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Weak-Star Convergence of Convex Sets

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Dedicated to the memory of Simon Fitzpatrick.

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We show that if a Banach space X is weakly compactly generated and C, C_n are weak-star-closed bounded convex nonempty subsets of the dual space X^*, then the support functionals $\delta^*_{C_n}$ converge to δ^*_C pointwise on X if and only if the sequence (C_n) is uniformly bounded with weak-star limit C.

Author's note. The authors were visiting Dalhousie University in 1988 during a seventeen-day labor dispute that left the Mathematics Department empty. During this period they occupied themselves writing the present paper, on a natural topic in variational analysis known elsewhere as "scalar convergence" [22, 16, 4, 17, 3, 18]. Although referenced in the literature [19, 2], this work was never published. The second author reproduces the 1988 manuscript here, essentially unedited, in tribute to the insightful and elegant mathematical vision of Simon Fitzpatrick: it was a privilege to work with Simon.

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1. Introduction

In any normed space it is possible to define various types of convergence for sequences of convex sets (see for example [16] for a review in the finite-dimensional case). One particularly useful idea was introduced in [12]. A sequence of sets (C_n) *converges Mosco* to a set C in a normed space E if

$$\begin{array}{l}\text{for each } x \in C \text{ there exist, for large } n,\ x_n \in C_n, \\ \text{such that } x_n \text{ converges in norm to } x,\end{array} \tag{1}$$

and

$$\begin{array}{l}\text{for any subsequence } x_{n_j} \in C_{n_j} \text{ with} \\ x_{n_j} \text{ converging weakly to } x,\ x \in C.\end{array} \tag{2}$$

Another way of expressing this is

$$\text{weak-}\limsup C_n \subset C \subset \text{strong-}\liminf C_n. \tag{3}$$

In [13] this notion of convergence is related to the convergence of the corresponding support functionals,

$$\delta^*_{C_n}(x^*) = \sup\{\langle x^*, x\rangle : x \in C_n\} \quad (x^* \in E^*).$$

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If F is a normed space and $f, f_n : F \to [-\infty, +\infty]$, then we say f_n *converges Mosco* to f if for each $x \in F$,

$$\limsup f_n(x_n) \le f(x) \le \liminf f_{n_j}(y_j) \quad \text{as } n, j \to \infty$$

for some sequence x_n converging to x in norm, and for any sequence y_j converging weakly to x and any subsequence (f_{n_j}). It is shown in the above paper that if E is reflexive and C and all C_n are closed, convex and nonempty, then C_n converges Mosco to C if and only if $\delta^*_{C_n}$ converges Mosco to δ^*_C.

Unfortunately $\delta^*_{C_n}$ can converge pointwise to δ^*_C without necessarily converging Mosco. For example, take $E = l_2$, $C_n = \{e_n\}$ (the nth unit vector), and $C = \{0\}$. Then $\delta^*_{C_n}(x^*) = (x^*)_n$ and $\delta^*_C(x^*) = 0$ for all $x^* \in l_2$, so clearly $\delta^*_{C_n} \to \delta^*_C$ pointwise. However, the sequence $(-e_n)$ converges to 0 weakly, and $\delta^*_{C_n}(-e_n) = -1$ for all n, so

$$0 = \delta^*_{C_n}(0) > \liminf \delta^*_{C_n}(-e_n) = -1,$$

and thus $\delta^*_{C_n}$ does not converge Mosco to δ^*_C. It is also easy to see that C_n does not converge Mosco to C (since e_n does not converge to 0 in norm). The question of when $\delta^*_{C_n} \to \delta^*_C$ pointwise thus remains open. This type of convergence is called $*$-convergence in [16], where it is shown that in finite dimensions, when C and all C_n are compact, convex and nonempty, it coincides with classical Kuratowski convergence:

$$\limsup C_n \subset C \subset \liminf C_n.$$

Our purpose here is to generalize this result to the infinite-dimensional case.

It is worth remarking that convergence of support functionals is closely connected with another type of convergence introduced in [21, 22]. We define $d(x, C) = \inf\{\|x - c\| : c \in C\}$ (and $d(x, \emptyset) = +\infty$). Then C_n *converges Wijsman* to C if $d(\cdot, C_n) \to d(\cdot, C)$ pointwise. In [5] it is shown that Mosco and Wijsman convergence coincide for sequences of convex sets if and only if the underlying space is reflexive with the dual norm having the Kadec property.

Consider the conjugate of the distance function:

$$\begin{aligned} d^*(x^*, C) &= \sup_{x\in E}\{\langle x^*, x\rangle - d(x, C)\} = \sup_{x\in E}\Big\{\langle x^*, x\rangle - \inf_{y\in C}\|x - y\|\Big\} \\ &= \sup_{y\in C,\ x\in E}\{\langle x^*, x\rangle - \|x - y\|\} = \sup_{y\in C,\ z\in E}\{\langle x^*, y + z\rangle - \|z\|\} \\ &= \sup_{y\in C}\Big\{\langle x^*, y\rangle + \sup_{z\in E}\{\langle x^*, z\rangle - \|z\|\}\Big\}. \end{aligned}$$

Now it is easy to check that

$$\begin{aligned} \sup_{z\in E}\{\langle x^*, z\rangle - \|z\|\} &= \begin{cases} 0 & (\|x^*\| \le 1) \\ +\infty & (\text{otherwise}) \end{cases} \\ &= \delta_{B^*}(x^*), \end{aligned}$$

the indicator function of B^*, the unit ball in E^*. Thus

$$d^*(x^*, C) = \delta^*_C(x^*) + \delta_{B^*}(x^*) \quad (x^* \in E^*).$$

Since δ^*_C is always positively homogeneous it follows that $\delta^*_{C_n} \to \delta^*_C$ pointwise if and only if $d^*(\cdot, C_n) \to d^*(\cdot, C)$ pointwise on B^*.

2. Weak-star convergence

Throughout the remainder of this paper X will be a Banach space and C and C_n ($n = 1, 2, \ldots$) will be weak-star-closed, bounded, nonempty, convex subsets of the dual space X^*. We say C_n converges weak-star to C if

$$\text{w}^*\text{-}\limsup C_n \subset C \subset \text{w}^*\text{-}\liminf C_n,$$

where the weak-star semilimits are taken in the sequential sense. More precisely, we make the following definition.

Definition 2.1. The sequence of sets C_n *converges weak-star* to C if

$$\begin{array}{l}\text{for each } x^* \in C \text{ there exist, for large } n,\ x_n^* \in C_n,\\ \text{such that } x_n^* \text{ converges weak-star to } x^*,\end{array} \tag{4}$$

and

$$\begin{array}{l}\text{if, for some subsequence } x_{n_j}^* \in C_{n_j},\\ x_{n_j}^* \text{ converges weak-star to } x^*, \text{ then } x^* \in C.\end{array} \tag{5}$$

We shall also need the following definition.

Definition 2.2. The Banach space X is *weakly compactly generated* (WCG) if there exists a weakly compact subset of X whose linear span is dense in X.

In particular, if X is either separable or reflexive then it is WCG (see for example [7]). We shall need the following two properties of WCG spaces.

Theorem 2.3 (Amir and Lindenstrauss [1]). *If the Banach space X is a subspace of a WCG space then the unit ball in X^* is weak-star sequentially compact.*

Theorem 2.4 (Davis, Figiel, Johnson and Pelczynski [6]). *A Banach space X is WCG if and only if there is a reflexive space Y and a one-to-one continuous linear operator $T : Y \to X$ with $T(Y)$ dense in X.*

We are now ready to proceed with one direction in our main result.

Theorem 2.5. *Suppose that X is WCG. Suppose further that the C_n are uniformly bounded and converge weak-star to C. Then $\delta_{C_n}^* \to \delta_C^*$ pointwise on X.*

Proof. Suppose for some $x \in X$, $\delta_{C_n}^*(x) \not\to \delta_C^*(x)$. Then for some $\epsilon > 0$, and for some subsequence (n_i), either

$$\delta_{C_{n_i}}^*(x) - \delta_C^*(x) > \epsilon \quad \text{for each } i \tag{6}$$

or

$$\delta_C^*(x) - \delta_{C_{n_i}}^*(x) > \epsilon \quad \text{for each } i. \tag{7}$$

Suppose first that (6) holds. Then there exist $x_{n_i}^* \in C_{n_i}$ for each i satisfying

$$\langle x_{n_i}^*, x\rangle > \epsilon + \delta_C^*(x). \tag{8}$$

Now by assumption $\cup_i C_{n_i}$ is bounded, and by Theorem 2.3, any closed bounded set in X^* is weak-star sequentially compact. Thus there exists a subsequence $x^*_{n_{i_j}}$ converging weak-star to some x^*_∞, and by (5), $x^*_\infty \in C$. However, from (8),

$$\langle x^*_\infty, x\rangle = \lim_j \langle x^*_{n_{i_j}}, x\rangle \geq \epsilon + \delta^*_C(x),$$

so $x^*_\infty \notin C$, which is a contradiction.

On the other hand, suppose (7) holds. Then for some $x^* \in C$,

$$\langle x^*, x\rangle > \epsilon + \delta^*_{C_{n_i}}(x) \quad \text{for each } i. \tag{9}$$

By (4) there exists a sequence $x^*_n \in C_n$ such that x^*_n converges weak-star to x^*. But then

$$\langle x^*_{n_i}, x\rangle \leq \delta^*_{C_{n_i}}(x) < \langle x^*, x\rangle - \epsilon \quad \text{for each } i,$$

which gives a contradiction as $i \to \infty$. □

Notice that in fact this result will hold whenever the unit ball in X^* is weak-star sequentially compact—for example if X is a subspace of a WCG space, or is smoothly normed or is a weak Asplund space (see [8]).

The following example shows that uniform boundedness of the C_n is necessary in Theorem 2.5.

Example 2.6. Let $X = l_2$, $C = \{0\}$, and $C_n = \{\lambda e_n : 0 \leq \lambda \leq n\}$, where e_n is the nth unit vector. X is reflexive, so it is certainly WCG. We claim C_n converges weak-star (in fact Mosco) to C.

Certainly (4) holds, since $0 \in C_n$ for each n. Suppose the sequence $(\lambda_{n_i} e_{n_i})$ is weak-star (and so weakly) convergent. It follows from the uniform boundedness principle [10, p. 135] that the sequence is bounded in norm, and so (λ_{n_i}) is a bounded sequence. Thus clearly $\lambda_{n_i} e_{n_i}$ converges weak-star to 0, as required. So C_n converges weak-star to C. Clearly $\delta^*_C(x) = 0$ for all $x \in l_2$. On the other hand,

$$\delta^*_{C_n}(x^*) = \sup\{\langle x, \lambda e_n\rangle : 0 \leq \lambda \leq n\} = \max\{n x_n, 0\}.$$

Thus if we define $x \in l_2$ by $x_n = 1/n$, then $\delta^*_C(x) = 0$, whereas $\delta^*_{C_n}(x) = 1$, for all n. Thus $\delta^*_{C_n} \not\to \delta^*_C$ pointwise.

Theorem 2.5 can also fail if the unit ball in X^* fails to be weak-star sequentially compact.

Example 2.7. Consider the subsets of l_1 defined by $C = \{0\}$, and $C_n = \{\lambda e_n : -1 \leq \lambda \leq 1\}$ for each n, where e_n is the nth unit vector. Then C and C_n are nonempty, convex, uniformly bounded, and both norm and weak-star closed (regarding l_1 as the dual of c_0). It is easily checked that $\delta^*_C(x) = 0$ and $\delta^*_{C_n}(x) = |x_n|$, for all $x \in l_\infty$. Thus $\delta^*_{C_n} \to \delta^*_C$ pointwise on c_0 but not on l_∞. It follows from Theorem 2.5 (since c_0 is WCG) that C_n converges weak-star to C in l_1, and this may easily be checked directly.

Suppose some subsequence of $(\lambda_n e_n)_1^\infty$ converges weakly in l_1, where $-1 \leq \lambda_n \leq 1$ for each n. Denote the subsequence by $(\lambda_{n_i} e_{n_i})$, and define $x \in l_\infty$ by

$$x_n = \begin{cases} (-1)^i \text{sign}(\lambda_{n_i}), & (n = n_i) \\ 0, & (\text{otherwise}). \end{cases}$$

Then $\langle x, \lambda_{n_i} e_{n_i}\rangle = (-1)^i |\lambda_{n_i}|$ for each i. Since this sequence is convergent, $\lambda_{n_i} \to 0$, so the subsequence converges to 0. Thus $C_n \to C$ weakly (in fact Mosco) in l_1. However, as we saw, $\delta^*_{C_n} \not\to \delta^*_C$ pointwise on l_∞. Thus Theorem 2.5 can fail in the weak case.

Now regard the sets C and C_n $(n = 1, 2, \ldots)$ as subsets of l^*_∞. Thus they are nonempty, convex, uniformly bounded, and weak-star closed. If any subsequence of $(\lambda_n e_n)_1^\infty$ converges weak-star in l^*_∞, where $-1 \le \lambda_n \le 1$ for each n, then the same argument shows it converges to 0. Thus $C_n \to C$ weak-star in l^*_∞, and yet as we know, $\delta^*_{C_n} \not\to \delta^*_C$ pointwise on l_∞. Thus Theorem 2.5 fails in this case, precisely because the unit ball in l^*_∞ is not weak-star sequentially compact.

3. The reflexive case

We now turn our attention to the converse of Theorem 2.5. The first step is an easy application of the uniform boundedness principle.

Proposition 3.1. *Suppose $\delta^*_{C_n} \to \delta^*_C$ pointwise on X. Then the C_n are uniformly bounded.*

Proof. For any $x \in X$, $\delta^*_{C_n}(x) \to \delta^*_C(x)$, so $(\delta^*_{C_n}(x))$ is a bounded sequence, and thus

$$\sup\{\langle x^*, x\rangle : x^* \in \cup_n C_n\} = \sup_n \delta^*_{C_n}(x) < +\infty.$$

The uniform boundedness principle then implies that $\cup_n C_n$ is bounded in X^* [10, p. 135] □

Checking the second half (5) of the definition of weak-star convergence follows from the Hahn-Banach Theorem.

Proposition 3.2. *Suppose $\delta^*_{C_n} \to \delta^*_C$ pointwise on X. If, for some subsequence $x^*_{n_j} \in C_{n_j}$, $x^*_{n_j} \to x^*$ weak-star, then $x^* \in C$.*

Proof. Suppose $x^* \notin C$. Then by the Hahn-Banach Theorem [10, p. 64], we can strongly separate x^* from C: there exists $x \in X$ and $\epsilon > 0$ with $\langle x^*, x\rangle > \delta^*_C(x) + \epsilon$. However, $\langle x^*_{n_j}, x\rangle \le \delta^*_{C_{n_j}}(x)$ for each j, and letting $j \to \infty$ gives $\langle x^*, x\rangle \le \delta^*_C(x)$, which is a contradiction. □

We will first prove the converse of Theorem 2.5 in the case where X is reflexive, and then apply Theorem 2.4 to extend to the WCG case. We will need the following lemma. We denote the closed unit ball in the normed space Y by B_Y, and the weak topology on Y by $\sigma(Y, Y^*)$.

Lemma 3.3. *Suppose the Banach space Y is reflexive and $(y_n)_1^\infty \subset Y$. Then $(B_Y \cap \text{cl span}(y_n), \sigma(Y, Y^*))$ is metrizable.*

Proof. Define $Z = \text{cl span}(y_n)$. Then Z is separable, so Z^* is weak-star separable [10, Ex. 2.22(a)]. Furthermore Z is a closed subspace of a reflexive space, so is reflexive [10, p. 126]. Thus Z^* is weakly separable, so separable [10, Ex. 2.22(b)], so $B_{Z^{**}}$ is weak-star metrizable [10, p. 72], or in other words B_Z is weakly metrizable. Now $B_Z = Z \cap B_Y$, and it is easy to check that $\sigma(Z, Z^*)$ is exactly $\sigma(Y, Y^*)$ restricted to Z, since any element of Z^* has an extension to an element of Y^*. The result follows. □

The next step is to show that the set of all weak limits of $x_n^* \in C_n$ is closed.

Lemma 3.4. *Suppose X is reflexive and $\delta^*_{C_n} \to \delta^*_C$ pointwise on X. Then the set*

$$S = \Big\{x^* \in X^* : \exists x_n^* \in C_n \text{ with } x_n^* \to x^* \text{ weakly}\Big\} \tag{10}$$

is closed, convex and bounded.

Proof. Since each C_n is convex, so is S, and the C_n are uniformly bounded (by K say) by Proposition 3.1, and KB_{X^*} is weakly closed [10, p. 66], $S \subset KB_{X^*}$. It remains to show S is norm closed.

Suppose $x_m^* \in S$ $(m = 1, 2, \ldots)$ and $\|x_m^* - x_\infty^*\| \to 0$ as $m \to \infty$. By definition, for each m there exist $x_{m,n}^* \in C_n$ $(n = 1, 2, \ldots)$ with $x_{m,n}^* \to x_m^*$ weakly as $n \to \infty$. Define $Z = \text{cl span}\{x_{m,n}^* : m, n = 1, 2, \ldots\}$. By Lemma 3.3, $(KB_{X^*} \cap Z, \sigma(X^*, X))$ is metrizable, by the metric ρ say, and note that $x_{m,n}^*$, x_m^* and x_∞^* lie in $KB_{X^*} \cap Z$ for each m, n.

Now $x_m^* \to x_\infty^*$ in norm as $m \to \infty$, so certainly $x_m^* \to x_\infty^*$ weakly and thus $\rho(x_m^*, x_\infty^*) \to 0$ as $m \to \infty$. Thus for each j we can pick m_j so that $\rho(x_{m_j}^*, x_\infty^*) < 2^{-j}$. Now pick an increasing sequence n_j $(j = 1, 2, \ldots)$ so that for all $n \geq n_j$, $\rho(x_{m_j,n}^*, x_{m_j}^*) < 2^{-j}$: this is possible because $x_{m_j,n}^* \to x_{m_j}^*$ weakly as $n \to \infty$. It now follows by the triangle inequality that $\rho(x_{m_j,n}^*, x_\infty^*) < 2^{-j+1}$ for all $n \geq n_j$, and for all j. Finally, for each n define $x_n^* = x_{m_j,n}^*$ whenever $n_j \leq n < n_{j+1}$. Then $x_n^* \in C_n$ and $\rho(x_n^*, x_\infty^*) < 2^{-j+1}$ for all $n \geq n_j$ and all j, so $x_n^* \to x_\infty^*$ weakly. Thus $x_\infty^* \in S$ as required. □

We are now ready to prove the converse of Theorem 2.5 in the reflexive case.

Theorem 3.5. *Suppose X is reflexive. If $\delta^*_{C_n} \to \delta^*_C$ pointwise on X then the sequence (C_n) is uniformly bounded with weak-star limit C.*

Proof. After Propositions 3.1 and 3.2, it only remains to show that (4) holds in the definition of weak-star convergence. If we denote by S the set in (10) again, what we need to show is $C \subset S$. (Notice in fact Proposition 3.2 already shows $S \subset C$.)

Suppose that x^* is an exposed point of C. In other words, for some $x \in X$, $\langle x^*, x\rangle = \delta^*_C(x)$, but for any $y^* \in C \setminus \{x^*\}$, $\langle y^*, x\rangle < \delta^*_C(x)$. Since $\delta^*_{C_n}(x) \to \delta^*_C(x)$, there exists a sequence $x_n^* \in C_n$ with $\langle x_n^*, x\rangle \to \delta^*_C(x) = \langle x^*, x\rangle$. We claim $x_n^* \to x^*$ weakly.

Suppose not. Then for some $y \in X$, some $\epsilon > 0$, and some subsequence (n_i), we have $\langle x_{n_i}^*, y\rangle \leq \langle x^*, y\rangle - \epsilon$, for each i. Now by Proposition 3.1, the sequence $(x_{n_i}^*)$ is bounded in the reflexive space X^*, so has a weakly convergent subsequence, $x_{n_{i_j}}^* \to x_\infty^*$ weakly as $j \to \infty$. From the above we know therefore that $\langle x_\infty^*, x\rangle = \langle x^*, x\rangle$, and $x_\infty^* \in C$ by Proposition 3.2, so $x_\infty^* = x^*$. But we also have $\langle x_\infty^*, y\rangle \leq \langle x^*, y\rangle - \epsilon$, which is a contradiction.

Thus we have shown that the exposed points of C are contained in S. Since S is closed and convex by Lemma 3.4, it follows that the closed convex hull of the exposed points of C is contained in S. Now, using results that may be found, for example, in [9], since X is reflexive, X^* has the Radon-Nikodym property, and thus C is the closed convex hull of its strongly exposed points. Since each strongly exposed point is exposed, we have $C \subset S$, as required. □

The results in this section may be compared to those in [4], where it is shown in the reflexive case that if $\delta^*_{C_n} \to \delta^*_C$ pointwise on X then the C_n are uniformly bounded and C is the closed convex hull of w- $\limsup C_n$.

4. The WCG case

We have now developed the tools to prove our main result. The idea is to use the Davis-Figiel-Johnson-Pelczynski factorization theorem (2.4) to extend Theorem 3.5 from the reflexive case to the WCG case.

Theorem 4.1. *Suppose X is WCG and C, C_n $(n = 1, 2, \ldots)$ are weak-star-closed, bounded, convex, nonempty subsets of X^*. Then $\delta^*_{C_n} \to \delta^*_C$ pointwise on X if and only if the sequence (C_n) is uniformly bounded with weak-star limit C.*

Proof. One direction has already been proved in Theorem 2.5. In the other direction, in view of Propositions 3.1 and 3.2, it only remains to show (4) holds in the definition of weak-star convergence: given $x^* \in C$, we need to find a sequence $x^*_n \in C_n$ with $x^*_n \to x^*$ weak-star.

By Theorem 2.4 there is a reflexive space Y and a one-to-one continuous linear operator $T : Y \to X$ with $T(Y)$ dense in X. Denote the adjoint map by $T^* : X^* \to Y^*$. Then T^* is continuous, linear and one-to-one [15, p. 95]. It is easy to see that T^* is weak-star-to-weak continuous: if a net $x^*_\alpha \to x^*_\infty$ weak-star in X^* then for any $y \in Y$,

$$\langle T^* x^*_\alpha, y\rangle = \langle x^*_\alpha, Ty\rangle \to \langle x^*_\infty, Ty\rangle = \langle T^* x^*_\infty, y\rangle,$$

so $T^* x^*_\alpha \to T^* x^*_\infty$ weak-star in Y^*.

Now define D, D_n $(n = 1, 2, \ldots)$ by $D = T^*C$, $D_n = T^*C_n$. The sets D and D_n are convex (by the linearity of T^*), bounded (since C and C_n are bounded and T^* is continuous) and closed (since C and C_n are weak-star compact and T^* is weak-star-to-weak continuous, so D and D_n are weakly closed and convex, so closed). Furthermore, by Proposition 3.1, $C \cup \bigcup_n C_n$ is bounded, so

$$\text{weak-star-cl}\left(C \cup \bigcup_n C_n\right)$$

is weak-star compact by the Alaoglu-Bourbaki theorem [10, p. 70]. It now follows by, for example, [20, 3.83], that

$$T^* : \text{weak-star-cl}\left(C \cup \bigcup_n C_n\right) \ \to \ \text{weak-cl}\left(D \cup \bigcup_n D_n\right) \tag{11}$$

is a weak-star-to-weak homeomorphism. Specifically, it is one-to-one and weak-star-to-weak continuous as above, and it is easily seen to be onto (as defined). Since its range is further weakly compact, it is a weak-star-to-weak homeomorphism.

Now by definition, for $y \in Y$,

$$\begin{aligned} \delta^*_D(y) &= \sup\{\langle y^*, y\rangle : y^* \in D\} = \sup\{\langle T^*x^*, y\rangle : x^* \in C\} \\ &= \sup\{\langle x^*, Ty\rangle : x^* \in C\} = \delta^*_C(Ty), \end{aligned}$$

and similarly $\delta^*_{D_n}(y) = \delta^*_{C_n}(Ty)$. By assumption therefore, $\delta^*_{D_n} \to \delta^*_D$ pointwise on Y.

Since Y is reflexive we can now apply Theorem 3.5. Since $T^*x^* \in D$, there exists a sequence $x_n^* \in C_n$ with $T^*x_n^* \to T^*x^*$ in Y^*. Since T^* is a weak-star-to weak homeomorphism, it follows from (11) that $x_n^* \to x^*$ weak-star in X^*, as required. □

A more general question than that of this paper would seek to relate the convergence of a sequence of functions f_n to f with the convergence of f_n^* to f^*. In [13] it is shown that if the underlying space X is reflexive and $f, f_n : X \to (-\infty, +\infty]$ are closed, convex and proper, then f_n converges Mosco to f if and only if f_n^* converges Mosco to f^* (see also [11]). The question of when $f_n^* \to f^*$ pointwise appears to be harder, although it is proved in [14] that under the same conditions, if $f_n \to f$ uniformly on all bounded sets then $f_n^*(x^*) \to f^*(x^*)$ for all $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. This condition seems difficult to check in practice.

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