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Boundedness, Differentiability and Extensions of Convex Functions

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Dedicated to the memory of Simon Fitzpatrick.

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We survey various boundedness, differentiability and extendibility properties of convex functions, and how they are related to sequential convergence with respect to various topologies in the dual space. It is also shown that if X/Y is separable then every continuous convex function on Y can be extended to a continuous convex function on X.

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1. Introduction

The genesis of the material in this note can be found in the first author's paper [2], where, among other things, it was shown that weak Hadamard and Fréchet differentiability coincide for continuous convex functions on Asplund spaces. This was expanded upon by the first author and M. Fabian in [3] where relationships between various forms of differentiability for convex functions were connected with sequential convergence of the related topologies in the dual space. In late 1993 whilst Simon Fitzpatrick was visiting Simon Fraser University, he played a key role in producing the paper [5] – which among other things connected boundedness properties of convex functions with sequential convergence of related topologies in the dual space. A few years later, S. Simons [28] produced examples of continuous convex functions whose biconjugates are not continuous and asked

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which classes of Banach spaces admit such examples. The answer, as shown in [8], was connected to sequential convergence in dual topologies and used techniques that had been developed in [5].

Our goal is to more thoroughly understand how properties of convex functions on Banach spaces are connected to sequential convergence with respect to various topologies in the dual space. To this end, the next two sections survey some of the key techniques and results in this topic. In the final section of this note, we build on ideas of Simon Fitzpatrick's (from [5]) to develop a new characterization concerning extensions of convex functions and use it to show that any continuous convex function on a Banach space Y can be extended to a continuous convex function on a Banach space X for which X/Y is separable. This answers a question implicitly found in [8, p. 1802].

We now introduce some of the notation that we will use in this article. We will work in real Banach spaces X, whose unit ball and unit sphere are denoted by B_X and S_X respectively. As in [22, p. 59] we say a *bornology* on X is a family of bounded sets whose union is all of X, which is closed under reflection through the origin and under multiplication by positive scalars, and the union of any two members of the bornology is contained in some member of the bornology. We will denote a general bornology by β, but our attention will focus on the following three bornologies: $\mathcal{F}$ the *Gâteaux* bornology of all finite sets; $\mathcal{W}$ the *weak Hadamard* bornology of weakly compact sets; and $\mathcal{B}$ the *Fréchet* bornology of all bounded sets. Given a bornology β on X, we will say a function $f : X \to \mathbb{R} \cup \{\infty\}$ is *β-differentiable* at x in the domain of f, if there is a $\phi \in X^*$ such that for each β-set S, the following limit exists uniformly for $h \in S$

$$\lim_{t\to 0} \frac{f(x+th)-f(x)}{t} = \phi(h).$$

In particular, we say f is *Gâteaux differentiable* at x if β is the Gâteaux bornology. Similarly for the weak-Hadamard and Fréchet bornologies. Also, given any bornology β on X, by τ_β we mean the topology on X^* of uniform convergence on β-sets. In particular, $\tau_{\mathcal{W}}$ is the Mackey topology of uniform convergence on weakly compact sets, usually denoted by $\mu(X^*, X)$ in the theory of locally convex spaces. Following [3], when we speak of the *Mackey topology* on X^*, we will mean $\mu(X^*, X)$. Also, for $\epsilon \geq 0$, the *ϵ-subdifferential* of f at x_0 in the domain of f is defined by

$$\partial_\epsilon f(x_0) := \{\phi \in X^* : \phi(x) - \phi(x_0) \leq f(x) - f(x_0) + \epsilon, \quad \text{for all } x \in X\}.$$

When $\epsilon = 0$ in the above, this is just the *subdifferential* of f at x_0, and is denoted by $\partial f(x_0)$.

2. Canonical Examples

We begin with constructions of convex functions that seem to be central to connecting their properties with linear topological properties in the dual. The following result is essentially from [3, 5].

Proposition 2.1. *Let $\{\phi_n\}_{n=1}^\infty \subset B_{X^*}$. Consider the functions from X into $\mathbb{R} \cup \{+\infty\}$ that are defined as follows*

$$f(x) := \sup_n\{\phi_n(x) - \frac{1}{n}, 0\} \qquad g(x) := \sup_n(\phi_n(x))^{2n} \qquad h(x) := \sum_{n=1}^\infty (\phi_n(x))^{2n}$$

Then f, g and h are lower semicontinuous convex functions. Moreover

(a) *f is β-differentiable at 0 if and only if $\phi_n \to_{\tau_\beta} 0$ and, if this is the case, f is Lipschitz on X.*

(b) *g and h are bounded on β-sets if and only if $\phi_n \to_{\tau_\beta} 0$ and, if this is the case, both functions are continuous.*

Proof. It is clear that the functions are lower semicontinuous and convex as sums and suprema of such functions. We outline the other implications.

(a) Because $f(0) = 0$, and $f \geq 0$, the only possibility is that $f'(0) = 0$ if f is differentiable at 0. If $\phi_n \not\to_{\tau_\beta} 0$, we can find a β-set W and infinitely many n such that $w_n \in W$ and $\phi_n(w_n) > 2$. Then for such n, $nf(\frac{1}{n}w_n) \geq 1$ from which it follows that f is not β-differentiable at 0.

Conversely, if $\phi_n \to_{\tau_\beta} 0$, then f is Lipschitz since $\{\phi_n\}$ is bounded. Moreover, given any $\epsilon > 0$ and any β-set W, there is an $n_0 \in \mathbb{N}$ such that $\phi_n(w) < \epsilon$ for all $n > n_0$ and $w \in W$. Now for t sufficiently small, it follows that $\phi_n(tw) - \frac{1}{n} \leq 0$ for all $n \leq n_0$ and all $w \in W$. Hence for sufficiently small t we have $f(tw) - f(0) \leq \epsilon|t|$ for all $w \in W$. Thus f is β-differentiable at 0 with $f'(0) = 0$.

(b) If $\phi_n \to_{\tau_\beta} 0$, it is straightforward to check that g and h are bounded on β-sets. As finite-valued lower semicontinuous convex functions, f and g are continuous (see e.g. [22, Proposition 3.3]). Conversely, if $\phi_n \not\to_{\tau_\beta} 0$, then we can find a β-set W such that $\phi_n(w_n) > 2$ for infinitely many n where $w_n \in W$. Then neither g nor h is bounded on W. □

We refer to the previous examples as "canonical" because they are natural constructions that capture the essence of how convex functions can behave when comparing various bornological notions of boundedness or differentiability. The next proposition follows from combining results from [3, 5, 6].

Proposition 2.2. *Let X be a Banach space. Then the following are equivalent.*

(a) *Mackey and norm convergence coincide sequentially in X^*.*

(b) *Every sequence of lower semicontinuous convex functions that converges to a continuous affine function uniformly on weakly compact sets converges uniformly on bounded sets to the affine function.*

(c) *Every continuous convex function that is bounded on weakly compact subsets of X is bounded on bounded subsets of X.*

(d) *Weak Hadamard and Fréchet differentiability agree for continuous convex functions.*

Proof. (a) $\Rightarrow$ (b): Suppose $\{f_n\}$ is a sequence of lower semicontinuous convex functions that converges uniformly on weakly compact sets to some continuous affine function A. By replacing f_n with $f_n - A$ we may assume that $A = 0$. Now suppose f_n does not converge to 0 uniformly on bounded sets. Thus there are $K > 0$, $\{x_k\}_{k\geq 1} \subset KB_X$ and $\epsilon > 0$ so that $f_{n_k}(x_k) > \epsilon$ for a certain subsequence $\{n_k\}$ of $\{n\}$ (using convexity and the fact that $f_{n_k}(0) \to 0$). Now let $C_k := \{x : f_{n_k}(x) \leq \epsilon\}$ and choose $\phi_k \in S_{X^*}$ such that $\sup_{C_k} \phi_k < \phi_k(x_k) \leq K$. We observe that ϕ_k do not converge to 0 in $\tau_{\mathcal{W}}$ by (a). Find a weakly compact set $C \subset X$ so that $\sup_C \phi_k > K$ for infinitely many k. We have $\sup_C \phi_k = \phi_k(c_k)$ for some $c_k \in C$, and $c_k \notin C_k$ (so $f_{n_k}(c_k) > \varepsilon$) for infinitely many k, which contradicts the uniform convergence to 0 of (f_n) on C.

Now (b) implies (d) follows because difference quotients are lower semicontinuous convex functions, and (d) implies (a) follows from Proposition 2.1.

Finally, (c) implies (a) follows from Proposition 2.1, so we conclude by establishing (a) implies (c). For this, we suppose (c) is not true. We can find then a continuous convex function f that is bounded on weakly compact subsets of X and not bounded on all bounded subsets of X. We may assume $f(0) = 0$ and we let $\{x_n\}$ be a bounded sequence such that $f(x_n) > n$, and let $C_n := \{x : f(x) \leq n\}$. By the separation theorem, choose $\phi_n \in S_{X^*}$ such that $\sup_{C_n} \phi_n < \phi_n(x_n)$. Now choose $K > 0$ such that $K > \phi_n(x_n)$ for all n. If $\phi_n \not\to_{\tau_W} 0$, then there is a weakly compact set $W \subset X$ and infinitely many n such that $\phi_n(w_n) > K$ and $w_n \in W$. In particular, $w_n \notin C_n$ for those n and so f is unbounded on W. Thus (a) is not true when (c) is not true. $\square$

The Banach spaces for which (a) in the previous proposition is true are precisely those that do not contain an isomorphic copy of ℓ_1 [3, 21]. We conclude this section with a bornological extension of Proposition 2.2 that combines results from [3, 5, 6].

Theorem 2.3. *Let X be a Banach space with bornologies $\beta_1 \subset \beta_2$. Then the following are equivalent.*

- (a) *τ_{β_1} and τ_{β_2} agree sequentially in X^*.*
- (b) *Every sequence of lower semicontinuous functions on X that converge to a continuous affine function uniformly on β_1-sets, converges uniformly on β_2-sets.*
- (c) *Every continuous convex function on X that is bounded on β_1-sets is bounded on β_2-sets.*
- (d) *β_1-differentiability agrees with β_2-differentiability for continuous convex functions on X.*
- (e) *β_1-differentiability agrees with β_2-differentiability for equivalent norms on X.*

Proof. The equivalence of (a) and (e) follows from [3, Theorem 1]. The equivalence of (a), (b), (c) and (d) is proved by naturally modifying the proof of Proposition 2.2. However, there is a subtlety in the proof of $(a) \Rightarrow (b)$. While the fact that $\{\phi_k\} \not\to 0$ in the norm topology was automatic, to show that $\{\phi_k\} \not\to 0$ in the β_2-topology one should additionally show that eventually $\sup_{C_k} \phi_k > \delta > 0$ for some $\delta > 0$. For this, let $F_n := \{x \in X : f_k(\pm x) \leq \epsilon \text{ for all } k \geq n\}$. Since $\{f_n\}$ converges pointwise to 0, $\bigcup_{n\geq 1} F_n = X$. The Baire category theorem ensures that $F_{\bar{n}}$ has nonempty interior for some $\bar{n} \in N$, and because $F_{\bar{n}}$ is a symmetric convex set, for some $\delta > 0$ we have that $\delta B_X \subset F_{\bar{n}}$. Consequently, for $n_k \geq \bar{n}$, $\sup_{C_k} \phi_k > \delta$. $\square$

In the next section we will delineate how this theorem applies in various classes of Banach spaces. To avoid excessive redundancy, we will highlight only conditions (a), (c) and (d) from Theorem 2.3 in our statements.

3. Characterizations of Various Classes of Spaces

In this section we provide a listing of various classifications of Banach spaces in terms of properties of convex functions. Many of the implications follow from Theorem 2.3 or dualization of the arguments upon which it is based. We will organize these results based upon when two of the following notions (Gâteaux, weak Hadamard or Fréchet) differentiability coincide for continuous convex functions on a space, and then for continuous weak*-lower semicontinuous functions on the space.

First, we consider when Gâteaux and Fréchet differentiability coincide for continuous convex functions.

Theorem 3.1. *For a Banach space X, the following are equivalent.*

(a) *X is finite dimensional.*
(b) *Weak* and norm convergence coincide sequentially in X^*.*
(c) *Every continuous convex function on X is bounded on bounded subsets of X.*
(d) *Gâteaux and Fréchet differentiability coincide for continuous convex functions on X.*

Proof. The equivalence of (a) and (b) is the decidedly nontrivial Josefson-Nissenzweig Theorem (see, for example, [11, p. 219]). The equivalence of (b) through (d) is a direct consequence of Theorem 2.3 with the Gâteaux and Fréchet bornologies. □

In particular, on every infinite dimensional Banach space there is a continuous convex function that is unbounded on a ball and that assertion is equivalent to the Josefson-Nissenzweig Theorem.

Next, we consider when Gâteaux and weak Hadamard differentiability coincide. As in [4], we will say a Banach space possess the *DP*-property* if weak* and Mackey convergence (uniform convergence on weakly compact subsets of X) coincide sequentially in X^*. Recall that a Banach space is said to be a *Grothendieck space* if weak* and weak convergence coincide sequentially in X^*. A Banach space is said to have the *Dunford-Pettis property* if $\langle x_n^*, x_n\rangle \to 0$ whenever $x_n \to_w 0$ and $x_n^* \to_w 0$. It is straightforward to verify that a Banach space has the Dunford-Pettis property if and only if weak* and Mackey convergence agree sequentially in X^*, so a space has DP*-property if it is a Grothendieck space with the Dunford-Pettis property. Hence the spaces $\ell_\infty(\Gamma)$ for any index set Γ have the DP*-property as they are $C(K)$ spaces for K Stonian and so Grothendieck (see [12, pp. 156, 179]) and they have the Dunford-Pettis property (see, for example, [13, Theorem 11.36]). On the other hand, trivially every space with the DP*-property has the Dunford-Pettis property; however, there are spaces with the DP*-property which are not Grothendieck, such as ℓ_1 (every Grothendieck separable space is reflexive); see the remarks after Theorem 3.2.

Theorem 3.2. *For a Banach space X, the following are equivalent.*

(a) *X has the DP*-property.*
(b) *Gâteaux and weak Hadamard differentiability coincide for all continuous convex functions on X.*
(c) *Every continuous convex function on X is bounded on weakly compact subsets of X.*

Proof. This is a direct consequence of Theorem 2.3 using the Gâteaux and weak Hadamard bornologies. □

Because ℓ_∞ has the DP*-property, the previous theorem applies in spaces where the relatively compact sets and relatively weakly compact sets form different bornologies.

Recall that a subset L of a Banach space X is called *limited* if every weak*-null sequence in X^* converges to 0 uniformly on L. Then $\mathcal{RK} \subset \mathcal{L} \subset \mathcal{B}$, where $\mathcal{RK}$ is the collection of the relatively compact subsets, $\mathcal{L}$ of the limited subsets and $\mathcal{B}$ of the bounded subsets. The Josefson-Nissenzweig Theorem says that in infinite dimensional Banach spaces, $\mathcal{L} \neq \mathcal{B}$. A Banach space is called *Gelfand-Phillips* if $\mathcal{RK} = \mathcal{L}$. If B_{X^*} is weak*-sequentially compact, then X is Gelfand-Phillips (for these results, see [11, p. 116, 224 and 238]), while ℓ_∞ is not Gelfand-Phillips. Moreover, for a given bornology β in X, τ_β and weak* agree sequentially if and only if $\beta \subset \mathcal{L}$. In particular, a Banach space has property DP^* if and only if $\mathcal{W} \subset \mathcal{L}$, where as before $\mathcal{W}$ denotes the bornology of weakly compact subsets of X. If a Banach space is DP^* and Gelfand-Phillips (for example, the space ℓ_1) then it is Schur, and every Schur space has the DP^* property.

We now turn to spaces where weak Hadamard and Fréchet differentiability coincide for continuous convex functions. Analogous to the previous result, these are not the spaces where the weak Hadamard and Fréchet bornologies coincide – but where the dual topologies they induce agree sequentially.

Theorem 3.3. *For a Banach space X, the following are equivalent.*

(a) $X \not\supset \ell_1$.
(b) *Mackey and norm convergence coincide sequentially in X^*.*
(c) *Weak Hadamard and Fréchet differentiability coincide for continuous convex functions on X.*
(d) *Every convex function on X bounded on weakly compact sets is bounded on bounded sets.*

Proof. See [3, Theorem 5] or [21] for the equivalence of (a) and (b). The equivalence of (b) through (d) is in Proposition 2.2. $\square$

We now consider analogous situations for weak*-lower semicontinuous convex functions. Recall that a Banach space has the *Schur property* if its weakly convergent sequences are norm convergent. Let us also recall that a function f is said to be *supercoercive* if $\lim_{\|x\|\to\infty} \dfrac{f(x)}{\|x\|} = \infty$, while f is said to be *cofinite* if its conjugate f^* is defined everywhere on X^*; see [1, pp. 623,624]. A convex function is *β-subdifferentiable* if $\lim_{t\to 0+} \frac{1}{t}\,[f(x+th)-f(x)]$ exists uniformly on $h \in S$, for every β-set S. For the bornology $\mathcal{B}$ of bounded sets, this concept has been studied in, for example, [9, 14, 16].

Theorem 3.4. *For a Banach space X, the following are equivalent.*

(a) *X has the Schur property.*
(b) *Gâteaux differentiability and Fréchet differentiability coincide for continuous weak*-lower semicontinuous convex functions on X^*.*
(c) *Each continuous weak*-lower semicontinuous convex function on X^* is bounded on bounded subsets of X^*.*
(d) *Every proper lower semicontinuous cofinite convex function on X is supercoercive.*
(e) *Gâteaux differentiability and weak Hadamard differentiability agree for Lipschitz functions on X.*
(f) *Gâteaux differentiability and weak Hadamard differentiability coincide for differences of Lipschitz convex functions on X.*

(g) *Every continuous convex function on X is weak Hadamard subdifferentiable.*

Proof. The equivalence of (a) through (c) follows by dualizing the proof of Theorem 2.3 (see e.g. [4, Theorem 4.1]). See [1, Theorem 3.6] for the equivalence of the supercoercivity assertion (e) with (c). It follows from the definitions involved that (a) implies each of (e), (f) and (g) which also uses the local Lipschitzian property of continuous convex functions. Also, (e) implies (f) is trivial, the more subtle results that (f) and (g) each imply (a) can be found in [7, Proposition 8]. □

The conditions (e), (f) and (g) deal with concepts that are outside the main focus of this note. However, we feel it is important to mention them, because they show the sharpness of Theorem 2.3 in various senses. For example, it follows from Theorem 3.2 that Gâteaux and weak Hadamard differentiability agree for continuous convex functions on ℓ_∞. However, these two notions of differentiability do not coincide even for differences of Lipschitz convex functions on ℓ_∞ by (g) of the previous theorem. See [7] for further results showing that continuous convex functions cannot be replaced by differences of continuous convex functions in Theorem 2.3 and that differentiability cannot be replaced with subdifferentiability – at least for certain important bornologies.

Theorem 3.5. *For a Banach space X, the following are equivalent.*

(a) *X has the Dunford-Pettis Property.*

(b) *Weak and Mackey convergence coincide sequentially in X^*.*

(c) *Gâteaux differentiability and weak Hadamard differentiability coincide for continuous weak*-lower semicontinuous convex functions on X^*.*

(d) *Each continuous weak*-lower semicontinuous convex function on X^* is bounded on weakly compact subsets of X^*.*

Proof. This is a dualization of Theorem 2.3; see e.g. [4, Theorem 4.2]. □

Our last result regarding classes of differentiability for weak*-lower semicontinuous functions is as follows.

Theorem 3.6. *For a Banach space X, the following are equivalent.*

(a) *Every sequence in X considered as a subset of X^{**} that converges uniformly on weakly compact subsets of X^*, converges in norm (i.e. Mackey convergence in X^{**} agrees with norm convergence for sequences in X).*

(b) *Weak Hadamard and Fréchet differentiability coincide for continuous weak*-lower semicontinuous convex functions on X^*.*

(c) *Every weak*-lower semicontinuous convex function on X^* that is bounded on weakly compact subsets of X^* is bounded on bounded subsets of X^*.*

Proof. We will sketch the proof of the equivalence of (a) and (c), because we have not seen this theorem elsewhere in the literature. (a) $\Rightarrow$ (c): Suppose that $f : X^* \to \mathbb{R}$ is a convex and weak*-lower semicontinuous function bounded on weakly compact subsets of X^*. We may and do assume $f(0) = 0$. Suppose that f is unbounded on KB_{X^*} for some $K > 0$. Let $C_n := \{x^* : f(x^*) \leq n\}$, a weak*-closed subset of X^*, $n \in \mathbb{N}$. Now there are $x_n \in S_X$ and $x_n^* \in KB_{X^*}$ so that $K \geq x_n(x_n^*) > \sup_{C_n} x_n$. From ($a$) it follows that $x_n \not\to_{\tau_{\mathcal{W}}} 0$. Then find a weakly compact set $W \subset X^*$ such that $\sup_W x_n > K$ for infinitely many n, and get that f is unbounded on W, a contradiction. To prove $(c) \Rightarrow (a)$, apply Proposition 2.1 with functionals $x_n \to_{\tau_\mathcal{W}} 0$ but $\|x_n\| \not\to 0$. The equivalence of (a) and (b) follow similarly from dualization of Theorem 2.3. $\square$

Note that the previous theorem applies to spaces X such that X does not have the Schur property and $X^* \supset \ell_1$: for example $X = \ell_1 \oplus \ell_2$. So this provides information that cannot be deduced from Theorem 3.4 or Theorem 3.3.

Finally, we will consider two further classes of spaces; first, Grothendieck spaces because of their significance to the continuity of bi-conjugate functions, and second, dual spaces with the Schur property.

Theorem 3.7. *For a Banach space X, the following are equivalent.*

(a) *X is a Grothendieck space.*
(b) *For each continuous convex function f on X, every weak*-lower semicontinuous convex extension of f to X^{**} is continuous.*
(c) *For each continuous convex function f on X, f^{**} is continuous on X^{**}.*
(d) *For each continuous convex function f on X, there is at least one weak*-lower semicontinuous convex extension of f to X^{**} that is continuous.*
(e) *For each Fréchet differentiable convex function f on X, there is at least one weak*-lower semicontinuous convex extension of f to X^{**} that is continuous.*

Proof. This proof again uses many ideas from Theorem 2.3 working with weak and weak* topologies in X^*. The details are available in [8, Theorem 2.1]. $\square$

Other characterizations of Grothendieck spaces concerning weak*-lower semicontinuous convex extensions that preserve points of Gâteaux differentiability are given in [15]. For further information on Grothendieck spaces and related spaces, see [11, 12, 17].

Theorem 3.8. *For a Banach space X, the following are equivalent.*

(a) *X^* has the Schur property.*
(b) *$X \not\supset \ell_1$ and X has the Dunford-Pettis property.*
(c) *If $f : X \to \mathbb{R}$ is a continuous convex function such that f^{**} is continuous, then f is bounded on bounded sets.*

Proof. See [11, p. 212] for the equivalence of (a) and (b). See [8, Proposition 2.4] for the equivalence of (a) and (c) which uses Theorem 3.4 and ideas as needed in Theorem 3.7. $\square$

4. Extension of Convex Functions

We now consider the question of extending convex functions to preserve continuity:

Question 4.1. Suppose Y is a closed subspace of a Banach space X. If $f : Y \to \mathbb{R}$ is a continuous convex function, is there a continuous convex function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_Y = f$? That is can f be extended to a continuous convex function on X?

First, we present an example showing that such extensions are not always possible.

Example 4.2. Let $Y = c_0$ or ℓ_p with $1 < p < \infty$. Let $f : Y \to \mathbb{R}$ be defined by $f(y) = \sum_{n=1}^{\infty}(e_n^*(y))^{2n}$ where e_n^* are the coordinate functionals. If Y is considered as a subspace of ℓ_∞, then f cannot be extended to a continuous convex function on ℓ_∞.

Proof. Because $e_n^* \to_{w^*} 0$, Example 2.1 shows that f is a continuous convex function. However, f is not bounded on the weakly compact set $\{2e_n\}_{n=1}^{\infty} \cup \{0\}$. Now ℓ_∞ is a space with the DP*-property and so Theorem 3.2 shows every continuous convex function on ℓ_∞ is bounded on weakly compact subsets of ℓ_∞. Therefore, f cannot be extended to a continuous convex function on ℓ_∞. □

We refer the reader to [8, Theorem 2.3] for a more general formulation of Example 4.2: such examples exist whenever we consider an extension from a Gelfand-Phillips space that is not Schur, to a superspace with the DP*-property. We should also point out that in the case $Y = c_0$, the preceding provides an example of a continuous convex function f whose biconjugate fails to be continuous; see [28]. Before proceeding, observe that there are natural conditions that can be imposed on f that allow it to be extended to any superspace. For example, if f is Lipschitz (just consider an infimal convolution with an appropriate multiple of the norm on X, i.e., $\widetilde{f}(x) := \inf\{f(y) + (L+1)\|y - x\|;\ y \in Y\}$, where L is the Lipschitz constant of f). More generally, the extension can be done if f is bounded on bounded sets (see for example [8, p. 1801]). However, our present goal is to find conditions on X and/or Y for which every continuous convex function on Y can be extended to a continuous convex function on X. A well-known natural condition where this is true is recorded as

Remark 4.3. Suppose Y is a complemented subspace of a Banach space X. Then every continuous convex function on Y can be extended to a continuous convex function on X.

Proof. Let $f : Y \to \mathbb{R}$ be continuous and convex. Then $\tilde{f}(x) := f(P(x))$, where $P : X \to Y$ is a continuous linear projection, is one such extension. □

In light of Example 4.2, the above remark doesn't extend to quasicomplements because c_0 is quasicomplemented in ℓ_∞; see [13, Theorem 11.42].

We were not aware of any existing result in the literature providing a positive answer to Question 4.1 even in the case X is separable when no additional restrictions are placed on the continuous convex function f and the closed subspace Y; this is why we consider that Corollary 4.10 below may have some interest. To address this, we will consider 'generalized canonical' examples which will allow us to, in some respects, capture the essence of all convex functions on the space. In this section, all nets $\{\phi_{n,\alpha}\}_{\alpha\in A_n, n\in\mathbb{N}}$, where A_n are nonempty sets, are directed by $(n,\alpha) \le (m,\beta)$ if and only if $n \le m$. Thus, $\phi_{n,\alpha} \to_{w^*} 0$ if for each $\epsilon > 0$ and $x \in X$, there exists $n_0 \in \mathbb{N}$ such that $|\phi_{n,\alpha}(x)| < \epsilon$ whenever $\alpha \in A_n$ and $n \ge n_0$.

Proposition 4.4. *Let $\{\phi_{n,\alpha}\} \subset X^*$ be a bounded net. Consider the lower semicontinuous convex functions $f : X \to \mathbb{R} \cup \{\infty\}$ that are defined as follows*

$$f(x) := \sup_{n,\alpha}\{\phi_{n,\alpha}(x) - a_{n,\alpha}, 0\} \qquad \text{and} \qquad g(x) := \sup_{n,\alpha} n(\phi_{n,\alpha}(x))^{2n}$$

where $b_n \le a_{n,\alpha} \le c_n$ and $b_n \downarrow 0$, $c_n \downarrow 0$. Then

(a) *f is β-differentiable at 0 if and only if $\phi_{n,\alpha} \to_{\tau_\beta} 0$. If this is the case, f is Lipschitz on X.*

(b) *g is bounded on β-sets if and only if $\phi_{n,\alpha} \to_{\tau_\beta} 0$. If this is the case, g is continuous.*

Proof. Follow the details of the proof of Proposition 2.1. □

We include the following fact for completeness, as we will have occasion to use it in what follows.

Lemma 4.5. *Let Y be a closed subspace of a Banach space X, and let $\epsilon \geq 0$. Suppose $f : Y \to \mathbb{R}$ is continuous and convex, and suppose $\tilde{f} : X \to \mathbb{R}$ is a continuous convex extension of f. If $\phi \in \partial_\epsilon f(y_0)$, then there is an extension $\tilde{\phi} \in X^*$ of ϕ such that $\tilde{\phi} \in \partial_\epsilon \tilde{f}(y_0)$.*

Proof. By shifting f, we may without loss of generality assume that $f(0) = -1$. Let $\phi \in \partial_\epsilon f(y_0)$ and let $a := \phi(y_0) - f(y_0) + \epsilon$. Then $f(y) - f(y_0) + \varepsilon \geq \phi(y - y_0)$ for all $y \in Y$. In particular, $f(0) - f(y_0) + \varepsilon \geq -\phi(y_0)$, so $a \geq 1$. Moreover, considering $(\phi, -1)$ as an element of $(Y \times \mathbb{R})^*$ we have $(\phi, -1)(y, t) = \phi(y) - t \leq \phi(y) - f(y) \leq a$ for all $(y, t) \in \operatorname{epi} f$. Now define the continuous sublinear function $\rho : X \times \mathbb{R} \to [0, \infty)$ by $\rho := a\mu_{\operatorname{epi} \tilde{f}}$ where $\mu_{\operatorname{epi} \tilde{f}}$ is the Minkowski functional of the epigraph of $\tilde{f}$. Then $(\phi, -1) \leq \rho$ on $Y \times \mathbb{R}$. According to the Hahn-Banach theorem, $(\phi, -1)$ extends to a continuous linear functional $(\tilde{\phi}, -1)$ on $X \times \mathbb{R}$ that is dominated by ρ. Therefore, $(\tilde{\phi}, -1)(x, t) \leq a$ if $(x, t) \in \operatorname{epi} \tilde{f}$ which implies $\tilde{\phi} \in \partial_\epsilon \tilde{f}(y_0)$. □

Corollary 4.6. *Suppose Y is a closed subspace of a Banach space X. Suppose $f : Y \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous convex functions such that $f \leq g|_Y$. If, for some $\varepsilon \geq 0$ we have $\phi \in \partial_\varepsilon f(y_0)$ then ϕ can be extended to a continuous linear functional $\tilde{\phi}$ such that $f(y_0) + \tilde{\phi}(x - y_0) \leq g(x) + \varepsilon$ for all $x \in X$.*

Proof. Let $\phi \in \partial_\varepsilon f(y_0)$. Then $\phi \in \partial_r g|_Y(y_0)$ where $r := g(y_0) - f(y_0) + \varepsilon$. Apply Lemma 4.5 to obtain $\tilde{\phi}$ such that

$$\tilde{\phi}(x) - \tilde{\phi}(y_0) \leq g(x) - g(y_0) + g(y_0) - f(y_0) + \varepsilon, \quad \text{for all } x \in X,$$

from which the conclusion is immediate. □

Lemma 4.7. *Let Y be a closed subspace of X, and suppose $f : Y \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous convex functions such that $f \leq g|_Y$. Then f can be extended to a continuous convex function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f} \leq g$.*

Proof. For each $y \in Y$, choose $\phi_y \in \partial f(y)$. Let $\tilde{\phi}_y$ be an extension as given by the Corollary 4.6. Now define $\tilde{f}(x) := \sup_{y \in Y} f(y) + \tilde{\phi}_y(x - y)$ for $x \in X$. □

The following theorem provides a useful condition for determining when every continuous convex function on a given subspace of a Banach space can be extended to the whole space.

Theorem 4.8. *Suppose Y is a closed subspace of a Banach space X. Then the following are equivalent.*

(*a*) *Every continuous convex function* $f : Y \to \mathbb{R}$ *can be extended to a continuous convex function* $\tilde{f} : X \to \mathbb{R}$.

(*b*) *Every bounded net* $\{\phi_{n,\alpha}\} \subset Y^*$ *that converges weak* to* 0 *can be extended to a bounded net* $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ *that converges weak* to* 0.

Proof. $(a) \Rightarrow (b)$. Suppose $\{\phi_{n,\alpha}\}$ is a bounded net in Y^* that converges weak* to 0, and without loss of generality suppose $\|\phi_{n,\alpha}\| \leq 1$ for all n, α. Now define

$$f(y) := \sup_{n,\alpha}(\phi_{n,\alpha}(y))^{2n}.$$

Then $f : Y \to \mathbb{R}$ is a continuous convex function (as in Proposition 4.4), so we extend it to a continuous convex function $\tilde{f} : X \to \mathbb{R}$. Now let $C_n := \{x \in X : \tilde{f}(x) \leq 2^{2n}\}$. Observe that $\tilde{f}(0) = 0$, and so the continuity of $\tilde{f}$ at 0 implies that there is an $\epsilon > 0$ so that $\tilde{f}(x) \leq 1$ for all $\|x\| \leq \epsilon$. Then

$$\epsilon B_X \subset C_n \qquad \text{and} \qquad C_n \cap Y \subset \{x : \phi_{n,\alpha}(x) \leq 2\}.$$

Define the sublinear function $p_n := 2\mu_{C_n}$. Then $\phi_{n,\alpha}(y) \leq p_n(y)$ for all $y \in Y$. By the Hahn-Banach theorem, extend $\phi_{n,\alpha}$ to $\tilde{\phi}_{n,\alpha}$ so that $\tilde{\phi}_{n,\alpha}(x) \leq p_n(x)$ for all $x \in X$. Then $\|\tilde{\phi}_{n,\alpha}\| \leq 2/\epsilon$. Now let us suppose that $\{\tilde{\phi}_{n,\alpha}\}$ does not converge weak* to 0. Then we can find $x_0 \in X$, a subsequence $\{n_k\}$ of $\{n\}$ and a sequence (α_k) such that $\tilde{\phi}_{n_k,\alpha_k}(x_0) > 2$ for all k. Thus $x_0 \notin C_n$ for infinitely many n, and so $\tilde{f}(x_0) > 2^{2n}$ for infinitely n. Thus $\tilde{f}(x_0) = \infty$ which contradicts the continuity of $\tilde{f}$.

$(b) \Rightarrow (a)$: Suppose $f : Y \to \mathbb{R}$ is a continuous convex function. Without loss of generality we may assume $f(0) = 0$. Now define $C_n := \{y \in Y : f(y) \leq n\}$. Because f is continuous, there is a $\delta > 0$ such that $\delta B_Y \subset C_n$ for each $n \in \mathbb{N}$. Thus we can write

$$C_n = \bigcap_{\alpha}\{y \in Y : \phi_{n,\alpha}(y) \leq 1\}$$

where $\|\phi_{n,\alpha}\| \leq 1/\delta$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$ (A_n can be chosen as a set with cardinality the density of Y). Also, $\{\phi_{n,\alpha}\}$ converges weak* to 0, otherwise there would be a $y_0 \in Y$, a subsequence $\{n_k\}$ of $\{n\}$ and a sequence $\{\alpha_k\}$ such that $\phi_{n_k,\alpha_k}(y_0) > 1$ for all k. Consequently, $y_0 \notin C_n$ for infinitely many n which would yield the contradiction $f(y_0) = \infty$. Thus, by the hypothesis of (b), $\{\phi_{n,\alpha}\}$ extends to a bounded net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges weak* to 0. Now define

$$g(x) := \sup n(\tilde{\phi}_{n,\alpha}(x))^{2n} + 1.$$

Then $g : X \to \mathbb{R}$ is a continuous convex function (Proposition 4.4). Moreover, $g(y) \geq f(y)$ for all $y \in Y$; this is because $g(x) \geq 1$ for all $x \in X$, and if $n - 1 < f(y) \leq n$ where $n \geq 2$, then $y \notin C_{n-1}$ and so $\phi_{n-1,\alpha_0}(y) > 1$ for some α_0 which implies $g(y) > (n-1) + 1 \geq f(y)$. According to Lemma 4.7, there is a continuous convex extension $\tilde{f} : X \to \mathbb{R}$ of f. $\square$

We now show that Theorem 4.8(b) is satisfied when X/Y is separable. This is a direct consequence of a theorem of Rosenthal's [26] as we now outline for completeness. Recall that a Banach space X is said to be *injective* if for each superspace Z of X, there is a continuous linear projection mapping Z onto X; in the event that there is a norm 1 linear projection from Z onto X for each superspace Z, then X is said to be *1-injective*; see Zippin's article [31] and [20, Section 2.f] for further information concerning this subject.

Theorem 4.9. *Let X be a Banach space, Y a closed subspace such that X/Y is separable. Let $\{\phi_{n,\alpha}\}_{\alpha\in A_n,n\in\mathbb{N}}$ be a weak*-null net in Y^* such that $\|\phi_{n,\alpha}\| \leq 1$ for all $\alpha \in A_n, n \in \mathbb{N}$. Then, for every $\epsilon > 0$ there exists a weak*-null net $\{\tilde{\phi}_{n,\alpha}\}_{\alpha\in A_n,n\in\mathbb{N}}$ of elements in X^* such that $\|\tilde{\phi}_{n,\alpha}\| \leq 2+\epsilon$ and $\tilde{\phi}_{n,\alpha}$ extends $\phi_{n,\alpha}$ for all $\alpha \in A_n, n \in \mathbb{N}$.*

Proof. Define a bounded linear operator $T : Y \to (\sum_{n=1}^{\infty} \ell_\infty(A_n))_{c_0}$ by $T(y) := (\{\phi_{n,\alpha}(y)\}_{\alpha\in A_n})_n$; then $\|T\| \leq 1$. Now using the following extension theorem of H. P. Rosenthal (see [26]): *Let $Z_1, Z_2, \dots$ be 1-injective Banach spaces, X, Y be Banach spaces with $Y \subset X$ and X/Y separable, and set $Z := (\sum_{i=1}^{\infty} Z_i)_{c_0}$. Then for every non-zero operator $T : Y \to Z$ and every $\epsilon > 0$, there exists $\hat{T} : X \to Z$ extending T with $\|\hat{T}\| < (2+\epsilon)\|T\|$.* According to this result, T defined above extends to $\hat{T} : X \to (\sum_{n=1}^{\infty} \ell_\infty(A_n))_{c_0}$ with $\|\hat{T}\| < 2+\epsilon$. Now let $e^*_{n,\alpha}$ denote the coordinate functional so that $e^*_{n,\alpha}(x) := x_\alpha$ for $x = (x_i)_{i\in A_n} \in \ell_\infty(A_n)$. Then $e^*_{n,\alpha}(T(y)) = \phi_{n,\alpha}(y)$ for all y and $\tilde{\phi}_{n,\alpha} = e^*_{n,\alpha} \circ \hat{T}$ extends $\phi_{n,\alpha}$. Because $\hat{T}(x) \in (\sum_{n=1}^{\infty} \ell_\infty(A_n))_{c_0}$, it follows that $\tilde{\phi}_{n,\alpha} \to_{w^*} 0$; moreover, $\|\tilde{\phi}_{n,\alpha}\| \leq 1\|\hat{T}\| < 2+\epsilon$. □

Our main application of Theorem 4.8 is

Corollary 4.10. *Suppose X is a Banach space and Y is a closed subspace of X such that X/Y is separable. Then every continuous convex function $f : Y \to \mathbb{R}$ can be extended to a continuous convex function $\tilde{f} : X \to \mathbb{R}$.*

Proof. Apply Theorem 4.8 and Theorem 4.9. □

Observe that Example 4.2 shows the previous corollary can fail if X/Y is not separable, it also shows it is not always possible to extend a continuous convex function from a separable closed subspace of a Banach space X to a continuous convex function on the whole space X. The following result provides a condition on X for which the latter is always possible.

Corollary 4.11. *Suppose Y is a separable closed subspace of a Banach space X, where X has a countably norming M-basis. Then every continuous convex function on Y can be extended to a continuous convex function on X.*

Proof. There is a separable subspace Y_1 of X such that $Y \subset Y_1$ and Y_1 is complemented in X; see [23]. Extend the continuous convex function to Y_1 by Theorem 4.10 and then use Remark 4.3 to extend it to X. □

Clearly, if Y is an injective Banach space, then any continuous convex function can be extended to any superspace according to Remark 4.3. Another class of spaces that allow extensions to superspaces is as follows.

Proposition 4.12. *Suppose Y is a $C(K)$ Grothendieck space. Then any continuous convex function $f : Y \to \mathbb{R}$ can be extended to a continuous convex function $f : X \to \mathbb{R}$ where X is any superspace of Y.*

Proof. Write $Y \subset X$. Then $Y^{**} \cong Y^{\perp\perp} \subset X^{**}$. According to [8, Theorem 2.1] (cf. Theorem 3.7), f can be extended to a continuous convex function on Y^{**}. Now, Y^{**} as the bidual of a $C(K)$ space is isomorphic to a $C(K)$ space where K is compact Stonian (i.e. K is extremely disconnected); see [27, p. 121]. Therefore, Y^{**} is injective; see [27, Theorem 7.10, p. 110]. According to Remark 4.3 the extension of f to $Y^{\perp\perp}$ can further be extended to X^{**} which contains X. □

Observe that the previous proposition doesn't work for general $C(K)$ spaces, e.g. $c_0 \subset \ell_\infty$, and doesn't work for reflexive Grothendieck, e.g. $\ell_2 \subset \ell_\infty$. More significantly, using some deep results in Banach space theory one can conclude that the above proposition applies to some cases where Y is not a complemented subset of X.

Remark 4.13. There are Grothendieck $C(K)$ spaces that are not injective.

Proof. Let X be Haydon's Grothendieck $C(K)$ space that does not contain ℓ_∞ [17]. Because $X \not\supset \ell_\infty$, X is not injective by a theorem of Rosenthal's ([24], or [20, Theorem 2.f.3]). □

We have focused on preserving continuity in our extensions. One could similarly ask whether extensions exist preserving a given point of differentiability. Again, negative examples in the same spirit of Example 4.2 have been constructed. We sketch one such example similar to [4, Example 3.8].

Example 4.14. Let $Y = c_0$ or ℓ_p with $1 < p < \infty$. Let $f : Y \to \mathbb{R}$ be defined by $f(y) := \sup\{e_n^*(y) - \frac{1}{n}, 0\}$ where e_n^* are the coordinate functionals. Then there is no continuous convex extension of f to ℓ_∞ that preserves the Gâteaux differentiability of f at 0.

Proof. This follows because Gâteaux and weak Hadamard differentiability coincide for continuous convex functions on ℓ_∞ (see Theorem 3.2); cf. Example 4.2. □

A positive result that is analogous to Theorem 4.8 is as follows.

Theorem 4.15. *Suppose Y is a closed subspace of a Banach space X. Then the following are equivalent.*

(a) *Every Lipschitz convex function $f : Y \to \mathbb{R}$ that is Gâteaux differentiable at some $y_0 \in Y$ can be extended to a Lipschitz convex function $\tilde{f} : X \to \mathbb{R}$ that is Gâteaux differentiable at y_0.*

(b) *Every bounded net $\{\phi_{n,\alpha}\} \subset Y^*$ that converges weak* to 0 can be extended to a bounded net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges weak* to 0.*

Proof. $(a) \Rightarrow (b)$: Let $\{\phi_{n,\alpha}\} \subset Y^*$ be a bounded net that converges weak* to 0. Define $f(x) := \sup\{\phi_{n,\alpha}(x) - \frac{1}{n}, 0\}$. Then f is a Lipschitz convex function that, according to Proposition 4.4, is Gâteaux differentiable at 0 (observe, too, that $f(0) = 0$ and $f(y) \geq 0$ for all $y \in Y$, so $f'(0) = 0$). Then extend f to a Lipschitz convex function $\tilde{f}$ that is Gâteaux differentiable at 0 with Gâteaux derivative $\tilde{f}'(0) = \phi$, where $\phi \in X^*$. Now $\phi|_Y = f'(0)$ which implies $\phi|_Y = 0$. Thus $\tilde{f} - \phi$ is a Lipschitz convex function extending f, and whose Gâteaux derivative is 0. Replacing $\tilde{f}$ with $\tilde{f} - \phi$, we can and do assume $\tilde{f}(x) \geq 0$ for all $x \in X$ and $\tilde{f}'(0) = 0$. Clearly $\phi_{n,\alpha} \in \partial_{\frac{1}{n}} f(0)$, thus by Lemma 4.5 there is an extension $\tilde{\phi}_{n,\alpha} \in X^*$ of $\phi_{n,\alpha}$ such that $\tilde{\phi}_{n,\alpha} \in \partial_{\frac{1}{n}} \tilde{f}(0)$. Thus $\|\tilde{\phi}_{n,\alpha}\| \leq K + 1/n$, where K is the Lipschitz constant for $\tilde{f}$. Moreover, $\tilde{f}(x) \geq g(x)$, where $g(x) = \sup_{n,\alpha}\{\tilde{\phi}_{n,\alpha}(x) - \frac{1}{n}, 0\}$.

The Gâteaux differentiability of $\tilde{f}$ at 0 now forces the Gâteaux differentiability of g at 0. Use again Proposition 4.4 to obtain the weak* convergence of $\{\tilde{\phi}_{n,\alpha}\}$ to 0.

$(b) \Rightarrow (a)$: By subtracting off a derivative and translating f, we need only to consider the case where $f'(0) = 0$ and $f(0) = 0$. For each $u \in Y$, fix $\phi_u \in \partial f(u)$, and define $a_{k,u} := \phi_u(u) - f(u) + \frac{1}{k}$. Then, using properties of subgradients, it follows that $f(y) = \sup\{\phi_{k,u}(y) - a_{k,u}, 0 : u \in Y, k \in \mathbb{N}\}$. Now, from the fact that f is Lipschitz (with Lipschitz constant L) we have $1/k \le a_{k,u} \le 2L\|u\| + 1/k$ for every $k \in \mathbb{N}$ and $u \in Y$. Put

$$A_n := \{(k,u) : k \in \mathbb{N}, u \in Y, \quad \text{such that } \frac{1}{n} \le a_{k,u} < \frac{1}{n-1}\}$$

for $n = 2, 3, \dots$ and

$$A_1 := \{(k,u) : k \in \mathbb{N}, u \in Y, \quad \text{such that } 1 \le a_{k,u}\}.$$

It is plain that $(n, 0) \in A_n$ and so A_n is nonempty, for every $n \in \mathbb{N}$. Moreover, $\mathbb{N} \times Y = \bigcup_{n=1}^{\infty} A_n$. To each $(n,(k,u)) \in \{n\} \times A_n$ we associate $\psi_{(n,(k,u))} := \phi_u$ and $b_{(n,(k,u))} := a_{k,u}$. Then $f(y) = \sup\{\psi_{(n,(k,u))}(y) - b_{(n,(k,u))}, 0 : (n,(k,u)) \in \bigcup_{n=1}^{\infty}\{n\} \times A_n\}$. According to Proposition 4.4, $\psi_{(n,(k,u))} \to_{w^*} 0$ because f is Gâteaux differentiable at 0. The Lipschitz property of f guarantees that $\{\psi_{(n,(k,u))}\}$ is bounded. According to (b), we can extend $\{\psi_{(n,(k,u))}\}$ to a bounded net $\{\tilde{\psi}_{(n,(k,u))}\}$ that converges weak* to 0. Then $\tilde{f}(x) = \sup\{\tilde{\psi}_{(n,(k,u))}(x) - b_{(n,(k,u))}, 0\}$ is a convex Lipschitz function that is Gâteaux differentiable at 0 by Proposition 4.4, and $\tilde{f}$ extends f. □

Let us remark that in contrast to this, Zizler ([32]) has shown that extensions of Gâteaux differentiable norms from a subspace of a separable space to a Gâteaux differentiable norm on the whole space are not always possible.

Finally, let us conclude by stating a bornological version that combines Theorems 4.8 and 4.15.

Theorem 4.16. *Suppose Y is a closed subspace of a Banach space X. Then the following are equivalent.*

- (a) *Every continuous convex function $f : Y \to \mathbb{R}$ bounded on β-sets can be extended to a continuous convex function $\tilde{f} : X \to \mathbb{R}$ that is bounded on β-sets in X.*
- (b) *Every Lipschitz convex function $f : Y \to \mathbb{R}$ that is β-differentiable at some point y_0 can be extended to a Lipschitz convex function $f : X \to \mathbb{R}$ that is β-differentiable at y_0.*
- (c) *Every bounded net $\{\phi_{n,\alpha}\} \subset Y^*$ that converges τ_β to 0 can be extended to a bounded net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges τ_β to 0.*

Let us mention that if β is the bornology of bounded sets, then (c) is always possible according to the Hahn-Banach theorem. Thus this recaptures the results: (i) a convex function that is bounded on bounded sets can always be extended to a convex function bounded on bounded sets; (ii) Lipschitz convex functions can be extended to a superspace while preserving a point of Fréchet differentiability.

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