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Linear Structures in the Set of Norm-Attaining Functionals on a Banach Space

Pradipta Bandyopadhyay
Stat–Math Division, Indian Statistical Institute,
203, B. T. Road, Kolkata 700 108, India
pradipta@isical.ac.in

Gilles Godefroy
Equipe d'Analyse, Université Paris VI, Case 186,
4 Place Jussieu, 75252 Paris Cedex 05, France
godefroy@math.jussieu.fr

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We show, among other results, that if the unit ball of the dual of a Banach space X is w^*-sequentially compact, the set of norm-attaining functionals contains a separable norm closed subspace M if and only if the dual M^* of M is the canonical quotient of X. We provide examples of spaces which cannot be renormed in such a way that the set of norm-attaining functionals become a linear space.

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1. Introduction

In recent years, quite some work has been done on the "lineability" of some natural subsets of Banach spaces, in other words, on the existence of linear subspaces of essentially non-linear subsets. We refer in particular to the paper [1] and references therein. The following terminology is by now standard:

Definition 1.1. A subset M of a topological vector space E is said to be *spaceable* in E if $M \cup \{0\}$ contains an infinite dimensional closed linear subspace.

In this note, we investigate the spaceability properties of the set $NA(X)$ of all norm attaining functionals on a Banach space X. This study is intimately related with isometric duality theory. If X is a dual space, then $NA(X)$ certainly contains its predual. It is also motivated by proximinality questions: indeed, an hyperplane $H = \ker x^*$ of X is proximinal in X if and only if $x^* \in NA(X)$. If a finite codimensional subspace Y of X is proximinal in X, then $Y^\perp \subseteq NA(X)$, and the converse also holds in some Banach spaces, but not always. We refer to [3, 19, 20, 28] for some recent progress on this question.

When a Banach space X has an infinite dimensional quotient which is isomorphic to a dual space, it is easy to construct an equivalent norm on X for which $NA(X)$ is spaceable. We show in this paper that the converse holds for Banach spaces whose dual unit ball is

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w^*-sequentially compact (Theorem 2.12), and that similar results are available in spaces which share some features of Asplund spaces. On the other hand, we also show that $NA(X)$ is not a linear space when a non-reflexive space X enjoys some form of the Radon-Nikodým Property (RNP) (Proposition 2.23, Proposition 2.24). Quite naturally, one of our main tools is James' characterization of weakly compact sets [25], and more precisely the combinatorial principle which lies in the proof of its separable case, namely Simons' inequality [32].

This work leads to several natural open questions, which are scattered throughout this note. Our hope is to stimulate research on these topics.

2. The Results

Notation 2.1. We work with *real* Banach spaces. For a Banach space X, we will denote by B_X and S_X respectively the closed unit ball and the unit sphere of X. We will denote by $NA(X)$ the set of all continuous linear functionals which attain their norm on B_X. We will identify any $x \in X$ with its canonical image in X^{**}.

We recall that a closed linear subspace Y is said to be a proximinal subspace of X if for every $x \in X$ there exists $y \in Y$ such that $\|x - y\| = d(x, Y)$.

We begin with some simple observations. The first one is an immediate application of James' theorem.

Lemma 2.2. *Let Y be a proximinal subspace of X. Then X/Y is reflexive if and only if $Y^\perp \subseteq NA(X)$.*

Proof. If X/Y is reflexive, every $x^* \in Y^\perp \simeq (X/Y)^*$ is norm attaining on X/Y. Since Y is proximinal in X, $x^* \in NA(X)$. Thus $Y^\perp \subseteq NA(X)$.

Conversely, suppose $Y^\perp \subseteq NA(X)$. Then every $x^* \in Y^\perp$ attains its norm on X/Y. By James' theorem, X/Y is reflexive. □

Corollary 2.3. *A linear subspace $M \subseteq NA(X)$ is w^*-closed if and only if M is reflexive.*

It follows that if there exists a proximinal subspace Y of X such that X/Y is infinite dimensional and reflexive, then $NA(X)$ is spaceable. In fact, the above argument actually proves something more.

Lemma 2.4. *Let Y be a proximinal subspace of X such that X/Y is isometrically isomorphic to a dual space Z^*. Then $Z \subseteq NA(X)$.*

Proof. If $X/Y \simeq Z^*$, then every $x^* \in Z$ is norm attaining on X/Y. Thus, as above, $Z \subseteq NA(X)$. □

It follows that if there exists a proximinal subspace Y of X such that X/Y is isometrically isomorphic to an infinite dimensional dual space, then $NA(X)$ is spaceable. We now show that in certain Asplund-like spaces, up to renorming, these are the only situations when $NA(X)$ is spaceable. We refer to [8] for the renorming theory of Banach spaces.

We will need the following renorming result.

Lemma 2.5. *Let Y be a closed linear subspace of a Banach space $(X, \|\cdot\|)$. Let $\|\cdot\|_1$ be an equivalent norm on X/Y. Then there is an equivalent norm $|||\cdot|||$ on X that coincides with $\|\cdot\|$ on Y, whose quotient norm on X/Y coincides with a positive multiple of $\|\cdot\|_1$, and which makes Y proximinal.*

Proof. Since $\|\cdot\|_1$ is an equivalent norm on X/Y, there exist $\alpha, \beta > 0$ such that $\alpha\|x+Y\|_1 \leq \|x+Y\| < \beta\|x+Y\|_1$ for all $x \in X \setminus Y$. Now define $|||x||| = \max\{\|x\|, \beta\|x+Y\|_1\}$. Clearly, this is an equivalent norm on X. It is also clear that $|||y||| = \|y\|$ for any $y \in Y$.

Now for any $x \in X \setminus Y$, since $\|x+Y\| < \beta\|x+Y\|_1$, there exists $y \in Y$ such that $\|x+Y\| \leq \|x+y\| < \beta\|x+Y\|_1$. It follows that $|||x+y||| = \beta\|x+Y\|_1 = |||x+Y|||$. Thus, $|||x+Y||| = \beta\|x+Y\|_1$ and Y is proximinal in the $|||\cdot|||$ norm. □

We are now ready for our main results.

Notation 2.6. If M is a norm closed linear subspace of X^*, we say that M^* is the canonical quotient of X if the restriction $S = Q|_X$ of the canonical quotient map $Q : X^{**} \to M^*$ to X is an isometry between $X/M_\perp$ and M^*; equivalently, $S(B_X)$ is norm dense in B_{M^*}.

The following lemma extends previous results in [14, 16, 30].

Lemma 2.7. *Let X be a Banach space such that B_{X^*} is w^*-sequentially compact. Let $M \subseteq NA(X)$ be a norm closed separable subspace. Then M^* is the canonical quotient of X.*

Proof. Let S be as above and let $B = S(B_X) \subseteq B_{M^*}$. Let $\tau_p(B)$ denote the topology of pointwise convergence on B.

Since $M \subseteq NA(X)$, for every $m \in M$ there is $m^* \in B$ such that $m^*(m) = \|m\|$. In other words, B is a boundary of B_{M^*}. Since B_{X^*} is w^*-sequentially compact, for any $(m_n) \subseteq B_M$, there is a subsequence (m_{n_k}) which is $\tau_p(B)$-convergent, and hence, weakly Cauchy ([32], see [17, Corollary 2]). Thus, $M \not\supseteq \ell_1$. Following the lines of the proof of [16, Theorem I.2], we now show that M^* is the canonical quotient of X.

If $\overline{B} \neq B_{M^*}$, there exists $F \in B_{M^{**}}$ and $m_0^* \in B_{M^*}$ such that $F(m_0^*) > \sup F(B)$. Let $\sup F(B) < \alpha < F(m_0^*)$. Let $C = \{m \in B_M : m_0^*(m) > \alpha\}$. Clearly $F \in \overline{C}^{w^*}$. Since M is separable and $M \not\supseteq \ell_1$, the compact space $B_{M^{**}}$ is "angelic" in the sense defined in [6], and thus there is, by [29], a sequence $\{m_n\} \subseteq C$ such that $\lim_{n\to\infty} m^*(m_n) = F(m^*)$ for all $m^* \in B$. Since B is a boundary of B_{M^*}, it follows from Simons' inequality [32] that there is $m \in co(\{m_n\}) \subseteq C$ such that $\alpha > \sup m(B)$. Since, clearly, $\overline{B}^{w^*} = B_{M^*}$, this implies $\alpha > \|m\|$. But this contradicts $m_0^*(m) > \alpha$. □

Remark 2.8.

(a) Clearly, every separable Banach space satisfies the hypothesis of Lemma 2.7. By [23], so does any Asplund space.

(b) Lemma 2.7 is false for $X = \ell_1(c)$. Take $M = C[0,1] \subseteq NA(\ell_1(c))$. Lemma 2.10 below implies that the space $\ell_1(c)$ is essentially a minimal example.

(c) It may happen that M^* is not even isomorphic to a quotient of X. Let $X = L^\infty[0,1]$ and $M \subseteq NA(X)$ be the space constructed (under (CH)) in [16, p. 183]. One has

$M = \ell_1(c)$ and $M^* = \ell_\infty(c)$ is not a quotient of L^∞, since $\mathrm{dens}(L^\infty) = c$ and $\mathrm{dens}(\ell_\infty(c)) = 2^c$.

The following lemma provides the same conclusion under weaker assumptions on X, but under some mild topological assumption of the set $S(B_X)$.

Definition 2.9. A metrizable space is analytic if it is a continuous image of a Polish space.

We recall that any Borel subset of a Polish space is analytic.

Lemma 2.10. *Let $X \not\supseteq \ell_1(c)$ and $M \subseteq NA(X)$ be a norm closed separable subspace. If $S(B_X) \subseteq B_{M^*}$ contains a w^*-analytic boundary, then M^* is the canonical quotient of X.*

Proof. Let $\mathcal{A} \subseteq S(B_X)$ be a w^*-analytic boundary. Let

$$\mathcal{K} = B_{M^{**}}|_{\mathcal{A}} \subseteq \mathcal{F}(\mathcal{A}, \mathbb{R}).$$

$\mathcal{K}$ is $\tau_p(\mathcal{A})$-compact and since $B_{M^{**}} = \overline{B_M}^{w^*}$, $\mathcal{K} \cap \mathcal{C}(\mathcal{A}, w^*)$ is $\tau_p(\mathcal{A})$-dense in $\mathcal{K}$. Now by [6], $\mathcal{K}$ is angelic or $\mathcal{K} \supseteq L \simeq \beta\mathbb{N}$. If $\mathcal{K}$ is angelic, then $\overline{co}(\mathcal{A}) = B_{M^*}$ as in Lemma 2.7 and thus $B_{M^*} = \overline{S(B_X)}$.

Assume now that $\mathcal{K} \supseteq L \simeq \beta\mathbb{N}$. We can lift L to a compact subset L_0 of $B_{M^{**}}$ with $L_0 \simeq \beta\mathbb{N}$ and since $\mathcal{A} \subseteq S(B_X)$, the restriction of S^* to L_0 is one-to-one. Therefore (B_{X^*}, w^*) contains $S^*(L_0) \simeq \beta\mathbb{N}$, and thus, $X \supseteq \ell_1(c)$ by [33], a contradiction. □

Example 2.11. If X is weakly $\mathcal{K}$-analytic, then $X \not\supseteq \ell_1(c)$ and $S(B_X)$ is w^*-analytic, hence Lemma 2.10 applies. Note that Lemma 2.7 applies as well in this case.

The next result provides, for a large class of Banach spaces, a simple characterization of the existence of an equivalent norm for which the set $NA(X)$ is spaceable.

Theorem 2.12. *Let X be a Banach space such that B_{X^*} is w^*-sequentially compact. Then the following are equivalent:*

(a) There is an equivalent norm $\|\cdot\|$ on X such that $NA(X, \|\cdot\|)$ is spaceable.
(b) there is an infinite dimensional quotient space of X which is isomorphic to a dual space.

Proof. *(b)* ⇒ *(a)*. This implication holds in any Banach space X. Let Y be a subspace of X such that X/Y is isomorphic to an infinite dimensional dual space Z^*. Let $\|\cdot\|_1$ be an equivalent dual norm on X/Y. Now Lemma 2.5 produces an equivalent norm $|||\cdot|||$ on X which makes Y proximinal and X/Y isometrically isomorphic to Z^*. Thus the result follows from Lemma 2.4.

(a) ⇒ *(b)*. If $NA(X)$ is spaceable, then it contains an infinite dimensional separable subspace M. Thus the implication follows immediately from Lemma 2.7. □

Question 2.13. Is Theorem 2.12 true in full generality?

Under a mild regularity assumption, Lemma 2.10 provides a positive answer. Indeed, one has:

Proposition 2.14. *Let X be any Banach space and $M \subseteq NA(X)$ be a norm closed infinite dimensional separable subspace such that $S(B_X)$ contains a w^*-analytic boundary. Then there is an infinite dimensional quotient space of X which is isomorphic to a dual space.*

Proof. If $X \not\supseteq \ell_1(c)$, apply Lemma 2.10. If $X \supseteq \ell_1(c)$, then $\ell_\infty(\mathbb{N})$ is a quotient of X. $\square$

Theorem 2.12 applies of course when X is separable. Under the stronger assumption that X^* is separable, more can be said.

Theorem 2.15. *Let X be a Banach space such that X^* is separable. Then the following are equivalent:*

(a) *There is an equivalent norm $\|\cdot\|$ on X such that $NA(X, \|\cdot\|)$ is spaceable.*
(b) *X^* contains an infinite dimensional reflexive subspace.*

Proof. (b) $\Rightarrow$ (a). As before, this implication holds in any Banach space X and follows from Lemma 2.2, modulo the renorming via Lemma 2.5.

(a) $\Rightarrow$ (b). If X^* is separable, then so is X, and we can apply Lemma 2.7. If $M \subseteq NA(X)$ then $M^* = X/M_\perp$ and thus $M^{**} \subseteq X^*$. Therefore M^{**} is separable. This implies that M contains an infinite dimensional reflexive subspace [26] (see [10, Theorem 4.1] for a more general result). $\square$

Remark 2.16. The above proof actually shows that if X^* is separable and $M \subseteq NA(X)$ is a norm closed infinite dimensional subspace, then M contains an infinite dimensional reflexive subspace. This fails in general for separable $X \not\supseteq \ell_1(\mathbb{N})$. Indeed, by [22], there exists Z not containing an infinite dimensional reflexive subspace, such that Z^* is separable and does not contain $\ell_1(\mathbb{N})$. Then we can take $X = Z^*$ and $M = Z \subseteq NA(X) \subseteq Z^{**}$.

It follows from [5] or [11] that when the above space M is not reflexive, one may replace "reflexive" by "quasi-reflexive of order 1" in Theorem 2.15.

Question 2.17. Is Theorem 2.15 true for any Asplund space?

Let us note that the answer is affirmative if X is a weakly compactly generated (WCG) Asplund space.

Proposition 2.18. *Let X be a WCG Asplund space. Then the following are equivalent:*

(a) *There is an equivalent norm $\|\cdot\|$ on X such that $NA(X, \|\cdot\|)$ is spaceable.*
(b) *X^* contains an infinite dimensional reflexive subspace.*

Proof. (a) $\Rightarrow$ (b). Since X is Asplund, we can apply Lemma 2.7. Thus, if $M \subseteq NA(X)$ is a closed separable subspace, $M^* = X/M_\perp$. Therefore M^* is a WCG Asplund space. Since M is separable and M^* is WCG, M^* is separable. Moreover, since M^* is Asplund, M^{**} is separable. Thus, as before, by [26], M contains an infinite dimensional reflexive subspace. $\square$

The answer is also affirmative under some additional isometric assumptions on the Asplund space.

Definition 2.19. [4] A point $x \in S_X$ is said to be an almost LUR point of B_X if for any $\{x_n\} \subseteq B_X$ and $\{x_n^*\} \subseteq B_{X^*}$, the condition

$$\lim_m \lim_n x_m^* \left(\frac{x_n + x}{2} \right) = 1$$

implies $\lim_n \|x_n - x\| = 0$. We say that a Banach space X is almost LUR if every point of S_X is an almost LUR point.

Proposition 2.20. *Let X be an Asplund space. Assume that the norm $\|\cdot\|_X$ on X satisfies one of the following conditions:*

- (i) $\|\cdot\|_X$ *is almost LUR.*
- (ii) *the dual norm on X^* is Gâteaux differentiable.*

Then the following are equivalent:

- (a) $NA(X)$ *is spaceable.*
- (b) *there exists a proximinal subspace Y of X such that X/Y is infinite dimensional and reflexive.*

Proof. $(b) \Rightarrow (a)$ follows from Lemma 2.2.

$(a) \Rightarrow (b)$. Let $M \subseteq NA(X)$ be a norm closed infinite dimensional subspace.

It has been proved in [2] that $x \in S_X$ is an almost LUR point if and only if it is strongly exposed by every functional that attains its norm at x. It follows that (i) holds if and only if

$$NA(X) = \{x^* \in X^* : \|\cdot\|_{X^*} \text{ is Fréchet smooth at } x^*\} \cup \{0\}$$

Thus, the norm on M is Fréchet smooth and hence, M is an Asplund space with the RNP. Therefore, by [26], there is $Z \subseteq M$ infinite dimensional and reflexive. We now argue as in [3]. Let $Y = Z_\perp$ and $Q : X \to X/Y$ be the quotient map. Since Z is reflexive and smooth, X/Y is strictly convex. If $\lambda \in S_{X/Y}$, there is $x^* \in Y^\perp$ with $\|x^*\| = x^*(\lambda) = 1$ and since X/Y is strictly convex, $\{\sigma \in S_{X/Y} : x^*(\sigma) = \|\sigma\| = 1\} = \{\lambda\}$. Now since $x^* \in Y^\perp \subseteq NA(X)$, there is $x \in X$ with $\|x\| = x^*(x) = 1$, and thus $Q(x) = \lambda$. Hence $Q(B_X) = B_{X/Y}$ and Y is proximinal in X.

Under (ii), M has a Gâteaux smooth norm and it follows from Bishop-Phelps theorem that M^* is the canonical quotient of X. Therefore, M^{**} is Gâteaux smooth and thus, the norm on M is very smooth. Hence, M is an Asplund space with RNP and the conclusion follows as before. □

Remark 2.21. (i) applies in particular when $\|\cdot\|_X$ is LUR. Note that under the assumption (i), the proximinal subspace Y is actually a Chebyshev subspace with a continuous metric projection [3].

When a Banach space X has a w-UR norm, then it is an Asplund space [24] and its dual X^* is (uniformly) Gâteaux smooth. Hence, (ii) applies in particular to spaces with a w-UR norm.

Coming to non-spaceability, the following isomorphic result implies that the Asplund spaces with the Dunford-Pettis Property are far from being dual spaces.

Proposition 2.22. *Let X be an Asplund space with the Dunford-Pettis Property. Then norm closed linear subspaces of $NA(X)$ are finite dimensional. Therefore, $NA(X)$ is not spaceable.*

Proof. It follows from [9, Exercise IX.4] that X^* has the Schur property.

Let $M \subseteq NA(X)$ be a norm closed subspace. Let $(x_n^*) \subseteq B_M$. By [23], (x_n^*) has a w^*-convergent subsequence $(x_{n_k}^*)$. As in the proof of Lemma 2.7, $(x_{n_k}^*)$ is weakly Cauchy, and hence, by the Schur property, norm convergent. Thus, B_M is norm compact, and hence, $\dim(M) < \infty$. □

Note that Proposition 2.22 applies in particular to $X = C(K)$ for a scattered compact set K, and more generally, to any space whose dual is isomorphic to $\ell^1(\Gamma)$ for some set Γ. In the case of isometric preduals of $\ell^1(\Gamma)$, the result can also be shown through Lemma 2.7 and the fact that such a space X has Pełczynski's property (V) [27].

It seems natural to conjecture that Proposition 2.22 applies to spaces which are isomorphic to polyhedral spaces (see [13] and references thereof for this notion). However, it is not known whether the dual of a separable polyhedral space can contain an infinite dimensional reflexive space [12].

We now investigate some cases where $NA(X)$ itself is not a linear space. The following proposition is essentially from [3]. We include the details for completeness. We refer to [7, 21] for more precise results on the topological properties of the set $NA(X)$.

Proposition 2.23. *Let X be a Banach space with the RNP or an almost LUR norm. Then* $\operatorname{span}(NA(X)) = X^*$. *In particular, if $NA(X)$ is a vector space, then X must be reflexive.*

Consequently, X is reflexive if and only if the intersection of any two proximinal hyperplanes in X is proximinal.

Proof. As noted above, if X is almost LUR, then,

$$NA(X) = \{x^* \in X^* : \text{the norm on } X^* \text{ is Fréchet smooth at } x^*\} \cup \{0\}.$$

It is well known that the points of Fréchet smoothness of a dual norm always forms a G_δ set in X^* and is contained in $NA(X)$. Moreover, if X has the RNP, this set is also dense. Thus, in either case, $NA(X)$ is residual. It now follows from Baire Category Theorem that $NA(X) - NA(X) = X^*$. Thus, $\operatorname{span}(NA(X)) = X^*$.

If $NA(X)$ is a vector space, $NA(X) = X^*$. By James' theorem, this implies X is reflexive.

Now, suppose the intersection of any two proximinal hyperplanes is proximinal. Let $x^*, y^* \in NA(X)$. Then $\ker x^*$ and $\ker y^*$ are proximinal hyperplanes in X. If $Y = \ker x^* \cap \ker y^*$ is proximinal in X, then $Y^\perp \subseteq NA(X)$. This implies $NA(X)$ is a linear subspace of X^*. □

In a dual space, Proposition 2.23 can be improved. Indeed we have

Proposition 2.24. *If $NA(X^*) \subseteq X^{**}$ is a vector space, then X is weakly sequentially complete.*

Proof. If X is not weakly sequentially complete, there exists $x_0^{**} \in X^{**} \setminus X$ of 1st Baire class. By [15, Theorem II.11], then there exists $x \in X$ such that $x^{**} = x_0^{**} + x \in NA(X^*) \setminus X$. Now if $x^{**} = w^*\text{-}\lim x_n$, there exists $x_0 \in \overline{co}(x_n)$ such that $x^{**} - x_0 \notin NA(X^*)$ ([31], see [18, Remark II.20.1]). This shows that $NA(X^*)$ is not a linear subspace. $\square$

Question 2.25. Does there exist a nonreflexive Banach space X such that $NA(X^*)$ is a vector space?

We conclude on a last open problem, which shows how little we know of the structure of the set of norm-attaining functionals.

Question 2.26. Let X be an infinite dimensional Banach space. Does there exist a linear space of dimension 2 contained in $NA(X)$?

It might even be that every infinite dimensional Banach space X contains a proximinal subspace of codimension 2, and this would immediately imply a positive answer to this last question.

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