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New Families of Convex Sets Related to Diametral Maximality

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Dedicated to the memory of Simon Fitzpatrick.

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Eggleston proved in a landmark monograph that, in every finite dimensional normed space, a bounded closed convex set with constant radius from its boundary is diametrically maximal. We show that this is no longer true in general and we characterize a set with constant radius by means of an equation involving its radius and diameter. A somewhat similar equation yields the definition of a constant difference set, a notion which turns out to be stronger than diametrically maximal but weaker than constant width. We investigate the interplay of these notions with the geometry of the underlying Banach space.

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1. Introduction

The notion of a diametrically maximal set was introduced by Meissner at the beginning of last century, as a counterpart to the older notion of a constant width set. A closed, bounded and convex set C in a Banach space is called *diametrically maximal* if, for every $x \notin C$, $\operatorname{diam}(\{x\} \cup C) > \operatorname{diam} C$; we say that C has *constant width* $d > 0$ if, for every $f \in X^*$ with $\|f\| = 1$, we have $\sup f(C - C) = d$. Sets with constant width are always diametrically maximal. The two notions coincide in any two dimensional space as well as in n-dimensional spaces with the Euclidean norm [9], but they fail to coincide in certain 3–dimensional spaces. In the case of infinite dimensional spaces, they coincide also in

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$(c_0(I), \|\cdot\|_\infty)$ [19]. In the case of $(C(K), \|\cdot\|_\infty)$, K a compact Hausdorff space, they coincide if and only if K is extremally disconnected [18]. Classical results on these topics, in finite dimensional spaces, can be found in the surveys by Chakerian and Groemer [6], Heil and Martini [15] and most recently by Martini and Swanepoel [17]. The interest of exploring these notions in spaces of infinite dimension originated with the pioneering papers by Franchetti [10], Behrends and Harmand [4] and Amir [1]. It has gained later renewed attention with the works by Behrends [5], Payá and Rodríguez-Palacios [20], [22] and Baronti and Papini [3], among others.

We are concerned in this paper, which is a sequel to [19], with the following question: Are diametrically maximal sets characterized by any relation between radius and diameter? Eggleston [8] observed that, in finite dimensional spaces, C is diametrically maximal if and only if it has "constant radius from its boundary". This notion is based on the concept of *the radius of C with respect to x*; which is defined for $x \in X$ as $r(x, C) = \sup\{\|x-y\| \colon y \in C\}$. A set C is said to be of *constant radius* if $r(x, C)$ is constant for $x \in \partial C$, the boundary of C. It is easily seen that this constant is the diameter $\operatorname{diam} C$ of C. (Since $r(x, C)$ has traditionally been called the radius of C with respect to x, we continue to use the term "radius" despite the fact that in this setting, the (constant) radius equals the diameter.) The family of all constant radius sets will be denoted by $\mathcal{CR}$. We prove that $C \in \mathcal{CR}$ if and only if, for every $x \in C$,

$$\operatorname{diam} C = r(x, C) + \operatorname{dist}(x, \partial C).$$

It is natural to ask whether $\mathcal{CR}$ coincides with $\mathcal{DM}$, the family of all diametrically maximal sets. We show that $\mathcal{DM} \subset \mathcal{CR}$ but the inclusion, in general, cannot be reversed. The above formula suggests considering the family $\mathcal{CD}$ of all sets satisfying

$$r(x, C) = \operatorname{diam} C + \operatorname{dist}(x, \partial C)$$

for all x *not* in C; these will be called *constant difference* sets (since the difference $r(x, C) - \operatorname{dist}(x, \partial C)$ is constant for all $x \notin C$). We prove that sets with constant width are always constant difference sets and these are diametrically maximal. However, in general, none of these inclusions can be reversed. Summarizing, if we denote by $\mathcal{CW}$ the family of constant width sets, we have

$$\mathcal{CW} \subsetneq \mathcal{CD} \subsetneq \mathcal{DM} \subsetneq \mathcal{CR}.$$

Note that the classes of sets $\mathcal{CD}$ and $\mathcal{CR}$ have been defined by functional equations using certain real–valued functions ϕ, all of which satisfy $|\phi(y) - \phi(x)| \le \|y - x\|$ for x, y in the domain of ϕ. Indeed, it is easily verified that $x \to r(x, C)$, as the supremum of convex functions $x \to \|x - y\|$, $y \in C$, is convex and has this property in X. It is well known that the same is true for $x \to \operatorname{dist}(x, C)$. On the other hand, it is also readily seen that $x \to \operatorname{dist}(x, \partial C)$ is *concave* for $x \in C$ and has the same continuity property.

When working with diametrically maximal sets, it is sometimes useful to know that they are intersections of closed balls. In fact, Eggleston observed in [9] that a set C with $\operatorname{diam} C = d$ is diametrically maximal if and only if it satisfies the *spherical intersection property*, that is, if and only if $C = \bigcap_{x \in C}(x + dB)$ where B is the (closed) unit ball. In contrast to this fact, we show that constant radius sets need not be intersections of balls. We also prove that K is an extremally disconnected compact space if and only if, in $(C(K), \|\cdot\|_\infty)$,

$$\mathcal{M} \cap \mathcal{CR} = \mathcal{DM}$$

where $\mathcal{M}$ denotes the family of all intersections of closed balls. Section 3 is devoted to constant difference sets, the two most important results being that $\mathcal{DM} = \mathcal{CD}$ in spaces $C(K)$ endowed with the usual sup norm and $\mathcal{CD} = \mathcal{CW}$ in spaces with the *Mazur intersection property*. Finally, the last section is devoted to an investigation of the behavior of the above families when regarding Hausdorff convergent sequences of such sets. We prove that all the above families having the word *constant* in their names ($\mathcal{CR}$, $\mathcal{CD}$ and $\mathcal{CW}$) are (topologically) closed. We do not know whether this is also true for the family of diametrically maximal sets. In this direction we can prove, for instance, that $\mathcal{DM}$ is closed in every $C(K)$ space with the sup norm. Let us recall that it is also unknown whether $\mathcal{M}$ is closed [13].

2. Sets with constant radius

It is elementary to show, using the Bishop–Phelps theorem and an $\varepsilon/3$ argument, that a bounded closed convex set C has constant width $d > 0$ if and only if $\sup f(C - C) = d$ for every *support* functional f of C. That is, C has constant width if and only if for every support point $x \in \partial C$ (the boundary of C) and every $f \in X^*$ with $\|f\| = 1$ and $f(x) = \sup f(C)$, we have $\sup f(C - C) = d$. Suppose that, instead of measuring C with all the possible parallel supporting hyperplanes, we want to measure it with balls. Namely, for every point $x \in X$, consider

$$r(x, C) = \sup \{\|x - y\| : y \in C\}$$

and say that C *has constant radius* if there is $d > 0$ such that $r(x, C) = d$ for every boundary point $x \in \partial C$. Notice that we can replace d by $\operatorname{diam} C$ in the above definition, since it is readily shown that $\operatorname{diam} C = \sup \{r(x, C) \colon x \in \partial C\}$. Also, it is equivalent to having $\operatorname{diam} C = r(x, C)$ for every support point, since the latter are dense in ∂C. The family of all sets with constant radius will be denoted by $\mathcal{CR}$. We begin by showing a couple of basic properties satisfied by the members of $\mathcal{CR}$ that have been already proved for the smaller class of diametrically maximal sets [19].

Proposition 2.1. *Symmetric constant radius sets are balls. In spaces of infinite dimension, constant radius sets are not compact.*

Proof. Suppose that C has constant radius with $\operatorname{diam} C = d > 0$ and, without loss of generality, that C is symmetric about 0. Let $r = r(0, C) \equiv \sup\{\|x\| \colon\ x \in C\}$. We will show that $C = rB$. Obviously, $C \subset rB$. Since C has constant radius, by using the reformulation in Proposition 3.1 (below), we know that $r(x, C) + \operatorname{dist}(x, \partial C) = d$ for all $x \in C$; taking $x = 0$, this yields $r + \operatorname{dist}(0, \partial C) = d$. Choose $x \in C$ such that $\|x\| > r - \varepsilon$. By symmetry, $-x \in C$, therefore $d \geq \|x - (-x)\| = 2\|x\| > 2r - 2\varepsilon$. It follows that $d \geq 2r$ and $\operatorname{dist}(0, \partial C) = d - r \geq r$, so $rB \subset C$.

To prove the second part of the proposition, suppose now that C is a compact convex subset of an infinite dimensional space X with $\operatorname{diam} C > 0$. Clearly, C has empty interior; moreover, C is necessarily separable, hence contains a dense sequence $\{x_n\}_{n=1}^{\infty}$, say. Let $x = \sum_{n=1}^{\infty} 2^{-n} x_n$; then $x \in C = \partial C$. Moreover, for any $y \in C$ we have

$$\|x - y\| = \|\sum 2^{-n}(x_n - y)\| \leq \sum 2^{-n}\|x_n - y\|.$$

Now, $\|x_n - y\| < \operatorname{diam} C$ for (infinitely) many n, so $\|x - y\| < \operatorname{diam} C$. Since the function $y \to \|x - y\|$ is continuous and C is compact, we conclude that $r(x, C) < \operatorname{diam} C$, hence C cannot be of constant radius. $\square$

As mentioned in the introduction, the property of constant radius was previously considered by Eggleston [8], primarily as a characterization of diametrically maximal sets in finite dimensional spaces. The next proposition recaptures this result and shows that, in infinite dimensional spaces, this equivalence is no longer true. Moreover, it shows that constant radius sets which are not diametrically maximal must necessarily have empty interior. Let us recall, on the other hand, that the existence in a Banach space of a set of constant radius with empty interior has geometric implications related to Jung's constant [19].

Proposition 2.2. *Every diametrically maximal set has constant radius. The converse, which is false in general, holds for sets with nonempty interior. In particular, the two notions coincide in finite dimensional spaces. They also coincide in Hilbert spaces.*

Proof. If C does not have constant radius, then for some $x \in \partial C$ there is $\varepsilon > 0$ such that $r(x, C) = \operatorname{diam} C - \varepsilon$ and there exists a point $z \notin C$ with $\|z - x\| < \varepsilon$. Hence, for all $y \in C$,

$$\|z - y\| \leq \|z - x\| + \|x - y\| < \varepsilon + r(x, C) = \operatorname{diam} C$$

so C is not diametrically maximal. Let us see now that the converse is true provided C has nonempty interior. To this end, recall first that the real function $X \ni x \to r(x, C)$ is convex. Pick an interior point $a \in C$ and consider, for any $w \notin C$, the two points $y, x \in \partial C$ such that w, y, x, a are in the same line. By hypothesis $r(y, C) = r(x, C) = \operatorname{diam} C$, since C has constant radius. Also, $r(a, C) < \operatorname{diam} C$ since a is an interior point. Finally, using the convexity of the function $r(\cdot, C)$ and the fact that w, y, x, a are in the same line, we get that $r(w, C) > \operatorname{diam} C$ thus implying that C is diametrically maximal.

In finite dimensional spaces, sets of constant radius have nonempty interior and, consequently, are diametrically maximal. Indeed, assume that $C \subset \mathbb{R}^n$ is a set with constant radius and *empty* interior. Then, C has nonempty interior in some m-dimensional subspace X of $\mathbb{R}^n$. If we consider an interior point $x \in C$ (in the relative topology of X), we know that $r(x, C) < \operatorname{diam} C$. But x is a boundary point of C in $\mathbb{R}^n$, hence the above inequality contradicts the assumption that C has constant radius.

Suppose, now, that X is a Hilbert space and that $x \in X \setminus C$. Let $y \in C$ be the nearest point in C to x. Since $r(y, C) = \operatorname{diam} C$, we can choose $z \in C$ such that $\|y - z\|^2 > (\operatorname{diam} C)^2 - \|x - y\|^2/2$. Note that since y is the nearest point in C to x, we must have $\langle x - y, z - y\rangle \leq 0$. Consequently

$$\begin{aligned} \|x - z\|^2 = \|x - y + y - z\|^2 &= \|x - y\|^2 + 2\langle x - y, y - z\rangle + \|y - z\|^2 \\ &\geq \|x - y\|^2 + (\operatorname{diam} C)^2 - \|x - y\|^2/2 \\ &> (\operatorname{diam} C)^2 \end{aligned}$$

so $\operatorname{diam}(C \cup \{x\}) > \operatorname{diam} C$.

Finally, we present an example of a set with constant radius which is not diametrically maximal. (By the previous result, such examples can be found only in infinite dimensional spaces.) In fact, the set C in the following example is not even an intersection of balls, hence is not diametrically maximal.

Consider the two functions $f, g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in [0,1/4) \cup (1/4,1/2] \\ 0 & \text{otherwise} \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x \in [0,1/2) \\ -1 & \text{otherwise.} \end{cases}$$

Then, the set $C = \{h \in C[0,1] : f(t) \leq h(t) \leq g(t) \quad \text{for all } t \in [0,1]\}$ is not an intersection of balls, since every ball containing C contains also the function $\phi(t) = -2t+1$. Nevertheless, it has constant radius 1. □

The question that now arises is whether a constant radius set which is also an intersection of balls is necessarily diametrically maximal. Next we show that, when restricted to the class of $C(K)$ spaces, where K is a compact Hausdorff space, the answer is affirmative if and only if K is extremally disconnected.

Proposition 2.3. *A compact Hausdorff space K is extremally disconnected if and only if every bounded closed convex subset of $C(K)$ with constant radius which is also an intersection of balls is diametrically maximal.*

Proof. Suppose that K is extremally disconnected and $C \subset C(K)$ is a closed, convex set of (positive) constant radius. If $C \subset C(K)$ is an intersection of balls, then $C = [f,g] = \{h \in C(K) \colon f(t) \leq h(t) \leq g(t)\}$ where $f, g \colon K \to \mathbb{R}$ are continuous functions ([18], Proposition 2.3). Consider the function $h = (1/2)(f+g) \in C$. We'll show that h lies in the interior of C, and hence, by Proposition 2.2, that C is diametrically maximal. Indeed, if $h \in \partial C$, then since C has constant radius, $r(h,C) = \operatorname{diam} C$. Let us check that this is impossible. Indeed,

$$r(h,C) = \max\{\|h-g\|, \|f-h\|\}.$$

Assume, for instance, that $r(h,C) = \|h-g\| = g(t_0) - h(t_0)$, for some $t_0 \in K$. Then $\operatorname{diam} C = r(h,C) = g(t_0) - h(t_0) = (1/2)(g(t_0) - f(t_0)) \leq (1/2) \operatorname{diam} C$, a contradiction since we are assuming $\operatorname{diam} C > 0$. The second case $r(h,C) = \|h-f\|$ is analogous.

To prove the converse, given a compact Hausdorff space K which is not extremally disconnected, we will construct a constant radius set which is an intersection of balls but is not diametrically maximal. There is an open set $G \subset K$ such that $\overline{G}$ is not open. Define $f, g \colon K \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1/2 & \text{if } x \in K \setminus G \\ 0 & \text{if } x \in G \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x \in K \setminus \overline{G} \\ -1 & \text{if } x \in \overline{G}. \end{cases}$$

It is easy to see that g is upper semicontinuous, f is lower semicontinuous and $f \leq g$. Then $C = [f,g]$ is an intersection of balls [19] and $\operatorname{diam} C = 1$. Let us check that C has constant radius. Consider $y \in S = (\overline{G} \setminus G) \cap \left(\overline{K \setminus \overline{G}}\right)$. The reader can try to draw a picture with $K = [0,1]$, $G = [0,1/2)$ and $y = 1/2$. For any $h \in C$ and $\varepsilon > 0$, there is $z \in G$ satisfying $|h(z)| < \varepsilon$. Actually, $C \subset \delta_y^{-1}(0)$, being δ_y the point evaluation map at y. Using Urysohn's lemma, we know the existence of $\phi \in C$ such that $\phi(z) = -1$. Then

$\|\phi-h\| \geq 1-\varepsilon$, thus showing that $r(h,C)=1$. However, C is not diametrically maximal. Indeed, take $w \in K \setminus \overline{G}$ and consider a Urysohn function $\psi\colon K \to \mathbb{R}$ satisfying $\psi(t)=0$ for every $t \in \overline{G}$, $\psi(w)=1$ and $0 \leq \psi(t) \leq 1$ for every $t \in K \setminus \overline{G}$. Then, the addition of ψ to C does not increase the diameter. □

3. Sets with constant difference

The purpose of this section is to introduce a new family of closed bounded convex sets satisfying a certain functional equation in which the diameter is the constant difference between the radius with respect to a point and the distance of that point from the set, thus its name. In order to motivate the definition, we begin by noting that the condition of constant radius can be reformulated as follows.

Proposition 3.1. *The set C has constant radius if and only if, for every $x \in C$,*

$$\operatorname{diam} C = r(x,C) + \operatorname{dist}(x,\partial C). \tag{1}$$

Proof. If this holds for all $x \in C$, then it holds for all $x \in \partial C$, and therefore C has constant radius. To prove the necessity, assume that $x \in C$ is an interior point. Let $\alpha = \operatorname{dist}(x,\partial C) > 0$ and fix $0 < \varepsilon < \alpha$. There is $c \in C$ such that $\|x-c\| \geq r(x,C) - \varepsilon/2$. Define $c' = x - (\alpha - \varepsilon/2)\|c-x\|^{-1}(c-x)$. Then $\operatorname{diam} C \geq \|c-c'\| = \|c-x\| + \|x-c'\| \geq r(x,C) + \alpha - \varepsilon$ hence, since ε can be chosen arbitrarily small, $\operatorname{diam} C \geq r(x,C)+\alpha$. To prove the reverse inequality, choose again $0 < \varepsilon < \alpha$ and $y \in \partial C$ satisfying $\|x-y\| < \operatorname{dist}(x,\partial C) + \varepsilon$. Then $r(x,C) + \|x-y\| \geq r(y,C) = \operatorname{diam} C$ and this implies that $r(x,C) + \operatorname{dist}(x,\partial C) > \operatorname{diam} C - \varepsilon$. □

Once we know that the property of constant radius can be expressed by equation (1) for all points $x \in C$, it is natural to investigate the implications of the analogous requirement

$$r(x,C) = \operatorname{diam} C + \operatorname{dist}(x,\partial C) \tag{2}$$

for every $x \notin C$. A closed, convex and bounded set C satisfying (2) will be said to have *constant difference* (or to be a *constant difference set*). Denote by $\mathcal{CD}$ the family of all closed, convex and bounded sets with constant difference.

Proposition 3.2. *Every set of constant width has constant difference and every constant difference set is diametrically maximal. There are counterexamples to each of the reverse implications; that is, $\mathcal{CW} \subsetneq \mathcal{CD} \subsetneq \mathcal{DM}$.*

Proof. If C has constant difference and $x \notin C$, then $\operatorname{diam} C < r(x,C) \leq \operatorname{diam}(C \cup \{x\})$, so C is diametrically maximal. To see that constant width sets have constant difference, first notice that for $x \notin C$, the inequality $r(x,C) \leq \operatorname{diam} C + \operatorname{dist}(x,\partial C)$ always holds. Indeed, if $x \notin C$, given $\varepsilon > 0$ there is $x \in \partial C$ such that $\|x-x'\| \leq \operatorname{dist}(x,\partial C)+\varepsilon$, so for every $c \in C$ we have $\|x-c\| \leq \|x-x'\| + \|x'-c\| \leq \operatorname{dist}(x,\partial C) + \varepsilon + \operatorname{diam} C$. Assume now that C has constant width; we must prove the equality in (2) for an arbitrary $x \notin C$. We proceed by contradiction: If equality does not hold in (2), there is $\varepsilon > 0$ such that

$$r(x,C) < \operatorname{diam} C + \operatorname{dist}(x,\partial C) - \varepsilon. \tag{3}$$

By using the separation theorem to separate the ball $x + \operatorname{dist}(x,\partial C)B$ from C, we can find $f \in X^*$ of norm one satisfying $f(x) \leq \inf f(C) - \operatorname{dist}(x,\partial C)$. On the other hand, since C has constant width and thus $\sup f(C-C) = \operatorname{diam} C$, we can find $x'' \in C$ such that $f(x'') > \inf f(C) + \operatorname{diam} C - \varepsilon$. Now

$$\begin{aligned} \|x - x''\| \geq f(x'') - f(x) &> \inf f(C) + \operatorname{diam} C - \varepsilon - \inf f(C) + \operatorname{dist}(x, \partial C) \\ &= \operatorname{diam} C + \operatorname{dist}(x, \partial C) - \varepsilon, \end{aligned}$$

which contradicts (3).

Let us present now a constant difference set which does not have constant width. Consider functions $f, g : [0,1] \to \mathbb{R}$ defined as follows: $g(x) = 0$ if $x \in [0, 1/2)$ and $g(x) = 1$ otherwise; $f(x) = -1$ if $x \in [0, 1/2]$ and $f(x) = 0$ otherwise. Let $C = [f, g] = \{h \in C[0,1] : f(t) \leq h(t) \leq g(t) \quad \text{for all} \quad t \in [0,1]\}$. Then C is diametrically maximal and moreover, since it satisfies (2), it has constant difference. However, C does not have constant width since the point evaluation map at $1/2$ equals 0 for every $h \in C$, while $\operatorname{diam} C = 1$.

Finally, let us prove the last part of the proposition. To show that diametrically maximal sets need not have constant difference, consider the following subspace of $C[0,1]$: $X = \{h \in C[0,1] : 2h(1/2) = h(1)\}$. Let $C' = C \cap X$ where $C = [f, g]$ is the previous example. Then, C' is diametrically maximal and $\operatorname{diam} C = 1$. However, if we take the function $\psi(t) = 2t$ which is in $X \setminus C'$, it is easy to check that $\operatorname{dist}(\psi, C') = 2 = r(\psi, C') < \operatorname{diam} C' + \operatorname{dist}(\psi, C')$, thus implying that C' does not have constant difference. $\square$

Suppose that K is a compact Hausdorff space. We say that $f, g \colon K \to \mathbb{R}$ form an *admissible pair* when: (a) they are lower and upper semicontinuous, respectively; (b) for every $x \in K$, $\liminf_{y \to x} g(y) \geq \limsup_{y \to x} f(y)$ [18]. For a semicontinuous function f on K, we let D_f denote the dense G_δ set of points of continuity of f. Admissible pairs can be used to describe intersections of balls in $C(K)$ spaces and consequently diametrically maximal sets, in particular. Precisely, the set $C \subset C(K)$ is a nonempty intersection of closed balls if and only if

$$C = [f, g] = \{h \in C(K) \colon f(x) \leq h(x) \leq g(x) \quad \text{for all } x \in K\} \tag{4}$$

where $f, g \colon K \to \mathbb{R}$ form an admissible pair [19]. The set $C \subset C(K)$ is diametrically maximal if and only if $C = [f, g]$ where f, g form an admissible pair and $g(x) - f(x) = \operatorname{diam} C$ for every $x \in D_f \cap D_g$ [19].

Proposition 3.3. *Every diametrically maximal set has constant difference in spaces $(C(K), \|\cdot\|_\infty)$, where K is any compact Hausdorff space.*

Proof. Let $C \subset C(K)$ be a diametrically maximal set with $\operatorname{diam} C = d > 0$. According to the preceding characterization, $C = [f, g]$ where f, g form an admissible pair and $g(t) - f(t) = d$ for each t in the dense G_δ–set $D_f \cap D_g$. We will show that, for every $h \notin C$,

$$\operatorname{dist}(h, C) = \sup\{\operatorname{dist}(h(t), [f(t), g(t)]) \colon t \in D_f \cap D_g\} \tag{5}$$

and, also,

$$r(h, C) = \operatorname{dist}(h, C) + d \tag{6}$$

that is, C has constant difference. Let us see first that (6) follows from (5). Indeed, given $\varepsilon > 0$, let $t \in D_f \cap D_g$ be such that

$$\operatorname{dist}(h, C) - \varepsilon < \operatorname{dist}(h(t), [f(t), g(t)])$$

and assume, for instance, that $h(t) < f(t)$ (the case $h(t) > g(t)$ is analogous). Since t is a point of continuity for g, there is $\phi \in C$ satisfying $g(t) - \phi(t) < \varepsilon$ (see Step 1 in Theorem 4.3 of [19]) and hence

$$\begin{aligned} r(h,C) \geq \|\phi - h\| &\geq \phi(t) - g(t) + g(t) - h(t) \\ &\geq -\varepsilon + g(t) - f(t) + f(t) - h(t) \\ &\geq -\varepsilon + d + \operatorname{dist}(h,C) - \varepsilon \end{aligned}$$

which proves (6), since it is always true that $r(h,C) \leq \operatorname{dist}(h,C) + d$. To prove (5), we pick $\varphi \in C = [f,g]$ and define a pair of sequences $\{f_n\}, \{g_n\} \subset C$ for which $\varphi \in [f_n, g_n] \subset [f,g]$ taking advantage of the continuity of f_n, g_n. First, for every $n \in \mathbb{N}$ and every $x \in K$, we consider

$$F_x^n = \{t \in K : \tilde{f}(x) + \frac{1}{n} < \varphi(t)\} \quad \text{and} \quad G_x^n = \{t \in K : \varphi(t) < \tilde{g}(x) - \frac{1}{n}\}$$

where $\tilde{f}(x) = \limsup_{y\to x} f(y)$ and $\tilde{g}(x) = \liminf_{y \to x} f(y)$. Then F_x^n and G_x^n are open sets (possibly empty), hence $K \setminus F_x^n$ and $K \setminus G_x^n$ are compact. If $x \in F_x^n$, there are open neighborhoods V_x^n and U_x^n of $K \setminus F_x^n$ and x, respectively, such that $V_x^n \cap U_x^n = \emptyset$. We may assume that U_x^n is chosen so that

$$\varphi(t) > \tilde{f}(x) + \frac{1}{n} \quad \text{and} \quad f(t) \leq \tilde{f}(x) + \frac{1}{n} \tag{7}$$

whenever $t \in U_x^n$. Let $\psi_x^n \colon K \to [0,1]$ be a Urysohn function with $\psi_x^n(t) = 0$ if $t \in V_x^n$ and $\psi_x^n(t) = 1$ if $t \in U_x^n$. Define

$$f_x^n(t) = (1 - \psi_x^n(t))\,\varphi(t) + \psi_x^n(t)\,(\tilde{f}(x) + \frac{1}{n}),\ t \in K. \tag{8}$$

Otherwise, if $x \notin F_x^n$, we set $f_x^n = \varphi$ and choose a neighborhood U_x^n of x such that

$$|\varphi(y) - \varphi(z)| < 1/n \tag{9}$$

whenever $y, z \in U_x^n$. Analogously, if $x \in G_x^n$, there is a pair of open neighborhoods M_x^n and N_x^n of $K \setminus G_x^n$ and x, respectively, such that $M_x^n \cap N_x^n = \emptyset$. We may assume, as before, that N_x^n is chosen so that

$$\varphi(t) < \tilde{f}(x) - \frac{1}{n} \quad \text{and} \quad g(t) \geq \tilde{g}(x) - \frac{1}{n} \tag{10}$$

whenever $t \in N_x^n$. Let $\phi_x^n \colon K \to [0,1]$ be another Urysohn function with $\phi_x^n(t) = 0$ if $t \in M_x^n$ and $\phi_x^n(t) = 1$ if $t \in N_x^n$. Define

$$g_x^n(t) = (1 - \phi_x^n(t))\,\varphi(t) + \phi_x^n(t)\,(\tilde{g}(x) - \frac{1}{n}) \tag{11}$$

Otherwise, if $x \notin G_x^n$, we set $g_x^n = \varphi$ and choose a neighborhood N_x^n of x satisfying (9) whenever $y, z \in N_x^n$. It is not difficult to verify, using (7) and (10), that $f \leq f_x^n$ and $g_x^n \leq g$. Finally, we consider for every $x \in K$ a neighborhood W_x^n contained in $U_x^n \cap N_x^n$ and satisfying

$$|h(y) - h(x)| < \frac{1}{n}, \tag{12}$$

$$f(y) \leq \tilde{f}(x) + \frac{1}{n} \quad \text{and} \quad g(y) \geq \tilde{g}(x) - \frac{1}{n} \tag{13}$$

for every $y \in W_x^n$. Since K is compact, there is a finite subset $\{x_1, ..., x_m\} \subset K$ such that $K \subset \cup_{i=1}^m W_{x_i}^n$. We now define

$$\begin{aligned} f_n &= \min\{\varphi, f_{x_1}^n, ..., f_{x_m}^n\} \\ g_n &= \max\{\varphi, g_{x_1}^n, ..., g_{x_m}^n\} \end{aligned}$$

It is clear that $f \leq f_n \leq \varphi \leq g_n \leq g$, so that $\varphi \in [f_n, g_n] \subset [f, g] = C$. Finally, consider

$$h_n(t) = \begin{cases} f_n(x) & \text{if } h(x) \leq f_n(x) \\ h(x) & \text{if } f_n(x) \leq h(x) \leq g_n(x) \\ g_n(x) & \text{if } g_n(x) \leq h(x) \end{cases}$$

which, by virtue of the pasting lemma, is a continuous function. Notice that $h_n = (f_n \vee h) \wedge g_n$. Also, $f_n \leq h_n \leq g_n$ and therefore $h_n \in C$. Our aim now is to estimate $\|h - h_n\| = \sup_{t \in K} |h(t) - h_n(t)|$. Take $t \in K$ such that $|h(t) - h_n(t)| > 0$ (if $|h(t) - h_n(t)| = 0$ for every $t \in K$, then $h_n = h$ which contradicts $h \notin C$). Let $x_j \in \{x_1, ..., x_m\}$ be such that $t \in W_{x_j}^n$. There are two possibilities: $h(t) < h_n(t)$ or $h(t) > h_n(t)$. In the first case we have $h_n(t) = f_n(t)$ and hence, using (12) and (13), we obtain

$$\begin{aligned} 0 \leq h_n(t) - h(t) &= f_n(t) - h(t) \\ &\leq \tilde{f}(x_j) + \frac{1}{n} - h(x_j) + \frac{1}{n} \\ &\leq \operatorname{dist}(\, h(x_j),\, [\tilde{f}(x_j), \tilde{g}(x_j)]\,) + \frac{2}{n}\,. \end{aligned}$$

For the second case when $h(t) > h_n(t)$, we use a similar argument, replacing f_n by g_n, $\tilde{f}$ by $\tilde{g}$ and other minor changes. To finish the proof of (5), we just need to check that

$$\sup_{t \in K} \{\operatorname{dist}(h(t), [\tilde{f}(t), \tilde{g}(t)])\} = \sup_{t \in D_f \cap D_g} \{\operatorname{dist}(h(t), [f(t), g(t)])\},$$

the inequality $\geq$ being obvious since $\tilde{f}(t) = f(t)$ and $\tilde{g}(t) = g(t)$ for every $t \in D_f \cap D_g$. The reverse inequality comes from the continuity of h and the following observation

$$\begin{aligned} \tilde{f}(x) &= \limsup_{y \to x} f(y) = \lim \sup_{y \to x\,,\, y \in D_f \cap D_g} f(y) \\ \tilde{g}(x) &= \liminf_{y \to x} g(y) = \lim \inf_{y \to x\,,\, y \in D_f \cap D_g} g(y) \end{aligned}$$

which, in turn, follows from the fact that f and g are lower and upper semicontinuous, respectively. □

We now prove that constant difference sets *do* have constant width in a wide class of Banach spaces: those satisfying the *Mazur Intersection Property* (MIP). Recall that, by definition, a normed space has the MIP provided every closed, convex and bounded set is an intersection of closed balls. This class includes spaces with a Fréchet differentiable norm (this was the seminal result by Mazur) and also certain spaces which are not Asplund spaces [16]. The reader is referred to [14] for a recent account on this property.

Proposition 3.4. *If X has the Mazur Intersection Property and C is a constant difference set, then C has constant width.*

Proof. We'll show that if C does not have constant width, then it does not have constant difference. First, if C does not have constant width, then there exist $\|f\| = 1$ and $\alpha > 0$ such that

$$\sup f(C - C) + \alpha < \operatorname{diam} C.$$

Those f for which this inequality holds form a relatively open subset of the dual unit sphere so, by the Giles, Gregory and Sims [12] characterization of the MIP, we may assume that f is a weak*-denting point of the dual unit ball. Using the characterization of weak*-denting points due to Chen and Lin [7], we can find a ball $x + \lambda B$ which contains C but does not intersect the hyperplane $H = \{y \in X : f(y) = \sup f(C) + \alpha\}$. We would like to have $\lambda > \operatorname{diam} C$ and this is possible: Choose a point z with $f(z) \leq \sup f(C)$ and $\operatorname{dist}(z, C) > 2 \operatorname{diam} C$, let $D = \operatorname{conv}(\{z\} \cup C)$ and apply the Chen-Lin result to find $x + \lambda B$ containing D and missing the hyperplane $\{y \in X : f(y) = \sup f(D) + \alpha = \sup f(C) + \alpha\} = H$ so $\sup f(x + \lambda B) = f(x) + \lambda < \sup f(C) + \alpha$. Now, if $f(x) \geq \inf f(C)$, then

$$\begin{aligned} f(x) + \lambda &> \inf f(C) + \operatorname{diam} C \\ &> \inf f(C) + \sup f(C - C) + \alpha \\ &= \sup f(C) + \alpha \end{aligned}$$

which is impossible, showing that $f(x) < \inf f(C)$ and therefore $x \notin C$. In fact, $\operatorname{dist}(x, C) \geq \inf f(C) - f(x)$, since, if $y \in C$, then $\|y - x\| \geq f(y) - f(x) \geq \inf f(C) - f(x)$. Consequently, if C were to be a constant difference set, then

$$\begin{aligned} \lambda \geq r(x, C) &= \operatorname{dist}(x, \partial C) + \operatorname{diam} C \\ &\geq \inf f(C) - f(x) + \operatorname{diam} C \\ &> \inf f(C) - f(x) + \sup f(C - C) + \alpha \\ &= \sup f(C) + \alpha - f(x) \\ &> f(x) + \lambda - f(x) = \lambda, \end{aligned}$$

a contradiction which completes the proof. □

Hilbert space does, of course, have the Mazur Intersection Property, but more is known: *In Hilbert space, diametrically maximal sets have constant width.* This follows easily from Theorem 1 in Polovinkin [21], which itself relies on his work with Balashov [2] showing that the unit ball of Hilbert space is what they call M–strongly convex.

4. Sequences of diametrically maximal sets

When dealing with new classes of convex sets, it is natural to investigate their basic topological properties. In particular, we are interested in this section in the question of whether the limit, in the Hausdorff metric, of a sequence of sets satisfying a certain property, still shares the same property. For instance, it is unknown whether the limit of a sequence of intersections of balls is again an intersection of balls (see [11] and [13]). In this section, we focus our attention on the families considered in previous sections. For three of them, the answer to the above question is positive.

Proposition 4.1. *The families $\mathcal{CW}$, $\mathcal{CR}$ and $\mathcal{CD}$ are (topologically) closed.*

Proof. The first step is observing that, whenever $\{A_n\}$ is a sequence of closed, convex and bounded sets which converges to A, then $\{\operatorname{diam} A_n\}$ converges to $\operatorname{diam} A$. Indeed, given $\varepsilon > 0$, there is $n_1 \in \mathbb{N}$ so that $A \subset A_n + \varepsilon B$ and $A_n \subset A + \varepsilon B$ for every $n \geq n_1$ and, therefore,

$$\operatorname{diam} A \leq \operatorname{diam} A_n + 2\varepsilon \leq (\operatorname{diam} A + 2\varepsilon) + 2\varepsilon.$$

Consider now $\{A_n\} \subset \mathcal{CW}$ converging to A and let $f \in X^*$ be a norm one functional. For simplicity, we may assume (taking a subsequence, if necessary) that $A \subset A_n + (1/n)B$ and $A_n \subset A + (1/n)B$ for every $n \in \mathbb{N}$. Thus, we have

$$\sup f(A_n - A_n) - 2/n \leq \sup f(A - A) \leq \sup f(A_n - A_n) + 2/n.$$

Observe that $\sup f(A_n - A_n) = \operatorname{diam} A_n$, since $A_n \in \mathcal{CW}$. This fact, together with $\operatorname{diam} A_n \to \operatorname{diam} A$, implies that $\sup f(A - A) = \operatorname{diam} A$, hence $A \in \mathcal{CW}$.

To prove the second part of the proposition, consider now a sequence $\{A_n\}$ of constant radius sets which converges to A. Choose an arbitrary support point $x \in \partial A$. We claim that there is a sequence $\{x_n \in \partial A_n\}$ such that $x_n \to x$, that is, $\operatorname{dist}(x, \partial A_n) \to 0$. Suppose, to the contrary, that there is a $\lambda > 0$ and a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $\operatorname{dist}(x, \partial A_{n_k}) \geq \lambda$ for every $k \in \mathbb{N}$. Consider the ball $x + (\lambda/2)B$. Since $A_{n_k} \to A$ and $x \in A$, there is k_0 such that $(x + (\lambda/2)B) \cap A_{n_k} \neq \emptyset$ for $k \geq k_0$. But $(x + (\lambda/2)B) \cap \partial A_{n_k} = \emptyset$ implies that

$$x + (\lambda/2)B \subset A_{n_k} \quad \text{for } k \geq k_0.$$

Consider now a norm–one functional $f \in X^*$ supporting A at x, that is, $f(x) = \sup f(A)$. Clearly, $\sup f(A_{n_k}) \geq \sup f(A) + \lambda/2$ for $k \geq k_0$, which contradicts the assumption that $A_{n_k} \to A$. Having proved the claim, choose $x_n \in \partial A_n$ such that $\{x_n\}$ converges to x. We can show that $r(x_n, A_n) \to r(x, A)$ simply by considering

$$r(x_n, A_n) \leq \|x_n - x\| + \operatorname{dist}(A_n, A) + r(x, A) \tag{14}$$

and the analogous inequality obtained replacing A by A_n and x by x_n in (14). Now, finally, since $r(x_n, A_n) = \operatorname{diam} A_n \to \operatorname{diam} A$, we conclude that $r(x, A) = \operatorname{diam} A$. This holds for every support point x of A and, consequently, the latter has constant radius.

For the third part of the proposition, suppose that $\{A_n\}$ is a sequence of constant difference sets which converges to A. We must check that (2) holds. The idea is simply to prove that, for each $x \notin A$,

$$\begin{array}{ccccc} \operatorname{dist}(x, A_n) & + & \operatorname{diam} A_n & = & r(x, A_n) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{dist}(x, A) & & \operatorname{diam} A & & r(x, A). \end{array}$$

If $x \notin A$, we may assume that $x \notin A_n$, either. We can show that $\operatorname{dist}(x, A_n) \to \operatorname{dist}(x, A)$ by using the triangle inequality

$$\operatorname{dist}(x, A_n) \leq \operatorname{dist}(x, A) + \operatorname{dist}(A, A_n) \tag{15}$$

and the analogous inequality obtained by interchanging A and A_n. In a similar manner, one can prove that $r(x, A_n) \to r(x, A)$ just using $r(x, A_n) \leq r(x, A) + \operatorname{dist}(A, A_n)$ and $r(x, A) \leq r(x, A_n) + \operatorname{dist}(A_n, A)$. $\square$

What about diametrically maximal sets? As in the case of the family of all intersections of balls, we do not know whether $\mathcal{DM}$ is closed, in contrast with the last result. The answer is affirmative, however, in finite dimensional spaces and Hilbert spaces, as shown in next corollary. The proof depends on the fact (Proposition 2.2) that $\mathcal{DM} = \mathcal{CR}$ in these spaces.

Corollary 4.2. *$\mathcal{DM}$ is closed: (i) in finite dimensional spaces; (ii) in spaces with Jung's constant less than 2 (in particular, in Hilbert spaces).*

Proof. Assume that $\{A_n\}$ is a sequence of diametrically maximal sets in X which converges to A. According to the above proposition, since $\{A_n\} \subset \mathcal{CR}$, the set A also has constant radius which, by virtue of Proposition 2.2, implies that A is diametrically maximal when X has finite dimension. For the second part, notice that constant radius sets always have nonempty interior when Jung's constant is less than 2 [19]. □

It is known that, in spaces $(C(K), \|\cdot\|_\infty)$, when K is extremally disconnected, $\mathcal{DM}$ is also closed. Indeed, in these spaces, diametrically maximal sets are balls or single points [10]. We improve this result by showing that $\mathcal{DM}$ is always closed in $C(K)$ spaces (even if K is not extremally disconnected).

Proposition 4.3. *$\mathcal{DM}$ is closed in $(C(K), \|\cdot\|_\infty)$, where K is any compact space.*

Proof. Let $\{A_n\} \subset C(K)$ be a sequence of diametrically maximal sets converging to A. We know, as shown in the proof of Proposition 4.1, that the sequence $\{\operatorname{diam} A_n\}$ converges to $\operatorname{diam} A = d$. Also, A is an intersection of closed balls since $\mathcal{M}$ is closed in these spaces [18]. Hence, according to (4), $A = [f, g]$, where $f, g\colon K \to \mathbb{R}$ form an admissible pair. By using the above mentioned characterization, the proof will be accomplished if we prove that $g(x) - f(x) = d$, for every x which is a point of continuity of f and g. It is convenient also to represent $A_n = [f_n, g_n]$, where f_n, g_n are admissible pairs for every $n \in \mathbb{N}$; again, according to (4). For every $\varepsilon > 0$ and n big enough, we have

$$[f, g] \subset [f_n, g_n] + \varepsilon B = [f_n, g_n] + [-\varepsilon, \varepsilon] = [f_n - \varepsilon, g_n + \varepsilon] \tag{16}$$

and, analogously,

$$[f_n, g_n] \subset [f - \varepsilon, g + \varepsilon] \tag{17}$$

where B denotes the closed unit ball, and the last equality in (16) follows from a more general result in [18]. According also to [18], g is continuous at x if and only if $g(x) = \sup\{h(x) : h \in C\}$ and f is continuous at x if and only if $f(x) = \inf\{h(x) : h \in C\}$. This fact, together with the analogous statements for g_n and f_n and the inclusions (16) and (17) yield

$$g(x) \le g_n(x) + \varepsilon \le g(x) + 2\varepsilon$$

$$f(x) \ge f_n(x) - \varepsilon \ge f(x) - 2\varepsilon$$

for every $x \in K$ point of continuity of g, g_n, f and f_n. Denote by D_g and D_f the set of points of continuity of g and f, respectively, and define D_{g_n} and D_{f_n} in a similar way. The set

$$D = (D_g \cap D_f) \cap \Big(\bigcap_n (D_{g_n} \cap D_{f_n})\Big)$$

is dense in K. Indeed, the set of points of continuity of a semicontinuous function on a compact space is residual. Moreover, D is the set of points of continuity of the upper semicontinuous function

$$F(x) = \sum_{n=0}^{\infty} (1/2^n)(g_n - f_n)(x)$$

where $f_0 = f$ and $g_0 = g$. Now, if $x \in D$, then $g_n(x) \to g(x)$, $f_n(x) \to f(x)$ and so $\operatorname{diam} A_n = g_n(x) - f_n(x) \to g(x) - f(x)$. Having in mind that $\operatorname{diam} A_n \to \operatorname{diam} A$, it is clear that $g(x) - f(x) = d$ for a dense set of points in K, thus for every point of continuity of g and f, as desired. □

The case of nested sequences. A sequence $\{A_n\}$ of sets is *nested* if $A_{n+1} \subset A_n$ for every $n \in \mathbb{N}$. As mentioned above, we don't know whether A is diametrically maximal when $A = \lim_n A_n$ and the sets A_n are diametrically maximal. However, what if $\{A_n\}$ is a nested sequence? This still seems to be too general a question and so we will restrict our attention to the following problem: Given a nested sequence $\{A_n\}$ of diametrically maximal sets, is their intersection $A = \cap_n A_n$ also diametrically maximal? We present first an example to show that the set A can, eventually, be empty. Then we will prove that, in $(C(K), \|\cdot\|_\infty)$, A is always nonempty and diametrically maximal.

Example 4.4. Let $X = \{f \in C[-1,1] : f(0) = \int_0^1 f$ *and* $f(1) = 0\}$ endowed with the sup norm. Consider A_n defined as those $f \in X$ satisfying $1 - 1/n \le f(x) \le 2$ for $x \in [-1,0]$ and $0 \le f(x) \le 1 + 1/n$ for $x \in [0,1]$. Then $\{A_n\}$ is a nested sequence of diametrically maximal sets with $\operatorname{diam} A_n = 1 + 1/n$ but $\cap_n A_n = \emptyset$.

Proposition 4.5. *Let $\{A_n\} \subset \mathcal{DM}$ be a nested sequence in $(C(K), \|\cdot\|_\infty)$, where K is a compact Hausdorff space. Then $A = \cap_n A_n$ is nonempty and diametrically maximal.*

Proof. We keep the notation of Proposition 4.3. According to (4), we represent $A_n = [f_n, g_n]$ where $f_n, g_n \colon K \to \mathbb{R}$ form admissible pairs of densely continuous functions, for every $n \in \mathbb{N}$, and $g_n(x) - f_n(x) = \operatorname{diam} A_n$ for every $x \in D_{f_n} \cap D_{g_n}$. Define $g = \inf g_n$ and $f = \sup f_n$. Then f and g are lower and upper semicontinuous, respectively, hence densely continuous, and $f(x) \le g(x)$ for every $x \in K$, because of the nested condition of $\{A_n\}$. Finally, in virtue of Proposition 2.2 of [18], $\{g_n(x)\}$ is a decreasing sequence and $\{f_n(x)\}$ is an increasing sequence whenever $x \in D = (D_g \cap D_f) \cap (\cap_n (D_{g_n} \cap D_{f_n}))$. Since D is dense in K and f is lower semicontinuous, we have $\limsup_{y\to x} f(y) = \limsup_{y\to x,\, y\in D} f(y)$ for every $x \in K$. Also, $\liminf_{y\to x} g(y) = \liminf_{y\to x,\, y\in D} g(y)$ since g is upper semicontinuous. Analogous statements for f_n, g_n and the fact that they form admissible pairs yield

$$\begin{aligned}
\limsup_{y\to x} f(y) &= \lim \sup_{y\to x,\, y\in D} f(y) \overset{(*)}{=} \lim_n (\lim \sup_{y\to x,\, y\in D} f_n(y)) \\
&= \lim_n (\limsup_{y\to x} f_n(y)) \le \lim_n (\lim \inf_{y\to x} g_n(y)) \\
&= \lim_n (\lim \inf_{y\to x,\, y\in D} g_n(y)) \overset{(**)}{=} \lim \inf_{y\to x,\, y\in D} g(y) \\
&= \lim \inf_{y\to x,\, y\in D_g} g(y)
\end{aligned}$$

for every $x \in K$. To prove the equality $(*)$, denote by $\alpha(x) = \limsup_{y\to x,\, y\in D} f(y)$ and $\beta(x) = \lim_n (\limsup_{y\to x,\, y\in D} f_n(y))$. Then, $\alpha(x) \ge \beta(x)$ for every $x \in K$, simply because $f(y) \geq f_n(y)$ for every $y \in D$ and every $n \in \mathbb{N}$. Suppose that the reverse inequality doesn't hold, so there exists $x_0 \in K$ satisfying $\alpha(x_0) - \beta(x_0) > 0$. Since $\{\operatorname{diam} A_n\}$ is a decreasing sequence of nonnegative numbers, it is convergent. Hence, if we fix $0 < \varepsilon < \frac{\alpha(x_0)-\beta(x_0)}{4}$, there is $n_0 \in \mathbb{N}$ such that

$$0 \leq \operatorname{diam} A_n - \operatorname{diam} A_m < \varepsilon \tag{18}$$

whenever $m \geq n \geq n_0$. Now, using the definition of $\beta(x_0)$, there is a neighborhood U of x_0 such that $f_{n_0}(y) \leq \beta(x_0)+\varepsilon$ if $y \in U \cap D$. On the other hand, using the definitions of f and $\alpha(x_0)$, there are $n_1 > n_0$ and $x_1 \in (U \cap D)$ satisfying $f_{n_1}(x_1) > \alpha(x_0) - \varepsilon$. Therefore, $g_{n_1}(x_1) > \operatorname{diam} A_{n_1} + \alpha(x_0) - \varepsilon$ which implies also $g_{n_0}(x_1) > \operatorname{diam} A_{n_1} + \alpha(x_0) - \varepsilon$ since $g_{n_0}(x_1) \geq g_{n_1}(x_1)$. But $f_{n_0}(x_1) \leq \beta(x_0) + \varepsilon$ and this yields

$$\operatorname{diam} A_{n_0} = g_{n_0}(x_1) - f_{n_0}(x_1) \geq \operatorname{diam} A_{n_1} + \alpha(x_0) - \beta(x_0) - 2\varepsilon$$

which leads to

$$\operatorname{diam} A_{n_0} - \operatorname{diam} A_{n_1} > \frac{\alpha(x_0) - \beta(x_0)}{2} > 2\varepsilon$$

in contradiction with (18). The proof of the equality $(**)$ is analogous and, for this reason, omitted. So far we have proved that that f, g form also an admissible pair and hence $[f, g] \neq \emptyset$ [18]. It is not difficult to check that $[f, g] = A$ and we just need to prove that $[f, g]$ is diametrically maximal. Let $d = \lim_n\{\operatorname{diam} A_n\}$. Then, for every point $x \in D$, we have $\operatorname{diam} A_n = g_n(x) - f_n(x) \to g(x) - f(x)$. Hence, since D is dense in K, $g(x) - f(x) = d$ for every $x \in D_f \cap D_g$. This implies, using the characterization obtained in [19], that $[f, g]$ is a diametrically maximal set with diameter d. □

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