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Star-Shaped Separability with Applications*

A. M. Rubinov
Centre for Informatics and Applied Optimization, SITMS, University of Ballarat, Ballarat, Victoria 3353, Australia

E. V. Sharikov
Centre for Informatics and Applied Optimization, SITMS, University of Ballarat, Ballarat, Victoria 3353, Australia

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We discuss the notion of a support collection to a star-shaped set at a certain boundary point and a weak separability of two star-shaped sets. Applications to some problems, including the minimization of a star-shaped distance, are given.

Keywords: Star-shaped sets, separability, distance function, star-shaped distance, best approximation

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1. Introduction

Separability of two convex sets is one of the fundamental facts of convex analysis that can be considered as a geometrical form of Hahn-Banach theorem. Some attempts to extend the notion of separability for star-shaped sets were undertaken in [4] and [3]. (Recall that a set is star-shaped if it can be represented as the union of a family of convex sets $(U_t)_{t\in T}$, such $\bigcap_{t\in T} U_t \neq \emptyset$.) The support collection of linear functions at a regular boundary point x of a star-shaped set $U \subset \mathbb{R}^n$ was defined there and the existence of this collection was proved. A separability of two star-shaped sets by means of m linearly independent linear functions (the so-called weak separability) was also defined and studied. In the current paper we introduce the notion of a conical support collection and discuss some properties of conical collections and also weak separability (Section 2). We also examine some applications of these notions (Sections 3–5). One of the main goals of this paper is to indicate some areas of research where the star-shaped separability can be used.

In Section 3 we study a "best approximation-like" problem for star-shaped sets: we introduce a star-shaped distance and consider the minimization of this distance over a star-shaped set. This is a non-convex optimization problem.

One of the most challenging questions that arise in modern optimization is the development of a theory of global minimization for some broad classes of non-convex optimization problems. The theory of local optimization is based on calculus and its sophisticated generalizations. Different tools should be used in global optimization. Since separability by

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a linear function has found applications in convex programming, it is natural to apply separability by a collection of linear functions in star-shaped optimization.

From a certain point of view, classical best approximation problems are the simplest convex nonlinear problems. Similarly, star-shaped best approximation problems are the simplest star-shaped optimization problems, so we start with best approximation. One of the interesting questions here is the description of functions with star-shaped level sets, in particular star-shaped distances. We chose the following way for such a description: it is known (see, for example, [3]) that each upper semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ can be represented as the pointwise infimum of a family of convex finite functions. In order to describe functions with the given properties, we need to indicate properties of a family of convex functions, such that the infimum over this family gives a required function. Such approach allows us to describe some star-shaped distances. In particular, the function of the form $d(x, y) = \inf_{t \in T} \|x - y\|_t$ is a star-shaped distance, if $(\|\cdot\|_t)_{t \in T}$ is a family of norms such that $c\|x\|_* \leq \|x\|_t \leq C\|x\|_*$ for all $x \in \mathbb{R}^n$, where $0 < c < C < +\infty$ and $\|\cdot\|_*$ is a fixed norm.

We show that characterization of best approximation can be done in terms of weak separability of star-shaped sets. A challenging problem is to describe separation collections of linear functions at least in simple cases. This is the topic of the further research.

In Section 4 we show that some problems involving the distance function to an arbitrary (not star-shaped) set can be studied by means of star-shaped analysis. In Section 5 we give another application of a conical support collection of linear functions. Using these collections we define a degree of strict non-convexity of a radiant set U near its regular boundary point. Examples demonstrate that this definition is not of a local nature. We study some simple properties of the degree of strict non-convexity.

We use the following notation:

$\mathbb{R}^n$ is the usual n-dimensional vector space with the topology of the coordinate-wise convergence,

$\mathbb{R} = \mathbb{R}^1$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

$[l, x] = \sum_{i=1}^n l_i x_i$ is the inner product of vectors l and x.

If $x^* \in (\mathbb{R}^n)^*$ is a linear function generated by a vector $l \in \mathbb{R}^n$ then we use notation $x^*(x) = [l, x]$.

We need the following definitions. Let $U \subset \mathbb{R}^n$ be a set. The set

$$\operatorname{kern} U = \{u \in U : u + \lambda(x - u) \in U \quad \text{for all } x \in U \text{ and } \lambda \in [0, 1]\}$$

is called the kernel of U. A set U is called star-shaped if $\operatorname{kern} U \neq \emptyset$. A star-shaped set U is called radiant if $0 \in \operatorname{kern} U$. If U is a star-shaped set and $u \in \operatorname{kern} U$ then $U - u$ is radiant.

Let $\mathcal{U}_n$ be the totality of all radiant sets $U \subset \mathbb{R}^n$ that are nontrivial in the sense that $U \neq \{0\}$. Let $U \in \mathcal{U}_n$. The function

$$\mu_U(x) = \inf\{\lambda > 0 : x \in \lambda U\} \quad (x \in \mathbb{R}^n)$$

is called the Minkowski gauge of U. (By definition, $\inf \emptyset = +\infty$.) If $U \in \mathcal{U}_n$ then $\mu_U(x) = 1$ if and only if $x \in \operatorname{bd} U$ and $\lambda x \notin U$ for all $\lambda > 1$. (Here and below $\operatorname{bd} U$ stands for the boundary of a set U.) It is known (see, for example [3]) that $0 \in \operatorname{int} \operatorname{kern} U$ if and only if μ_U is Lipschitz.

2. Support collections and weak separability

We start with the following definition (see [4] and [3], Subsection 5.4.3).

Definition 2.1. Let $U \subset \mathbb{R}^n$ and let $x \neq 0$ belong to the closure $\operatorname{cl} U$ of the set U. A collection of linearly independent vectors $\ell = (l_1, \dots, l_m)$ is called a support collection to U at x if $[l_i, x] = 1\,(i = 1, \dots, m)$ and

$$\min_{i=1,\dots,m} [l_i, u] < 1 \quad \text{for all } u \in U, u \neq x. \tag{1}$$

Equalities $[l_1, x] = \ldots = [l_m, x]$ are used here only for normalization. It is important that $[l_i, x] > 0$ for all i.

Let $U \subset \mathbb{R}^n$ be a set and $x \in U$. Recall the Bouligand cone $\Gamma(x, U)$ consists of all vectors z such that for each $\alpha_0 > 0$ and $\varepsilon > 0$ there exist v and $\alpha > 0$ such that $\|v - z\| < \varepsilon$, $\alpha < \alpha_0$ and $x + \alpha v \in U$.

For $U \in \mathcal{U}_n$ consider the set

$$\Delta(U) = \{x \in U : \mu_U(x) = 1 \text{ and } x \notin \Gamma(x, U)\}. \tag{2}$$

The inclusion $\Delta(u) \subset \operatorname{bd} U$ holds. A point $x \in U$ is called a regularly boundary point of U if $x \in \Delta(U)$.

Remark 2.2. If $0 \in \operatorname{int} \operatorname{kern} U$ then μ_U is a Lipschitz function. In such a case (see [3], Propositions 5.15 and 5.17) $\operatorname{bd} U = \{x \in \mathbb{R}^n : \mu_U(x) = 1\}$ and (see [3], Corollary 5.6) $x \notin \Gamma(x, U)$ for all $x \in \operatorname{bd} U$. Hence $\Delta(U) = \operatorname{bd}(U)$.

The following result holds.

Theorem 2.3 (see [3], Theorem 5.7). *Let $U \in \mathcal{U}_n$ be a closed set and let $x \in \Delta(U)$. Then there exists a support collection $\ell = (l_1, \dots, l_n)$ to the set U at the point x.*

We present a geometric interpretation of a support collection ℓ. For each collection $\ell = \{l_1, \dots, l_m\}$ of m vectors consider the cone $T^\ell = \{y : [l_i, y] \geq 0,\ i = 1, \dots, m\}$. Then ℓ is a support collection to a set U at a point $x \in \operatorname{cl} U$ if and only if vectors $(l_1, \dots, l_m)$ are linearly independent, $U \cap (x + T^\ell) = \{x\}$ and $[l_i, x] = 1$ for all $i = 1, \dots, m$. If $U \cap (x + T^\ell) = \{x\}$ and $x \in \operatorname{int} T^\ell$ then $[l_i, x] > 0$ and the collection $\ell' = (l'_1, \dots, l'_m)$ with $l'_i = l_i/[l_i, x]$ is support at x. If vectors $(l_1, \dots, l_m)$ are linearly independent then the cone T^ℓ does not contain straight lines if and only if $m = n$.

Sometimes it is convenient to consider a weaker object than a support collection. First we recall the following definition.

Definition 2.4. Vectors $l_1, \dots, l_m \in \mathbb{R}^n$ are said to be conically independent if conditions

$$\alpha_1 l_1 + \cdots + \alpha_m l_m = 0, \quad \alpha_i \geq 0 \quad \text{for all } i = 1, \dots, m$$

imply that $\alpha_1 = \cdots = \alpha_m = 0$.

Conical independence of the collection $\ell = (l_1, \dots, l_m)$ means that $-l_i$ does not belong to the cone spanned by $(l_k)_{k \neq i}$ for all i.

Definition 2.5. Let $U \subset \mathbb{R}^n$ be a closed set and $x \in U, x \neq 0$. A collection of vectors $\ell = (l_1, \dots, l_m)$ is called a conical support collection to U at x if $[l_i, x] = 1\,(i = 1, \dots, m)$ and

$$\min_{i=1,\dots,m} [l_i, u] < 1 \quad \text{for all } u \in U, u \neq x. \tag{3}$$

A conical support collection $\ell = (l_1, \dots, l_m)$ at x consists of conically independent vectors. Indeed, let $\sum_{i=1}^m \alpha_i l_i = 0$ where $\alpha_i \geq 0$ for all $i = 1, \dots, m$. Then $\sum_{i=1}^m \alpha_i [l_i, x] = \sum_{i=1} \alpha_i = 0$, hence $\alpha_i = 0,\ i = 1, \dots, m$.

It is clear that each support collection is a conical support collection. It follows from Definition 2.1, that a support collection cannot contain more than n vectors, on the other hand a conical collection can contain an arbitrary finite number of vectors.

Let $\ell = (l_1, \dots, l_m)$ be a conical support collection and $T^\ell = \{y : [l_i, y] \geq 0,\ i = 1, \dots, m\}$ be the cone generated by this collection. Then $\operatorname{int} T^\ell$ is nonempty and contains the cone $\{y : [l_i, y] > 0\ i = 1, \dots, m\}$. It is known (see, for example, Proposition 5.32 and Remark 5.12 in [3]) that for each convex cone Q with $\operatorname{int} Q \neq \emptyset$ there exists a collection $\ell' = (l'_1, \dots, l'_n)$ of n linearly independent vectors such that $[l'_i, x] = 1$ for all $i = 1, \dots, n$ and $T^{\ell'} \subset \operatorname{int} Q$. It follows from this that existence of a conical support collection to U at x implies existence of a support collection to U at x. However, the number of vectors in these collections can be different.

We now discuss some properties of conical support collections.

Proposition 2.6. *Let U be a closed radiant set, and $x_0 \in \operatorname{bd} U$. Let $\ell = (l_1, \dots, l_m)$ be a conical support collection at x_0 and*

$$U_i = \{x \in U : [l_i, x] \leq 1\} = U \cap H_i, \quad \text{where } H_i = \{x \in \mathbb{R}^n : [l_i, x] \leq 1\}. \tag{4}$$

Then $\bigcup_{i=1,\dots,m} U_i = U$ and

1) *for each $i = 1, \dots, m$ the set U_i is a nonempty radiant set and* $\operatorname{kern} U_i \supset \operatorname{kern} U \cap H_i$*; the Minkowski gauge μ_{U_i} of U_i has the form*

$$\mu_{U_i}(x) = \max(\mu_U(x), [l_i, x]); \tag{5}$$

2) *for each $x \in U$ there exists i such that $R_x \cap U \subset U_i$ (here $R_x = \{\lambda x : \lambda \geq 0\}$);*
3) *let*

$$V = \bigcap_{i=1,\dots,m} U_i = U \cap \left(\bigcap_{i=1,\dots,m} H_i \right). \tag{6}$$

Then $V \in \mathcal{U}_n$ and

$$\mu_V(x) = \max_i \max(\mu_U(x), [l_i, x]). \tag{7}$$

Proof. *1*) Let $y \in \operatorname{kern} U$ and $[l_i, y] \leq 1$. Let $u \in U_i$, that is, $u \in U$ and $[l_i, u] \leq 1$. For each $\lambda \in [0, 1]$ we have $\lambda y + (1 - \lambda)u \in U$ and $[l_i, \lambda y + (1 - \lambda)u] \leq 1$. This means that $y \in \operatorname{kern} U_i$. We showed that $\operatorname{kern} U_i \supset (\operatorname{kern} U) \cap H_i$. It follows from this that $0 \in \operatorname{kern} U_i$, hence U_i is a nonempty radiant set. It is well-known (see, for example [3]) that the

Minkowski gauge of the intersection of a finite number of sets is equal to the maximum of the Minkowski gauges of these sets. On the other hand, $\mu_{H_i}(x) = \max(0, [l_i, x])$, where μ_{H_i} is the Minkowski gauge of the half-space H_i. This implies (5) and also (7).

2) Let $x \in U$ and $\lambda_x = \sup\{\lambda \geq 0 : \lambda x \in U\}$. If $\lambda_x < +\infty$ then $\lambda_x x \in U$. Let i be the index such that $\lambda_x x \in U_i$. Then $R_x \cap [0, \lambda_x]x = R_x \cap U \subset U_i$. Assume now that $\lambda_x = +\infty$. Then $R_x \cap U = R_x$. For each $i = 1, \ldots, m$ consider the set $\Lambda_i = \{\lambda \geq 0 : \lambda x \in U_i\}$. There exists at least one i such that Λ_i is unbounded. It easy to check that $R_x = \{\lambda x : \lambda \in \Lambda_i\}$, hence $R_x \subset U$.

3) V is radiant as the intersection of radiant sets. Since $[l_i, x_0] = 1$ for all i, it follows that $x_0 \in V$, therefore $V \neq \{0\}$. Hence $V \in \mathcal{U}_n$. □

We need the following definition (see [4] and also ([3], Definition 5.17).

Definition 2.7. Let U and V be subsets of $\mathbb{R}^n$ and $\ell = (l_i)_{i=1,\ldots,m}$ be a collection of linearly independent vectors. The sets U, V are said to be weakly separated by vectors $(l_i)_{i=1,\ldots,m}$ if for each pair $u \in U, v \in V$ there exists $i \in I$ such that $[l_i, u] \leq [l_i, v]$. We say that U, V are conically weakly separated if there exists a collection ℓ of conically independent vectors with the indicated property.

Proposition 2.8. *Let U and V be conically weakly separated by vectors $(l_i)_{i=1}^m$ and* $\operatorname{int} U \neq \emptyset$. *Then $V \cap \operatorname{int} U = \emptyset$.*

Proof. First we show that $U - V \neq \mathbb{R}^n$. Consider the superlinear function $q(x) = \min_{i=1,\ldots,m}[l_i, x]$. Weak separability of the sets U and V means that $q(u - v) \leq 0$ for all $u \in U, v \in V$. Let $\bar\partial q(0) = \{l \in \mathbb{R}^n : [l, x] \geq q(x)$ for all $x \in \mathbb{R}^n\}$ be the superdifferential of q at zero. Then $\bar\partial q(0)$ coincides with convex hull $S = \{l = \sum_{i\in 1}^m \alpha_i l_i : \alpha_i \geq 0\ (i = 1, \ldots, m), \sum_{i=1}^m \alpha_i = 1\}$ of vectors $(l_i)_{i=1}^m$. Since these vectors are conically independent we conclude that $0 \notin S$. Then there exists $x \in \mathbb{R}^n$ such that $0 < \inf_{l\in S}[l, x] = q(x)$. This means that $x \notin U - V$.

Assume that there exists $z \in (\operatorname{int} U) \cap V$. Let $B_\varepsilon(z) \subset U$ be a neighborhood of z. Then $B_\varepsilon(z) - z \subset U - V$ is a neighborhood of zero. Since q is positive homogeneous and $q(x) \leq 0$ for $x \in U - V$ it follows that $q(x) \leq 0$ for all $x \in \mathbb{R}^n$, which is a contradiction. □

Remark 2.9. Let $l_1, \ldots, l_m$ be a collection of vectors in $\mathbb{R}^n$ such that $0 \in \operatorname{co}(l_1, \ldots, l_m)$. Then $q(x) = \min_{i=1,\ldots,m}[l_i, x] \leq 0$ for all $x \in \mathbb{R}^n$. This implies the following assertion: let $U, V \subset \mathbb{R}^n$ be two arbitrary sets. Then for each $u \in U, v \in V$ there exists $i \in \{1, \ldots, m\}$ such that $[l_i, u] \leq [l_i, v]$. A collection $\ell = (l_i)_{i=1}^m$ does not depend on sets U, V. Note that there exist $n + 1$ vectors $(l_i)_{i=1}^{n+1}$ such that $0 \in \operatorname{co}(l_1, \ldots, l_{n+1})$.

For weak separability we consider collections $(l_i)_{1=1}^m$ of no more than n vectors and these vectors are linearly independent. It can be shown (see Proposition 2.11 below) that this collection can be chosen as a support collection to a certain set Z at a certain point $\bar z$. This means that this collection enjoys an additional property: $[l_i, \bar z] > 0$, $i = 1, \ldots, m$ and also that strict inequalities can be used instead of nonstrict ones.

A conical collection $(l_i)_{i=1}^m$ can contain more then n vectors. However, $0 \notin \operatorname{co}(l_1, \ldots, l_m)$ for such a collection.

Under some additional assumptions it can be proved that if $(\operatorname{int} U) \cap V = \emptyset$ then U and

V can be weakly separated (see [4, 3]).

Theorem 2.10 (see Theorem 5.8 in [3]). *Let U and V be star-shaped sets such that* $\operatorname{int}\operatorname{kern} U \neq \emptyset$ *and* $(\operatorname{int} U) \cap V = \emptyset$. *Then U and V are weakly separated.*

Assume that $U \cap V \neq \emptyset$. The proof of the given theorem (Theorem 5.8 in [3]) is based on the following construction, which is a modification of the construction from [4]. Let $u \in \operatorname{int}\operatorname{kern} U$, $v \in \operatorname{kern} V$. Consider the point $z = v - u$ and the set $Z = (U-u)-(V-v) = U - V + z$. Then Z is a radiant set and $0 \in \operatorname{int}\operatorname{kern} Z$. It can be shown that either $z \notin \operatorname{cl} Z$ or $z \in \operatorname{bd}\operatorname{cl} Z$. Since $U \cap V \neq \emptyset$ it follows that $0 \in U - V$, hence $z \in Z$. This implies that z is a boundary point of $\operatorname{cl} Z$. Since $z \in \operatorname{int}\operatorname{kern} Z$ it follows that $z \in \Delta(Z)$. Then there exists a support collection $\ell = (l_1, \dots, l_n)$ to $\operatorname{cl} Z$ at the point z. It is easy to check that ℓ weakly separate U and V. It follows from the aforesaid that the following statement holds.

Proposition 2.11. *Let U and V be star-shaped sets such that* $\operatorname{int}\operatorname{kern} U \neq \emptyset$, *the set $U \cap V$ is nonempty and the set* $(\operatorname{int} U) \cap V$ *is empty. Let* $z \in \operatorname{kern} V - \operatorname{int}\operatorname{kern} U$ *and* $Z = U - V + z$. *Then there exists a support collection* $\ell = (l_1, \dots, l_m)$, $(m \leq n)$ *to Z at z and this collection weakly separate sets U and V. In other words, the following holds:*

1) $[l_1, z] = \dots = [l_m, z] = 1$;

2) *for each* $u \in U, v \in V$ *with* $u \neq v$ *there exists i such that* $[l_i, u] < [l_i, v]$.

We only comment the assertion *2*). If $u \in U$, $v \in V$ and $u \neq v$, then $u - v + z \neq z$, hence there exists i such that $[l_i, u - v + z] < 1 = [l_i, z]$.

3. Star-shaped distance and its minimization

The following well-known corollary of Hahn-Banach theorem is a classical result of the approximation theory. Let U be a convex subset of a normed space X and $x \notin U$ and let $\bar{u} \in U$ be best approximation of x by elements of U, that is, $r := \min\{\|u - x\| : u \in U\} = \|\bar{u} - x\|$. Then there exists a continuous linear function l such that $l(u) \leq l(\bar{u}) \leq l(v)$ for all $u \in U$ and $v \in B(x, r) = \{y : \|x - y\| \leq r\}$. We can present this result in the following form. An element $\bar{u}$ is the best approximation of x by U if and only if there exists a continuous linear function l such that

$$0 = (-l, l)(\bar{u}, \bar{u}) = \min\{(-l, l)(u, v) : (u, v) \in U \times B(x, r)\} \tag{8}$$

If U is strictly convex then in addition to (8) the following holds:

$$((u, v) \in U \times V,\ (u, v) \neq (\bar{u}, \bar{u})) \implies (-l, l)(u, v) > 0. \tag{9}$$

We now give a version of (8)–(9) for star-shaped sets in $\mathbb{R}^n$. We assume that $\mathbb{R}^n$ is equipped with its usual topology of the pointwise convergence. Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be a star-shaped set and $x \notin U$, let $r = \inf\{\|u - x\| : u \in U\}$. Then the intersection $U \cap \{v : \|x - v\| < r\}$ is empty, so the sets U and $\{v : \|x - v\| \leq r\}$ can be weakly separated. We do not need to have exactly a norm in order to prove this result, so we consider a more general situation. First consider a function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\rho(x, y + \alpha(x - y)) \leq \rho(x, y), \quad x, y \in \mathbb{R}^n, \alpha \in [0, 1]. \tag{10}$$

It is easy to check that the function ρ enjoys this property if and only if the "balls" $B(x,r) = \{y : \rho(x,y) \le r\}$ are star-shaped with $x \in \operatorname{kern} B(x,r)$ for all $r > 0$. We need to have star-shaped balls $B(x,r)$ such that

(1) $x \in \operatorname{int}\operatorname{kern} B(x,r)$ for all $r > 0$.
(2) the inequality $\rho(x,y) < r$ holds for interior points of $B(x,r)$.

The following definition takes into account these requirements:

Definition 3.1. A function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is called a star-shaped distance if

(i) $\rho(x,x) = 0$ for all $x \in \mathbb{R}^n$ and $\rho(x,y) > 0$ for all $x, y \in \mathbb{R}^n$ $x \ne y$.
(ii) for each $x \in \mathbb{R}^n$ and $r > 0$ there exists a neighborhood V of x such that

$$\rho(x, y + \alpha(x' - y)) \le \alpha r + (1-\alpha)\rho(x,y), \quad y \in \mathbb{R}^n, x' \in V, \alpha \in [0,1]. \tag{11}$$

(iii) for each $x \in \mathbb{R}^n$ the function $\rho_x(y)$ defined by

$$\rho_x(y) = \rho(x,y), \quad y \in \mathbb{R}^n \tag{12}$$

has no local maxima.

(iv) for each $x \in \mathbb{R}^n$ the function ρ_x defined by (12) is continuous.

Let $x, y \in \mathbb{R}^n$. Then (11) with $r = \rho(x,y)$ and $x' = x$ implies (10), so sets $B(x,r)$ are star-shaped for all $x \in \mathbb{R}^n$ and $r > 0$. On the other hand (11) implies $x \in \operatorname{int}\operatorname{kern} B(x,r)$. Indeed, let $r > 0$, V be a neighborhood of x that is considered in (11) and let $\rho(x,y) \le r$. Then

$$\rho(x, \alpha x' + (1-\alpha)y) \le \alpha r + (1-\alpha)\rho(x,y) \le \alpha r + (1-\alpha) r = r,$$

so $V \subset \operatorname{kern} B(x,r)$. This means that $x \in \operatorname{int}\operatorname{kern} B(x,r)$.

Proposition 3.2. *Let ρ be a star-shaped distance. Then* $\operatorname{int} B(x,r) = \{v \in \mathbb{R}^n : \rho(x,v) < r\}$ *for each $x \in \mathbb{R}^n$ and $r > 0$.*

Proof. Since ρ_x is continuous, the set $\{v : \rho(x,v) < r\}$ is open. This implies $\{v : \rho(x,v) < r\} \subset \operatorname{int} B(x,r)$. Let $\rho(x,v) = r$ and $v \in \operatorname{int} B(x,r)$. Then there exists a neighborhood V of zero such that $v + V \subset \operatorname{int} B(x,r)$. For all $v' \in V$ we have $\rho(x, v' + v) \le r = \rho(x,v)$. This means that v is a local maximum of ρ_x which contradicts (iii). □

We now give an example of a star-shaped distance.

Proposition 3.3. *Let $(f_t)_{t\in T}$ be an equicontinuous family of convex functions $f_t : \mathbb{R}^n \to \mathbb{R}_+$ such that $f_t(0) = 0$ and $\inf_{t\in T} f_t(x) > 0$ for $x \ne 0$. Then function $\rho(x,y) = \inf_{t\in T} f_t(x-y)$ is a star-shaped distance in $\mathbb{R}^n$.*

Proof. We need to check that (i)–(iv) hold.

(i) It follows from properties of the family $(f_t)_{t\in T}$ that $\rho(x,x) = 0$ and $\rho(x,y) > 0$ for $x \ne y$.

(ii) Let us check (11). Let $x \in \mathbb{R}^n$ and $r > 0$. Since $(f_t)_{t\in T}$ is equicontinuous at zero it follows that there exists a neighborhood V_0 of zero such that $f_t(v) \le r$ for all $v \in V_0$ and

$t \in T$. Let $V = x - V_0$ be a neighborhood of x. For any $x' \in V$ it holds:

$$\begin{aligned}\rho(x, y+\alpha(x'-y)) &= \inf_{t\in T} f_t(x-y-\alpha(x'-y)) = \inf_{t\in T} f_t((1-\alpha)(x-y)+\alpha(x-x')) \\ &\le \inf_{t\in T}((1-\alpha)f_t(x-y)+\alpha f_t(x-x')) \le (\inf_{t\in T}(1-\alpha)f_t(x-y))+\alpha r \\ &= (1-\alpha)\rho(x,y)+\alpha r.\end{aligned}$$

Thus (11) is valid.

(iii) We need to check that for each $x, y \in \mathbb{R}^n$ and small $\varepsilon' > 0$ there exists a direction v such that

$$\rho(x, y+\varepsilon' v) - \rho(x, y) > 0. \tag{13}$$

Let $z = x - y$. Then $\rho(x, y) = \inf_{t\in T} f_t(z)$. If $z = 0$ then (13) trivially holds, so we assume that $z \ne 0$. Consider the number $\varepsilon = \dfrac{\varepsilon'}{1+\varepsilon'}$ and put $u = \dfrac{z}{1-\varepsilon}$. Then for each $t \in T$ we have

$$f_t(z) = f_t((1-\varepsilon)u) = f_t((1-\varepsilon)u+\varepsilon 0) \le (1-\varepsilon)f_t(u) \le f_t(u) - \varepsilon \inf_{\tau\in T} f_\tau(u) = f_t\left(\frac{z}{1-\varepsilon}\right) - \varepsilon\chi$$

where $\chi = \inf_{\tau\in T} f_\tau(u) > 0$. Since $\dfrac{1}{1-\varepsilon} = 1+\varepsilon'$ this implies the following:

$$\rho(x, y) = \inf_{t\in T} f_t(z) \le \inf_{t\in T} f_t((1+\varepsilon')z) - \varepsilon\chi < \rho(x, y-\varepsilon' z).$$

Thus (13) holds.

(iv) Since the family $(f_t)_{t\in T}$ is equicontinuous it follows that $\rho_x(y) = \inf_{t\in T} f_t(x-y)$ is continuous. □

Example 3.4. Consider a family $(\|\cdot\|_t)_{t\in T}$ of norms for which there exist numbers $0 < c < C < +\infty$ such that

$$c\|x\|_* \le \|x\|_t \le C\|x\|_*, \tag{14}$$

for all $t \in T$, where $\|\cdot\|_*$ is a fixed norm. The right inequality in (14) shows that family $(\|\cdot\|_t)_{t\in T}$ is equicontinuous, the left inequality shows that $\inf_{t\in T}\|x\|_t > 0$ for all $x \ne 0$. Hence the function $d(x, y) = \inf_{t\in T}\|x-y\|_t$ is a star-shaped distance.

Theorem 3.5. *Let ρ be a star-shaped distance on $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a radiant set. Let $x \notin U$, $\bar{u} \in U$ and $r = \rho(x, \bar{u})$. Then*

1) *If $r = \min_{u\in U} \rho(x, u)$ then there exists m linearly independent vectors $l_1, \dots, l_m$ such that:*
   (i) $[l_1, x] = \ldots = [l_m, x] = 1$.
   (ii) for each $u \in U$ and $v \in B(x, r)$ with $u \ne v$ there exists an index i such that $[l_i, u] < [l_i, v]$.
2) *If there exist m conically independent vectors l_i such that the condition (ii') below holds then $r := \rho(x, \bar{u}) = \min_{u\in U} \rho(x, u)$. Here*
   (ii') $U \times B(x, r) = \bigcup_{i=1}^m (U \times B(x, r))_i$ where

$$(U \times B(x, r))_i = \{(u, v) \in U \times B(x, r) : [l_i, u] \le [l_i, v]\}.$$

(Condition (ii′) means that for every pair (u, v) with $u \in U$ and $v \in B(x, r)$ there exists i such that $[l_i, u] \le [l_i, v]$.)

Proof. *1*) Let $r := \rho(x, \bar{u}) = \min_{u \in U} \rho(x, u)$. It follows from the properties of the star-shaped distance that the set $B(x, r)$ is star-shaped and $x \in \operatorname{int} \operatorname{kern} B(x, r)$. The intersection $U \cap B(x, r)$ contains $\bar{u}$, hence nonempty. The intersection $U \cap \operatorname{int} B(x, r) = \emptyset$. Indeed, in view of Proposition 3.2 we have $\operatorname{int} B(x, r) = \{v : \rho(x, v) < r\}$. On the other hand $U \subset \{u : \rho(x, u) \ge r\}$.

Consider the set $Z = U - B(x, r)$. Since $0 \in \operatorname{kern} U$ and $x \in \operatorname{int} \operatorname{kern} B(x, r)$ it follows that $\bar{z} := -x \in \operatorname{kern} U - \operatorname{int} \operatorname{kern} B(x, r)$. Then (see Proposition 2.11) there exists m linearly independent vectors $l'_1, \dots, l'_m$ such that $[l'_1, -x] = \ldots = [l'_m, -x] = 1$ and for each $u \in U$, $v \in B(x, r)$ with $u \ne v$ there exists i such that $[l'_i, v] < [l'_i, u]$. Thus (i) and (ii) hold for $l_i = -l'_i$.

2) Let (ii') hold. Let $q(x) = \min_{i=1,\dots,m} [l_i, x]$. Then (ii') is equivalent to

$$q(u - v) \le 0 \quad \text{for all } u \in U,\ v \in B(x, r). \tag{15}$$

We claim that

$$U \cap \operatorname{int} B(x, r) = \emptyset. \tag{16}$$

Indeed, assume that there exists $u \in U$ and a neighborhood V of zero such that $u - V \subset B(x, r)$. In view of (15) we get $q(v) \le 0$ for all $v \in V$. It follows from positive homogeneity of q that $q(x) \le 0$ for all $x \in \mathbb{R}^n$, hence $0 \in \bar{\partial} q = \operatorname{co}\{l_1, \dots, l_m\}$. This contradicts the conical independence of vectors $l_1, \dots, l_m$.

Combining (16) and Proposition 3.2 we get $U \subset \{u \in \mathbb{R}^n : \rho(x, u) \ge r\}$. Since $\bar{u} \in U$ and $r = \rho(x, \bar{u})$ it follows that $r = \min_{u \in U} \rho(x, \bar{u})$. □

Theorem 3.5 can be considered as a version of (8)–(9) for $X = \mathbb{R}^n$. If U is a convex set and $\rho(x, y) = \|x - y\|$ and if we replace strict inequalities in (ii) with nonstrict ones, then (8) follows from Theorem 3.5 with $m = 1$. We cannot take $m = 1$ for convex sets if we use strict inequalities. However Theorem 3.5 holds with $m = 1$ for a strictly convex set U.

4. Star-shapedness and distance to a closed set

In this section we demonstrate that star-shapedness can be used in the study of arbitrary (not necessarily star-shaped) sets.

First we consider an arbitrary closed subset U of $\mathbb{R}^n$ with $0 \in U$. Let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^n$ and

$$d_U(x) = \inf\{\|x - u\| : u \in U\}, \quad x \in \mathbb{R}^n$$

be the distance function generated by this norm. Let β_U be the function defined on $\mathbb{R}^n$ by

$$\beta_U(x) = \|x\| - d_U(x). \tag{17}$$

Note that $\beta_U(x) = \|x\|$ for $x \in U$; if $x \notin U$ then $\beta(x) < \|x\|$. The sets $\{x \in \mathbb{R}^n : \beta(x) \le c\} = \{x \in \mathbb{R}^n : d_U(x) \ge \|x\| - c\}$, $c > 0$ can be useful for examination of the distance function. We will study these sets from the point of view of star-shapedness.

We need some preliminaries.

A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is called increasing-along-rays (IAR) if for each $x \neq 0$ the function of one variable $f_x(t) = f(tx)$ is increasing (that is $f_x(t_1) \geq f_x(t_2)$ for $t_1 \geq t_2$ on $[0, +\infty)$). (The definition of IAR function in more general situation was introduced in [1].) Note that $f(0) = \min_{x \in X} f(x)$ for an IAR function f. It has been proved in [5], (see also [1]) that a function f is IAR if and only if its level sets $S_r(f) := \{x \in X : f(x) \leq r\}$ are radiant for all $r \geq f(0)$.

Let

$$(d_U)^{\uparrow}_H(x, x) = \limsup_{\alpha \to +0, v \to x} \frac{d_U(x + \alpha v) - d_U(x)}{\alpha}$$

be the upper Hadamard directional derivative of d_U at a point x in the direction x. It is easy to check that $(d_U)^{\uparrow}_H(x, x) \leq \|x\|$. Indeed, since the distance d_U is Lipschitz continuous with the Lipschitz constant $L = 1$ it follows that $(d_U)^{\uparrow}_H(x, x) \leq \limsup_{v \to x} \|v\| = \|x\|$.

Theorem 4.1. *Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ be a point such that $\|x_0\| > (d_U)^{\uparrow}_H(x_0, x_0)$ and let $V = \{x \in \mathbb{R}^n : \|x\| - d_U(x) \leq \|x_0\| - d_U(x_0)\}$. Then there exists m linearly independent vectors $l_1, \ldots, l_m$ such that*

1) $[l_1, x_0] = \ldots = [l_m, x_0] = 1$*;*

2) *for each $x \in V$ there exists i such that $[l_i, x] \leq 1$.*

The sets $V_i = \{x \in V : [l_i, x] \leq 1\}$ are star-shaped for all i.

Proof. First we show that the function β_U defined by (17) is increasing-along-rays. Let $x \in \mathbb{R}^n$ and $\lambda > \mu \geq 0$. Then

$$d_U(\lambda x) = \inf_{u \in U} \|\lambda x - u\| = \inf_{u \in U} \|(\lambda - \mu)x + \mu x - u\| \leq (\lambda - \mu)\|x\| + d_U(\mu x),$$

hence

$$\beta_U(\lambda x) = \lambda\|x\| - d_U(\lambda x) \geq \mu\|x\| - d_U(\mu x) = \beta_U(\mu x),$$

so β_U is IAR. It follows from this that level sets $S_r(\beta_U) = \{x : \beta_U(x) \leq r\}$ of β_U are radiant for all $r \geq \beta(0) = 0$.

Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ be a given point and let $r = \beta(x_0) = \|x_0\| - d_U(x_0) \geq 0$. Then

$$V := S_r(\beta_U) = \{x \in \mathbb{R}^n : \|x\| - d_U(x) \leq \|x_0\| - d_U(x_0)\} \tag{18}$$

is a radiant set. Since $x_0 \neq 0$ it follows that $V \in \mathcal{U}_n$. We need to show that x_0 is a regularly boundary point, that is, $x_0 \in \Delta(V)$, where $\Delta(V) = \{x \in V : \mu_V(x) = 1, x \notin \Gamma(x, V)\}$ is the set defined for $V = S_r(\beta_U)$ by (2). Let us calculate $\mu_V(x_0)$. The inclusion $x_0 \in V$ implies $\mu_V(x_0) \leq 1$. Let us check that $\mu_V(x_0) \geq 1$ and hence $\mu_V(x_0) = 1$.

Since V is a radiant set and $0 \neq x_0 \in V$ then it is sufficiently to check that $\lambda x_0 \notin V$ for $\lambda > 1$. If $\lambda x_0 \in V$ for some $\lambda > 1$ then $(x_0 + \alpha x_0) \in V$ for all $\alpha \in (0, \lambda - 1)$, that is

$$\|x_0 + \alpha x_0\| - d_U(x_0 + \alpha x_0) \leq \|x_0\| - d_U(x_0) \quad \text{for all } \alpha \in (0, \lambda - 1).$$

This fact implies $\|x_0\| \leq (d_U)^{\uparrow}_H(x_0, x_0)$, which is impossible. Hence $\mu_V(x_0) = 1$.

Assume that $x_0 \in \Gamma(x_0, V)$. Then there exist sequences $v_k \to x_0$ and $\alpha_k \to 0$ such that $x_0 + \alpha_k v_k \in V$, that is

$$\|x_0 + \alpha_k v_k\| - \|x_0\| \leq d_U(x_0 + \alpha_k v_k) - d_U(x_0). \tag{19}$$

Let $p(x) = \|x\|$. Since $p'(x, x) = \|x\|$ it follows from (19) that $\|x_0\| \leq (d_U)^{\uparrow}_H(x_0, x_0)$, which is impossible. Hence $x_0 \notin \Gamma(x_0, V)$.

Applying Theorem 2.3 and Proposition 2.6 we conclude that the desired result holds. □

We now consider bounded subsets of $\mathbb{R}^n$.

Proposition 4.2. *Let U be a bounded subset of $\mathbb{R}^n$. Then the set* $\operatorname{hyp} d_U$ *is star-shaped. (Here* $\operatorname{hyp} d_U = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \lambda \leq d_U(x)\}$.*)*

Proof. Let $c > 0$ be a number such that $\|u\| \leq c$ for all $u \in U$. We will show that $(0, -c) \in \operatorname{kern} \operatorname{hyp} d_U$. Let $t \leq d_U(x)$ and $\alpha \in (0, 1)$. Then

$$\begin{aligned} -\alpha c + (1-\alpha)t &\leq -\alpha c + (1-\alpha) d_U(x) = \inf_{u \in U}(\|(1-\alpha)x - (1-\alpha)u\| - \alpha c) \\ &= \inf_{u \in U}(\|(1-\alpha)x - u + \alpha u\| - \alpha c) \leq \inf_{u \in U}(\|(1-\alpha)x - u\| + \alpha\|u\| - \alpha c) \\ &\leq \inf_{u \in U}\|(1-\alpha)x - u\| = d_U((1-\alpha)x. \end{aligned}$$

Hence

$$\alpha(0, -c) + (1-\alpha)(x, t) = ((1-\alpha)x, -\alpha c + (1-\alpha)t) \in \operatorname{hyp} d_U,$$

which means $(0, -c) \in \operatorname{kern} \operatorname{hyp} d_U$. □

Corollary 4.3. *Let U be a bounded subset of $\mathbb{R}^n$ and $0 \in U$. Then the sets* $\operatorname{epi} \|\cdot\|$ *and* $\operatorname{hyp} d_U$ *are weakly separated. (Here* $\operatorname{epi} \|\cdot\| = \{(x, \lambda) : \lambda \geq \|x\|\}$.*)*

Proof. Since $0 \in U$ then $d_U(x) \leq \|x\|$, that is $\operatorname{int} \operatorname{epi} \|\cdot\| \cap \operatorname{hyp} d_U = \emptyset$. It follows from convexity of norm that $\operatorname{int} \operatorname{kern} \operatorname{epi} \|\cdot\| = \operatorname{int} \operatorname{epi} \|\cdot\| \neq \emptyset$. Thus we can apply Theorem 2.10. □

5. Degree of strict non-convexity

Consider a radiant set U, which is non strictly convex. It is interesting to classify its boundary points in terms of their strict non-convexity. Conical support collections can be used for such a classification.

Definition 5.1. A positive integer m is called the degree of strict non-convexity of a set $U \in \mathcal{U}_n$ near a point $x \in \Delta(U)$ if there exists a conical support collection ℓ that consists of m conically independent vectors and there is no a support collection of $m-1$ conically independent vectors. We denote the degree of strict non-convexity by $\operatorname{nsc}(x, U)$.

A point $x \in \Delta(U)$ will be called a point of strict convexity of U if $\operatorname{nsc}(x, U) = 1$. We now present some simple illustrative examples.

Example 5.2. 1) Let $U \subset \mathbb{R}^2$ be a polyhedron with $0 \in \operatorname{int} U$. Then $\operatorname{nsc}(x, U) = 1$ for each vertex x of U and $\operatorname{nsc}(x, U) = 2$ for a point $x \in U$, which is not a vertex.

2) Let $U = U_1 \cup U_2$, where U_1 and U_2 are circles:

$$U_1 = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1)^2 + x_2^2 \leq 1\}, \quad U_2 = \{(x_1, x_2) : (x_1 + 1)^2 + x_2^2 \leq 1\}.$$

Then U is a radiant set. Consider all boundary points of U. Let $x = (0, 0)$. Then $\mu_U(x) = 0$ so $x \notin \Delta(U)$ so the degree of strict non-convexity is not defined at this point. Let $x = (x_1, x_2)$ be a boundary point of U with either $x_1 < -1$ or $x_1 > 1$. Then $\operatorname{nsc}(x, U) = 1$, so such points are points of strict convexity. Let $x = (x_1, x_2)$ be a boundary point with either $x_1 \in [-1, 0)$ or $x_1 \in (0, 1]$. Then $\operatorname{nsc}(x, U) = 2$.

3) Let $U = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \leq 1\}$. Then U is radiant and $\operatorname{nsc}(x, U) = 2$ for each boundary point x of U.

The second example shows that degree of strict non-convexity of U at x is a global notion: it is possible that $\operatorname{nsc}(x, U) > 1$ and the intersection of a set U with a small neighborhood of x is strictly convex (this means that U is locally strictly convex at x).

We now present a simple assertion about the degree of strict non-convexity.

Proposition 5.3. *Let $U_1, \ldots, U_k$ be strictly convex subsets of $\mathbb{R}^n$ such that $0 \in \operatorname{int} U_i$ for all i. Let $U = \bigcup_i U_i$ and $x \in \operatorname{bd} U$. Then nsc $(x, U) \leq k$.*

Proof. It easy to check that $\operatorname{kern} U \supset \cap_i \operatorname{int} U_i$, therefore $0 \in \operatorname{int} kerU$. In view of Remark 2.2 we conclude that each boundary point of U belongs to $\Delta(U)$, therefore $x \in \Delta(U)$. Since $x \in \operatorname{bd} U$ then $x \notin \operatorname{int} U_i$ for all i. It is well known from convex analysis that there exist vectors $l_1, \ldots, l_k \in \mathbb{R}^n \backslash \{0\}$ such that $[l_i, u] < [l_i, x]$ for all i and $u \in U_i$ $(u \neq x)$. Since $l_i \neq 0$ and $0 \in \operatorname{int} U_i$ it follows that $[l_i, x] > 0$ for all i. Denote $l'_i = l_i / [l_i, x]$. Then $[l'_i, x] = 1$ and $[l'_i, u] < 1$ for all i and $u \in U_i$ $(u \neq x)$. This implies $\min_i [l'_i, u] < 1$ for all $u \in U$ $(u \neq x)$. $\square$

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