Journal of Convex Analysis
Volume 13 (2006), No. 3+4, 799–822

Coderivative Calculus and Robust Lipschitzian Stability of Variational Systems

Boris S. Mordukhovich*
Department of Mathematics, Wayne State University,
Detroit, MI 48202, USA
boris@math.wayne.edu

Dedicated to the memory of Simon Fitzpatrick.

Received: March 2, 2005

The paper deals with a broad class of parametric variational systems in infinite-dimensional spaces and mainly concerns their robust Lipschitzian stability with respect to parameter perturbations. We develop a local sensitivity analysis for such systems based on advanced tools of generalized differentiation. A special attention is paid to variational systems arising as solution maps to variational inequalities, problems of parametric optimization, and their extensions. A number of the results obtained are new even in finite-dimensional settings.

Keywords: Variational analysis, generalized differentiation, coderivatives, robust Lipschitzian stability, parametric variational systems

2000 Mathematics Subject Classification: 49J52, 58C06, 90C31

1. Introduction

This paper mainly concerns variational analysis of the so-called *generalized equations* given by

$$0 \in f(y) + Q(y), \tag{1}$$

where f is a single-valued mapping while Q is a set-valued mapping between Banach spaces. For convenience we use the terms *base* and *field* referring to the single-valued and set-valued part of (1), respectively. Generalized equations were introduced by Robinson [24] as an extension of standard equations with no multivalued part. It has been well recognized that this model provides a convenient framework for the unified study of *optimal solutions* in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, mechanical equilibria, game theory, etc. In particular, generalized equations (1) reduce to the classical *variational inequalities*:

$$\text{find } y \in \Omega \quad \text{with } \langle f(y), v - y\rangle \ge 0 \quad \text{for all } v \in \Omega \tag{2}$$

when $Q(y) = N(y;\Omega)$ is the normal cone mapping generated by a convex set Ω. The classical *complementarity* problem corresponds to (2) when Ω is the nonnegative orthant

*Research was partly supported by the National Science Foundation under grant DMS-0304989 and by the Australian Research Council under grant DP-0451168.

ISSN 0944-6532 / $ 2.50

in $I\!R^n$. It is well known that the latter form covers sets of optimal solutions with corresponding Lagrange multipliers, or sets of KKT (Karush-Kuhn-Tucker) vectors, satisfying first-order necessary optimality conditions in problems of nonlinear programming.

Observe that the variational inequality (2) can be written in form (1) with the subdifferential mapping $Q(y) = \partial\varphi(y)$ for $\varphi(y) = \delta(y;\Omega)$. Thus the generalized equation model (1) covers also natural generalizations of variational inequalities when φ is not an indicator function and may even be nonconvex; the latter case relates to the so-called "hemivariational inequalities".

The primary goal of this paper is to study a certain *robust stability* of generalized equations (1) and their specifications under perturbations of the initial data. For these purposes we consider a parametric version of (1) given in the form

$$0 \in f(x,y) + Q(x,y) \tag{3}$$

with a perturbation parameter x, where y is usually called the *decision* variable. It seems naturally to label (3) as *parametric variational systems*, since this model describes sets of optimal solutions in parameter-dependent variational and related problems. The central question of local sensitivity analysis for (3) is to clarify how the following *solution map*

$$S(x) := \big\{ y \in Y \big|\, 0 \in f(x,y) + Q(x,y) \big\} \tag{4}$$

depends on the parameter x while (x,y) vary around the reference point $(\bar{x},\bar{y}) \in \operatorname{gph} S$. Note that, in contrast to the standard framework, model (3) covers the general case of variational systems when the field Q may *depend on the perturbation parameter*, which particularly includes the so-called "quasi-variational inequalities" and has drawn a strong attention in recent publications; see, e.g., [9, 15, 18, 25] and the references therein.

Our main objective is *Lipschitzian stability* of solution maps (4) in the sense introduced by Aubin [1] who called such Lipschitzian behavior of set-valued mapping "pseudo-Lipschitz property." As suggested in [15], it seems to be more appropriate to use the term "Lipschitz-like" referring to this kind of Lipschitzian behavior, which is probably the most proper extensions of the classical Lipschitz continuity to set-valued mapping (while "pseudo" means "false"; cf. Rockafellar and Wets [28], where it is called "Aubin property" without specifying its Lipschitzian nature). It is well known that Aubin's Lipschitz-like property of an arbitrary set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces is equivalent to metric regularity as well as to linear openness of its inverse $F^{-1}\colon Y \rightrightarrows X$. These *robust* (i.e., stable with respect to perturbations) properties play a fundamental role in variational analysis and its applications; see [28] and the recent books by Borwein and Zhu [5] and by Mordukhovich [16, 17] for extended expositions, thorough developments, and applications of the key issues of variational analysis in finite-dimensional and infinite-dimensional spaces.

To establish robust Lipschitzian stability of variational systems (4), we employ advanced tools of generalized differentiation satisfying full calculi. Working in infinite-dimensional spaces and using a dual-space approach to generalized differentiation, we are based on *sequential* limiting constructions, which may be generally different from topological/net ones; see the next section. This has been well understood after the work by Simon Fitzpatrick (see, e.g., his paper with Borwein [2]) who made fundamental contributions to many aspects of infinite-dimensional analysis. Among others, I greatly benefited from numerous discussions and personal communications with him, and I am honored to dedicate this paper to his memory.

The rest of the paper consists of three sections. In Section 2 we present basic definitions and necessary preliminaries from variational analysis and generalized differentiation needed for establishing our main results. Section 3 is devoted to *computing* and/or *estimating coderivatives* of general variational systems (4) and their various specifications important for applications. Finally, in Section 4 we derive *sufficient* as well as *necessary and sufficient* conditions for robust Lipschitzian stability of such systems, with computing and estimating the *exact bounds* of Lipschitzian moduli, based on coderivative characterizations of the Lipschitz-like property in variational analysis. Some of the results of this section are extensions of those previously obtained in [13] for perturbed generalized equations (3) with $Q = Q(y)$ in finite-dimensional spaces.

Throughout the paper we use standard notation, with special symbols introduced where they are defined. Unless otherwise stated, all spaces considered are Banach whose norms are always denoted by $\|\cdot\|$. For any space X we consider its dual space X^* equipped with the weak* topology w^*, where $\langle\cdot,\cdot\rangle$ stands for the canonical pairing. For multifunctions $F\colon X \rightrightarrows X^*$ the expression

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \;\Big|\; \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \quad \text{for all } k \in I\!N\Big\}$$

signifies the *sequential* Painlevé-Kuratowski upper/outer limit with respect to the norm topology in X and the weak* topology in X^*; $I\!N := \{1, 2, \ldots\}$. Recall that $F\colon X \rightrightarrows Y$ is *positively homogeneous* if $0 \in F(0)$ and $F(\alpha x) \subset \alpha F(x)$ for all $x \in X$ and $\alpha > 0$. The *norm* of a positive homogeneous set-valued mapping is defined by

$$\|F\| := \sup\big\{\|y\| \;\big|\; y \in F(x) \text{ and } \|x\| \le 1\big\}. \tag{5}$$

2. Robust Lipschitzian Properties and Generalized Differentiation

Recall that a set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces is (locally) *Lipschitz-like* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\ell \ge 0$ if there are neighborhood U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell\|x - u\| I\!B_Y \quad \text{for all } x, u \in U, \tag{6}$$

where $I\!B_Y$ stands for the closed unit ball in Y. This is exactly Aubin's "pseudo-Lipschitz" property introduced in [1]. We call the infimum of all such moduli $\{\ell\}$ by the *exact Lipschitzian bound* of F around $(\bar{x}, \bar{y})$ and denote it by $\operatorname{lip} F(\bar{x}, \bar{y})$. If $V = Y$ in (6), the above property reduces to the local Lipschitz continuity of F around $\bar{x}$ with respect to the Pompieu-Hausdorff distance on 2^Y, and for single-valued mappings $F = f\colon X \to Y$ it agrees with the classical local Lipschitz continuity. For general set-valued mappings F the defined Lipschitz-like property can be viewed as a localization of Lipschitzian behavior around the given point $(\bar{x}, \bar{y})$ in the *graph* of F.

We are able to provide *complete dual characterizations* of the Lipschitz-like property (and hence the classical local Lipschitzian behavior) using appropriate constructions of generalized differentiation. Let us recall some basic definitions and facts needed in what follows.

The reader can find more details, history, and discussions in the book by Mordukhovich [16]. Note that the constructions presented below, in spite of (actually due to) their *nonconvexity*, enjoy *full calculi* based mainly on *extremal/variational principles.*

Given a nonempty subset Ω of a Banach space X and a number $\varepsilon \geq 0$, consider first the collection of (Fréchet) ε-*normals* to Ω defined by

$$\widehat{N}_\varepsilon(x;\Omega) := \Big\{x^* \in X^* \Big|\ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \leq \varepsilon\Big\} \quad \text{for } x \in \Omega$$

and by $\widehat{N}_\varepsilon(x;\Omega) := \emptyset$ for $x \notin \Omega$, where $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. Then the *basic/limiting normal cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega). \tag{7}$$

When the space X is *Asplund* (i.e., its every separable subspace has a separable dual; see Phelps' book [23] for more details and other characterizations) and when the set Ω is closed around $\bar{x}$, one can equivalently replace $\widehat{N}_\varepsilon(\cdot;\Omega)$ in (7) by $\widehat{N}(x;\Omega) := \widehat{N}_0(x;\Omega)$. It follows from (7) that $\widehat{N}(\bar{x};\Omega) \subset N(\bar{x};\Omega)$, while the equality therein postulates the *normal regularity* of Ω at $\bar{x}$.

Given a set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces and given any $\varepsilon \geq 0$, we form the ε-*coderivative sets* for F at $(x,y) \in \operatorname{gph} F$ by

$$\widehat{D}^*_\varepsilon F(x,y)(y^*) := \big\{x^* \in X^* \big|\ (x^*, -y^*) \in \widehat{N}_\varepsilon((x,y); \operatorname{gph} F)\big\} \tag{8}$$

and then define our basic *normal coderivative* $D^*_N F(\bar{x},\bar{y})$ and *mixed coderivative* $D^*_M F(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ by, respectively,

$$D^*_N F(\bar{x},\bar{y})(\bar{y}^*) := \mathop{\mathrm{Lim\,sup}}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ y^* \xrightarrow{w^*} \bar{y}^* \\ \varepsilon \downarrow 0}} \widehat{D}^*_\varepsilon F(x,y)(y^*), \tag{9}$$

$$D^*_M F(\bar{x},\bar{y})(\bar{y}^*) := \mathop{\mathrm{Lim\,sup}}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ y^* \to \bar{y}^* \\ \varepsilon \downarrow 0}} \widehat{D}^*_\varepsilon F(x,y)(y^*). \tag{10}$$

Thus both coderivatives (9) and (10) are positively homogeneous set-valued mappings from Y^* to X^*. They can be equivalently described in the sequential limiting form: $x^* \in D^*_N F(\bar{x},\bar{y})(y^*)$ if and only if there are sequences $\varepsilon_k \downarrow 0$, $(x_k,y_k) \to (\bar{x},\bar{y})$, and $(x^*_k, y^*_k) \xrightarrow{w^*} (x^*, y^*)$ with $(x_k,y_k) \in \operatorname{gph} F$ and $x^*_k \in \widehat{D}^*_{\varepsilon_k} F(x_k,y_k)(y^*_k)$, while the only difference in the description of $x^* \in D^*_M(\bar{x},\bar{y})(y^*)$ is that the weak* convergence $y^*_k \xrightarrow{w^*} y^*$ above is replaced by the norm convergence $\|y^*_k - y^*\| \to 0$. Note that we can put $\varepsilon = \varepsilon_k = 0$ in these definitions and descriptions when the spaces X and Y are Asplund and the graph of F is locally closed around $(\bar{x},\bar{y})$.

Clearly $D^*_M F(\bar{x},\bar{y})(y^*) \subset D^*_N F(\bar{x},\bar{y})(y^*)$ for all $y^* \in Y^*$. We say that F is *coderivatively normal* (resp. *strongly coderivatively normal*) at $(\bar{x},\bar{y})$ if

$$\|D^*_M F(\bar{x},\bar{y})\| = \|D^*_N F(\bar{x},\bar{y})\| \ \Big(\text{resp. } D^*_M F(\bar{x},\bar{y})(y^*) = D^*_N F(\bar{x},\bar{y})(y^*)\Big),$$

where the norm of positively homogeneous mappings is defined in (5). The latter property obviously holds when $\dim Y < \infty$ and also when F is *normally regular* at $(\bar{x}, \bar{y})$, i.e.,

$$\widehat{D}^* F(\bar{x}, \bar{y})(y^*) = D^*_N F(\bar{x}, \bar{y})(y^*) \quad \text{whenever } y^* \in Y^*,$$

which particularly includes set-valued mapping of convex graph as well as single-valued mappings f strictly differentiable at $\bar{x}$ with the derivative $\nabla f(\bar{x})$; in the latter case

$$\widehat{D}^* f(\bar{x})(y^*) = D^*_M f(\bar{x})(y^*) = D^*_N f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\}, \quad y^* \in Y^*. \tag{11}$$

More involved conditions ensuring (strong) coderivative normality of mappings (as well as examples of violating these properties) are presented in [16]; see, in particular, Proposition 4.9 therein and [15, Proposition 3.2] summarizing such conditions. In what follows we use the *common symbol* D^* for both coderivatives when there is no need to distinguish between them.

Given an extended-real-valued function $\varphi\colon X \to \overline{I\!R} := [-\infty, \infty]$ finite at $\bar{x}$, define its (first-order) *basic subdifferential* $\partial\varphi(\bar{x})$ and *singular subdifferential* $\partial^\infty\varphi(\bar{x})$ at $\bar{x}$ by

$$\partial\varphi(\bar{x}) := D^* E_\varphi(\bar{x}, \varphi(\bar{x}))(1), \;\; \partial^\infty\varphi(\bar{x}) := D^* E_\varphi(\bar{x}, \varphi(\bar{x}))(0) \tag{12}$$

via the coderivative of the associate epigraphical multifunction $E_\varphi(x) := \big\{\nu \in I\!R \big|\; \nu \geq \varphi(x)\big\}$; see [16, 19] for equivalent analytic representations. Note that the singular subdifferential $\partial^\infty\varphi(\bar{x})$ reduces to zero for locally Lipschitzian functions. If a mapping $f\colon X \to Y$ is Lipschitz continuous around $\bar{x}$, then there are *scalarization formulas*

$$D^*_M f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}) \quad \text{and} \quad D^*_N f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}), \quad y^* \in Y^*, \tag{13}$$

relating the coderivatives (9) and (10) and the basic subdifferential (12) of the scalarized real-valued function $\langle y^*, f\rangle(x) := \langle y^*, f(x)\rangle$. The first formula in (13) holds in general Banach spaces X and Y, while the second one requires the Asplund property of X and in addition the *strict Lipschitzian* property of f at $\bar{x}$, which means that there is a neighborhood V of the origin in X such that the sequence

$$\frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in I\!N,$$

contains a norm convergent subsequence whenever $v \in V$, $x_k \to \bar{x}$, and $t_k \downarrow 0$. The latter property happens to be equivalent to the basic version of “compactly Lipschitz" behavior originally introduced by Thibault [29]. Of course, it is automatic for locally Lipschitzian mappings with values in finite-dimensional spaces.

One of the most essential ingredients of variational analysis in *infinite dimensions*, in comparison with its finite-dimensional counterpart, is the necessity to impose certain “normal compactness" properties that are automatic in finite dimensions. The following *sequential* normal compactness properties are widely used in this paper; see [16, 17, 21] and the references therein for more discussions, calculus, and applications.

A set-valued mapping $F\colon X \rightrightarrows Y$ is *sequentially normally compact* (SNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if for any sequences $(\varepsilon_k, x_k, y_k, x^*_k, y^*_k) \in [0, \infty) \times (\operatorname{gph} F) \times X^* \times Y^*$ satisfying

$$\varepsilon_k \downarrow 0, \;\; (x_k, y_k) \to (\bar{x}, \bar{y}), \;\; x^*_k \in \widehat{D}^*_{\varepsilon_k} F(x_k, y_k)(y^*_k) \tag{14}$$

one has $(x_k^*, y_k^*) \overset{w^*}{\to} (0,0) \Longrightarrow \|(x_k^*, y_k^*)\| \to 0$ as $k \to \infty$. A set-valued mapping F is *partially sequentially normally compact* (PSNC) at $(\bar{x}, \bar{y})$ if for any above sequences satisfying (14) one has

$$\Big[x_k^* \overset{w^*}{\to} 0 \ \text{ and } \ \|y_k^*\| \to 0\Big] \Longrightarrow \|x_k^*\| \to 0 \quad \text{as } k \to \infty.$$

We may equivalently put $\varepsilon_k = 0$ in the above properties if both spaces X and Y are Asplund while the mapping F is closed-graph around $(\bar{x}, \bar{y})$. Finally, a set $\Omega \subset X$ is *SNC* at $\bar{x} \in \Omega$ if the constant mapping $F(x) \equiv \Omega$ satisfies this property.

Note that the SNC property of sets and mappings are closely related to (but generally different from) the *compactly epi-Lipschitzian* (CEL) property in the sense of Borwein and Strójwas [4]; see [8, 10, 11, 16] for comprehensive studies in this direction. For closed convex sets $\Omega \subset X$ in normed spaces the CEL (and hence SNC) property holds [3] *if and only if* the affine hull of Ω is a closed *finite-codimensional* subspace of X with $\operatorname{ri}\Omega \neq \emptyset$. On the other hand, every *Lipschitz-like* mapping $F\colon X \rightrightarrows Y$ between Banach spaces is always *partially* SNC at $(\bar{x}, \bar{y})$, and hence it is fully SNC at this point when $\dim Y < \infty$. This is a consequence of the following fundamental result [16, Theorem 4.10] (see also [15] and the references therein) that provides *dual coderivative characterizations* of the Lipschitz-like property for general set-valued mappings between infinite-dimensional spaces.

Theorem 2.1 (dual characterizations of robust Lipschitzian behavior). *Let $F\colon X \rightrightarrows Y$ be a mapping between Banach spaces that is closed-graph around $(\bar{x}, \bar{y})$. Consider the properties:*

(a) F is Lipschitz-like around $(\bar{x}, \bar{y})$.
(b) F is PSNC at $(\bar{x}, \bar{y})$ and $\|D_M^* F(\bar{x}, \bar{y})\| < \infty$.
(c) F is PSNC at $(\bar{x}, \bar{y})$ and $D_M^* F(\bar{x}, \bar{y})(0) = \{0\}$.

Then (a) ⇒ (b) ⇒ (c) while these properties are equivalent if both X and Y are Asplund. Moreover, one has the estimates

$$\|D_M^* F(\bar{x}, \bar{y})\| \le \operatorname{lip} F(\bar{x}, \bar{y}) \le \|D_N^* F(\bar{x}, \bar{y})\| \tag{15}$$

for the exact Lipschitzian bound of F around $(\bar{x}, \bar{y})$, where the upper estimate holds if $\dim X < \infty$ and Y is Asplund. Thus

$$\operatorname{lip} F(\bar{x}, \bar{y}) = \|D_M^* F(\bar{x}, \bar{y})\| = \|D_N^* F(\bar{x}, \bar{y})\| \tag{16}$$

if in addition F is coderivatively normal at $(\bar{x}, \bar{y})$.

Our subsequent strategy in this paper is as follows: to apply the characterizations of Theorem 2.1 to general solution maps (4) for the variational systems under consideration and their remarkable specifications. To proceed in this way, we need to employ generalized differential and SNC calculi for the constructions involved in the above characterizations and finally obtain efficient conditions for robust Lipschitzian stability of variational systems in terms of their initial data. Prior the implementation of this procedure in the next two sections, let us recall the definition and some properties of the *second-order* subdifferential/generalized Hessian construction for extended-real-valued functions needed in what follows; see [14, 16, 18] for more details.

Given $\varphi\colon X \to \overline{I\!R}$ finite at $\bar{x}$ and given a basic subgradient $\bar{y} \in \partial\varphi(\bar{x})$ from (12), we define the *second-order subdifferential* of φ at $\bar{x}$ *relative to* $\bar{y}$ by

$$\partial^2\varphi(\bar{x},\bar{y})(u) := D_N^*(\partial\varphi)(\bar{x},\bar{y})(u), \quad u \in X^{**}, \tag{17}$$

i.e., as the normal coderivative (9) of the first-order subdifferential mapping $\partial\varphi\colon X \rightrightarrows X^*$. More precisely, construction (17) is naturally to call the *normal* second-order subdifferential (since it is generated by the normal coderivative) and to denote by $\partial_N^2\varphi(\bar{x},\bar{y})(u)$, but we simplify the name and notion in this paper, which does not employ the “mixed" counterpart of (17). If φ is twice continuously differentiable around $\bar{x}$, then

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})^*u\big\} \quad \text{for all } u \in X^{**},$$

where $\nabla^2\varphi(\bar{x})$ stands for the classical second-order derivative operator that particularly happens to be symmetric in finite-dimensional spaces.

3. Computing and Estimating Coderivatives of Solution Maps

We begin with verifying efficient conditions that allow us to derive *exact formulas* (equalities) for computing coderivatives of solutions maps $S\colon X \rightrightarrows Y$ to the variational systems defined in Section 1. These conditions impose the *strict differentiability* assumption on the base mapping f in (3) at the reference point and consist of two independent parts: the result of part (i) requires the *surjectivity* of the partial derivative $\nabla_x f(\bar{x},\bar{y})$ in the general Banach space setting, while those of part (ii) require instead that the field Q is *normally regular* and the spaces in question are Asplund. To formulate the theorem, we fix a point $(\bar{x},\bar{y})$ satisfying the generalized equation (3) and introduce the *adjoint generalized equation* important in what follows:

$$0 \in \nabla f(\bar{x},\bar{y})^*z^* + D_N^*Q(\bar{x},\bar{y},\bar{z})(z^*) \quad \text{with } \bar{z} := -f(\bar{x},\bar{y}). \tag{18}$$

Theorem 3.1 (exact formulas for coderivatives of solution maps under surjectivity or regularity conditions). *Let S be given in* (4)*, where $f\colon X\times Y \to Z$ is strictly differentiable at $(\bar{x},\bar{y})$ with $\bar{z} = -f(\bar{x},\bar{y}) \in Q(\bar{x},\bar{y})$. The following hold:*

(i) *Assume that X,Y,Z are Banach, that $\nabla_x f(\bar{x},\bar{y})$ is surjective, and that $Q = Q(y)$. Then*

$$D_N^*S(\bar{x},\bar{y})(y^*) = \Big\{x^* \in X^* \Big|\ \exists z^* \in Z^* \quad \textit{with } x^* = \nabla_x f(\bar{x},\bar{y})^*z^*, \\ -y^* \in \nabla_y f(\bar{x},\bar{y})^*z^* + D_N^*Q(\bar{y},\bar{z})(z^*)\Big\}.$$

(ii) *Assume that X,Y,Z are Asplund and that Q is locally closed-graph around $(\bar{x},\bar{y},\bar{z})$ and normally regular at this point. Suppose also that either* $\dim Z < \infty$ *or Q is SNC at $(\bar{x},\bar{y},\bar{z})$. Then S is normally regular at $(\bar{x},\bar{y})$ and*

$$D^*S(\bar{x},\bar{y})(y^*) = \Big\{x^* \in X^* \Big|\ \exists z^* \in Z^* \quad \textit{with } \Big(x^* - \nabla_x f(\bar{x},\bar{y})^*z^*, \\ -y^* - \nabla_y f(\bar{x},\bar{y})^*z^*\Big) \in D_N^*Q(\bar{x},\bar{y},\bar{z})(z^*)\Big\}$$

provided that the adjoint generalized equation (18) *admits only the trivial solution* $z^* = 0$.

Proof. We prove assertions (i) and (ii) in a parallel way. Observe that the graph of the solution map S in (4) is represented as the *inverse image*

$$\operatorname{gph} S = g^{-1}(\Theta) := \big\{(x,y) \in X \times Y \big|\, g(x,y) \in \Theta\big\} \quad \text{with } \Theta := \operatorname{gph} Q, \tag{19}$$

where the mapping g is defined by

$$g(x,y) := \big(y, -f(x,y)\big) \quad \text{if } Q = Q(y); \tag{20}$$

$$g(x,y) := \big(x, y, -f(x,y)\big) \quad \text{if } Q = Q(x,y). \tag{21}$$

In case (20) we easily conclude that $\nabla g(\bar{x},\bar{y})$ is surjective *if and only if* $\nabla_x f(\bar{x},\bar{y})$ is surjective, which is the main assumption of (i). Then the calculus rule from [22, Corollary 3.9] for basic normals to inverse images gives

$$N((\bar{x},\bar{y}); g^{-1}(\Theta)) = \nabla g(\bar{x},\bar{y})^* N(g(\bar{x},\bar{y}); \Theta) \tag{22}$$

in general Banach spaces. Combining this with the representation

$$D_N^* F(\bar{x},\bar{y})(y^*) = \big\{x^* \in X^* \big|\, (x^*, -y^*) \in N((\bar{x};\bar{y}); \operatorname{gph} F)\big\} \tag{23}$$

of the normal coderivative for an arbitrary set-valued mapping $F\colon X \rightrightarrows Y$, which follows from (7) and (9), and computing $\nabla g(\bar{x},\bar{y})$ via $\nabla f(\bar{x},\bar{y})$ in (20), we arrive at the exact formula for $D_N^* S(\bar{x},\bar{y})$ in assertion (i) under the assumptions made.

To prove (ii), we cannot use the above calculus rule for inverse images, since $\nabla g(\bar{x},\bar{y})$ is *never surjective* in (21). To proceed in this case, we apply [19, Corollary 6.9], which ensures (22) in the Asplund space setting provided that

$$N((\bar{x},\bar{y}); \operatorname{gph} \Theta) \cap \ker \nabla g(\bar{x},\bar{y})^* = \{0\} \tag{24}$$

and that either Θ is SNC at $g(\bar{x},\bar{y})$ or $\dim Z < \infty$. Moreover, the normal regularity for Θ at $g(\bar{x},\bar{y})$ implies this property for $g^{-1}(\Theta)$ at $(\bar{x},\bar{y})$. Computing now $\nabla g(\bar{x},\bar{y})$ in (22) and (24) for g given in (21), we observe that the qualification condition (24) is equivalent to the triviality of solutions to the adjoint generalized equation (18), while the SNC and normal regularity properties for $\Theta = \operatorname{gph} Q$ at $g(\bar{x},\bar{y})$ are equivalent to the corresponding properties for Q at $(\bar{x},\bar{y},\bar{z})$ due to representation (23). This concludes the proof of (ii). $\square$

Remark 3.2 (partial adjoint generalized equations). When $Q = Q(y)$ and f is strictly differentiable at $(\bar{x},\bar{y})$, it is convenient to consider the following *partial adjoint generalized equation*

$$0 \in \nabla_y f(\bar{x},\bar{y})^* z^* + D_N^* Q(\bar{y},\bar{z})(z^*) \tag{25}$$

with $\bar{z} = -f(\bar{x},\bar{y}) \in Q(\bar{y})$. In this setting z^* is a solution to the (full) adjoint generalized equation (18) *if and only if* it satisfies the partial one (25) together with $z^* \in \ker \nabla_x f(\bar{x},\bar{y})^*$, where the latter requirement is redundant when $\nabla_x f(\bar{x},\bar{y})$ is surjective. Thus the qualification condition of Theorem 3.1 on the triviality of solutions to (18) reduces for $Q = Q(y)$ to the triviality of those solutions to (25), which *belong to the kernel* of $\nabla_x f(\bar{x},\bar{y})^*$.

We can get various consequences of Theorem 3.1 when the field Q of the generalized equation (3) is given in special forms allowing us to evaluate/estimate the normal coderivative $D^*_N Q$. Let us present efficient results for the case of *convex-graph* multifunctions Q.

Given $Q\colon X \times Y \rightrightarrows Z$ and $f\colon X \times Y \to Z$ strictly differentiable at $(\bar{x}, \bar{y})$, consider the *linearized* set-valued operator $L\colon X \times Y \rightrightarrows Z$ with

$$L(x,y) := f(\bar{x},\bar{y}) + \nabla_x f(\bar{x},\bar{y})(x-\bar{x}) + \nabla_y f(\bar{x},\bar{y})(y-\bar{y}) + Q(x,y) \tag{26}$$

as well as, in the case of $Q = Q(y)$, the *partial linearized* operator $\widetilde{L}\colon Y \rightrightarrows Z$ defined by

$$\widetilde{L}(y) := f(\bar{x},\bar{y}) + \nabla_y f(\bar{x},\bar{y})(y-\bar{y}) + Q(y). \tag{27}$$

Corollary 3.3 (computing coderivatives for generalized equations with convex-graph fields). *Let $(\bar{x},\bar{y})$ satisfy the generalized equation (3), where $f\colon X \times Y \to Z$ is strictly differentiable at $(\bar{x},\bar{y})$ and where the graph of $Q\colon X \times Y \rightrightarrows Z$ is convex. The following hold for the coderivatives of the solution map (4):*

(*i*) *Assume that X, Y, Z are Banach, that $\nabla_x f(\bar{x},\bar{y})$ is surjective, and that $Q = Q(y)$. Then S is normally regular at $(\bar{x},\bar{y})$, and one has*

$$D^*S(\bar{x},\bar{y})(y^*) = \Big\{\nabla_x f(\bar{x},\bar{y})^* z^* \Big|\ -(y^*,z^*) \in N((0,0); \operatorname{rge} \widetilde{M})\Big\},$$

where $\widetilde{M}\colon Y \rightrightarrows Y \times Z$ is defined by

$$\widetilde{M}(y) := \big(y-\bar{y}, \widetilde{L}(y)\big)$$

with $\widetilde{L}$ given in (27), and where $\operatorname{rge} F$ stands for the range of a set-valued mapping F.

(*ii*) *Assume that X, Y, Z are Asplund and that Q is locally closed-graph around $(\bar{x},\bar{y},\bar{z})$ with $\bar{z} = -f(\bar{x},\bar{y})$. Suppose also that either Z is finite-dimensional or Q is SNC at $(\bar{x},\bar{y},\bar{z})$, and that*

$$N(0; \operatorname{rge} L) = \{0\}, \tag{28}$$

where the mapping L is given in (26). Then S is normally regular at $(\bar{x},\bar{y})$, and one has

$$D^*S(\bar{x},\bar{y})(y^*) = \Big\{x^* \in X^* \Big|\ \exists z^* \in Z^* \quad \text{with} \quad (x^*, -y^*, -z^*) \in N((0,0,0); \operatorname{rge} M)\Big\},$$

where $M\colon X \times Y \rightrightarrows X \times Y \times Z$ is defined by

$$M(x,y) := \big(x-\bar{x}, y-\bar{y}, L(x,y)\big).$$

Proof. Due to (23) and to the fact that the normal cone (7) reduces for convex sets to the classical normal cone of convex analysis, we rewrite the adjoint generalized equation (18) as

$$\big\langle \nabla f(\bar{x},\bar{y})^* z^*, (x,y)-(\bar{x},\bar{y})\big\rangle + \big\langle z^*, f(\bar{x},\bar{y}) + z\big\rangle \geq 0 \quad \text{for } (x,y,z) \in \operatorname{gph} Q.$$

This is equivalent to

$$\big\langle z^*, L(x,y) + z\big\rangle \ge 0 \quad \text{whenever } (x,y,z) \in \operatorname{gph} Q. \tag{29}$$

The latter means that $\bar{w} = 0$ is an optimal solution to the *convex minimization problem*:

$$\text{minimize } \langle z^*, w\rangle \quad \text{subject to } w \in \Omega := \operatorname{rge} L.$$

Employing the *generalized Fermat rule* $0 \in \partial\varphi(\bar{w})$ as a *necessary and sufficient* condition for minimization of the convex function $\varphi(w) := \langle z^*, w\rangle + \delta(w;\Omega)$ and then using the Moreau-Rockafellar subdifferential sum rule from convex analysis, we conclude that (29) is equivalent to the inclusion $-z^* \in N(0; \operatorname{rge} L)$. Thus the adjoint generalized equation (18) has only the trivial solution *if and only if* the qualification condition (28) holds.

To justify the coderivative representations in (i) and (ii) under the assumptions made, we involve similar arguments applied to the corresponding representations of Theorem 3.1. Since convex-graph mappings are normally regular at every point of their graph, we conclude that the solution map (4) is normally regular at $(\bar{x},\bar{y})$ under the assumptions of this corollary. □

The qualification condition (28) obviously holds if $0 \in \operatorname{int}(\operatorname{rge} L)$, which is actually equivalent to (28) if the range of L is locally closed around $\bar{w} = 0$ and SNC at this point. Note that, due to convexity, the SNC property of the sets $\operatorname{rge} L$ and $\operatorname{gph} Q$ can be characterized via their *finite codimensionality* by [3].

Let us mention a special case of (3) when Q is given by

$$Q(x,y) := \begin{cases} E & \text{if } (x,y) \in \Omega, \\ \emptyset & \text{otherwise,} \end{cases} \tag{30}$$

where $E \subset Z$ and $\Omega \subset X \times Y$ are closed convex sets. In this case the interiority condition $0 \in \operatorname{int}(\operatorname{rge} L)$ reduces to

$$0 \in \operatorname{int}\Big\{ f(\bar{x},\bar{y}) + \nabla f(\bar{x},\bar{y})\big(\Omega - (\bar{x},\bar{y})\big) + E \Big\}$$

When $Q = Q(y)$ in (30), the qualification condition (28) automatically holds under the *Robinson regularity condition*

$$0 \in \operatorname{int}\Big\{ f(\bar{x},\bar{y}) + \nabla_y f(\bar{x},\bar{y})\big(\Omega - \bar{y}\big) + E \Big\}.$$

In case (30) the coderivative formulas from Corollary 3.3 can be modified accordingly.

Next we obtain efficient conditions under which the equalities in Theorem 3.1 turn into *upper estimates* for coderivatives of solution maps (4) with *no surjectivity* and/or *normal regularity* assumptions made above. Moreover, we consider general cases of *nonsmooth bases* f in (3).

Theorem 3.4 (upper coderivative estimates for generalized equations). *Let $(\bar{x}, \bar{y})$ satisfy* (3), *where X, Y, Z are Asplund, $f\colon X \times Y \to Z$ is continuous around $(\bar{x},\bar{y})$, and the graph of Q is closed around $(\bar{x},\bar{y},\bar{z})$ with $\bar{z} = -f(\bar{x},\bar{y})$. Then*

$$D^*S(\bar{x},\bar{y})(y^*) \subset \Big\{ x^* \in X^* \,\Big|\, \exists z^* \in Z^* \quad \textit{with} \quad (x^*, -y^*) \in D^*_N f(\bar{x},\bar{y})(z^*) + D^*_N Q(\bar{x},\bar{y},\bar{z})(z^*) \Big\} \tag{31}$$

for both coderivatives $D^* = D^*_N, D^*_M$ *of the solution map* (4) *at* $(\bar{x}, \bar{y})$ *provided that either one of the following conditions holds:*

(*a*) Q *is SNC at* $(\bar{x}, \bar{y}, \bar{z})$*, and* $(x^*, y^*, z^*) = (0, 0, 0)$ *is the only triple satisfying*

$$(x^*, y^*) \in D^*_N f(\bar{x}, \bar{y})(z^*) \cap \big(- D^*_N Q(\bar{x}, \bar{y}, \bar{z})(z^*)\big); \tag{32}$$

the latter is equivalent to

$$\Big[0 \in \partial\langle z^*, f\rangle(\bar{x}, \bar{y}) + D^*_N Q(\bar{x}, \bar{y}, \bar{z})(z^*)\Big] \Longrightarrow z^* = 0 \tag{33}$$

if f *is strictly Lipschitzian at* $(\bar{x}, \bar{y})$*.*

(*b*) f *is Lipschitz continuous around* $(\bar{x}, \bar{y})$*,* Z *is finite-dimensional, and the qualification condition* (33) *is satisfied.*

Proof. Taking into account the graph representation (19) for the solution map S from (4), we apply [20, Theorem 4.4] ensuring the upper estimate

$$N((\bar{x}, \bar{y}); g^{-1}(\Theta)) \subset \Big\{ D^*_N g(\bar{x}, \bar{y})(z^*) \Big| \; z^* \in N(g(\bar{x}, \bar{y}); \Theta) \Big\} \tag{34}$$

of the basic normal cone to inverse images for mappings between Asplund spaces provided that the qualification condition

$$N(\bar{z}; \Theta) \cap \ker D^*_N g(\bar{x}, \bar{y}) = \{0\} \tag{35}$$

holds and that either Θ is SNC at $g(\bar{x}, \bar{y})$ or g^{-1} is PSNC at $(\bar{w}, \bar{x}, \bar{y})$ with $\bar{w} := (\bar{x}, \bar{y}, \bar{z})$. By the structure of g in (21) we have

$$g(x, y) = (x, y, 0) + (0, 0, -f(x, y)),$$

which easily gives the equality

$$D^*_N g(\bar{x}, \bar{y})(x^*, y^*, z^*) = (x^*, y^*) + D^*_N f(\bar{x}, \bar{y})(-z^*)$$

due to $D^*_N(-f)(\bar{x}, \bar{y})(z^*) = D^*_N f(\bar{x}, \bar{y})(-z^*)$ and elementary coderivative calculus. Observe that the qualification condition (35) is equivalent in this setting to the triviality $(x^*, y^*, z^*) = (0, 0, 0)$ for every triple satisfying (32). The latter reduces to the subdifferential form (33) due to the coderivative scalarization formula from [19, Theorem 5.2]. Similarly we can derive the coderivative inclusion (31) from the one in (34).

It remains to check that the assumptions of the theorem implies the fulfillment of the above SNC/PSNC conditions needed for the validity of (34). Since $\Theta = \operatorname{gph} Q$, we only need to show that the PSNC property of g^{-1} at $(\bar{w}, \bar{x}, \bar{y})$ holds if f is Lipschitz continuous around $(\bar{x}, \bar{y})$ while Z is finite-dimensional. To proceed, we first observe that the *Fréchet coderivative* $\widehat{D}^* f(\bar{x})$ defined by (8) as $\varepsilon = 0$ admits, for single-valued and locally Lipschitzian mapping $f\colon X \to Y$ between Banach spaces, the following subdifferential representation:

$$\widehat{D}^* f(\bar{x})(y^*) = \widehat{\partial}\langle y^*, f\rangle(\bar{x}), \quad y^* \in Y^*, \tag{36}$$

via the *Fréchet subdifferential* of $\varphi\colon X \to \overline{I\!R}$ defined by

$$\widehat{\partial}\varphi(\bar{x}) := \Big\{x^* \in X^* \Big| \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0\Big\}. \tag{37}$$

Taking this into account, we conclude by the structure of g in (21) that the PSNC property of g^{-1} at $(\bar{w}, \bar{x}, \bar{y})$ means in this setting that for every sequences $(x_k, y_k) \to (\bar{x}, \bar{y})$, $(u_k^*, v_k^*) \xrightarrow{w^*} (0, 0)$, and

$$(x_k^*, y_k^*) - (u_k^*, v_k^*) \in \widehat{\partial}\langle -z_k^*, f\rangle(x_k, y_k) \quad \text{with } \|(x_k^*, y_k^*, z_k^*)\| \to 0$$

one has $\|(u_k^*, v_k^*)\| \to 0$ as $k \to \infty$. The latter can be easily derived from (37). □

Many important applications of variational systems (3) relate to the case when $Q = \partial\varphi$ is a *subdifferential operator* generated by a l.s.c. function φ. In this case we have $D_N^* Q(\bar{x}, \bar{y}) = \partial^2\varphi(\bar{x}, \bar{y})$ by construction (17) of the *second-order subdifferential*, and hence one can use advantages of the second-order subdifferential calculus. Borrowing mechanical terminology, we label φ as *potential.*

As mentioned in Section 1, potentials φ are convex and parameter-independent in the classical settings of variational inequalities and complementarity problems. In the case of nonconvex and parameter-independent potentials the corresponding generalized equations relate to the so-called *hemivariational inequalities* (HVIs), which are conventionally considered in terms of Clarke subgradients for Lipschitz continuous functions. For convenience we use this terminology also in the case of our *basic subgradients* for l.s.c. *parameter-independent* potentials.

Our main attention is paid to general classes of variational systems (3), where the *parameter-dependent* field $Q = Q(x, y)$ is given in two *composite* forms involving the basic first-order subdifferential. For convenience we call generalized equations with subdifferential fields by *generalized variational inequalities* (GVIs).

The first class of GVIs under consideration concerns fields with *composite potentials* of the type $\varphi \circ g$, where $g\colon X \times Y \to W$ and $\varphi\colon W \to \overline{I\!R}$ are mappings between Banach spaces. On the other words, we study solutions maps given in the composite form

$$S(x) := \Big\{y \in Y \Big| \, 0 \in f(x, y) + \partial(\varphi \circ g)(x, y)\Big\}. \tag{38}$$

Note that the range space for f and $Q = \partial(\varphi\circ g)$ in (38) is either $X^* \times Y^*$ when $g = g(x, y)$, or Y^* when $g = g(y)$. Let us start with computing coderivatives of general HVIs with composite (parameter-independent) potentials under certain *surjectivity* conditions.

Following [22], we say that a closed linear subspace L of a Banach space X is w^*-*extensible* if every sequence $\{v_k^*\} \subset X^*$ with $v_k^* \xrightarrow{w^*} 0$ as $k \to \infty$ contains a subsequence $\{v_{k_j}^*\}$ such that each $v_{k_j}^*$ can be extended to $x_j^* \in X^*$ with $x_k^* \xrightarrow{w^*} 0$ as $j \to \infty$. As shown in [22], this property is automatic if *either* L is *complemented* in X, *or* the closed unit ball of X^* is *weak* sequentially compact.* On the other hand, the w^*-extensibility property may not hold even in some classical Banach spaces, e.g., for $L = c_0$ with $X = \ell^\infty$; see [22].

Theorem 3.5 (computing coderivatives of solution maps to HVIs with composite potentials). *Let X, Y, and W be Banach spaces, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ for S*

defined in (38) *with* $g\colon Y \to W$ *and* $\varphi\colon W \to \overline{I\!R}$. *Put* $\bar{q} := -f(\bar{x},\bar{y}) \in \partial(\varphi \circ g)(\bar{y})$ *and assume the following:*

(a) $f\colon X \times Y \to Y^*$ *is strictly differentiable at* $(\bar{x},\bar{y})$ *with the surjective partial derivative* $\nabla_x f(\bar{x},\bar{y})\colon X \to Y^*$.

(b) g *is continuously differentiable around* $\bar{y}$ *with the surjective derivative* $\nabla g(\bar{y})\colon Y \to W$, *and the mapping* ∇g *from* Y *to the corresponding space* $\mathcal{L}(Y,W)$ *of linear continuous operators is strictly differentiable at* $\bar{y}$.

Let $\bar{v} \in W^*$ *be a unique functional satisfying the relations*

$$\bar{q} = \nabla g(\bar{y})^*\bar{v} \quad \textit{and} \quad \bar{v} \in \partial\varphi(\bar{w}) \quad \textit{with } \bar{w} := g(\bar{y}).$$

Then one has the inclusion

$$\begin{aligned} D_N^* S(\bar{x},\bar{y})(y^*) \subset \Big\{ x^* \in X^* \Big| \; &\exists u \in Y^{**} \quad \textit{with } x^* = \nabla_x f(\bar{x},\bar{y})^* u, \\ -y^* \in \nabla_y f(\bar{x},\bar{y})^* u + \nabla^2\langle \bar{v}, g\rangle(\bar{y})^* u &+ \nabla g(\bar{y})^* \partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{y})^{**} u\big) \Big\}, \end{aligned} \tag{39}$$

which becomes an equality if the range of $\nabla g(\bar{y})^*$ *is* w^**-extensible in* Y^*, *in particular, when either this subspace is complemented in* Y^* *or the closed unit ball of* Y^{**} *is weak** *sequentially compact.*

Proof. It follows from Theorem 3.1(i) and the construction of $\partial^2(\varphi \circ g)$ that

$$\begin{aligned} D_N^* S(\bar{x},\bar{y})(y^*) = \Big\{ x^* \in X^* \Big| \; &\exists u \in Y^{**} \quad \text{with } x^* = \nabla_x f(\bar{x},\bar{y})^* u, \\ &-y^* \in \nabla_y f(\bar{x},\bar{y})^* u + \partial^2(\varphi \circ g)(\bar{y},\bar{q})(u) \Big\} \end{aligned}$$

under the assumptions in (a). Now applying the second-order chain rule for $\partial^2(\varphi\circ g)$ in the inclusion form of [22, Theorem 4.2], we arrive at (39) if both (a) and (b) are assumed. The equality case in (39) follows from the one in [22, Theorem 4.2] under the w^*-extensibility assumption. □

Next we obtain *upper coderivative estimates* for solution maps to GVIs with composite potentials (38) *depending on the parameter* x under essentially less restrictive assumptions on the mappings f and g in comparison with Theorem 3.5. To proceed, one may combine the upper coderivative estimates for general variational systems from Theorem 3.4 with general second-order subdifferential chain rules from [14]. We are not going to present here the most general results in this direction, restricting for simplicity our consideration to finite-dimensional spaces. Moreover, we confine our study to the remarkable class of so-called "amenable" functions that are overwhelmingly encountered in finite-dimensional composite optimization.

Recall [28] that $\psi\colon Z \to \overline{I\!R}$ is *strongly amenable* at $\bar{z}$ if there is a neighborhood U of $\bar{z}$ on which ψ can be represented in the form $\psi = \varphi \circ g$ with a twice continuous differentiable mapping $g\colon U \to I\!R^m$ and a proper l.s.c. convex function $\varphi\colon I\!R^m \to \overline{I\!R}$ satisfying the qualification condition

$$\partial^\infty\varphi(g(\bar{z})) \cap \ker \nabla g(\bar{z})^* = \{0\}.$$

Theorem 3.6 (coderivative estimates for solution maps to GVI's with amenable potentials). *Let $(\bar{x},\bar{y}) \in \operatorname{gph} S$ for S defined in (38) by mappings between finite-dimensional spaces. Assume that f is continuous around $(\bar{x},\bar{y})$, that the potential $\psi := \varphi \circ g$ is strongly amenable at $(\bar{x},\bar{y})$ satisfying the second-order qualification condition*

$$\partial^2\varphi(\bar{w},\bar{v})(0) \cap \ker \nabla g(\bar{x},\bar{y})^* = \{0\} \quad \text{for all } \bar{v} \in M(\bar{x},\bar{y}), \tag{40}$$

where $\bar{w} := g(\bar{x},\bar{y})$ and where the set $M(\bar{x},\bar{y})$ is given by

$$M(\bar{x},\bar{y}) := \Big\{\bar{v} \in W^* \Big|\ \bar{v} \in \partial\varphi(\bar{w}), \quad \nabla g(\bar{x},\bar{y})^*\bar{v} = -f(\bar{x},\bar{y})\Big\}.$$

Then one has the inclusion

$$D^*S(\bar{x},\bar{y})(y^*) \subset \Big\{x^* \in X^* \Big|\ \exists u \in X \times Y \quad \text{with } (x^*,-y^*) \in D^*f(\bar{x},\bar{y})(u)$$
$$+ \bigcup_{\bar{v}\in M(\bar{x},\bar{y})} \Big[\nabla^2\langle \bar{v}, g\rangle(\bar{x},\bar{y})(u) + \nabla g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{x},\bar{y})u\big)\Big]\Big\}.$$

Proof. It follows from Theorem 3.4 with $Q(x,y) = \partial(\varphi \circ g)$ that the coderivative upper estimate

$$D^*S(\bar{x},\bar{y})(y^*) \subset \Big\{x^* \in X^* \Big|\ \exists u \in X \times Y \ \text{ with}$$
$$(x^*,-y^*) \in D^*f(\bar{x},\bar{y})(u) + \partial^2(\varphi \circ g)(\bar{x},\bar{y},\bar{q})(u)\Big\}$$

holds for the solution set S from (38), where $\bar{q} := -f(\bar{x},\bar{y})$. Note that the subdifferential mapping $\partial(\varphi \circ g)$ is always SNC in finite dimensions. Since the potential $\varphi \circ g$ is strongly amenable and the second-order qualification condition (40) is assumed, we have from the second-order subdifferential chain rule established in [14, Corollary 4.3] that

$$\partial^2(\varphi \circ g)(\bar{x},\bar{y},\bar{q})(u)$$
$$\subset \bigcup_{\bar{v}\in M(\bar{x},\bar{y})} \Big[\nabla^2\langle \bar{v}, g\rangle(\bar{x},\bar{y})(u) + \nabla g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{x},\bar{y})u\big)\Big], \quad u \in X.$$

Combining the two latter inclusions, we arrive at the conclusion of the theorem. □

Observe that the second-order qualification condition (40) is automatic if, in particular, either φ is continuously differentiable around $g(\bar{x},\bar{y})$ and its derivative is locally Lipschitzian, or the derivative operator $\nabla g(\bar{x},\bar{y})\colon X \times Y \to W$ is surjective.

The second class of parametric GVIs under consideration involves *composite fields* of the form $Q(x,y) = \partial\varphi \circ g$ with $g\colon X \times Y \to W$ and $\varphi\colon W \to \overline{\mathbb{R}}$. Solution maps for such GVIs are given by

$$S(x) := \Big\{y \in Y \Big|\ 0 \in f(x,y) + (\partial\varphi \circ g)(x,y)\Big\}, \tag{41}$$

where $f\colon X \times Y \to W^*$. Such systems include, in particular, perturbed *implicit complementarity* problems of the type: find $y \in Y$ satisfying

$$f(x,y) \geq 0, \quad y - g(x,y) \geq 0, \quad \langle f(x,y), y - g(x,y)\rangle = 0,$$

where the inequalities are understood in the sense of some order on Y. Problems of this kind frequently arise in a broad spectrum of mathematical models involving various types of economic and mechanical equilibria; see, e.g., [9, 12, 18] and the references therein. The following theorem summarizes results on computing and estimating coderivatives of solution maps (41).

Theorem 3.7 (coderivatives of solution maps to GVIs with composite fields). *Let* $(\bar{x},\bar{y}) \in \operatorname{gph} S$ *for* S *defined in* (41) *with some mappings* $g\colon X\times Y \to W$, $f\colon X\times Y \to W^*$, *and* $\varphi\colon W \to \overline{I\!R}$ *between Banach spaces. Denote* $\bar{w} := g(\bar{x},\bar{y})$ *and* $\bar{q} := -f(\bar{x},\bar{y}) \in \partial\varphi(\bar{w})$. *The following assertions hold:*

(i) *Assume that* f *is strictly differentiable at* $(\bar{x},\bar{y})$ *with the surjective partial derivative* $\nabla_x f(\bar{x},\bar{y})$ *and that* $g = g(y)$ *is strictly differentiable at* $\bar{y}$ *with the surjective derivative* $\nabla g(\bar{y})$. *Then one has*

$$D_N^*S(\bar{x},\bar{y})(y^*) = \Big\{x^* \in X^* \Big|\ \exists u \in W^{**} \quad \text{with } x^* = \nabla_x f(\bar{x},\bar{y})^*u,\ -y^* \in \nabla_y f(\bar{x},\bar{y})^*u + \nabla g(\bar{y})^*\partial^2\varphi(\bar{w},\bar{q})(u)\Big\}.$$

(ii) *Assume that* X *and* Y *are Asplund, that* $W = I\!R^m$, *that* $g\colon X\times Y \to I\!R^m$ *is continuous while* $f\colon X\times Y \to I\!R^m$ *are Lipschitz continuous around* $(\bar{x},\bar{y})$, *that the graph of* $\partial\varphi\colon I\!R^m \rightrightarrows I\!R^m$ *is closed around* $(\bar{w},\bar{q})$ *(which is automatic when* φ *is either locally continuous or amenable), that the qualification conditions*

$$\partial^2\varphi(\bar{w},\bar{q})(0) \cap \ker D^*g(\bar{x},\bar{y}) = \{0\}, \tag{42}$$

$$\Big[0 \in \partial\langle u,f\rangle(\bar{x},\bar{y}) + D^*g(\bar{x},\bar{y}) \circ \partial^2\varphi(\bar{w},\bar{q})(u)\Big] \Longrightarrow u = 0$$

are fulfilled, and that either g *is locally Lipschitzian around* $(\bar{x},\bar{y})$ *or* X *and* Y *are finite-dimensional. Then one has the inclusion*

$$D_N^*S(\bar{x},\bar{y})(y^*) \subset \Big\{x^* \in X^* \Big|\ \exists u \in I\!R^m \quad \text{with}\ (x^*,-y^*) \in \partial\langle u,f\rangle(\bar{x},\bar{y}) + D^*g(\bar{x},\bar{y}) \circ \partial^2\varphi(\bar{w},\bar{q})(u)\Big\}. \tag{43}$$

Proof. It follows from Theorem 3.4(c) with $Q(x,y) = (\partial\varphi \circ g)(x,y)$ and from the corresponding relations in (11) and (13) that

$$D_N^*S(\bar{x},\bar{y})(y^*) \subset \Big\{x^* \in X^* \Big|\ \exists u \in I\!R^m \quad \text{with}\ (x^*,-y^*) \in \partial\langle u,f\rangle(\bar{x},\bar{y}) + D_N^*(\partial\varphi \circ g)(\bar{x},\bar{y},\bar{q})(u)\Big\} \tag{44}$$

in case (ii) under the qualification condition

$$\Big[0 \in \partial\langle u,f\rangle(\bar{x},\bar{y}) + D_N^*(\partial\varphi \circ g)(\bar{x},\bar{y},\bar{q})(u)\Big] \Longrightarrow u = 0. \tag{45}$$

Moreover, (44) holds as equality, by Theorem 3.1, under the assumptions in (i).

To establish the coderivative formulas in (i) and (ii), it remains to employ the appropriate coderivative chain rules to represent the normal coderivative $D_N^*(\partial\varphi \circ g)$ of the composition. Under the surjectivity assumption on $\nabla g(\bar{y})$ in (i), we have

$$D_N^*(\partial\varphi \circ g)(\bar{x},\bar{y},\bar{q})(u) = \nabla g(\bar{y})^*\partial^2\varphi(\bar{w},\bar{q})(u)$$

by [22, Theorem 3.10] in general Banach spaces, which thus implies the equality representation for the solution map coderivative $D^*_N S(\bar{x},\bar{y})(y^*)$ in (i). To proceed in case (ii), we employ the coderivative chain rule from [20, Theorem 5.6], which ensures the inclusion

$$D^*_N(\partial\varphi\circ g)(\bar{x},\bar{y},\bar{q})(u)\subset D^*g(\bar{x},\bar{y})\circ\partial^2\varphi(\bar{w},\bar{q})(u) \tag{46}$$

under the second-order qualification condition (42). Substituting (46) into (44) and (45), we get (43) under the assumptions in (ii) and complete the proof of the theorem. □

4. Verifiable Conditions for Robust Lipschitzian Stability

In this section we obtain efficient conditions for robust Lipschitzian stability (in the sense of the fulfillment of the Lipschitz-like property) of solution maps to the general parametric variational systems under consideration and their remarkable specifications. Our sensitivity analysis is based on the dual characterizations from Theorem 2.1, the representations for coderivatives of solutions maps developed in Section 3, and available calculus results on the preservation of the SNC/PSNC properties under various compositions of sets and mappings in infinite-dimensional spaces. In this way we derive sufficient conditions (as well as necessary and sufficient conditions in some settings) for robust Lipschitzian stability of solution maps together with the corresponding formulas for computing/estimating the exact Lipschitzian bounds.

Let us start with *characterizations* of robust Lipschitzian stability for variational systems described by generalized equations under some *surjectivity/regularity* assumptions.

Theorem 4.1 (characterizations of Lipschitzian stability for regular generalized equations). *Let S be the solution map* (4), *where $f\colon X\times Y\to Z$ is strictly differentiable at $(\bar{x},\bar{y})\in\operatorname{gph}S$, where $Q\colon X\times Y\rightrightarrows Z$ is locally closed-graph around $(\bar{x},\bar{y},\bar{z})$ with $\bar{z}:=-f(\bar{x},\bar{y})$ and SNC at this point, and where the spaces X,Y are Asplund. The following hold:*

(i) *Assume that Z is Banach, that $\nabla_x f(\bar{x},\bar{y})$ is surjective, and that Q does not depend on x. Then S is Lipschitz-like around $(\bar{x},\bar{y})$ if the partial adjoint generalized equation* (25) *has only the trivial solution. This condition is also necessary for the Lipschitz-like property of S around $(\bar{x},\bar{y})$ if S is strongly coderivatively normal at $(\bar{x},\bar{y})$, in particular, when Y is finite-dimensional. If in addition the space X is finite-dimensional, then*

$$\operatorname{lip}S(\bar{x},\bar{y})=\sup\Big\{\|\nabla_x f(\bar{x},\bar{y})^*z^*\|\;\Big|\;\begin{array}{l}\exists y^*\in D^*_N Q(\bar{y},\bar{z})(z^*)\quad\textit{with}\\ \|\nabla_y f(\bar{x},\bar{y})^*z^*+y^*\|\le 1\end{array}\Big\}. \tag{47}$$

(ii) *Assume that Z is Asplund and that Q is normally regular at $(\bar{x},\bar{y},\bar{z})$. Then S is normally regular at $(\bar{x},\bar{y})$, and the condition*

$$\Big[(x^*,0)\in\nabla f(\bar{x},\bar{y})^*z^*+D^*Q(\bar{x},\bar{y},\bar{z})(z^*)\Big]\Longrightarrow x^*=z^*=0 \tag{48}$$

is sufficient for the Lipschitz-like property of S around $(\bar{x},\bar{y})$. This condition is also necessary for the Lipschitz-like property of S provided that the adjoint general-

ized equation (18) *has only the trivial solution. If in addition the space X is finite-dimensional, then*

$$\operatorname{lip} S(\bar{x},\bar{y}) = \sup \Big\{ \|x^*\| \;\Big|\; \exists z^* \in Z^* \quad \text{with} \Big(x^* - \nabla_x f(\bar{x},\bar{y})^* z^*, \\ -y^* - \nabla_y f(\bar{x},\bar{y})^* z^* \Big) \in D^*Q(\bar{x},\bar{y},\bar{z})(z^*),\;\; \|y^*\| \le 1 \Big\}. \tag{49}$$

In particular, for $Q = Q(y)$ the solution map S is Lipschitz-like around $(\bar{x},\bar{y})$ if the partial adjoint generalized equation (25) *has only the trivial solution. This condition is also necessary for the Lipschitz-like property of S provided that* (25) *admits only the trivial solution on the subspace* $\ker \nabla_x f(\bar{x},\bar{y})^*$. *If in addition* $\dim X < \infty$, *then* $\operatorname{lip} S(\bar{x},\bar{y})$ *is computed by formula* (47).

Proof. In what follows we apply criterion (c) of Theorem 2.1 and the exact bound formula (16) to the solution map (4) in the Asplund space setting. To proceed in this way, one needs to employ the coderivative formulas from Theorem 3.1 together with appropriate results of the SNC calculus.

Let us first prove assertion (i). By Theorem 3.1(i) one has

$$D_N^* S(\bar{x},\bar{y})(0) = \Big\{ \nabla_x f(\bar{x},\bar{y})^* z^* \quad \text{with } z^* \in Z^* \text{ satisfying (25)} \Big\}, \tag{50}$$

from which and the surjectivity of $\nabla_x f(\bar{x},\bar{y})$ we conclude that $D_N^* S(\bar{x},\bar{y}) = \{0\}$ *if and only if* the partial adjoint generalized equation (25) has only the trivial solution. Further, the representation

$$\operatorname{gph} S = \big\{ (x,y) \in X \times Y \,\big|\, g(x,y) \in \operatorname{gph} Q \big\} \quad \text{with } g(x,y) = \big(y, -f(x,y)\big)$$

yields by [22, Corollary 5.3] that, in the general Banach space setting, S is SNC at $(\bar{x},\bar{y})$ *if and only if* Q is SNC at $(\bar{y},\bar{z})$ provided that $\nabla g(\bar{x},\bar{y})$ is *surjective*. Since the latter condition is equivalent to the surjectivity of $\nabla_x f(\bar{x},\bar{y})$ and since $D_M^* S(\bar{x},\bar{y})(y^*) \subset D_N^* S(\bar{x},\bar{y})(y^*)$ with the equality for strongly coderivatively normal mappings, we arrive at all the conclusions of (i) on the Lipschitz-like property of S with the exact bound formula (47).

To justify (ii), we observe that condition (48) implies, in particular, that the (full) adjoint generalized equation (18) has only the trivial solution. Then Theorem 3.1(ii) ensures that S is normally regular at $(\bar{x},\bar{y})$ and that $D^* S(\bar{x},\bar{y})$ is computed by the formula therein. Thus one has

$$D^* S(\bar{x},\bar{y})(0) = \Big\{ x^* \in X^* \;\Big|\; \exists z^* \in Z^* \quad \text{with} \Big(x^* - \nabla_x f(\bar{x},\bar{y})^* z^*, \\ -\nabla_y f(\bar{x},\bar{y})^* z^* \Big) \in D^*Q(\bar{x},\bar{y},\bar{z})(z^* \Big\}.$$

Hence condition (48) implies that $D^* S(\bar{x},\bar{y})(0) = \{0\}$. Furthermore, by [22, Theorem 5.4] and representation (19) with g defined in (20), we conclude that S is SNC at $(\bar{x},\bar{y})$ under the assumptions made. Now Theorem 2.1 ensures the Lipschitz-like property of S around $(\bar{x},\bar{y})$ and the exact bound formula in (ii). It follows from the above arguments that condition (48) is also *necessary* for the Lipschitz-like property of S provided that (18) has only the trivial solution.

It remains to justify the last conclusion of the theorem for $Q = Q(y)$. Observe that in this case the generalized equation (18) has only the trivial solution if and only if

$$\ker \nabla_x f(\bar{x}, \bar{y})^* \cap \Big\{ z^* \in Z^* \quad \text{satisfying (25)} \Big\} = \{0\}.$$

Using this together with (50), we complete the proof of the theorem. □

Corollary 4.2 (characterizations of Lipschitzian stability for generalized equations with convex-graph fields). *Let S be the solution map* (4) *under the common assumptions of Theorem* 4.1*, let the graph of Q be convex, and let the mappings $M, \widetilde{M}, L$ be defined in Corollary* 3.3*. The following assertions hold:*

(i) *Assume that Z is Banach, that $\nabla_x f(\bar{x}, \bar{y})$ is surjective, and that Q is independent of x. Then the condition*

$$(0, z^*) \in N((0,0); \operatorname{rge} \widetilde{M}) \Longrightarrow z^* = 0$$

is necessary and sufficient for the Lipschitz-like property of S around $(\bar{x}, \bar{y})$. Moreover, in this case

$$\operatorname{lip} S(\bar{x}, \bar{y}) = \sup \Big\{ \|\nabla_x f(\bar{x}, \bar{y})^* z^*\| \ \Big| \ -(y^*, z^*) \in N((0,0); \operatorname{rge} \widetilde{M}), \ \ \|y^*\| \le 1 \Big\}$$

if X is finite-dimensional.

(ii) *Assume that Z is Asplund. Then the condition*

$$(x^*, 0, z^*) \in N((0,0,0); \operatorname{rge} M) \Longrightarrow x^* = z^* = 0$$

is sufficient for the Lipschitz-like property of S around $(\bar{x}, \bar{y})$ being also necessary for this property if $N(0; \operatorname{rge} L) = \{0\}$. In this case

$$\operatorname{lip} S(\bar{x}, \bar{y}) = \sup \Big\{ \|x^*\| \ \Big| \ \begin{array}{l} \exists (y^*, z^*) \in Y^* \times Z^* \quad \textit{with} \\ (x^*, -y^*, -z^*) \in N((0,0,0); \operatorname{rge} M), \ \ \|y^*\| \le 1 \end{array} \Big\}$$

if X is finite-dimensional.

Proof. If follows from Theorem 4.1 due to the coderivative representation for convex-graph mappings; see the proof of Corollary 3.3. It can be also derived directly from Theorem 2.1 and Corollary 3.3 similarly to the proof of Theorem 4.1. □

Remark 4.3 (basic normals versus Clarke normals in Lipschitzian stability). Observe that, due to [19, Theorem 8.11], the results of Theorem 4.1(ii) do not distinguish between the usage of our basic normal cone (7) and the Clarke normal cone N_C [6] to the graph of Q provided that the basic normal cone $N((\bar{x}, \bar{y}, \bar{z}); \operatorname{gph} Q)$ is weak* closed (this is the case, in particular, when either X, Y, Z are finite-dimensional or the graph of Q is convex as in Corollary 4.2). On the contrary, Theorem 4.1(i) *strikingly does.* Indeed, a counterpart of Theorem 4.1(i) with the normal coderivative $D_N^* Q(\bar{y}, \bar{z})(z^*)$ replaced by the Clarke normal cone

$$\Big\{ (y^*, z^*) \in Y^* \times Z^* \Big| \ (y^*, -z^*) \in N_C((\bar{y}, \bar{z}); \operatorname{gph} Q) \Big\}$$

obviously provides a *sufficient* condition for the Lipschitz-like property of the solution map (4) at $(\bar{x}, \bar{y})$; cf. [1, 26]. However, the latter condition is *far removed from necessity* and actually *does not hold at all* for a large class of set-valued mappings Q. Let us present *two examples* explicitly demonstrating this phenomenon.

First consider the parametric generalized equation

$$0 \in x + [-y, y] \quad \text{with } x, y \in I\!R.$$

In this case $Q(y) = [-y, y]$, and one may directly check that

$$N((0,0); \operatorname{gph} Q) = \Big\{ (v, u) \in I\!R^2 \Big|\ |u| = |v| \Big\} \quad \text{and} \quad N_C((0,0); \operatorname{gph} Q) = I\!R^2.$$

Hence $D^*Q(0,0)(u) = \{-u, u\}$ and the condition $D^*Q(0,0)(0) = \{0\}$ is obviously fulfilled characterizing Lipschitzian stability of (4), while its Clarke counterpart

$$\Big[\big(-\nabla_y f(\bar{x}, \bar{y}) z^*, -z^* \big) \in N_C((\bar{y}, \bar{z}); \operatorname{gph} Q) \Big] \Longrightarrow z^* = 0 \tag{51}$$

does not hold although the solution map $S(x) = \{y \in I\!R |\ -x \in [-y, y]\}$ is clearly Lipschitz-like around the reference point $(0, 0)$.

The second example concerns the classical framework of perturbed *variational inequalities* (actually *complementarity problems*):

$$\text{find } \ y \geq 0 \quad \text{with } (ay + x)(v - y) \geq 0 \quad \text{for all } v \geq 0, \tag{52}$$

where $a \in I\!R$ is a given number and $x \in I\!R$ is a perturbation parameter. This example can be written in the generalized equation form (3) with

$$f(x, y) := ay + x \quad \text{and} \quad Q(y) := \begin{cases} 0 & \text{if } y > 0, \\ I\!R_- & \text{if } y = 0, \\ \emptyset & \text{if } y < 0. \end{cases}$$

It is easy to see that $Q(y) = N(y; \Omega) = \partial\delta(y; \Omega)$ for $\Omega := I\!R_+$, and therefore the (nonconvex) graph of Q is computed by

$$\operatorname{gph} Q = \Big\{ (y, z) \in I\!R^2 \Big|\ y \geq 0, \ \ z \leq 0, \ \ yz = 0 \Big\}.$$

The basic normal cone to this graph is computed by

$$N((0,0); \operatorname{gph} Q) = \Big\{ (v, u) \in I\!R^2 \Big|\ v \leq 0, \ \ u \geq 0 \Big\},$$

which gives the coderivative expression

$$D^*Q(0,0)(u) = \begin{cases} 0 & \text{if } u > 0, \\ I\!R & \text{if } u = 0, \\ I\!R_- & \text{if } u < 0. \end{cases}$$

This allows us to conclude by Theorem 4.1(i) that the solution map to (52) is Lipschitz-like around $(0, 0)$ *if and only if* $a > 0$. On the other hand, one has $N_C((0,0); \operatorname{gph} Q) = I\!R^2$

for the Clarke normal cone, and hence the sufficient condition (51) carries *no information* about Lipschitzian stability of the perturbed variational inequality (52).

It turns out that the situation in the above examples is typical for a sufficiently *broad class of variational systems* including the classical variational inequalities and complementarity problems. Considering the case when $Q\colon I\!\!R^n \rightrightarrows I\!\!R^n$ is a *graphically Lipschitzian* mapping (in the sense of Rockafellar [27, 28]) of dimension n around $(\bar{y},\bar{z})$ (this includes maximal monotone relations, particularly *subdifferential mappings* $Q=\partial\varphi$ for convex and other nice functions), we derive from [27] that $N_C((\bar{y},\bar{z});\operatorname{gph} Q)$ is a *subspace* of $I\!\!R^{2n}$ having dimension *at least* n. It is easy to check that in this setting the sufficient condition (51) implies that the dimension of the subspace $N_C((\bar{y},\bar{z});\operatorname{gph} Q)$ is *exactly* n, and hence the graph of Q is *strictly smooth* [27] at $(\bar{y},\bar{z})$. Moreover, if $Q=\partial\varphi$ with a proper l.s.c. convex function on $I\!\!R^n$, then the latter property corresponds to some second-order differentiability of φ, which is very close to the classical contents; cf. [27]. Hence condition (51) involving Clarke normals cannot actually cover standard settings of variational inequalities and complementarity problems in finite dimensions, where φ is the indicator function of a convex set. In contrast to this, we present here *characterizations* and/or *efficient sufficient conditions* for Lipschitzian stability of such and more general variational systems in terms of our basic normals and second-order subdifferentials.

Now we examine variational systems (4) with *no* surjectivity and/or regularity assumptions on the initial data. The following theorem gives *sufficient* conditions for robust Lipschitzian stability with an *upper estimate* of the exact Lipschitzian bound. For simplicity we consider only the case when the base f is assumed to be strictly Lipschitzian at the reference point.

Theorem 4.4 (Lipschitzian stability for nonregular generalized equations). *Let S be the solution map* (4)*, where $f\colon X\times Y\to Z$ is strictly Lipschitzian at $(\bar{x},\bar{y})\in\operatorname{gph} S$, where $Q\colon X\times Y\rightrightarrows Z$ is locally closed-graph around $(\bar{x},\bar{y},\bar{z})$ with $\bar{z}=-f(\bar{x},\bar{y})$ and SNC at this point, and where the spaces X,Y,Z are Asplund. Assume further that*

$$\Big[(x^*,0)\in\partial\langle z^*,f\rangle(\bar{x},\bar{y})+D_N^*Q(\bar{x},\bar{y},\bar{z})(z^*)\Big]\Longrightarrow x^*=z^*=0. \tag{53}$$

Then S is Lipschitz-like around $(\bar{x},\bar{y})$. If in addition $\dim X<\infty$, we have the estimate

$$\operatorname{lip} S(\bar{x},\bar{y})\le\sup\Big\{\|x^*\|\;\Big|\;\begin{array}{l}\exists z^*\in Z^*\quad\text{with } (x^*,-y^*)\in\partial\langle z^*,f\rangle(\bar{x},\bar{y})\\ +D_N^*Q(\bar{x},\bar{y},\bar{z})(z^*),\quad \|y^*\|\le 1\Big\}.\end{array} \tag{54}$$

Proof. Observe that the assumptions made in this theorem imply the fulfillment of all the assumptions in Theorem 3.4. Hence we have the coderivative inclusion (31) with

$$D_N^*f(\bar{x},\bar{y})(z^*)=\partial\langle z^*,f\rangle(\bar{x},\bar{y}),\quad z^*\in Z^*,$$

by the normal scalarization formula, and thus $D_M^*S(\bar{x},\bar{y})(0)=\{0\}$. By Theorem 2.1 it remains to show that S is PSNC at $(\bar{x},\bar{y})$.

Let us prove that S is actually SNC at $(\bar{x},\bar{y})$ due to local Lipschitz continuity of f around this point in addition to (53) and the SNC property of Q at $(\bar{x},\bar{y},\bar{z})$. To furnish this, we apply [20, Theorem 3.8] to the inverse image

$$\operatorname{gph} S=g^{-1}(\operatorname{gph} Q)\quad\text{ with } g(x,y)=\big(x,y,-f(x,y)\big).$$

The only thing one needs to check is that g is PSNC at $(\bar{x},\bar{y})$ if f is Lipschitz continuous around this point. Indeed, taking sequences

$$(x_k^*,y_k^*)\in\widehat{D}^*g(x_k,y_k)(u_k^*,v_k^*,z_k^*)\quad\text{with }(x_k^*,y_k^*)\xrightarrow{w^*}(0,0)\quad\text{and }\|(u_k^*,v_k^*,z_k^*)\|\to 0$$

and using the scalarization formula (36) for the Fréchet coderivative, we get

$$(x_k^*,y_k^*)=(u_k^*,v_k^*)+(\hat{x}_k^*,\hat{y}_k^*)\quad\text{with }(\hat{x}_k^*,\hat{y}_k^*)\in\widehat{\partial}\langle -z_k^*,f\rangle(x_k,y_k)$$

due to the representation $g(x,y)=(x,y,0)+(0,0,-f(x,y))$ and the elementary sum rule with a smooth summand for Fréchet subgradients. This implies that $(\hat{x}_k^*,\hat{y}_k^*)\xrightarrow{w^*}(0,0)$, and hence $\|(\hat{x}_k^*,\hat{y}_k^*)\|\to 0$ by the PSNC property of locally Lipschitzian mappings. Thus $\|(x_k^*,y_k^*)\|\to 0$ as well, i.e., g is PSNC at $(\bar{x},\bar{y})$. □

Our next theorem gives efficient conditions ensuring robust Lipschitzian stability and exact bound formulas of subdifferential variational systems (38) with *composite potentials.*

Theorem 4.5 (Lipschitzian stability for variational systems with composite potentials). *Let $(\bar{x},\bar{y})\in\operatorname{gph}S$ for S defined in* (38)*, where $f\colon X\times Y\to X^*\times Y^*$ with $\bar{q}:=-f(\bar{x},\bar{y})$, where $g\colon X\times Y\to W$ with $\bar{w}:=g(\bar{x},\bar{y})$, and where $\varphi\colon W\to\overline{I\!R}$. The following assertions hold:*

(*i*) *Suppose that W is Banach, X is Asplund while $Y=I\!R^m$, that $g=g(y)$, and that assumptions* (*a*) *and* (*b*) *of Theorem* 3.5 *are fulfilled with $\bar{v}$ defined therein. Then S is Lipschitz-like around $(\bar{x},\bar{y})$ if and only if $u=0\in I\!R^m$ is the only vector satisfying*

$$0\in\nabla_y f(\bar{x},\bar{y})^*u+\nabla^2\langle\bar{v},g\rangle(\bar{y})^*u+\nabla g(\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{y})u\big).\tag{55}$$

In in addition X is finite-dimensional, then one has

$$\begin{aligned}&\operatorname{lip}S(\bar{x},\bar{y})=\sup\Big\{\|\nabla_x f(\bar{x},\bar{y})^*u\|\quad\textit{with}\\&-y^*\in\nabla_y f(\bar{x},\bar{y})^*u+\nabla^2\langle\bar{v},g\rangle(\bar{y})^*u+\nabla g(\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{y})u\big),\ \ \|y^*\|\le 1\Big\}.\end{aligned}\tag{56}$$

(*ii*) *Suppose that the potential $\varphi\circ g$ in* (38) *is strongly amenable, that all the spaces in question are finite-dimensional, that f is locally Lipschitzian around $(\bar{x},\bar{y})$, and that the qualification conditions* (40) *and*

$$\begin{aligned}\Big[(x^*,0)\in\partial\langle u,f\rangle(\bar{x},\bar{y})+\bigcup_{\bar{v}\in M(\bar{x},\bar{y})}\big[\nabla^2\langle\bar{v},g\rangle(\bar{x},\bar{y})(u)\\+\nabla g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{x},\bar{y})u\big)\big]\Big]\Longrightarrow x^*=u=0\end{aligned}\tag{57}$$

are fulfilled, where the set $M(\bar{x},\bar{y})$ is defined in Theorem 3.6*. Then S is Lipschitz-like around $(\bar{x},\bar{y})$ with the exact bound estimate*

$$\begin{aligned}&\operatorname{lip}S(\bar{x},\bar{y})\le\sup\Big\{\|x^*\|\,\Big|\ \exists u\in X\times Y\ \ \textit{with }(x^*,-y^*)\in\partial\langle u,f\rangle(\bar{x},\bar{y})\\&+\bigcup_{\bar{v}\in M(\bar{x},\bar{y})}\big[\nabla^2\langle\bar{v},g\rangle(\bar{x},\bar{y})(u)+\nabla g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{x},\bar{y})u\big)\big],\ \ \|y^*\|\le 1\Big\}.\end{aligned}\tag{58}$$

Proof. To prove (i), we use the coderivative representation (39) in Theorem 3.5 for $D^*_N S(\bar{x},\bar{y}) = D^*_M S(\bar{x},\bar{y})$, which holds as *equality* due to the finite dimensionality of Y. Moreover, the graph of S is SNC at $(\bar{x},\bar{y})$, since it is the inverse image of $\operatorname{gph} Q$ under a strictly differentiable mapping with the surjective derivative, where $Q = \partial(\varphi \circ g)\colon Y \rightrightarrows Y^*$ is automatically SNC; cf. the proof of Theorem 4.1(i). Thus the condition $D^*_M S(\bar{x},\bar{y})(0) = \{0\}$ reduces to (55), which is therefore *necessary and sufficient* for the Lipschitz-like property of S by Theorem 2.1. This also implies the exact bound formula (56).

To justify (ii), we employ the coderivative inclusion

$$D^*S(\bar{x},\bar{y})(y^*) \subset \Big\{x^* \in X^* \Big|\, \exists u \in X \times Y \quad \text{with } (x^*,-y^*) \in \partial\langle u, f\rangle(\bar{x},\bar{y}) + \bigcup_{\bar{v}\in M(\bar{x},\bar{y})} \Big[\nabla^2\langle \bar{v}, g\rangle(\bar{x},\bar{y})(u) + \nabla g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{v})\big(\nabla g(\bar{x},\bar{y})u\big)\Big]\Big\} \tag{59}$$

held by Theorem 3.6, by the scalarization formula for the Lipschitzian base f, and by the qualification condition (57). Moreover, the latter condition ensures that $D^*S(\bar{x},\bar{y})(0) = \{0\}$ by (59), and hence S is Lipschitz-like around $(\bar{x},\bar{y})$ with the exact bound estimate (58) due to the upper estimate (15) from Theorem 2.1. □

The last theorem of this section gives sufficient, as well as necessary and sufficient, conditions for robust Lipschitzian stability of variational systems (41) with *composite fields*. For simplicity we consider only the case of strictly differentiable mappings f and g in (41).

Theorem 4.6 (Lipschitzian stability for variational systems with composite fields). *Let $(\bar{x},\bar{y}) \in \operatorname{gph} S$ with $\bar{w} := g(\bar{x},\bar{y})$ and $\bar{q} := -f(\bar{x},\bar{y})$, where S is defined in (41), where X and Y are Asplund while W is Banach, and where $g\colon X\times Y \to W$ and $f\colon X \times Y \to W$ are strictly differentiable at $(\bar{x},\bar{y})$. The following assertions hold:*

(i) *Assume that $g = g(y)$ with the surjective derivative $\nabla g(\bar{y})$, and that the partial derivative $\nabla_x f(\bar{x},\bar{y})$ of f is also surjective. Then the condition*

$$\Big[0 \in \nabla_y f(\bar{x},\bar{y})^*u + \nabla g(\bar{y})^*\partial^2\varphi(\bar{w},\bar{q})(u)\Big] \Longrightarrow u = 0.$$

is sufficient for the Lipschitz-like property of S around $(\bar{x},\bar{y})$ being also necessary for this property if S is strongly coderivatively normal at $(\bar{x},\bar{y})$. Furthermore,

$$\operatorname{lip} S(\bar{x},\bar{y}) = \sup\Big\{\|\nabla_x f(\bar{x},\bar{y})^*u\| \,\Big|\, \exists z^* \in \partial^2\varphi(\bar{w},\bar{q})(u) \quad \textit{with} \quad \|\nabla_y f(\bar{x},\bar{y})^*u + \nabla g(\bar{y})^*z^*\| \le 1\Big\}$$

provided that $\dim X < \infty$ and that $\partial\varphi$ is SNC at $(\bar{w},\bar{q})$.

(ii) *Assume that $W = I\!R^m$, that the graph of $\partial\varphi$ is closed around $(\bar{w},\bar{q})$, and that*

$$\partial^2\varphi(\bar{w},\bar{q})(0) \cap \ker \nabla g(\bar{x},\bar{y})^* = \{0\}, \quad \partial^2\varphi(\bar{w},\bar{q})(0) \subset \ker \nabla_x g(\bar{x},\bar{y})^*, \tag{60}$$

$$\Big[0 \in \nabla_y f(\bar{x},\bar{y})^*u + \nabla_y g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{q})(u)\Big] \Longrightarrow u = 0. \tag{61}$$

Then S is Lipschitz-like around $(\bar{x},\bar{y})$. Furthermore,

$$\operatorname{lip} S(\bar{x},\bar{y}) \le \sup\Big\{\|x^*\| \,\Big|\, \exists u \in I\!R^m,\ y^* \in \nabla_y g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{q})(u) \quad \textit{with} \quad x^* - \nabla_x f(\bar{x},\bar{y})^*u \in \nabla_x g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{w},\bar{q})(u),\ \ \|\nabla_y f(\bar{x},\bar{y})^*u + y^*\| \le 1\Big\}.$$

if in addition $\dim X < \infty$.

Proof. To justify (i), we use the coderivative representation from Theorem 3.7(i) and then apply Theorem 2.1 observing that the SNC property of $\partial\varphi$ at $(\bar{w}, \bar{q})$ yields the one for S at $(\bar{x}, \bar{y})$ due to

$$\operatorname{gph} S = \big\{(x,u) \in X \times Y \big|\ g(x,y) \in \operatorname{gph}(\partial\varphi \circ g)\big\}$$

and the corresponding results of SNC calculus from [22, Corollary 5.3] and [20, Theorem 3.8].

To prove (ii), we apply the coderivative inclusion from Theorem 3.7(ii) held under the qualification conditions (60), and then use the basic characterization of the pseudo-Lipschitzian property from Theorem 2.1, which gives (61). It remains to observe that, as shown in the proof of Theorem 3.7(ii), the composition $\partial\varphi \circ g$ is SNC at $(\bar{x}, \bar{y}, \bar{q})$ under the assumptions made. Hence S is SNC at $(\bar{x}, \bar{y})$, which completes the proof of the theorem. □

To conclude this paper, we observe that practical implementations of the obtained results on robust Lipschitzian stability for variational systems of types (38) and (41), which are the most interesting in applications, require computing/estimating the *second-order subdifferentials* of potentials φ presented in the above theorems. This have been efficiently done for various classes of extended-real-valued functions particularly arising in frameworks of complementarity problems, parametric mathematical programs (including those with equilibrium constraints), mechanical and economic equilibria, etc.; see, e.g., [7, 12, 18, 30] and the references therein.

References

[1] J.-P. Aubin: Lipschitz behavior of solutions to convex minimization problems, Math. Oper. Res. 9 (1984) 87–111.

[2] J. M. Borwein, S. P. Fitzpatrick: Weak* sequential compactness and bornological limit derivatives, J. Convex Analysis 2 (1995) 59–68.

[3] J. M. Borwein, Y. Lucet, B. S. Mordukhovich: Compactly epi-Lipschitzian convex sets and functions in normed spaces, J. Convex Analysis 7 (2000) 375–393.

[4] J. M. Borwein, H. M. Strójwas: Tangential approximations, Nonlinear Anal., Theory Methods Appl. 9 (1985) 1347–1366.

[5] J. M. Borwein, Q. J. Zhu: Techniques of Variational Analysis, CMS Books in Mathematics 20, Springer, New York (2005).

[6] F. H. Clarke: Optimization and Nonsmooth Analysis, CMS Series of Monographs and Advanced Texts, Wiley, New York (1983).

[7] A. L. Dontchev, R. T. Rockafellar: Characterizations of strong regularity for variational inequalities over polyhedral convex sets, SIAM J. Optim. 7 (1996) 1087–1105.

[8] M. Fabian, B. S. Mordukhovich: Sequential normal compactness versus topological normal compactness in variational analysis, Nonlinear Anal., Theory Methods Appl. 54 (2003) 1057–1067.

[9] F. Facchinei, J.-S. Pang: Finite-Dimensional Variational Inequalities and Complementary Problems, Vol. I and Vol. II, Springer Series in Operations Research, Springer, New York (2003).

[10] A. D. Ioffe: Codirectional compactness, metric regularity and subdifferential calculus, in: Constructive, Experimental and Nonlinear Analysis, M. Théra (ed.), CMS Conf. Proc. 27, AMS, Providence (2000) 123–163.

[11] A. Jourani, L. Thibault: Verifiable conditions for openness, metric regularity of multivalued mappings, Trans. Amer. Math. Soc. 347 (1995) 1225–1268.

[12] M. Kočvara, J. V. Outrata: Optimization problems with equilibrium constraints and their numerical solutions, Math. Progr. 101 (2004) 119–149.

[13] B. S. Mordukhovich: Lipschitzian stability of constraint systems and generalized equations, Nonlinear Anal., Theory Methods Appl. 22 (1994) 173–206.

[14] B. S. Mordukhovich: Calculus of second-order subdifferentials in infinite dimensions, Control and Cybernetics 31 (2002) 557–573.

[15] B. S. Mordukhovich: Coderivative analysis of variational systems, J. Global Optim. 28 (2004) 347–362.

[16] B. S. Mordukhovich: Variational Analysis and Generalized Differentiation I. Basic Theory, Grundlehren der Mathematischen Wissenschaften 330, Springer, Berlin (2006).

[17] B. S. Mordukhovich: Variational Analysis and Generalized Differentiation II. Applications, Grundlehren der Mathematischen Wissenschaften 331, Springer, Berlin (2006).

[18] B. S. Mordukhovich, J. V. Outrata: Second-order subdifferentials and their applications, SIAM J. Optim. 12 (2001) 139–169.

[19] B. S. Mordukhovich, Y. Shao: Nonsmooth sequential analysis in Asplund spaces, Trans. Amer. Math. Soc. 348 (1996) 1235–1280.

[20] B. S. Mordukhovich, B. Wang: Extensions of generalized differential calculus in Asplund spaces, J. Math. Anal. Appl. 272 (2002) 164–186.

[21] B. S. Mordukhovich, B. Wang: Calculus of sequential normal compactness in variational analysis, J. Math. Anal. Appl. 282 (2003) 63–84.

[22] B. S. Mordukhovich, B. Wang: Restrictive metric regularity and generalized differential calculus in Banach spaces, Int. J. Maths. Math. Sci. 50 (2004) 2650–2683.

[23] R. R. Phelps: Convex Functions, Monotone Operators and Differentiability, Second Edition, Lecture Notes in Mathematics Series 1364, Springer, Berlin (1993).

[24] S. M. Robinson: Generalized equations and their solutions I: Basic theory, Math. Progr. Study 10 (1979) 128–141.

[25] S. M. Robinson: Constraint nondegeneracy in variational analysis, Math. Oper. Res. 28 (2003) 201–232.

[26] R. T. Rockafellar: Lipschitzian properties of multifunctions, Nonlinear Anal., Theory Methods Appl. 9 (1985) 867–885.

[27] R. T. Rockafellar: Maximal monotone relations and the second derivatives of nonsmooth functions, Ann. Inst. H. Poincaré, Anal. Non Linéaire 2 (1985) 167–184.

[28] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Grundlehren der Mathematischen Wissenschaften 317, Springer, Berlin (1998).

[29] L. Thibault: Sous-différentiels de fonctions vectorielles compactement lipschitziennes, C. R. Acad. Sci., Paris, Sér. A 286 (1978) 995–998.

[30] J. J. Ye: Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints, SIAM J. Optim. 10 (2000) 943–962.