"), costs to get some speed, costs to stop, and cancel your speed to be able to stay there. Inhibition costs are the costs to forget and leave what you are doing now to do something else. Excitation Costs (costs to allow adrenaline invasion) are often necessary to start some new action.

Costs of building new actions increase with the Degree of Dissimilarity with past actions. They are well known by Psychologists (see Ehrlich, 1975) who insist on the fact that similar actions (contiguous actions, mediated by intermediary actions...) following a given action are much easier to be done that disconnected actions belonging to different fields of competences.

Dynamic Costs to change actions concern Costs of Reactivity which increase with the speed the agent wants to built or do some new action (costs to do quickly and to stop quickly, i.e., Inhibition Costs as seen before). They are very important and somewhat ignored by economists. Reactivity is one of the more important feature of modern organizations, which follows Mass Production and Economies of Varieties aspects. Reactive processes are now essential to be able to produce new products to benefit quickly of some opportunity before the entry of rivals.

Some references are not cited in the text. We give them for an introduction to some subjects related to our "Worthwhile to Move Approach".

References

[1] S. Adly, H. Attouch, A. Cabot: Finite time stabilization of nonlinear oscillators subject to dry friction, in: Nonsmooth Mechanics and Analysis: Theoretical and Numerical Advances,

P. Alart, O. Maisonneuve, R. T. Rockafellar (eds.), Advances in Mechanics and Mathematics 12, Springer (2006) 289–304.

[2] P. Aghion, P. Bolton, C. Harris, B. Jullien: Optimal learning by experimentation, Rev. Econ. Stud. 58 (1991) 621–654.

[3] F. Alvarez: On the minimizing property of a second-order dissipative system in Hilbert space, SIAM J. Control. Optim. 38(4) (2000) 1102–1119.

[4] F. Alvarez, H. Attouch, J. Bolte, P. Redont: A second order gradient like dissipative system with Hessian-driven damping. Application to optimization and mechanics, J. Math. Pures Appl. 81 (2002) 747–779.

[5] H. Attouch, A. Soubeyran: Towards stable routines: Improving and satisficing enough by exploration-exploitation on an unknown landscape, Working paper, GREQAM (2004)

[6] H. Attouch, A. Soubeyran: From procedural rationality to routines: A worthwhile to move approach of satisficing with not too much sacrificing, mimeo (2005).

[7] H. Attouch, X. Goudou, P. Redont: The heavy ball with friction method. I: The continuous dynamical system, Commun. Contemp. Math. 2(1) (2000) 1–34.

[8] H. Attouch, M. Teboulle: Regularized Lotka-Volterra dynamical system as continuous proximal-like method in optimization, J. Optim. Theory Appl. 121(3) (2004) 541–580.

[9] J. P. Aubin, A. Lesne: Constructing and exploring wells of energy landscapes, Working paper (2004).

[10] J. P. Aubin: Applied Abstract Analysis, John Wiley, New York (1977).

[11] J. P. Aubin, I. Ekeland: Applied Non Linear Analysis, Wiley-Interscience (1983).

[12] J. P. Aubin, A. Cellina: Differential Inclusions, Springer (1984).

[13] A. Berthoz: La Decision, Odile Jacob (2003).

[14] A. Berthoz: Le Sens du Mouvement, Odile Jacob (1997).

[15] J. Bolte: Sur les Systèmes Dynamiques Dissipatifs de Type Gradient. Applications en Optimisation, Thèse, Université Montpellier II (2003).

[16] C. Boujon: L'Inhibition, Solal (2002).

[17] J. Conlisk: Why bounded rationality, J. Econ. Literature 34 (1996) 669–700.

[18] R. M. Cyert, J. G. March: A Behavioral Theory of the Firm, Prentice Hall, Englewood Cliffs (1963).

[19] E. Dar-El: Human Learning: From Learning Curves to Learning Organizations, Kluwer Academic Publishers, Boston (2000).

[20] S. Dasgupta: Multidisciplinary creativity: The case of Herbert A. Simon, Cognitive Science 27(5) (2003) 683–707.

[21] J. Dutta: Optimality Conditions in Approximate Optimization, Universitat Autonoma de Barcelona (2002).

[22] I. Ekeland: On the variational principle, J. Math. Anal. Appl. 47 (1974) 325–353.

[23] J. C. Gittins: Multi Armed Bandit Allocations Indices, Wiley, New York (1989).

[24] X. Goudou, J. Munier: Asymptotic behaviour of solutions of a gradient-like integrodifferential Volterra inclusion, Advances in Mathematical Sciences and Applications 15 (2005) 509–525.

[25] X. Goudou, J. Munier: The heavy ball with friction method: The quasiconvex case, Working paper, Université Montpellier II (2005).

[26] A. Haraux, M. Jendoubi: Convergence of solutions of second-order gradient-like systems with analytic nonlinearities, J. Differ. Equations 144 (1998) 313–320.

[27] G. Le Boterf: Construire les Compétences Individuelles et Collectives, Editions d'Organisation, Paris (2001).

[28] H. Leibenstein: The Collected Essays of H. Leibenstein, Vol. 2, K. Button (ed.), Elgar, Algershot (1989).

[29] B. Leloup: L'Incertitude de Deuxième Ordre en Economie: Le Compromis Exploration-Exploitation", Thèse Nouveau Régime, Ecole Normale Supérieure de Cachan, GRID-CREA (2002).

[30] D. Levinthal, M. Warglien: Landscape design: Designing for local action in complex worlds, Organization Science 10(3) (1999) 342–357.

[31] C. Levy-Leboyer: La Motivation dans L'Entreprise, Editions d'Organisation (2003).

[32] A. Y. Lewin, C. P. Long, T. N. Carroll: The coevolution of new organizational forms, Organization Science 10(5) (1999) 535–550.

[33] E. A. Locke, G. P. Lathan: A Theory of Goal Setting and Task Performance, Prentice Hall, Englewood Cliffs (1990).

[34] J. Martinez-Legaz, A. Soubeyran: Learning from errors, UAB, Working paper LEA, Revised version (2005); Satisficing by Rejection: Using Metha Heuristic of Elimination (2002).

[35] R. Nelson, S. Winter: An evolutionary theory of economic change, in: Resources Firms and Strategies, Oxford Management Readers, N. J. Foss (ed.), Oxford (1997) 82–102.

[36] M. Pingle, R. Day: Modes of economizing behavior: Experimental evidence, J. Econ. Behav. Organ. 29 (1996) 191–209.

[37] S. Postrel, R. Rumelt: Incentives, Routines and Self Command, Oxford University Press (1992).

[38] M. Rant: Organizational Fit and Organization, Environment Co-Evolution, European Applied Business Conference, Venice (2003).

[39] J. Robson: The biological basis of economic behavior, J. Econom. Literature 34 (2001) 11–33.

[40] R. Rockafellar, R. Wets: Variational Analysis, Springer, Berlin (2004).

[41] M. Rothschild: A two armed bandit theory of market pricing, J. Econom. Theory 9 (1974) 185–202.

[42] R. Rumelt: Inertia and transformation, mimeo (1990).

[43] J. P. Samose, P. Sarrazin, F. Cury: La Fixation de But: Une technique pour surmonter l'anxiété et augmenter la confiance en soi, mimeo (1998).

[44] H. Simon: Satisficing, in: The New Palgrave, A Dictionary of Economics, J. Eatwell, M. Milgate, P. Newman (eds.), London, Mac Millan (1987) 243–245.

[45] H. Simon: A behavioral model of rational choice, Quarterly Journal of Economics 69 (1955) 99–118.

[46] B. Sinclair, Desgagné, A. Soubeyran: A theory of routines as mindsavers, CIRANO Working Paper 2000s-52 (2000).

[47] J. Sobel: Economists' models of learning, J. Econom. Theory 94 (2000) 241–261.

[48] V. H. Vroom: Work and Motivation, Wiley, New York (1964).