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A Unified Theory for Metric Regularity of Multifunctions

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Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

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We survey a large bunch of results on metric regularity of multifunctions that appeared during the last 25 years. The tools used for this survey rely on a new variational method introduced in [9][1] and further developped in [10] and independently in [35, 36] which provides a characterization of metric regularity. It allows us to give a unified point of view both for primal results (based on some notion of tangent cones) and dual ones (based on some notion of normal cones). For most known results on metric regularity, a simple proof is given along with some slight improvements for some of them.

1. Introduction

The theory of metric regularity, whose origin dates back to [44] and [28], for mappings and further for multimappings is one of the cornerstones of nonsmooth analysis. A decisive step has been done in this theory by the systematic use of Ekeland's variational principle. However, there are some cases such as the Ursescu-Robinson Theorem and the finite dimensional case in which one can give metric regularity results without using the variational principle. We refer to [36] for its accurate bibliographical and historical comments. The first time in which these kinds of techniques has been used for metric regularity of mappings and multimappings seems to be [31] and [1]. Another important contribution is [33] in which the use of the Ekeland principle leads to sufficient conditions for metric regularity based both on tangent and normal cones. Recently, the use in [9] of the so-called strong slope of De Giorgi, Marino and Tosques opened the way to a unified theory of metric regularity involving both criteria invoking tangent and normal cones. This unified theory which is purely metric was developped in [10] and independently in the deep and complete survey [36]. The powerfullness of this unifying theory lies in the fact that it provides a purely metric characterization of metric regularity which allows Ioffe to write in [36] "*. . . the theory of metric regularity and the parallel Lipschitz theory of set-valued maps do not need anything like subdifferentials, directional derivatives, coderivatives and tangent cones*". The common feeling of mathematicians is that the metric regularity problem is linked to the error bounds one. In fact these two problems are equivalent (see [9, 10]) and the obtained characterization of metric regularity is a consequence of the characterizations of local and global error bounds obtained for the first time in [9, Remark 3.2] and in [8, Theorem 2.4]. The paper is organized as follows. After recalling in Sections 2 and 3 the characterization of metric regularity in complete metric

[1]The paper [9] appeared in 2002 but the corresponding preprint dates back to 1998-99 and was quoted and used in [35, 36].

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spaces, we review in Section 4 a large sample of results on metric regularity among those which appeared during the last 25 years.

Let us begin with some notations that will be used throughout this paper. We let X be a metric space endowed with the metric d, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. For $U \subset X$ and $r \in (0, +\infty]$ (resp., $r \in (0, +\infty[)$, we denote by $B_r(U)$ (resp., $\bar{B}_r(U)$) the open (resp., closed) r-neighborhood of U:

$$B_r(U) := \{x \in X : \ d(x, U) < r\}, \quad \bar{B}_r(U) := \{x \in X : \ d(x, U) \leq r\},$$

where

$$d(x, U) := \inf\{d(x, y) : \ y \in U\},$$

with the convention that $d(x, \emptyset) = +\infty$ (according to the general convention $\inf \emptyset = +\infty$). If $U = \{x\}$, we simply write $B_r(x)$, $\bar{B}_r(x)$ and $B_X = B_1(0)$, $\bar{B}_X = \bar{B}_1(0)$. For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \cup \{+\infty\}$, we let

$$[f{\leq}\alpha] := \{x \in X : \ f(x) \leq \alpha\}, \quad [f{<}\beta] := \{x \in X : \ f(x) < \beta\}$$

denote respectively the closed and open sublevel sets of f, and if $\alpha < \beta$, we further let

$$[\alpha{<}f{<}\beta] := [f{<}\beta] \setminus [f{\leq}\alpha]$$

denote the "slice" between α and β. If $\beta = +\infty$, we shall rather write:

$$[f{>}\alpha] := [\alpha{<}f{<}{+\infty}], \quad \operatorname{dom} f := [f{<}{+\infty}],$$

and say, as usual, that f is *proper* if $\operatorname{dom} f \neq \emptyset$. Given a subset C of a metric space (X, d), we further denote by $d_C(x)$ or $d(x, C)$ the distance from x to C that is $d_C(x) = \inf_{z \in C} d(x, z)$ and by $e(C, D)$ the Hausdorff-Pompeiu excess of C into D defined by $e(C, D) = \sup_{x \in C} d(x, D)$ with the conventions $e(\emptyset, D) = 0$ and $e(C, \emptyset) = +\infty$ whenever $C \neq \emptyset$. We shall further denote by i_S the indicatior function of a subset $S \subset X$ defined by $i_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$

2. The strong slope

In this section we shall give a characterization of metric regularity of multifunctions in the complete metric space setting. This characteristic conditions will be given in terms of the strong slope of De Giorgi, Marino and Tosques introduced in [20]. The first issue in which strong slope is used aimed at error bounds and metric regularity results seems do be [9]. The method initated in the quoted paper was further used and developped in [35, 36, 10]. We stress the fact that the strong slope allows a complete characterization of metric regularity. As we recall in this section, the strong slope can be estimated both in terms of suitable abstract subdifferential or directional derivatives. When applied to metric regularity, we shall also check in Section 4 that our abstract results encompass most of the results on this topic.

2.1. The strong slope

The notion of strong slope was introduced by De Giorgi, Marino, and Tosques in [20] aimed at finding solutions of abstract evolution equations in metric spaces.

2.1.1. Defining the slope

Definition 2.1. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and let $x \in \operatorname{dom} f$. Set:

$$|\nabla f|(x) := \begin{cases} 0 & \text{if } x \text{ is a local minimum of } f, \\ \limsup\limits_{y \xrightarrow{\neq} x} \dfrac{f(x) - f(y)}{d(x,y)} & \text{otherwise.} \end{cases}$$

For $x \notin \operatorname{dom} f$, let $|\nabla f|(x) := +\infty$. The nonnegative extended real number $|\nabla f|(x)$ is called the *strong slope* of f at x.

The main tool which is needed for our purposes is, of course, Ekeland's variational principle [24], of which we now recall an appropriate version, as well as an essential consequence in terms of the strong slope.

Theorem 2.2. *Let X be complete, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a (proper) lower semicontinuous function, and let $\bar{x} \in X$, $\sigma > 0$, and $r > 0$, be such that:*

$$f(\bar{x}) \leq \inf_X f + \sigma r\,.$$

Then, there exists $x \in \bar{B}_r(\bar{x})$ such that $f(x) \leq f(\bar{x})$ and

$$f(x) < f(y) + \sigma d(x,y) \text{ for every } y \in X \setminus \{x\}\,. \tag{1}$$

Corollary 2.3. *Let X be complete, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and let $\bar{x} \in X$, $\sigma > 0$, and $r > 0$, be such that:*

$$f(\bar{x}) < \inf_{\bar{B}_r(\bar{x})} f + \sigma r\,.$$

Then, there exists $x \in B_r(\bar{x})$ such that $f(x) \leq f(\bar{x})$ and $|\nabla f|(x) < \sigma$.

Proof. Let $0 < \sigma' < \sigma$ and $0 < r' < r$ be such that

$$f(\bar{x}) \leq \inf_{\bar{B}_r(\bar{x})} f + \sigma' r'\,.$$

Applying Theorem 2.2 with $X := \bar{B}_r(\bar{x})$, we find $x \in \bar{B}_{r'}(\bar{x}) \subset B_r(\bar{x})$ with $f(x) \leq f(\bar{x})$ and $|\nabla f_{|\bar{B}_r(\bar{x})}|(x) = |\nabla f|(x) \leq \sigma' < \sigma$, as follows from (1) and the definition of the strong slope. □

The next simple proposition illustrates the role played by the strong slope in existence results.

Proposition 2.4. *Let X be complete, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, U be a subset of X, $\alpha \in \mathbb{R}$, and $\sigma, \rho > 0$. Assume that $U \cap [f < \alpha + \sigma\rho] \neq \emptyset$ and that:*

$$\inf_{B_\rho(U) \cap [\alpha < f < \alpha + \sigma\rho]} |\nabla f| \geq \sigma\,.$$

Then, $[f \leq \alpha] \neq \emptyset$.

Proof. Assume, for a contradiction, that $[f{\le}\alpha] = \emptyset$, and let $\bar{x} \in U \cap [\alpha{<}f{<}\alpha{+}\sigma\rho]$. We have:

$$f(\bar{x}) < \alpha + \sigma\rho \le \inf_X f + \sigma\rho\,,$$

and, according to Corollary 2.3, we find $x \in B_\rho(\bar{x}) \subset B_\rho(U)$ with $\alpha < f(x) \le f(\bar{x}) < \alpha + \sigma\rho$ and $|\nabla f|(x) < \sigma$, contradicting our assumption. □

2.1.2. Computing the slope

In this subsection, we consider a Banach space X endowed with a norm $\|\cdot\|$, with topological dual X^*, d_* denoting the metric associated with the norm of X^*. In the case where the function f is Fréchet differentiable at x, it is readily seen that

$$|\nabla f|(x) = \|Df(x)\|_* \tag{2}$$

We further consider an "abstract" *subdifferential operator* ∂, which associates to any lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, and any point $x \in X$, a subset $\partial f(x)$ of the (topological) dual X^* of X, in such a way that $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom} f$, and the following two properties are satisfied:

(P1) if f is convex, then

$$\partial f(x) \subset \{\xi \in X^* :\ f(y) \ge f(x) + \langle \xi, y - x\rangle \text{ for all } y \in X\}; \tag{3}$$

(P2) whenever $\bar{x} \in \operatorname{dom} f$ is a local minimum point of $f + g + h$ where f is lower semicontinuous and g, h are Lipschitz continuous near $\bar{x}$ with h convex then, for every $\varepsilon > 0$ there exist x, y, $z \in X$, $\xi \in \partial f(x)$, and $\zeta \in \partial g(y)$, $\chi \in \partial h(z)$ such that

$$\|x - \bar{x}\| \le \varepsilon\,,\ \|y - \bar{x}\| \le \varepsilon\,,\ \|z - \bar{x}\| \le \varepsilon\,,\ \ f(x) \le f(\bar{x}) + \varepsilon\,,\ \text{ and }\ \|\xi + \zeta + \chi\|_* \le \varepsilon\,.$$

It follows that if the subdifferential satisisfies (P1) and if the Banach space X is ∂-trustworthy in the sense of Ioffe, then (P2) is satisfied on X.

Given a subdifferential ∂, one is also interested in the associated limiting subdifferential $\bar{\partial} f(x)$ defined as the set of $\xi \in X^*$ for which there exist sequences $(x_n)_{n\in\mathbb{N}}$ with $((x_n, f(x_n)) \to (x, f(x))$ and $(\xi_n)_{n\in\mathbb{N}} \subset X^*$ $*$-weakly converging to ξ such that $\xi_n \in \partial f(x_n)$ for all $n \in \mathbb{N}$. It is clear that $\bar{\partial}$ satisfies (P2) on X whenever it is the case for ∂.

Example 2.5. Let us list some basic subdifferential. Each of them clearly satisfies (P1).

1. The Clarke-Rockafellar subdifferential satisfies (P2) on every Banach space as easily shown by its definition. Let us recall that the Clarke-Rockafellar subdifferential $\partial f(x)$ at $x \in \operatorname{dom} f$ of a lower semicontinuous function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the set of $\xi \in X^*$ such that $\langle \xi, u\rangle \le f^\uparrow(x; u)$ for all $u \in X$ where

$$f^\uparrow(x;u) = \sup_{\varepsilon>0} \inf_{\eta>0} \sup_{(z,f(z),t)\in B_\eta(x)\times B_\eta(f(x))\times(0,\eta)} \inf_{v\in B_\varepsilon(u)} t^{-1}(f(z+tv) - f(z)).$$

2. The Dini subdifferential ∂^- is defined at $x_0 \in \operatorname{dom} f$ by

$$\partial^- f(x_0) = \{\xi \in X^* : \langle \xi, u\rangle \le f'(x_0; u) \text{ for all } u \in X\}, \tag{4}$$

where $f'(x_0;u) = \liminf\limits_{\begin{cases} t \downarrow 0 \\ v \to u \end{cases}} \dfrac{f(x_0+tv)-f(x_0)}{t}$. This subdifferential satisfies (P2) on every finite dimensional X (see e.g. [32, Theorem 2]).

3. The Fréchet subdifferential $\hat{\partial} f(x)$, that is the set of $\xi \in X^*$ such that

$$\liminf_{z \to x} \|z-x\|^{-1}(f(z)-f(x)-\langle \xi, z-x\rangle) \geq 0$$

satisfies (P2) on every Asplund space (see [25]). It follows that when X is asplund, the Mordukhovich subdifferential which in that case is the limiting Fréchet subdifferential, also satisfies (P2). One also uses, for $\varepsilon > 0$, the ε-Fréchet subdifferential $\hat{\partial}^\varepsilon f(x)$ defined as the set of $\xi \in X^*$ such that

$$\liminf_{z \to x} \|z-x\|^{-1}(f(z)-f(x)-\langle \xi, z-x\rangle) \geq -\varepsilon.$$

Then it is noteworthy that

$$0 \in \hat{\partial}^\varepsilon f(x) \text{ if and only if } |\nabla f|(x) \leq \varepsilon. \tag{5}$$

4. The Ioffe subdifferential ∂_A satisfies (P2) on every Banach space (see e.g. [36, Theorem 1]). It is defined for Lipschitz functions by

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}} \operatorname*{Lim\,sup}_{x \xrightarrow{f} x_0} \partial_L^- f(x) \tag{6}$$

where $\mathcal{F}$ is the family of finite dimensional subspaces of X,

$$\partial^- f_L(x_0) = \{\xi \in X^* : \langle \xi, u\rangle \leq f'(x_0;u) \text{ for all } u \in L\},$$

the Lim sup being taken with respect to the weak* topology and $x \xrightarrow{f} x_0$ means $(x, f(x)) \to (x_0, f(x_0))$. If f is lower semicontinuous and $x_0 \in \operatorname{dom} f$, this definition extends to

$$\partial_A f(x_0) = \{\xi \in X^* : (\xi, -1) \in \bigcup_{L>0} \partial_A d_{\operatorname{epi} f}(x, f(x))\}.$$

5. A bornology β is a collection of bounded convex symmetric subsets of X which is closed under multiplication by scalars and is such that $X = \bigcup_{B \in \beta} B$ and such that the union of two elements of β is contained in some element of β. A function f is said to be β-differentiable at x_0 whenever there is $\nabla^\beta f(x_0) \in X^*$ such that for any $B \in \beta$, we have $\lim_{t \to 0} t^{-1}(f(x_0+tu)-f(x_0)-t\langle \nabla^\beta f(x_0), u\rangle) = 0$ uniformly for $u \in B$. We say that the function f is β-smooth at x_0 whenever $\nabla^\beta f$ exists near x_0 and the mapping $\nabla^\beta f$ is continuous when X^* is endowed with the uniform convergence on elements of the bornology β. Then the β subdifferential $\partial_\beta f(x_0)$ is the set of $\xi \in X^*$ such that for all $\varepsilon > 0$ and for all $B \in \beta$, there exists $\eta > 0$ such that $t^{-1}(f(x_0+tu)-f(x_0)) - \langle \xi, u\rangle > -\varepsilon$ for all $t \in (0, \eta)$ and for all $u \in B$. This subdifferential satisfies (P2) whenever there exists a β-differentiable Lipschitz bump function, that is a Lipschitz function φ with values on $[0,1]$ with $\varphi(0) = 1$ whose support is contained in $\bar{B}$ and such that each maximizing sequence converges to 0 (see e.g. [36, 2, Theorem 1]).

6. Given a bornology on X, the β-viscosity subdifferential $D^\beta f(x_0)$ of a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x_0 \in \operatorname{dom} f$ is the set of $\xi \in X^*$ for which there exists a locally Lipschitz function g which is β-smooth at x_0 and such that $f - g$ attains a local minimum at x_0 along with $\nabla^\beta g(x_0) = \xi$. Such a subdifferential satifies (P2) whenever X admits an equivalent norm which is β-smooth away from 0 (see e.g. [13, Theorem 2.9]).
7. If $0 < s \leq 1$, a function $g : X \to \mathbb{R}$ is of class $C^{1,s}$ if it is (Gâteaux-)differentiable on X and its differential Dg is s-Hölder continuous, i.e., there is a constant $C \geq 0$ such that
$$\|Dg(x) - Dg(y)\| \leq C\|x - y\|^s \text{ for all } x, y \in X\,.$$
The Banach space X has a $s+1$-power modulus of smoothness if X has an equivalent differentiable norm $\|\cdot\|$ for which the function $g := \frac{1}{s+1}\|\cdot\|^{s+1}$ is of class $C^{1,s}$. The Banach spaces with a power modulus of smoothness are the superreflexive spaces. Equipped with their natural norm, Hilbert spaces have a power modulus of smoothness t^2, and L^p spaces, $p > 1$, have a power modulus of smoothness $t^{\min\{p,2\}}$. Then the s-Hölder smooth subdifferential is of f at $x \in \operatorname{dom} f$ is the set $\partial^s f(x)$ of $\xi \in X^*$ such that there exists η, $\sigma > 0$ such that
$$f(y) - f(x) \geq \langle \xi, y - x\rangle - \sigma\|y - x\|^{1+s} \text{ for all } y \in B_\eta(x).$$
In other words
$$\partial^s f(x) := \{\xi \in X^* :\ \liminf_{y \to x} \frac{f(y) - f(x) - \langle \xi, y - x\rangle}{\|y - x\|^{1+s}} > -\infty\}\,,$$
and the associated viscosity subdifferential is the set $\tilde{\partial}^s f(x)$ of those $D\varphi(x)$ such that φ is of class $C^{1,s}$ and x is a local minimum of $f - \varphi$. It is clear that $\tilde{\partial}^s f(x) \subset \partial^s f(x)$. It is easily seen, by using [22, Lemma 1.2.5], that the viscosity subdifferential $\tilde{\partial}^s f(x)$ satisfies (P2) whenever X has a power modulus of smoothness t^{s+1}. When $s = 1$ and X is Hilbert, then ∂^s coincides with Rockafellar's proximal subdifferential ∂^π [55].

The two following propositions provide useful information for estimating the slope from below.

Proposition 2.6. *Let X be a Banach space and ∂ be a subdifferential operator which satisfies* (P1) *and* (P2) *on X. Then, for every lower semicontinuous function $h : X \to \mathbb{R} \cup \{+\infty\}$, for every function g Lipschitz continuous near $x \in \operatorname{dom} f$, we have, setting $f = g + h$:*
$$|\nabla f|(x) \geq \liminf_{(z,y,f(z),f(y)) \to (x,x,f(x),f(x))} d_*(0, \partial g(z) + \partial h(y))\,.$$

Proof. Let $x \in X$. We may assume that x is not a local minimum point of f (for, otherwise, the result readily follows from (P2)), and that $|\nabla f|(x) < +\infty$ (so that $x \in \operatorname{dom} f$). Given $\sigma > |\nabla f|(x)$ and $\varepsilon > 0$, let $r > 0$ be such that
$$f(x) \leq f(y) + \sigma\|y - x\| \text{ for every } y \in B_r(x)\,,$$
so that the function $h + g + \sigma\|\cdot - x\|$ attains a finite local minimum at x. From property (P2), we find y, z $u \in X$ and $\xi \in \partial h(y)$, $\zeta \in \partial(\sigma\|\cdot - x\|)(u)$ and $\chi \in \partial g(z)$ such that
$$\|y - x\| \leq \varepsilon\,, \quad h(y) \leq h(x) + \varepsilon\,, \quad \text{and} \quad \|\xi + \chi + \zeta\|_* \leq \varepsilon\,.$$

From property (P1), $\|\zeta\|_* \leq \sigma$, so that $\|\xi + \chi\|_* \leq \sigma + \varepsilon$, and the conclusion follows since $\varepsilon > 0$ is arbitrary, taking into account the lower semicontinuity of f. $\square$

Applying the previous proposition with $g = 0$ we get:

Proposition 2.7. *Let X be a Banach space and ∂ be a subdifferential operator such that properties* (P1) *and* (P2) *are satisfied on X. Then, for every lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and every $x \in X$, we have:*

$$|\nabla f|(x) \geq \liminf_{(y,f(y))\to(x,f(x))} d_*(0, \partial f(y)) \,.$$

Proposition 2.7 tells us that if $\inf_{y\in\Omega} d_*(0, \partial f(y)) \geq \tau$ where Ω is an open subset of X for some $\tau > 0$, then $\inf_{x\in\Omega} |\nabla f|(x) \geq \tau$.

The strong slope can also be estimated, just using the definitions, with various notions of directional derivative. For example, for $x \in \operatorname{dom} f$, let us consider

$$f'(x; u) := \liminf_{\substack{t\searrow 0\\ v\to u}} \frac{f(x + tv) - f(x)}{t}$$

the contingent derivative.

Proposition 2.8. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function defined on a Banach space X and let $x \in \operatorname{dom} f$. We have*

$$|\nabla f|(x) \geq \sup_{\|u\|\leq 1} (-f'(x; u)). \tag{7}$$

Moreover, if X is finite dimensional, we have

$$|\nabla f|(x) = \max_{\|u\|\leq 1} (-f'(x; u)). \tag{8}$$

Proof. Assuming that x is a local minimum, then $f'(x; u) \geq 0$ for all $u \in X$, thus $|\nabla f|(x) \geq \sup_{\|u\|\leq 1}(-f'(x; u)$. Otherwise, fix $u \in X \setminus \{0\}$ with $\|u\| \leq 1$. If $-f'(x; u) \leq 0$, the inequality $-f'(x; u) \leq |\nabla f|(x)$ is obvious. Suppose that $-f'(x; u) > 0$ and choose sequences $t_n \downarrow 0$ and $v_n \to u$ such that

$$-f'(x; u) := \lim_{n\to\infty} \frac{f(x) - f(x + t_n v_n)}{t_n} > 0.$$

Then

$$-f'(x; u) \leq \lim_{n\to\infty} \frac{f(x) - f(x + t_n v_n)}{t_n\|u\|} = \lim_{n\to\infty} \frac{f(x) - f(x + t_n v_n)}{t_n\|v_n\|}$$

yielding

$$-f'(x; u) \leq \limsup_{y \xrightarrow{\neq} x} \frac{f(x) - f(y)}{\|x - y\|} = |\nabla f|(x),$$

from which we get (7). Assume now that X is finite dimensional. If x is a local minimum, then $|\nabla f|(x) = 0 = -f'(x; 0)$. Otherwise, let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to x

with $x_n \neq x$ such that $|\nabla f|(x) = \limsup_{n\to\infty} \dfrac{f(x) - f(x_n)}{\|x - x_n\|}$. Setting $t_n = \|x_n - x\|$ and $u_n = t_n^{-1}(x_n - x)$, there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$ converging to some $u \in X$ with $\|u\| = 1$. Thus we get

$$-|\nabla f|(x) = \liminf_{n\to\infty} t_n^{-1}(f(x + t_n u_n) - f(x)) \geq f'(x, u),$$

hence (8). □

Estimates from above of the slope are also available in some special cases. These estimates will be useful when aimed at seeking for necessary conditions for metric regularity of multifunctions.

Proposition 2.9. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a closed proper function defined on a Banach space X and let $x \in \operatorname{dom} f$. We have*

a) $d_*(0, \hat{\partial} f(x)) \geq |\nabla f|(x)$, *where $\hat{\partial}$ denotes the Fréchet subdifferential;*

b) $d(0, \partial^s f(x)) \geq |\nabla f|(x)$ *whenever ∂^s denotes the s-Hölder smooth subdifferential.*

c) $d_*(0, \partial^- f(x)) \geq |\nabla f|(x)$ *whenever X is finite dimensional and ∂^- denotes the Dini subdifferential.*

Proof. *a)* We may assume that x is not a local minimum, since in that case the right member of the inequality in *a)* is equal to 0. Let $\xi \in \hat{\partial} f(x)$ and let $\varepsilon > 0$. We can find $r > 0$ such that $f(y) - f(x) \geq \langle \xi, y - x\rangle - \varepsilon\|y - x\|$ for all $y \in B_r(x)$, yielding

$$\frac{f(x) - f(y)}{\|y - x\|} \leq \|\xi\|_* + \varepsilon \text{ for all } y \in B_r(x),$$

and then $\limsup_{y \xrightarrow{\neq} x} \dfrac{f(x) - f(y)}{\|x - y\|} \leq \|\xi\|_*$ for all $\xi \in \hat{\partial} f(x)$, from which we get

$$d_*(0, \hat{\partial} f(x)) \geq |\nabla f|(x).$$

b) follows from *a)* since $\partial^s f(x) \subset \hat{\partial} f(x)$.

c) From Proposition 2.8, putting $\tau := |\nabla f|(x)$, we can find a unit vector $u \in X$ such that $f'(x; u) = -\tau$. Thus if $\xi \in \partial^- f(x)$, we have $-\tau = f'(x; u) \geq \langle \xi, u\rangle$, so that $\|\xi\|_* \geq \langle \xi, -u\rangle \geq \tau$. □

Remark 2.10. If $f : X \longmapsto \mathbb{R} \cup \{+\infty\}$ is a closed proper convex function defined on a Banach space X, then

$$|\nabla f|(x) = d_*(0, \partial f(x)) \text{ for all } x \in \operatorname{dom} f. \tag{9}$$

Indeed, we get from part a) of Proposition 2.9 that $d_*(0, \partial f(x)) \geq |\nabla f|(x)$. Now if $0 < \sigma < d_*(0, \partial f(x))$, then, x is not a minimum point of the function $z \mapsto f(z) + \sigma\|x - z\|$ for, otherwise, from standard convex calculus, we have $0 \in \partial f(x) + \sigma \bar{B}_*$, where $\bar{B}_*$ denotes the closed unit ball of X^*, contradicting the choice of σ. Let thus $z \in X$ be such that $f(z) + \sigma\|x - z\| < f(x)$, so that for every $\lambda \in]0, 1]$:

$$\frac{f(x) - f(x + \lambda(z - x))}{\lambda\|x - z\|} > \sigma\,,$$

showing that $|\nabla f|(x) \geq \sigma$, whence $|\nabla f|(x) \geq d_*(0, \partial f(x))$.

2.1.3. Parametric error bounds

When dealing with parametric family of multifunctions, a parametric error bound result may be useful. Let us begin with a result of [10] which is of independant interest.

Theorem 2.11. *Let X be complete, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, U be a subset of X, $\alpha \in \mathbb{R}$, and $\sigma, \rho > 0$. Assume that $U \cap [f{<}\alpha + \sigma\rho] \neq \emptyset$ and that*

$$\inf_{B_\rho(U)\cap[\alpha<f<\alpha+\sigma\rho]} |\nabla f| \geq \sigma\,.$$

Then, $[f{\leq}\alpha] \neq \emptyset$, and the following local error bound holds

$$\sigma\, d(x, [f{\leq}\alpha]) \leq (f(x) - \alpha)^+ \text{ for all } x \in U \cap [f{<}\alpha + \sigma\rho].$$

Proof. The fact that $[f{\leq}\alpha] \neq \emptyset$ is established in Proposition 2.4. We may now assume that $U \cap [\alpha{<}f{<}\alpha + \sigma\rho] \neq \emptyset$. Assume further, for a contradiction, that for some $\bar{x} \in U \cap [\alpha{<}f{<}\alpha + \sigma\rho]$ we have:

$$f(\bar{x}) - \alpha < \sigma d(\bar{x}, [f{\leq}\alpha])\,.$$

Let $0 < r := \min\{d(\bar{x}, [f{\leq}\alpha]), \rho\}$, and $g := (f - \alpha)^+$, so that

$$g(\bar{x}) < \inf_{\bar{B}_r(\bar{x})} g + \sigma r\,.$$

Applying Corollary 2.3, we find $x \in B_r(\bar{x})$ with $g(x) \leq g(\bar{x})$ and $|\nabla g|(x) < \sigma$. Then, $x \in B_\rho(U) \cap [\alpha{<}f{<}\alpha + \sigma\rho]$ and $|\nabla f|(x) = |\nabla g|(x) < \sigma$: a contradiction. □

The following definition is classical.

Definition 2.12. Let X be a complete metric space, let P be a topological space and let $f : X \times P \longrightarrow \mathbb{R} \cup \{+\infty\}$. We say that f is epi-upper semicontinuous at (x_0, p_0) if

$$(\text{e-}\limsup_{p\to p_0} f_p)(x_0) \leq f_{p_0}(x_0)$$

where

$$(\text{e-}\limsup_{p\to p_0} f_p)(x_0) = \sup_{\varepsilon>0} \inf_{N\in\mathcal{N}(p_0)} \sup_{p\in N} \inf_{x\in B_\varepsilon(x_0)} f_p(x),$$

denoting by $\mathcal{N}(p_0)$ the family of neighborhoods of p_0 and by $f_p = f(\cdot, p)$.

Observing that $(\text{e-}\limsup_{p\to p_0} f_p)(x_0) \leq \limsup_{p\to p_0} f_p(x_0)$, it follows that f is epi-upper semicontinuous at (x_0, p_0) whenever the function $p \mapsto f_p(x_0)$ is upper semicontinuous at p_0. In the case $f(x, p) = i_{C_p}(x)$, where $(C_p)_{p\in P}$ is a family of subsets of X, the function f is epi-upper semicontinuous at (x_0, p_0) with $x_0 \in C_{p_0}$ if and only if $x_0 \in \operatorname{Lim\,inf}_{p\to p_0} C_p$. Such a condition is clearly weaker than the requirement $\limsup_{p\to p_0} f_p(x_0) \leq f_{p_0}(x_0)$ which amounts to $x_0 \in C_p$ for p close to p_0. Let us recall that the closed set $\operatorname{Lim\,inf}_{p\to p_0} C_p$ is the set of those $x \in X$ such that for each neighborhood V of x, the set of $p \in P$ such that $V \cap C_p \neq \emptyset$ is a neighborhood of p_0. In other words $x \in \operatorname{Lim\,inf}_{p\to p_0} C_p$ if and only if there exists a mapping $p \mapsto x_p$ defined near x with values in X such that $\lim_{p\to p_0} x_p = x$ and $x_p \in C_p$ near p_0.

Theorem 2.13. *Let X be a complete metric space, let P be a topological space and let $f : X \times P \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $(x_0, p_0) \in X \times P$. Assume that*

a) $f_{p_0}(x_0) \leq 0$;

b) f_p *is lower semicontinuous for all* p *near* p_0;

c) f *is epi-upper semicontinuous at* (x_0, p_0);

d) *there exist neighborhoods* U_0 *of* x_0, N_0 *of* p_0 *and* $\tau > 0$

such that

$$\inf_{\begin{cases}(x,p) \in U_0 \times N_0 \\ f(x,p) > 0\end{cases}} |\nabla f_p|(x) \geq \tau.$$

Then there exist neighborhoods V of x_0 and N of p_0 such that $[f_p {\leq} 0] \neq \emptyset$ for all $p \in N$, and for all $\gamma \geq 0$,

$$\tau\, d(x, [f_p {\leq} \gamma]) \leq (f_p(x) - \gamma)^+ \textit{ for all } (x,p) \in V \times N.$$

Proof. Let $\rho > 0$ be such that $B_{4\rho}(x_0) \subset U_0$. From assumption c), we can find $N \in \mathcal{N}(p_0)$ such that $N \subset N_0$ and $\sup_{p \in N} \inf_{x \in B_\rho(x_0)} f_p(x) < \tau\rho$. Setting $U = B_\rho(x_0)$, we get, for all $p \in N$ that $U \cap [f_p {<} \tau\rho] \neq \emptyset$ and

$$\inf_{x \in B_\rho(U) \cap [0 < f_p < \tau\rho]} |\nabla f_p|(x) \geq \tau \text{ for all } p \in N.$$

Thus we obtain from Theorem 2.11 that for all $p \in N$ we have $[f_p {\leq} 0] \neq \emptyset$ and

$$\tau\, d(x, [f_p {\leq} 0]) \leq f_p(x)^+ \text{ for all } x \in U \cap [f_p {<} \tau\rho] \text{ and } p \in N,$$

yielding $[f_p {\leq} 0] \cap B_{2\rho}(x_0) \neq \emptyset$ for all $p \in N$.

Now we claim that $\tau\, d(x, [f_p {\leq} \gamma]) \leq (f_p(x) - \gamma)^+$ for all $x \in V := B_\rho(x_0)$ for all $p \in N$ and for all $\gamma \geq 0$. If not, we can find $x \in B_\rho(x_0)$, $p \in N$ and $\gamma \geq 0$ such that $0 < f_p(x) - \gamma < \tau\, d(x, [f_p {\leq} \gamma])$. Setting $r = d(x, [f_p {\leq} \gamma])$, we get $0 < r < 3\rho$ and $g(x) < \inf_{\bar{B}_r(x)} g + \tau r$ where $g = (f_p - \gamma)^+$. From Corollary 2.3, there exists $y \in B_r(x)$ such that $|\nabla g|(y) < \tau$. As $d(y,x) < d(x, [f_p {\leq} \gamma])$, we get $f_p(y) > \gamma$, so that $|\nabla f_p|(y) = |\nabla g|(y) < \tau$ and $y \in B_{4\rho}(x_0)$, a contradiction. □

Remark 2.14. Theorem 2.13 fails to be true without epi-upper semicontinuity of f at (x_0, p_0) as shown by the following example. Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x,p) = \begin{cases} e^x & \text{if } p \neq 0 \\ x^+ & \text{if } p = 0, \end{cases}$$

so that f is lower semicontinuous on $\mathbb{R} \times \mathbb{R}$ and $f(0,0) = 0$. For all $p \in \mathbb{R}$ and for all $x \in [f_p {>} 0] \cap (-1,1)$, we derive from (2) that $|\nabla f_p|(x) > r$ with $r = e^{-1}$. Thus we get from (5) that for all $p \in \mathbb{R}$ and for all $x \in [f_p {>} 0] \cap (-1,1)$ we have $0 \notin \hat{\partial}^\varepsilon f_p(x)$ for all $\varepsilon \in (0, r)$ where $\hat{\partial}^\varepsilon f_p(x)$ is the ε-Fréchet subdifferential of f_p at x. However $[f_p {\leq} 0] = \emptyset$ for all $p \neq 0$ and then f provides a counterexample to [15, Theorem 2].

In the previous theorem, we saw that the fact for the slope to be bounded away from 0 on the complement of a level set in a neighborhood of some point yields a local error bound result for all greater level sets. The powerfulness of the strong slope lies in the fact that the converse is true as shown in the following proposition which is [9, Remark 3.2].

Proposition 2.15. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function defined on a metric space X, U be a subset of X, and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \cup \{+\infty\}$ with $\alpha < \beta$. Assume that, for some $\tau > 0$ we have*

$$\tau\, d(x, [f{\le}\gamma]) \le (f(x) - \gamma)^+ \text{ for all } \gamma \ge \alpha \text{ and for all } x \in U \cap [f{<}\beta].$$

Then

$$\inf_{U \cap [\alpha < f < \beta]} |\nabla f| \ge \tau.$$

Proof. Let $\sigma \in (0, \tau)$, let $x \in U \cap [\alpha{<}f{<}\beta]$, and set $\gamma_n := f(x) - 1/n$ for $n \in \mathbb{N}$ large enough so that $\gamma_n > \alpha$. For each $n \in \mathbb{N}$, let $x_n \in [f{\le}\gamma_n]$ be such that $f(x) - \gamma_n > \sigma d(x, x_n)$. Then, we have:

$$0 < d(x, x_n) \le \frac{f(x) - \gamma_n}{\sigma} \;\to\; 0 \quad \text{as } n \to \infty\,,$$

so that x is not a local minimum of f, and

$$\frac{f(x) - f(x_n)}{d(x, x_n)} \ge \frac{f(x) - \gamma_n}{d(x, x_n)} \ge \sigma\,,$$

showing that $|\nabla f|(x) \ge \sigma$, and the conclusion follows by letting σ increase to τ. □

3. Metric regularity in complete metric spaces

We further consider a *multifunction* $F \subset X \times Y$ identified to its graph: for $x \in X$ and $y \in Y$ we respectively set:

$$F(x) := \{y \in Y : \ (x, y) \in F\}\,, \quad F^{-1}(y) := \{x \in X : \ (x, y) \in F\}\,.$$

For $z \in Y$, we define a function $f_z : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by:

$$f_z(x, y) := d(z, y) + i_F(x, y) = \begin{cases} d(z, y) & \text{if } (x, y) \in F \\ +\infty & \text{otherwise,} \end{cases}$$

so that f_z is lower semicontinuous if (and only if) F is closed (F is a *closed-graph* multifunction). Moreover, for every $\gamma \ge 0$, we have:

$$[f_z{\le}\gamma] = \bigcup_{y \in \bar{B}_\gamma(z)} (F^{-1}(y) \times \{y\}) \tag{10}$$

(where, of course, $F^{-1}(y) \times \{y\} = \emptyset$ if $F^{-1}(y) = \emptyset$).

As usual, we say that the multifunction F is metrically regular near $(x_0, y_0) \in F$ if there exist $\tau > 0$ and a neighborhood W of (x_0, y_0) such that

$$\tau\, d(x, F^{-1}(y)) \le d(y, F(x)) \text{ for all } (x, y) \in W. \tag{11}$$

We shall need the following definition

Definition 3.1. For $z \in X$, let $d_z : X \to \mathbb{R}$ be defined by $d_z(x) := d(z,x)$. We say that the metric space X is *coherent* at $z \in X$ if

$$|\nabla d_z|(x) = 1 \text{ for every } x \neq z\,.$$

If this is true for every $z \in X$, we just say that X *is coherent*.

Clearly, if X is a convex subset of a normed vector space, then X (with the metric associated with the norm) is coherent. In general, since d_z is 1-Lipschitzian, so that $|\nabla d_z| \leq 1$, the fact that a complete metric space X is coherent at z is equivalent to:

$$(d(z,x) - \gamma)^+ \geq d(x, \bar{B}_\gamma(z)) \text{ for all } x \in X \text{ and for all } \gamma \geq 0\,, \tag{12}$$

according to [8, Theorem 2.4].

We now consider two metric spaces X and Y (we shall use the same notation d for both metrics), and, for $\delta > 0$, the product space $X \times Y$ as endowed with the metric:

$$d_\delta((x,y),(x',y')) := \max\{d(x,x'), \delta d(y,y')\}\,.$$

Accordingly, if $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, we shall let $|\nabla_\delta f|$ denote the strong slope of f with respect to the metric d_δ. The next theorem is a slight variant of [10, Theorem 5.3].

Theorem 3.2. *Let X and Y be complete metric spaces, $F \subset X \times Y$ be a closed multifunction, $(x_0, y_0) \in F$, let Z be a subset of Y, let V and W be neighborhoods of x_0 and y_0, respectively, $\sigma > 0$, and $0 < \delta \leq 1/\sigma$.*

(a) *Assume that:*

$$|\nabla_\delta f_z|(x,y) \geq \sigma \text{ for all } (x,y,z) \in V \times W \times Z\,,\ y \neq z\,,$$

where $f_z(x,y) = d(z,y) + i_F(x,y)$. Then, there exists $\varepsilon > 0$ such that:

$$d(z, F(x)) \geq \sigma d(x, F^{-1}(z)) \text{ for all } (x,z) \in B_\varepsilon(x_0) \times (B_\varepsilon(y_0) \cap Z)\,,$$

(in particular $B_\varepsilon(y_0) \cap Z \subset F(X)$).

(b) *Conversely, assume that Y is coherent and that:*

$$d(z, F(x)) \geq \sigma d(x, F^{-1}(z)) \text{ for all } (x,z) \in V \times W\,. \tag{13}$$

Then, there exists $r > 0$ such that

$$|\nabla_\delta f_z|(x,y) \geq \sigma \text{ for all } (x,y,z) \in B_r(x_0) \times B_r(y_0) \times B_r(y_0)\,,\ y \neq z\,.$$

Proof. (a) Let $\rho > 0$ be such that $B_{2\rho}(x_0,y_0) \subset V \times W$, and let $z \in B_{\sigma\rho}(y_0) \cap Z \subset W$. Applying Theorem 2.11 to f_z with $U := B_\rho(x_0,y_0)$, $\alpha := 0$, and the given σ and ρ, since $(x_0,y_0) \in U \cap [f_z{<}\sigma\rho]$ and $|\nabla_\delta f_z|(x,y) \geq \sigma$ for every $(x,y) \in [f_z{>}0] \cap B_\rho(U)$, we obtain that, for all $z \in B_{\sigma\rho}(y_0) \cap Z \subset W$:

$$f_z(x,y) \geq \sigma d_\delta((x,y),[f_z{\leq}0]) \text{ for all } (x,y) \in [f_z{<}\sigma\rho] \cap B_\rho(x_0,y_0)\,. \tag{14}$$

Let $0 < r \le \rho$ be such that $B_r(x_0, y_0) \cap F \subset [f_z{<}\sigma\rho]$ for every $z \in B_{\sigma r}(y_0)$. Then, taking the definitions of f_z and of d_δ, the fact that $\sigma\delta \le 1$, and (10), into account, (14) yields:

$$d(z, y) \ge \sigma d(x, F^{-1}(z)) \text{ for all } (x, y) \in B_r(x_0, y_0) \cap F\,, \text{ for all } z \in B_{\sigma r}(y_0) \cap Z\,, \quad (15)$$

and in particular:

$$d(z, y_0) \ge \sigma d(x_0, F^{-1}(z)) \text{ for all } z \in B_{\sigma r}(y_0) \cap Z\,. \quad (16)$$

Assume now that the conclusion does not hold. Then, there exist sequences $(x_n, y_n) \subset F$ and $(z_n) \subset Z$ such that $d(x_n, x_0) \to 0$, $d(z_n, y_0) \to 0$, and:

$$d(z_n, y_n) < \sigma d(x_n, F^{-1}(z_n))\,. \quad (17)$$

Taking (16) into account yields:

$$d(y_n, y_0) - d(z_n, y_0) < \sigma d(x_n, x_0) + \sigma d(x_0, F^{-1}(z_n)) \le \sigma d(x_n, x_0) + d(z_n, y_0)\,,$$

which shows that $d(y_n, y_0) \to 0$, so that (17) contradicts (15) for large n.

(*b*) Let $r > 0$ be such that $B_r(x_0) \subset V$ and $B_{3r}(y_0) \subset W$. Let $z \in B_r(y_0)$ be fixed, let then $\gamma \ge 0$ and $(x, y) \in F \cap (B_r(x_0) \times B_r(y_0)) \cap [f_z{>}\gamma]$, so that $\gamma < d(z, y) \le 2r$. Then, for every $y' \in \bar{B}_\gamma(z) \subset V$, we have:

$$d(y', y) \ge \sigma d(x, F^{-1}(y'))$$

(in particular, $F^{-1}(y') \ne \emptyset$ for every such y'), hence:

$$d(y', y) \ge \sigma d_\delta((x, y), F^{-1}(y') \times \{y'\})\,,$$

and finally, since y' is arbitrary in $\bar{B}_\gamma(z)$:

$$d(y, \bar{B}_\gamma(z)) \ge \sigma d_\delta((x, y), [f_z{\le}\gamma])\,.$$

According to (12), we have:

$$f_z(x, y) - \gamma \ge \sigma d_\delta((x, y), [f_z{\le}\gamma])\,.$$

Letting $U := B_r(x_0) \times B_r(y_0)$, we thus have:

$$\inf_{\gamma>0} \inf_{(x,y)\in U\cap[f_z>\gamma]} \frac{f_z(x, y) - \gamma}{d_\delta((x, y), [f_z{\le}\gamma])} \ge \sigma\,,$$

and the conclusion follows from Proposition 2.15 applied to f_z with $\alpha := 0$ and $\beta := +\infty$, since z is arbitrary in $B_r(y_0)$. $\square$

4. Metric regularity in Banach spaces

In this section we survey a large selection of results on metric regularity and we show how they enter in the general framework described in Section 3. The following result is the well-known Ursescu-Robinson Theorem ([57, 54]).

4.1. The Ursescu-Robinson Theorem

Let us recall that the core of a convex set $C \subset X$ is the set core(C) of those $x \in C$ such that $X = (0, +\infty)(C - x)$.

Theorem 4.1. *Let $F \subset X \times Y$ be a closed convex multifunction where X, Y are Banach spaces. Assume that $y_0 \in \operatorname{core} F(X)$, then F is metrically regular near any $(x_0, y_0) \in F$.*

Proof. Let us consider, for $z \in Y$, the closed convex function $f_z : X \times Y \longmapsto \mathbb{R} \cup \{+\infty\}$ defined by $f_z(x, y) = \|z - y\| + i_F(x, y)$. As $y_0 \in \operatorname{core} F(X)$, Baire's Theorem yields r_1, $r_2 > 0$ such that $\bar{B}_{r_1}(y_0) \subset \operatorname{cl}(F(\bar{B}_{r_2}(x_0))$. Let $0 < \sigma < r_1/r_2$, let $\delta = r_2/r_1 < 1/\sigma$ and let $\varepsilon = (1 + \sigma)^{-1}(r_1 - r_2\sigma) \in (0, r_1)$. Let $(x, y) \in [f_z{>}0] \cap (B_\varepsilon(x_0) \times B_\varepsilon(y_0)) \cap F$ and let $(\xi, \zeta) \in \partial f_z(x, y)$ so that $(\xi, \zeta) = (\xi_2, \zeta_1 + \zeta_2)$ with $\|\zeta_1\|_* = 1$ and $(\xi_2, \zeta_2) \in N_F(x, y)$. As $\bar{B}_{r_1 - \varepsilon}(y) \subset \bar{B}_{r_1}(y_0)$ we can find, for any $v \in (r_1 - \varepsilon)\bar{B}_Y$, a sequence $((x_n, y_n))_{n \in \mathbb{N}} \subset F$ such that $(y_n)_{n \in \mathbb{N}}$ converges to $y - v$ and $(x_n)_{n \in \mathbb{N}} \subset \bar{B}_{r_2}(x_0)$. Now we get, endowing $X \times Y$ with the norm $\|(u, v)\|_\delta = \max(\|u\|, \delta\|v\|)$,

$$\|(x - x_n, y - y_n)\|_\delta \|(\xi, \zeta)\|_* \geq \langle \xi_2, x - x_n \rangle + \langle \zeta_1, y - y_n \rangle + \langle \zeta_2, y - y_n \rangle \geq \langle \zeta_1, y - y_n \rangle,$$

yielding $\langle \zeta_1, v \rangle \leq (r_2 + \varepsilon)\|(\xi, \zeta)\|_*$ and then $\|(\xi, \zeta)\|_* \geq \dfrac{r_1 - \varepsilon}{r_2 + \varepsilon} = \sigma$. Thus we derive from (9) that, for all $z \in Y$,

$$\inf_{[f_z>0] \cap (B_\varepsilon(x_0) \times B_\varepsilon(x_0))} |\nabla_\delta f_z| \geq \sigma,$$

and then the conclusion of the theorem follows from Theorem 3.2 (a) applied with $Z = Y$. □

4.2. Normal conditions

Given a multifunction $F \subset X \times Y$ and a subdifferential operator ∂, the coderivative $D^*F(x, y)$ at a point $(x, y) \in F$ is the multifunction $D^*F(x, y) \subset Y^* \times X^*$ defined by $D^*F(x, y) = \{(\zeta, \xi) \in Y^* \times X^* : (\xi, -\zeta) \in N_F(x, y)\}$ where $N_F(x, y) = \partial i_F(x, y)$ is the normal cone associated to the subdifferential operator ∂. We shall further denote by S_{Y^*} the unit sphere in Y^*. Given a metrix space Z, a subset $S \subset Z$ and $z_0 \in \operatorname{cl} S$, the notation $z \xrightarrow{S} z_0$ will mean z goes to z_0 in S.

4.2.1. Characterization using coderivatives

The following basic result is a mixing of the main results of [33, Section 6] and [47].

Theorem 4.2. *Let X, Y be Banach spaces, let ∂ be a subdifferential satisfying $(P1)$ and $(P2)$ on $X \times Y$ and let $F \subset X \times Y$ be a closed multifunction. Assume that*

$$\liminf_{(x,y) \xrightarrow{F} (x_0, y_0)} d_*(0, D^*F(x, y)(S_{Y^*})) > \tau > 0\,.$$

Then there exists a neighborhood W of (x_0, y_0) such that

$$\tau\, d(x, F^{-1}(y)) \leq d(y, F(x)) \text{ for all } (x, y) \in W. \tag{18}$$

Conversely, assuming that (18) holds true, then

$$\liminf_{(x,y) \xrightarrow{F} (x_0, y_0)} d_*(0, D^*F(x, y)(S_{Y^*})) \geq \tau,$$

whenever ∂ is the Fréchet subdifferential or the s-Hölder-smooth subdifferential or the Dini subdifferential where X and Y are finite dimensional.

Proof. Let us set as usual $f_z(x, y) = \|z-y\|+i_F(x, y) = g(x, y)+i_F(x, y)$ and let us endow $X \times Y$ with the norm $\|(x, y)\| = \max(\|x\|, \tau^{-1}\|y\|)$ whose dual norm is $\|(\xi, \zeta)\|_* = \|\xi\|_* + \tau\|\zeta\|_*$. We can find an open neighborhood W_0 of (x_0, y_0) such that $d_*(0, D^*F(x, y)(S_{Y^*})) > \tau$ for all $(x, y) \in W_0 \cap F$. Let $z \in Y$ and $(x, y) \in W_0 \cap F \cap [f_z{>}0]$ and let (x_1, y_1), $(x_2, y_2) \in W_0 \cap F$, $\xi_1 \in \partial g(x_1, y_1)$, $(\xi_2, \zeta_2) \in N_F(x_2, y_2)$. Assume that (x_1, y_1) is close enough to (x, y) in order that $\|z - y_1\| > 0$. We have $\xi_1 = (0, \zeta_1)$ with $\|\zeta_1\|_* = 1$ so that

$$\|(0, \zeta_1) + (\xi_2, \zeta_2)\|_* = \|\xi_2\|_* + \tau\|\zeta_1 + \zeta_2\|_* \geq \|\xi_2\|_* - \tau\|\zeta_2\|_* + \tau \geq \tau$$

since $\xi_2 \in D^*F(x_2, y_2)(-\zeta_2)$. Thus we derive from Proposition 2.6 that $|\nabla f_z|(x, y) \geq \tau$ for all $(x, y) \in W_0 \cap [f_z{>}0]$ and then the conclusion of the theorem follows from Theorem 3.2.

Assume now that (18) holds true. We derive from Theorem 3.2 that there exists $r > 0$ such that $|\nabla_\delta f_z|(x, y) \geq \tau$ for all $(x, y, z) \in B_r(x_0) \times B_r(y_0) \times B_r(y_0)$ such that $f_z(x, y) > 0$ with $\delta = 1/\tau$. For any of the three subdifferentials involved, we have $\partial g(x, y)+\partial i_F(x, y) \subset \partial f_z(x, y)$. Now let $(x, y) \in F \cap (B_r(x_0) \times B_r(y_0))$, let $\zeta \in S_{Y^*}$ and let $\xi \in D^*F(x, y)(\zeta)$, so that $(\xi, -\zeta) \in N_F(x, y)$. Given $\varepsilon > 0$, we can find by the Bishop-Phelps Theorem (see e.g. [19, Theorem 7.2]) $\chi \in S_{Y^*}$ and $v \in S_Y = \{w \in Y : \|w\| = 1\}$ such that $\|\zeta - \chi\|_* \leq \tau^{-1}\varepsilon$ and $\langle \chi, v\rangle = 1$. Let $t > 0$ small enough in order that $z = y - tv \in B_r(y_0)$. We have $(0, \chi) \in \partial g(x, y)$ so that $(\xi, \chi - \zeta) \in \partial f_z(x, y)$ and then, by Proposition 2.9,

$$\|(\xi, \chi - \zeta)\|_* = \|\xi\|_* + \tau\|\chi - \zeta\|_* \geq |\nabla_\delta f_z|(x, y) \geq \tau,$$

from which we get $\|\xi\|_* \geq \tau - \varepsilon$ and then $\|\xi\|_* \geq \tau$ by letting ε go to 0. $\square$

4.2.2. The Borwein-Zhu sufficient condition

Given a metric space X, a subset $A \subset X$ and a mapping h defined near $x_0 \in A$ with values in a metric space space Y, we say that (h, A) is metrically regular near x_0 whenever there exist $\tau > 0$ and neighborhoods V of x_0 and W of $y_0 = h(x_0)$ such that $\tau\, d(x, h^{-1}(y) \cap A) \leq d(y, h(x))$ for all $(x, y) \in (V \cap A) \times W$. This amounts to say that the multifunction $\{(x, h(x)) : x \in A\}$ is metrically regular near (x_0, y_0).

Lemma 4.3. *Let X be a Banach space, let $A \subset X$ be a closed set and let h be a continuous mapping defined near $x_0 \in A$ with values in a normed space Y. Then (h, A) is metrically regular near x_0 if and only if there exist neighborhoods V of x_0 and W of y_0 such that*

$$\inf_{\begin{cases} (x, y) \in (V \cap A) \times W \\ h(x) \neq y \end{cases}} |\nabla f_y|(x) > 0, \tag{19}$$

where $f_y(x) = \|h(x) - y\| + i_A(x)$.

Proof. Observing that $[f_y{\leq}0] = h^{-1}(y) \cap A$ and that $f(x, \cdot)$ is continuous for all $x \in X$, we derive from Theorem 2.13 that (h, A) is metrically regular near x_0. Conversely, assume that (h, A) is metrically regular near x_0, so that we can find $r > 0$, $\rho > 0$ such that

$\tau\, d(x, h^{-1}(y) \cap A) \le \|h(x) - y\|$ all $(x, y) \in (B_r(x_0) \cap A) \times B_{2\rho}(y_0)$. Let $V \subset B_r(x_0)$ and $W \subset B_{2\rho}(y_0)$ be such that $\|h(x)-y\| < \rho$ for all $(x, y) \in V \times W$ and let $(x, y) \in (V \cap A) \times W$ be such that $h(x) \ne y$. Setting $\gamma_n = f_y(x) - n^{-1}$, we get $\gamma_n > 0$, $f_y(x) > \gamma_n$ and $\tau - n^{-1} > 0$ eventually. Thus there exist $\theta_n \in (0, 1)$ and $z_n \in Y$ such that $\|z_n - y\| = \gamma_n$ and $z_n = y + \theta_n(h(x) - y)$, from which we get $z_n \in B_{2\rho}(y_0)$. As $h(x) \ne z_n$, we can find $x_n \in h^{-1}(z_n) \cap A$ such that

$$(\tau - n^{-1})\|x - x_n\| \le \|z_n - h(x)\| = (1 - \theta_n)\|h(x) - y\| = f_y(x) - f_y(x_n) = f_y(x) - \gamma_n.$$

It follows that $(x_n)_{n\in\mathbb{N}}$ converges to x and that

$$\tau \le \limsup_{n\to\infty} \frac{f_y(x) - f_y(x_n)}{\|x - x_n\|} \le |\nabla f_y|(x),$$

yielding the conclusion of the lemma. □

Theorem 4.4. *Let X, Y be Banach spaces with β-smooth norms, let $A \subset X$ be a closed set, and let h be defined and continuous near $x_0 \in A$. Assume that h is strictly differentiable at x_0 and that (h, A) is not metrically regular near x_0. Then*

$$\liminf_{x \xrightarrow{A} x_0} d_*(0, \Lambda^*(S_{Y^*}) + N_A^\beta(x)) = 0 \tag{20}$$

where $\Lambda = Dh(x_0)$ and where $S_{Y^*} = \{\zeta \in Y^* : \|\zeta\|_* = 1\}$ is the unit sphere in Y^*.

Proof. Let $\varepsilon > 0$, $\delta > 0$ and let $0 < \eta < \delta$ such that $h - \Lambda$ is ε-Lipschitzian on $B_{2\eta}(x_0)$. From Lemma 4.3, we can find $x \in B_\eta(x_0) \cap A \cap [f_y{>}0]$ and $y \in B_\eta(y_0)$ such that $|\nabla f_y|(x) < \varepsilon$, so that the function $f_y(\cdot) + \varepsilon\|y - \cdot\|$ admits a local minimum at x. Setting $g(z, y) = \|h(z) - y\|$ and applying [13, Theorem 2.9], we can find $x_1 \in B_\eta(x_0)$, $x_2 \in B_\eta(x_0) \cap A$ and $\xi_1 \in D^\beta g_y(x_1)$, $\xi_2 \in N_A^\beta(x_2)$ such that $\|h(x_1) - y\| > 0$ and $\|\xi_1 + \xi_2\|_* \le \varepsilon$. Let $u \in \bar{B}_X$ so that we can find $t_u > 0$ such that

$$t^{-1}(\|h(x_1 + tu) - y\| - \|h(x_1) - y\|) \ge \langle \xi_1, u\rangle - \varepsilon \ge -\langle \xi_2, u\rangle - 2\varepsilon \text{ for all } t \in (0, t_u).$$

Let us set $\zeta_t(u) = \nabla^\beta(\|\cdot\|)(h(x_1 + tu) - y)$ which is well defined for all $t > 0$ small enough. As $\|\zeta_t(u)\|_* = 1$, we have for all $t > 0$ small enough

$$\langle \zeta_t(u), t^{-1}(h(x_1 + tu) - y - (h(x_1) - y)) - \Lambda(u)\rangle \le \varepsilon,$$

hence $t^{-1}(\|(h(x_1 + tu) - y\| - (\|(h(x_1) - y\|) \le \langle \zeta_t(u), \Lambda(u)\rangle + \varepsilon$ for all $t \in (0, t_u)$. As $\zeta_t(u)$ weakly converges to $\zeta = \nabla^\beta(\|\cdot\|)(h(x_1) - y)$, we get $\langle \Lambda^*(\zeta) + \xi_2, u\rangle \ge -3\varepsilon$ for all $u \in \bar{B}_X$ yielding $\|\Lambda^*(\zeta) + \xi_2\|_* \le 3\varepsilon$ and then $d_*(0, \Lambda^*(S_{Y^*}) + N_A^\beta(x_2)) \le 3\varepsilon$, so that $\liminf_{x \xrightarrow{A} x_0} d_*(0, \Lambda^*(S_{Y^*}) + N_A^\beta(x)) = 0$. □

Remark 4.5. In [13, Theorem 4.3] the preceding result is obtained with (20) replaced by $\liminf_{x \xrightarrow{A} x_0} d_*(0, \Lambda^*(S_{Y^*}) + \tilde{N}_A^\beta(x)) = 0$ with $\tilde{N}_A^\beta(x) = \bigcup_{L>0} L\, D^\beta d_A(x)$. As $N_A^\beta(x) \subset \tilde{N}_A^\beta(x)$, Theorem 4.4 extends the quoted result.

4.2.3. Pointwise criteria

Given a subdifferential operator ∂ and a multifunction $F \subset X \times Y$, we say that F is ∂-codirectionally compact at $(x_0, y_0) \in F$ whenever for any sequence $((x_n, y_n))_{n\in\mathbb{N}} \subset F$ converging to (x_0, y_0) and for any sequence $((\xi_n, \zeta_n))_{n\in\mathbb{N}} \subset X^* \times Y^*$ with $\xi_n \in D^*F(x_n, y_n)(\zeta_n)$ such that $(\xi_n)_{n\in\mathbb{N}}$ converges to 0, then $(\zeta_n)_{n\in\mathbb{N}}$ converges to 0 whenever it $*$-weakly converges to 0. It is clear that any closed multifunction is ∂-codirectionally compact at each $(x_0, y_0) \in F$ for any subdifferential operator whenever Y is finite dimensional. The next theorem captures the main results of [46, 48, 53].

Theorem 4.6. *Let X, Y be Banach spaces and let $F \subset X \times Y$ be a closed multifunction which is $\hat{\partial}$-codirectionally compact at $(x_0, y_0) \in F$ where $\hat{\partial}$ is the Fréchet subdifferential.*

a) *Assume that X and Y are Asplund and that*

$$Ker\, D^*F(x_0, y_0) = \{0\},$$

where $D^*F(x_0, y_0)$ is the Mordukhovich (limiting Fréchet) coderivative and

$$Ker\, D^*F(x_0, y_0) = \{\zeta \in Y^* : 0 \in D^*F(x_0, y_0)(\zeta)\}.$$

Then F is metrically regular near (x_0, y_0).

b) *Conversely, if X is finite dimensional and if F is metrically regular near (x_0, y_0), then $Ker\, D^*F(x_0, y_0) = \{0\}$.*

Proof. *a)* We claim that $\liminf_{(x,y)\xrightarrow{F}(x_0,y_0)} d_*(0, \hat{D}^*F(x, y)(S_{Y^*})) > 0$ where $\hat{D}^*F(x, y)$ stands for the Fréchet coderivative, which will give the conclusion of the theorem by Theorem 4.2. Indeed, if not, we can find sequences $((x_n, y_n))_{n\in\mathbb{N}} \subset F$ converging to (x_0, y_0) and $((\xi_n, \zeta_n))_{n\in\mathbb{N}} \subset X^* \times Y^*$ with $(\zeta_n)_{n\in\mathbb{N}} \subset S_{Y^*}$, $\xi_n \in \hat{D}^*F(x_n, y_n)(\zeta_n)$ such that $(\xi_n)_{n\in\mathbb{N}}$ converges to 0. Denoting also by $(\zeta_n)_{n\in\mathbb{N}}$ a subsequence that $*$-weakly converges to some $\zeta \in Y^*$, we get $\zeta \in \operatorname{Ker} D^*F(x_0, y_0)$ thus $\zeta = 0$. It follows that $(\zeta_n)_{n\in\mathbb{N}}$ converges to 0 contradicting the fact that $\|\zeta_n\|_* \equiv 1$.

b) Conversely, assume that $\operatorname{Ker} D^*F(x_0, y_0) \neq \{0\}$, so that there exists $\zeta \in S_{Y^*}$ such that $(0, \zeta) \in N_F(x_0, y_0)$. Thus we can find sequences $((x_n, y_n))_{n\in\mathbb{N}} \subset F$ converging to (x_0, y_0) and $((\xi_n, \zeta_n))_{n\in\mathbb{N}} \subset X^* \times Y^*$ such that $(\xi_n)_{n\in\mathbb{N}}$ converges to 0 and $(\zeta_n)_{n\in\mathbb{N}}$ $*$- weakly converges to ζ. Considering a subsequence still denoted by $((\xi_n, \zeta_n))_{n\in\mathbb{N}}$ such that $(\|\zeta_n\|_*)_{n\in\mathbb{N}}$ converges to some $c > 0$, we derive that $(\hat{\xi}_n, \hat{\zeta}_n) \in \hat{N}_F(x_n, y_n)$ where $\hat{\xi}_n = \|\zeta_n\|^{-1}\xi_n$ and $\hat{\zeta}_n = \|\zeta_n\|^{-1}\zeta_n \in S_{Y^*}$. As $(\hat{\xi}_n)_{\in\mathbb{N}}$ converges to 0, we get $\liminf_{(x,y)\xrightarrow{F}(x_0,y_0)} d_*(0, \hat{D}^*F(x, y)$ $(S_{Y^*})) = 0$ thus F is not metrically regular near (x_0, y_0) by applying again Theorem 4.2. $\square$

As a consequence, we derive the nice characterization given by Mordukhovich in [46].

Theorem 4.7. *Let X, Y be finite dimensional spaces and let $F \subset X \times Y$ be a closed multifunction. Then F is metrically regular near $(x_0, y_0) \in F$ if and only if*

$$Ker\, D^*F(x_0, y_0) = \{0\},$$

where $D^*F(x_0, y_0)$ is the Mordukhovich (limiting Fréchet) coderivative.

Proof. Follows immediately from Theorem 4.6 since finite dimensional spaces are Asplund and each closed multifunction $F \subset X \times Y$ is ∂-codirectionally compact at each $(x_0, y_0) \in F$ for any subdifferential operator. $\square$

4.3. Tangencial conditions

Let us list the definitions of the tangent cones and directional derivatives we shall use in this section. Given a subset $C \subset X$ of a normed space and $\varepsilon > 0$, the ε-contingent cone to C at $x_0 \in \mathrm{cl}\,(C)$ is the set $T_C^\varepsilon(x_0)$ of those $u \in X$ such that there exist sequences $(u_n)_{n\in\mathbb{N}} \subset \bar{B}(u, \varepsilon\|u\|)$ and $(t_n)_{n\in\mathbb{N}} \subset (0,+\infty)$ converging to 0 such that $x_0 + t_n u_n \in C$ for all $n \in \mathbb{N}$. We also need a notion of ε-directional derivative namely, given $f : X \to \mathbb{R} \cup \{+\infty\}$, $\varepsilon > 0$, $x_0 \in \mathrm{dom}\, f$ and $u \in X$:

$$f'_\varepsilon(x_0; u) = \liminf_{t\downarrow 0} \Big(\inf_{v\in \bar{B}(u,\varepsilon\|u\|)} \frac{f(x_0+tv) - f(x_0)}{t} \Big).$$

Observe that these two notions depend on the norm and that $(i_C)'_\varepsilon(x_0;\cdot) = i_{T_C^\varepsilon(x_0)}$. Then, it is clear that

$$(1+\varepsilon)\|u\||\nabla f|(x_0) \geq -f'_\varepsilon(x_0; u). \tag{21}$$

Moreover, it is also clear that

$$f'_\varepsilon(x_0; u) \leq g'(x_0; u) + K\varepsilon\|u\| + h'_\varepsilon(x_0; u) \tag{22}$$

whenever $f = g + h$ with $g : X \to \mathbb{R}$ convex and Lipschitz continuous of rank K near x_0.

The contingent cone (or Bouligand cone, see [16, p. 32]) to C at $x_0 \in \mathrm{cl}\,(C)$ is the set $T_C(x_0)$ of $u \in X$ for which there exist sequences $(u_n)_{n\in\mathbb{N}}$ converging to u and $(t_n)_{n\in\mathbb{N}} \subset (0,+\infty)$ converging to 0 such that $x_0 + t_n u_n \in C$ for all $n \in \mathbb{N}$. In other words $T_C(x_0) = \bigcap_{\varepsilon>0} T_C^\varepsilon(x_0)$ and we have $i'_C(x_0;\cdot) = i_{T_C(x_0)}$ where $f'(x;u) := \liminf\limits_{\substack{t\searrow 0\\ v\to u}} \dfrac{f(x+tv)-f(x)}{t}$.

Given a subset $C \subset X$ of a normed space, the Clarke tangent cone to C at $x_0 \in \mathrm{cl}\,(C)$ is the closed convex cone $T_C^\uparrow(x_0)$ defined by

$$T_C^\uparrow(x_0) = \{u \in X : \lim_{\begin{cases} x \xrightarrow{C} x_0 \\ t \downarrow 0 \end{cases}} t^{-1} d(x+tu, C) = 0\}.$$

4.3.1. The Borwein-Aubin-Frankowska Theorem

The next theorem which is [3, Theorem 2.3] is also a consequence of [11, Theorem 3.1].

Theorem 4.8. *Let X, Y be Banach spaces such that Y is finite dimensional, let $F \subset X \times Y$ be a closed multifunction, and let $(x_0, y_0) \in F$. Assume that $T_F^\uparrow(x_0,y_0)(X) = Y$. Then F is metrically regular near (x_0, y_0).*

Proof. Let us endow Y with a norm such that $\bar{B}_Y = \mathrm{co}\,(v_1, \cdots, v_N)$ for some $N \in \mathbb{N}$ and some $v_1, \cdots, v_N \in Y$. From our assumption, we can find $\tau > 0$ and $u_1, \cdots, u_N \in X$ such that $(u_i, v_i) \in T_F^\uparrow(x_0, y_0)$ and $\|u_i\| \leq \tau^{-1}\|v_i\|$, $i = 1, \cdots, N$. Let us endow $X \times Y$ with the norm $\|(x,y)\| = \max(\|x\|, \tau^{-1}\|y\|)$ and let us set

$$K = \mathrm{pos}\,((u_1,v_1), \cdots, (u_N,v_N)) = \sum_{i=1}^N \mathbb{R}_+(u_i, v_i)$$

so that K is a closed convex cone contained in $T_F^{\uparrow}(x_0, y_0)$ and $\tau \bar{B}_Y \subset K(\bar{B}_X)$. Given $\varepsilon \in (0,1)$, we can find, by Ascoli's Theorem and by the definition of the Clarke tangent cone, a neighborhood W of (x_0, y_0) and $\eta > 0$ such that $d((u,v), t^{-1}(F-(x,y)) < \varepsilon$ for all $(u,v) \in K \cap \bar{B}_{X\times Y}$, for all $(x,y) \in F \cap W$ and for all $t \in (0,\eta)$. For $z \in Y$, let us define a lower semicontinuous function $f_z : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$f_z(x,y) = \|z-y\| + i_F(x,y),$$

and let $z \in Y$ and let $(x,y) \in F \cap W$ be such that $f_z(x,y) > 0$. Setting $v = \tau \dfrac{z-y}{\|y-z\|}$, we can find $u \in \bar{B}_X$ such that $(u,v) \in K \cap \bar{B}_{X\times Y}$. Thus there exists, for all $t \in (0,\eta)$ a vector $(u_t, v_t) \in X \times Y$ such that $(x,y) + t(u_t, v_t) \in F$ and $\|(u_t,v_t) - (u,v)\| < \varepsilon$, so that

$$f_z((x,y) + t(u_t,v_t)) - f_z(x,y) \le \|y - z + tv\| - \|y-z\| + t\tau\varepsilon \le -t\tau + t\tau\varepsilon,$$

yielding $(1+\varepsilon)|\nabla f_z|(x,y) \ge \tau(1-\varepsilon)$. Thus the conclusion of the theorem follows from Theorem 3.2 applied with $Z = Y$. $\square$

In fact the Borwein-Aubin-Frankowska Theorem can be deduced from a more general result exposed in the next subsection.

4.3.2. Equi-circatangency

The results of this subsection are taken from [6].

Definition 4.9. Let X be a Banach space and let $C \subset X$ be a subset of X. We say that a cone K is equi-circatangent to C at $x_0 \in \operatorname{cl}(C)$ whenever

$$\lim_{\begin{cases} x \xrightarrow{C} x_0 \\ t \downarrow 0 \end{cases}} e\Big(K \cap \bar{B}_X, \frac{C-x}{t}\Big) = 0.$$

Thus if K is equi-circatangent to C at x_0 we have $K \subset T_C^{\uparrow}(x_0)$.

Theorem 4.10. *Let X, Y be Banach spaces, let $F \subset X \times Y$ be a closed multifunction, and let $(x_0, y_0) \in F$. Assume that there exists a cone $K \subset X \times Y$ such that K is equi-circatangent to F at (x_0,y_0) and there exists $\tau > 0$ such that $\tau \bar{B}_Y \subset K(\bar{B}_X)$. Then for all $\sigma \in (0,\tau)$, there exists a neighborhood W of (x_0,y_0) such that*

$$\sigma\, d(x, F^{-1}(y)) \le d(y, F(x)) \text{ for all } (x,y) \in W.$$

Proof. For $z \in Y$, let us define a lower semicontinuous function $f_z : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$f_z(x,y) = \|z-y\| + i_F(x,y).$$

Let us endow $X \times Y$ with the norm $\|(x,y)\| = \max(\|x\|, \tau^{-1}\|y\|)$ and let $\varepsilon \in (0,1)$ be such that $\sigma \le \tau \dfrac{1-\varepsilon}{1+\varepsilon}$. We can find a neighborhood W of (x_0,y_0) and $\eta > 0$ such that $d((u,v), t^{-1}(F-(x,y))) < \varepsilon$ for all $(u,v) \in K \cap \bar{B}_{X\times Y}$, $(x,y) \in F \cap W$ and $t \in (0,\eta)$. Now let $z \in Y$ and let $(x,y) \in F \cap W$ be such that $f_z(x,y) > 0$. Setting $v = \tau\|y-z\|^{-1}(z-y)$, we can find $u \in \bar{B}_X$ such that $(u, v) \in K$. As $(u, v) \in \bar{B}_{X\times Y}$, there exists, for all $t \in (0, \eta)$ a vector $(u_t, v_t) \in X \times Y$ such that $(x, y) + t(u_t, v_t) \in F$ and $\|(u_t, v_t) - (u, v)\| < \varepsilon$, so that

$$f_z((x, y) + t(u_t, v_t)) - f_z(x, y) \le \|y - z + tv\| - \|y - z\| + t\tau\varepsilon \le -t\tau + t\tau\varepsilon,$$

yielding $(1 + \varepsilon)|\nabla f_z|(x, y) \ge \tau(1 - \varepsilon)$ and then $|\nabla f_z|(x, y) \ge \sigma$. Thus the conclusion of the theorem follows from Theorem 3.2 applied with $Z = Y$. □

Remark 4.11. a) By the Ursescu-Robinson theorem, the assumptions of Theorem 4.10 are fulfilled whenever the equi-circatangent cone K is closed and convex and satisfies $K(X) = Y$.

b) In [5], Aubin and Frankowska define a closed set C to be uniformly sleek at $x_0 \in A$ if $\lim_{x \xrightarrow{C} x_0} e(T_C(x_0) \cap \bar{B}_X, T_C(x)) = 0$. In that case the contingent and the Clarke tangent cone at x_0 do coincide. One easily checks (see [6, Proposition 2.4]) that, assuming C to be uniformly sleek at x_0, the Clarke tangent cone $T_C^{\uparrow}(x_0)$ is then equi-circatangent to C at x_0. Nevertheless, there exist (see [6]) closed sets $C \ni x_0$ whose Clarke tangent cone is equi-circatangent to C at x_0 which are not uniformly sleek at x_0. It follows that Theorem 4.10 extends the results of [5] based on uniform sleekness.

4.3.3. ε-Contingent cone

The next theorem is a slight improvement of [7, Theorem 3.2].

Theorem 4.12. *Let X, Y be Banach spaces, let $F \subset X \times Y$ be a closed multifunction, and let $(x_0, y_0) \in F$. Assume that there exist $\varepsilon > 0$, $\tau > 0$, a neighborhood W_0 of (x_0, y_0) and $\gamma \ge 0$ such that $\tau - \gamma - \varepsilon(\tau + \gamma) > 0$ and*

$$\tau \bar{B}_Y \subset T_F^{\varepsilon}(x, y)(\bar{B}_X) + \gamma \bar{B}_Y \text{ for all } (x, y) \in W_0 \cap F. \tag{23}$$

Then there exists a neighborhood W of (x_0, y_0) such that

$$\sigma\, d(x, F^{-1}(y)) \le d(y, F(x)) \text{ for all } (x, y) \in W,$$

where $\sigma = \dfrac{\tau - \gamma - \varepsilon(\tau + \gamma)}{(1 + \varepsilon)(1 + \tau^{-1}\gamma)} > 0.$

Proof. Let us endow $X \times Y$ with the norm $\|(x, y)\| = \max(\|u\|, \tau^{-1}\|v\|)$, and let us define, for $z \in Y$ a lower semicontinuous function $f_z : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ by $f_z(x, y) = \|z - y\| + i_F(x, y)$. Now let $z \in Y$ and let $(x, y) \in W_0 \cap F$ such that $f_z(x, y) > 0$. Setting $v = \tau\|y - z\|^{-1}(z - y)$, we can find $(u, w) \in T_F^{\varepsilon}(x, y)$ such that $\|u\| \le 1$ and $\|w - v\| \le \gamma$. Relying on (22) and observing that $(x, y) \longmapsto \|z - y\|$ is τ-Lipschitzian, we get

$$\begin{aligned}(f_z)'_{\varepsilon}((x, y), (u, w)) &\le \|\cdot\|'(y - z; w) + \tau\varepsilon(\|(u, w)\| + (i_F)'_{\varepsilon}((x, y), (u, w)) \\ &\le \|\cdot\|'(y; v) + \gamma + \tau\varepsilon(1 + \tau^{-1}\gamma),\end{aligned}$$

thus $(f_z)'_{\varepsilon}((x, y), (u, w)) \le -\tau + \gamma + \varepsilon(\tau + \gamma)$, from which we get by (21) that $|\nabla f_z|(x, y) \ge \sigma$. Thus the conclusion of the theorem follows from Theorem 3.2 applied with $Z = Y$. □

A very slight modification of the proof of Theorem 4.12 yields [2, Theorem 2]:

Theorem 4.13. *Let X, Y be Banach spaces, let $F \subset X \times Y$ be a closed multifunction, and let $(x_0, y_0) \in F$. Assume that there exist $\tau > 0$, a neighborhood W_0 of (x_0, y_0) and $\gamma \in [0, \tau)$ such that*

$$\tau \bar{B}_Y \subset T_F(x,y)(\bar{B}_X) + \gamma \bar{B}_Y \text{ for all } (x,y) \in W_0 \cap F. \tag{24}$$

Then there exists a neighborhood W of (x_0, y_0) such that

$$\frac{\tau - \gamma}{1 + \tau^{-1}\gamma} d(x, F^{-1}(y)) \leq d(y, F(x)) \text{ for all } (x,y) \in W.$$

Proof. For all $\varepsilon \in \left(0, \dfrac{\tau - \gamma}{\tau + \gamma}\right)$, and for all $z \in Y$ we have (23) with W_0 independent of ε. For all $(x,y) \in W_0 \cap [f_z{>}0]$, we know from the proof of Theorem 4.12 that $|\nabla f_z|(x,y) \geq \dfrac{\tau - \gamma - \varepsilon(\tau + \gamma)}{(1+\varepsilon)(1 + \tau^{-1}\gamma)}$. Letting ε decrease to 0, we derive that $|\nabla f_z|(x,y) \geq \dfrac{\tau - \gamma}{1 + \tau^{-1}\gamma}$, hence the conclusion of the theorem follows by applying Theorem 3.2 with $Z = Y$. □

Remark 4.14. a) One can have $\gamma = 0$ in (24). It follows that if, for some $\tau > 0$ and some neighborhood W_0 of (x_0, y_0) we have

$$\tau \bar{B}_Y \subset T_F(x,y)(\bar{B}_X) \text{ for all } (x,y) \in W_0 \cap F. \tag{25}$$

Then there exists a neighborhood W of (x_0, y_0) such that $\tau\, d(x, F^{-1}(y)) \leq d(y, F(x))$ for all $(x,y) \in W$. Conversely, assuming that F is metrically regular near (x_0, y_0), then there exists $\tau > 0$, $\eta > 0$ and neighborhoods U of x_0 and V of y_0 such that for all $(x,y) \in (U \times V) \cap F$, $v \in \bar{B}_Y$ and for all $t \in (0, \eta)$, we have $y + \tau t v \in F(x + t u_t)$ for some $u_t \in \bar{B}_X$, from which we get

$$\tau \bar{B}_Y \subset \bigcap_{t \in (0,\eta)} t^{-1}(F(x + t\bar{B}_X) - y) \text{ for all } (x,y) \in (U \times V) \cap F. \tag{26}$$

Then for all $(x,y) \in (U \times V) \cap F$, $v \in \tau \bar{B}$ and for $t > 0$ small enough, there exists $u_t \in \bar{B}$ such that $(x + t u_t, y + t v) \in F$. It follows that assuming Y to be finite dimensional, there exists $(u, v) \in T_F(x,y)$ with $\|u\| \leq 1$, thus condition (25) is in fact a characterization of metric regularity of F near (x_0, y_0).

b) Assume that assumption (25) is replaced by the weaker one: there exists some $\tau > 0$ and some neighborhood W_0 of (x_0, y_0) such that

$$\tau \bar{B}_Y \subset (\overline{\text{co}}\, T_F(x,y))(\bar{B}_X) \text{ for all } (x,y) \in W_0 \cap F. \tag{27}$$

Then, it is shown in [23] that F is metrically regular near (x_0, y_0) whenever X and Y are finite dimensional. In fact this conclusion still holds under the weaker condition that X and Y are Asplund spaces. Indeed it is easily shown (see e.g. [53, Lemma 3.1]) that (27) leads to $\|\xi\|_* \geq \tau \|\zeta\|_*$ for all $(\xi, \zeta) \in T_F(x,y)^-$ and for all $(x,y) \in W_0$ where $T_F(x,y)^-$ denotes the negative polar cone of $T_F(x,y)$. As the Fréchet normal cone $\hat{N}_F(x,y)$ is contained in $T_F(x,y)^-$, it then follows from Theorem 4.2 that, assuming X and Y to be Asplund spaces, then the multifunction F is metrically regular near (x_0, y_0).

c) Assume that there exist $\tau > 0$ and a neighborhood W of (x_0, y_0) such that

$$\tau \bar{B}_Y \subset \operatorname*{Lim\,sup}_{t \downarrow 0} \frac{F(x + t\bar{B}_X) - y}{t} \text{ for all } (x, y) \in W \cap F, \tag{28}$$

where $\operatorname{Lim\,sup}_{t\downarrow 0} S_t = \bigcap_{\eta>0} \operatorname{cl}\left(\bigcup_{t\in(0,\eta)} S_t\right)$. Let us endow $X \times Y$ with the norm $\|(u, v)\| = \max(\|u\|, \tau^{-1}\|v\|)$ and let us set, for any $z \in Y$, $f_z(x, y) = \|z - y\| + i_F(x, y)$. For any $(x, y) \in [f_z{>}0] \cap W$, we can find sequences $(t_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ converging to 0, $(u_n)_{n\in\mathbb{N}} \subset \bar{B}_X$ and $(v_n)_{n\in\mathbb{N}} \subset Y$ converging to $v = \tau \dfrac{z - y}{\|z - y\|}$ such that $(x + t_n u_n, y + t_n v_n) \in F$ for all $n \in \mathbb{N}$. It then follows, for n large enough (so that $1 - t_n\tau\|z - y\|^{-1} \geq 0$)

$$f_z(x + t_n u_n, y + t_n v_n) \leq \|y + t_n v - z\| + t_n\|v_n - v\| \leq f_z(x, y) - t_n\tau + t_n\|v_n - v\|$$

thus, for n large enough in order that $v_n \neq 0$,

$$\max(1, \tau^{-1}\|v_n\|) \frac{f_z(x, y) - f_z((x, y) + t_n(u_n, v_n))}{t_n\|(u_n, v_n)\|} \geq \tau - \|v_n - v\|$$

yielding $|\nabla f_z|(x, y) \geq \tau$. Thus we get from Theorem 3.2 applied with $Z = Y$ that F is metrically regular near (x_0, y_0), this is [27, Theorem 6.1]. In fact (26) shows that (28) is a characterization of metric regularity.

d) In fact Theorem 4.10 follows from Theorem 4.12. Indeed, we know from [6, Proposition 2.3] that a cone K is equi-circatangent to a set $C \subset X$, X Banach, at $x_0 \in \operatorname{cl} C$ if and only if, for all $\varepsilon > 0$, there exists a neighborhood V of x_0 such that $K \subset \bigcap_{x\in V\cap C} T_C^{\varepsilon}(x)$.

4.3.4. The Ursescu Theorem

Theorem 4.15. *Let X, Y be Banach spaces and let $F \subset X \times Y$ be a closed multifunction. Assume that there exists an open neighborhood W of $(x_0, y_0) \in F$ and $\tau > 0$ such that*

$$\tau B_Y \cap T_F(x, y)(X) \subset cl\left(T_F(x, y)(B_X)\right) \textit{ for all } (x, y) \in W \cap F.$$

Then there exists neighborhoods U of x_0 and V of y_0 such that

$$\tau\, d(x, F^{-1}(y)) \leq d(y, F(x)) \textit{ for all } (x, y) \in U \times (V \cap Z).$$

where $Z = \displaystyle\bigcap_{(x,y)\in W\cap F} (y + \operatorname{cl}(T_F(x, y)(X)))$.

Proof. Let us endow $X \times Y$ with the norm $\|(x, y)\| = \max(\|x\|, \tau^{-1}\|y\|)$ and let us consider, for any $z \in Z$, the function $f_z : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by $f_z(x, y) = \|z - y\| + i_F(x, y)$. Let $(x, y) \in F \cap W$ and $z \in Z$ be such that $(x, y) \in [f_z{>}0]$. Given $\varepsilon \in (0, \tau)$ and setting $v = (\tau - \varepsilon)\|y - z\|^{-1}(z - y)$, we can find $w' \in T_F(x, y)(X)$ such that $\|w' - v\| \leq \varepsilon$ and $u \in \bar{B}_X$, $w \in Y$ such that $(u, w) \in T_F(x, y)$ and $\|w' - w\| \leq \varepsilon$. Given sequences $(t_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ converging to 0 and $((u_n, w_n))_{n\in\mathbb{N}} \subset X \times Y$ converging to (u, w) such that $(x, y) + t_n(u_n, w_n) \in F$, we have

$$\begin{aligned} \frac{f_z(x, y) - f_z(x + t_n u_n, y + t_n w_n)}{t_n} &\geq \frac{\|y - z\| - \|y + t_n v - z\|}{t_n} - \|w_n - w\| - 2\varepsilon \\ &\geq \tau - 3\varepsilon, \end{aligned}$$

yielding $\|(u,w)\| |\nabla f_z|(x,y) \geq \tau - 3\varepsilon$ and then, by letting ε go to 0, $|\nabla f_z|(x,y) \geq \tau$. Thus the conclusion of the theorem follows from Theorem 3.2. $\square$

Remark 4.16. In [58, Theorem 2] Ursescu proves a partial converse to Theorem 4.15. Namely, assuming that X is finite dimensional and that, for some subset $Z \subset Y$, there exist neighborhoods U of x_0 and V of y_0 such that

$$\tau\, d(x, F^{-1}(y)) \leq d(y, F(x)) \text{ for all } (x,y) \in U \times (V \cap Z),$$

then it follows from the quoted theorem that, for any $\sigma \in (0, \tau)$,

$$\sigma B_Y \cap T_Z(y) \subset T_F(x,y)(B_X) \text{ for all } (x,y) \in F \cap (U \times V).$$

In particular, if X is finite dimensional and if F is metrically regular near (x_0, y_0) (which corresponds to the case $Z = Y$), then $\sigma B_Y \subset T_F(x,y)(B_X)$ for all $\sigma \in (0, \tau)$ and for all $(x,y) \in F \cap (U \times V)$.

4.4. Parametric results

4.4.1. Tangencial results

Given Banach spaces X, Y and a metric space P, we say that a mapping $h(\cdot,\cdot)$ defined in a neighborhood of $(x_0, p_0) \in X \times P$ with values in Y is partially strictly differentiable in x at (x_0, p_0) whenever there exists a linear continuous mapping $\phi \in L(X,Y)$ such that for all $\varepsilon > 0$, there exist neighborhoods V_0 of x_0 and U_0 of p_0 such that

$$\|h(x,p) - h(z,p) - \phi(x-z)\| \leq \varepsilon \|x - z\| \text{ for all } (x,z,p) \in V_0 \times V_0 \times U_0.$$

Observe that we then have $\phi = Dh_{p_0}(x_0)$ where $h_p(\cdot) = h(\cdot, p)$. The following theorem is [11, Theorem 4.1].

Theorem 4.17. *Let X, Y be Banach spaces, let P be a topological space, let $A \subset X$ be a closed convex set, and let h be defined and continuous near $(x_0, p_0) \in A \times P$ with values in Y. Assume that h is partially strictly differentiable in x at (x_0, p_0) and that*

$$0 \in \operatorname{core}\big(Dh_{p_0}(x_0)(A - x_0)\big). \tag{29}$$

Then there exist $\tau > 0$ and neighborhoods U of x_0, V of $y_0 = h(x_0, p_0)$ and N of p_0 such that

$$\tau\, d(x, h_p^{-1}(y) \cap A) \leq \|h_p(x) - y\| \textit{ for all } (x,y,p) \in (A \cap U) \times V \times N.$$

Proof. From assumption (29), we can find $\tau > 0$ and an open neighborhood $\tilde{U}_0$ of x_0 such that $\tau \bar{B}_Y \subset \operatorname{cl}\big(\phi((A - x) \cap \bar{B}_X)\big)$ for all $x \in \tilde{U}_0$ where $\phi = Dh_{p_0}(x_0)$. Given $\varepsilon \in (0, \tau/2)$, we can find open neighborhoods $\hat{U}_0$ of x_0 and N_0 of p_0 such that

$$\|h(x,p) - h(z,p) - \phi(x-z)\| \leq \varepsilon \|x - z\| \text{ for all } (x,z,p) \in \hat{U}_0 \times \hat{U}_0 \times N_0.$$

Let us set $W_0 = \tilde{U}_0 \cap \hat{U}_0$ and $f_{p,y}(x) = \|h(x,p) - y\| + i_A(x)$, so that $[f_{p,y} \leq 0] = h_p^{-1}(y) \cap A$. Let $(p,y) \in N_0 \times Y$ and $x \in [f_{p,y} > 0] \cap A \cap W_0$. Setting $\hat{w} = \tau \dfrac{y - h(x,p)}{\|h(x,p) - y\|}$, we can find

$w \in Y$ and $v \in \bar{B}_X$ such that $\|w - \hat{w}\| \le \varepsilon$, $w = \phi(v)$ and $v \in (A - x) \cap \bar{B}_X$. We then have, for all $t > 0$ small enough

$$\|h(x + tv, p) - y\| \le \|h(x, p) - y + t\hat{w}\| + 2t\varepsilon \le \|h(x, p) - y\| - t\tau + 2t\varepsilon.$$

Thus we have $|\nabla f_{p,y}|(x) \ge \tau - 2\varepsilon$ for all $(p, y) \in N_0 \times Y$ and $x \in [f_{p,y} > 0] \cap W_0$ and then the conclusion of the theorem follows from Theorem 2.13 applied with the parameter space $P \times Y$, observing that the function $(p, y) \mapsto f_{p,y}(x_0)$ is continuous. □

A multifunction $F \subset P \times X$ between topological spaces is said to be lower semicontinuous at $(p_0, x_0) \in F$ whenever $x_0 \in \operatorname{Lim\,inf}_{p \to p_0} F(p)$. This is equivalent to the fact that the function $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x, p) = i_F(p, x) = i_{F(p)}(x)$ is epi-upper semicontinuous at (x_0, p_0).

In [9, Corollary 5.5], one can find a parametric metric regularity result involving a condition based on the ε-contingent derivative of the multifunctions.

Theorem 4.18. *Let X, Y be Banach spaces, P a metric space, and let*

$$F \subset P \times (X \times Y)$$

be a closed-valued multifunction which is lower semicontinuous at $(p_0, (x_0, y_0)) \in F$. Assume that there exist $\varepsilon > 0$, $\tau > 0$, neighborhoods W_0 of (x_0, y_0), U_0 of p_0 and $\gamma \ge 0$ such that $\tau - \gamma - \varepsilon(\tau + \gamma) > 0$ and

$$\tau \bar{B}_Y \subset T^{\varepsilon}_{F_p}(x, y)(\bar{B}_X) + \gamma \bar{B}_Y \text{ for all } (x, y) \in W_0 \cap F \text{ and for all } p \in U_0. \tag{30}$$

Then there exists neighborhoods U of x_0, W of y_0, N of p_0 such that $F_p^{-1}(z) \ne \emptyset$ for all $(p, z) \in N \times W$ and such that

$$\sigma\, d(x, F_p^{-1}(z)) \le d(z, F_p(x)) \text{ for all } (x, z, p) \in U \times W \times N,$$

where $\sigma = \dfrac{\tau - \gamma - \varepsilon(\tau + \gamma)}{(1 + \varepsilon)(1 + \tau^{-1}\gamma)}$.

Proof. For $(p, z) \in P \times Y$, let us define $f_{p,z} : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ by $f_{p,z}(x, y) = \|z - y\| + i_{F_p}(x, y)$. Observe that the functions $f_{p,z}$ are lower semicontinuous, that $f_{p_0,y_0}(x_0, y_0) = 0$ and that the function $f(x, y), (p, z)) = f_{p,z}(x, y)$ is epi-upper semicontinuous at $((x_0, y_0), (p_0, y_0))$ due to the lower semicontinuity of F at $(p_0, (x_0, y_0))$ and to the Lipschitz continuity of the norm. Let us endow $X \times Y$ with the norm $\|(u, v)\| = \sup(\|u\|, \tau^{-1}\|v\|)$ and let $(p, z) \in U_0 \times Y$ and $(x, y) \in W_0 \cap F_p$ be such that $f_{p,z}(x, y) > 0$. From the proof of Theorem 4.12, we obtain that $|\nabla f_{p,z}|(x, y) \ge \sigma$. Applying Theorem 2.13 yields the existence of neighborhoods $\hat{N}$ of p_0, $\hat{W}$, $\hat{V}$ of y_0 and $\hat{U}$ of x_0 such that $[f_{p,z} \le 0] \ne \emptyset$ for all $(p, z) \in \hat{N} \times \hat{W}$ and such that $\sigma\, d((x, y), [f_{p,z} \le 0]) \le f_{p,z}(x, y)$ for all $((x, y), (p, z)) \in \hat{U} \times \hat{V} \times \hat{N} \times \hat{W}$ from which we get

$$\sigma\, d(x, F_p^{-1}(z)) \le \|z - y\| \text{ for all } (p, (x, y)) \in F \cap (\hat{N} \times \hat{U} \times \hat{V}) \text{ and } z \in \hat{W}. \tag{31}$$

Assume now that the conclusion of the theorem fails, so that there exists a sequence $((x_n, z_n, p_n))_{n \in \mathbb{N}}$ converging to (x_0, y_0, p_0) and $y_n \in F_{p_n}(x_n)$ such that $d(z_n, y_n) < \sigma\, d(x_n,$

$F_{p_n}^{-1}(z_n))$. Considering a mapping $p \mapsto (x_p, y_p)$ such that $(x_p, y_p) \in F_p$ and $\lim_{p\to p_0}(x_p, y_p) = (x_0, y_0)$, we have

$$d(y_n, y_{p_n}) - d(z_n, y_{p_n}) < \sigma d(x_n, x_{p_n}) + \sigma\, d(x_{p_n}, F_{p_n}^{-1}(z_n)) \le \sigma d(x_n, x_{p_n}) + \sigma\, d(z_n, y_{p_n}),$$

so that $(y_n)_{n\in\mathbb{N}}$ converges to y_0 contradicting (31). □

4.4.2. Normal conditions

The next result is [9, Corollary 5.7].

Theorem 4.19. *Let X, Y be Banach spaces, let P be a topological space, let ∂ be a subdifferential satisfying $(P1)$ and $(P2)$ on $X \times Y$ and let $F \subset P \times (X \times Y)$ be a closed valued multifunction which is lower semicontinuous at some $(p_0, (x_0, y_0)) \in F$. Assume that*

$$\liminf_{(p,x,y)\xrightarrow{F}(p_0,x_0,y_0)} d_*(0, D^*F_p(x,y)(S_{Y^*})) > \tau > 0\,.$$

Then there exist neighborhoods W of (x_0, y_0) and N of p_0 such that

$$\tau\, d(x, F_p^{-1}(y)) \le d(y, F_p(x)) \text{ for all } ((x,y),p) \in W \times N.$$

Proof. Let us introduce the closed proper function $f : (X\times Y)\times(P\times Y) \longrightarrow \mathbb{R}\cup\{+\infty\}$ defined by $f((x,y),(p,z)) = f_{(p,z)}(x,y) = \|z-y\| + i_{F_p}(x,y) = g(x,y) + i_{F_p}(x,y)$ and let us endow $X \times Y$ with the norm $\|(x,y)\| = \max(\|x\|, \tau^{-1}\|y\|)$ whose dual norm is $\|(\xi,\zeta)\|_* = \|\xi\|_* + \tau\|\zeta\|_*$. We can find open neighborhoods U_0 of x_0, V_0 of y_0 and N_0 of p_0 such that $d_*(0, D^*F_p(x,y)(S_{Y^*})) > \tau$ for all $(p,x,y) \in (N_0 \times U_0 \times V_0) \cap F$. Let $p \in N_0$, $z \in Y$ and $(x,y) \in (U_0 \times V_0) \cap [f_{p,z}{>}0] \cap F_p$. Let $(p,z) \in N_0 \times Y$ and let (x_1, y_1), $(x_2, y_2) \in (U_0 \times V_0) \cap F_p$, $\xi_1 \in \partial g(x_1, y_1)$, $(\xi_2, \zeta_2) \in N_{F_p}(x_2, y_2)$. Assume that (x_1, y_1) is close to (x, y) in order that $\|z - y_1\| > 0$. We have $\xi_1 = (0, \zeta_1)$ with $\|\zeta_1\|_* = 1$ so that

$$\|(0,\zeta_1) + (\xi_2,\zeta_2)\|_* = \|\xi_2\|_* + \tau\|\zeta_1 + \zeta_2\|_* \ge \|\xi_2\|_* - \tau\|\zeta\|_2 + \tau \ge \tau$$

since $\|\xi_2\|_* \ge \tau\|\zeta\|_2$. Thus we derive from Proposition 2.6 that $|\nabla f_{p,z}(x,y) \ge \tau$ for all $(p,z) \in N_0 \times Y$ and for all $(x,y) \in [f_{p,z}{>}0] \cap (U_0 \times V_0)$. As F is lower semicontinuous at $(p_0, (x_0, y_0))$ and as the norm is Lipschitzian, it follows that f is epi-upper semicontinuous at $((x_0, y_0), (p_0, y_0))$. Applying Theorem 2.13, there exists neighborhoods $\hat{U}$ of x_0, $\hat{V}$, $\hat{W}$ of y_0 and $\hat{N}$ of p_0 such that $\tau\, d(x, F_p^{-1}(z)) \le \|z - y\|$ for all $(p,(x,y)) \in F \cap (\hat{N} \times \hat{U} \times \hat{V})$ and for all $z \in \hat{W}$ from which one easily gets the conclusion of the theorem as in Theorem 4.18. □

Remark 4.20. Parametric results based on coderivatives can be found in a less general setting in [43, 49] under a coderivative condition which seems to be difficult to check. Our condition seems to be more natural since it a uniform version, with respect to the parameter, of the condition used for a single multifunction. Moreover, our condition turns out to be necessary when using one of the subdifferentials of Proposition 2.9.

As a concluding remark, let us observe that some metric regularity results do not enter in the framework developped in this article. This is the case for the results based on the Brouwer's fixed point theorem, or equivalently the invariance of the domain, such as [29, 40, 41].

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