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# Extended Sums and Extended Compositions of Monotone Operators*

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*Dedicated to the memory of Simon Fitzpatrick.*

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We study extended notions for sums of monotone operators and for precompositions of monotone operators with continuous linear mappings. First, we establish some new properties related to the notion of extended sum recently proposed by Revalski and Théra, among them the monotonicity of this sum provided the operators involved are maximal monotone. Then, we show that the equivalence which exists between usual sums and compositions remains valid also for the extended operations. This allows us to obtain some known and new results for extended compositions via the corresponding properties of extended sums.

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## 1. Introduction

Since the very beginning of the study of monotone operators, an important question to investigate has been what is the operator obtained by summing up (in the usual Minkowski sense) two (or more) maximal monotone operators. Such a sum is always a monotone operator, but not necessarily a maximal one. This phenomenon is well known, for example, when one considers the sum of subdifferentials of proper convex lower semicontinuous functions: without additional assumptions this sum may not be maximal (equivalently,

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may be strictly smaller than the subdifferential of the sum of the functions). The investigation of this issue has been developed in two directions: the first one has been to find appropriate sufficient conditions under which the sum of two maximal monotone operators is maximal monotone – see, e.g., [18, 25] and the references therein; the second one has consisted in searching for possible notions of generalized sums of operators which are more likely to be maximal than the usual sum. Such attempts have been made in [11, 12] (for example, the so-called *parallel sum*) and more recently in [1], where the authors introduced and studied the concept of *variational sum* of monotone operators in Hilbert spaces based on the Yosida regularization of the operators involved. The latter notion naturally extends to the case of reflexive spaces ([21]). For example, it turned out that, in such a setting, the variational sum of subdifferentials of proper convex lower semicontinuous functions is always (without any qualification condition) a maximal monotone operator.

Another concept of generalized sum, called *extended sum*, was proposed in [20, 21]. It was based on the notion of enlargement of monotone operators (see below the precise definitions) and works in any Banach space. The extended sum of maximal monotone operators has turned out to be maximal monotone in some cases where the usual sum is not necessarily maximal – for instance, again, in the case of subdifferentials of convex functions.

Closely related (and in a certain sense equivalent) to the operation of sum of monotone operators is the precomposition of monotone operators with continuous linear mappings. Here we observe the same phenomenon as in the sum problem: the precomposition of a maximal monotone operator with a continuous linear mapping is monotone but not necessarily maximal monotone. Not surprisingly, the two directions of investigations mentioned above for the sum of operators have been also explored for the composition problem – see, e.g., [14, 23] for sufficient conditions and [7, 15, 17] for generalized compositions.

Let us recall that there is a sort of equivalence between the operations of (usual) sum and composition: any precomposition of a (maximal) monotone operator with a continuous linear mapping can be expressed as a sum of (maximal) monotone operators and, vice versa, the sum of (maximal) monotone operators can be given the form of precomposition of a (maximal) monotone operator with a continuous linear mapping.

The aim of this paper is twofold: after some preliminary notions and results in Section 2, we study first, in Section 3, the extended sum of monotone operators. We establish some new properties of this sum, among them the fact that it is always monotone, provided both operators are maximal. This allows to derive easily some known results about the extended sum, originally proved in a direct way. The question of coincidence of the usual and the extended sum under qualification conditions is also studied in this section. Then, in Section 4, we show that the equivalence between the operations of usual sums and compositions described above fully extends to the operations of extended sums and compositions. This allows one to obtain known and new results about compositions from the companion ones for sums of operators.

## 2. Preliminary notions and results

Throughout $X$ will denote a real Banach space and $X^*$ its continuous dual. With slight abuse of notation we will use the same symbol $\|\cdot\|$ for the norm in $X$ and $X^*$. As usual, $w^*$ will stand for the weak star topology in $X^*$. The pairing between $X$ and $X^*$ is designated by $\langle \cdot, \cdot \rangle$.

Given a set-valued operator $T : X \rightrightarrows X^*$ its *graph* is the set

$$\mathrm{Gr}(T) := \{(x, x^*) \in X \times X^* : x^* \in Tx\}.$$

The projection of the graph of $T$ on $X$ is the *domain* $\mathrm{Dom}\, T$ of $T$

$$\mathrm{Dom}\, T := \{x \in X : \ Tx \neq \emptyset\},$$

and the projection on $X^*$ is its *range* $\mathrm{R}(T)$

$$\mathrm{R}(T) := \bigcup \{Tx : x \in \mathrm{Dom}\, T\}.$$

A set-valued operator $T : X \rightrightarrows X^*$ is said to be *monotone* if it satisfies:

$$\langle y^* - x^*, y - x\rangle \geq \ 0 \quad \text{for every two pairs } (x, x^*), (y, y^*) \in \mathrm{Gr}(T).$$

The monotone operator $T : X \rightrightarrows X^*$ is called *maximal monotone* if its graph is not contained properly in the graph of any other monotone operator between $X$ and $X^*$. Equivalently, a monotone operator $T$ is maximal monotone if every pair $(y, y^*)$ which is *monotonically related* to $\mathrm{Gr}(T)$ (i.e. $\langle x^* - y^*, x - y\rangle \geq 0$ for any $(x, x^*) \in \mathrm{Gr}(T)$) belongs to $\mathrm{Gr}(T)$. It is easily seen that every maximal monotone operator has convex $w^*$-closed values in $X^*$ and a norm closed graph in $X \times X^*$.

Among the best known examples of maximal monotone operators are the subdifferentials of proper convex lower semicontinuous functions. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex extended real-valued function in $X$. Recall that $f$ is *proper* if its *effective domain* $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$ is nonempty. Given $\varepsilon \geq 0$, the *$\varepsilon$-subdifferential* $\partial_\varepsilon f$ of $f$ at $x \in X$ is the set:

$$\partial_\varepsilon f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle - \varepsilon \quad \text{for every } y \in X\}.$$

The case $\varepsilon = 0$ gives the *subdifferential* of $f$ at $x$, which is denoted by $\partial f(x)$. If $f$ is proper convex and lower semicontinuous, then for any $\varepsilon > 0$ we have $\partial_\varepsilon f(x) \neq \emptyset$ for each $x \in \mathrm{dom}\, f$, while $\partial_\varepsilon f(x) = \emptyset$ if $x \notin \mathrm{dom}\, f$. On the contrary, the subdifferential $\partial f$ does not have to be nonempty at each point of the domain of $f$. A classical result of Rockafellar [24] asserts that the subdifferential of any proper convex lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ on a Banach space $X$ is a maximal monotone operator.

Motivated by the above notion of $\varepsilon$-subdifferential, a concept of enlargement for an arbitrary monotone operator $T : X \rightrightarrows X^*$ was proposed: given $\varepsilon > 0$, let $T^\varepsilon : X \rightrightarrows X^*$ be defined as follows:

$$T^\varepsilon x := \{x^* \in X^* : \langle y^* - x^*, y - x\rangle \geq -\varepsilon \quad \text{for every } (y, y^*) \in \mathrm{Gr}(T)\}. \tag{1}$$

This operator was mentioned in [13] without study. A systematic investigation of its properties has been done in a series of papers by Burachik et al. – see, e.g., [3, 4, 5, 27]. It is immediately seen from the definition that $T^\varepsilon$ has always convex $w^*$-closed values and that it is really an extension of $T$ because of the monotonicity of $T$. Moreover, $T^{\varepsilon_1} \subset T^{\varepsilon_2}$, provided $0 \leq \varepsilon_1 \leq \varepsilon_2$. It is also worth noting that $T$ is maximal monotone if and only if

$T = T^0 = \bigcap_{\varepsilon>0} T^\varepsilon$. In the particular case of the subdifferential of a proper convex lower semicontinuous function $f$ one has $\partial_\varepsilon f(x) \subset \partial^\varepsilon f(x)$ and the inclusion can be strict (for example, for $f(x) = x^2/2$, $x \in \mathbb{R}$, see, e.g., [13, 3]).

Further, to any monotone operator $T : X \rightrightarrows X^*$, we associate the *Fitzpatrick function* $\varphi_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi_T(x, x^*) := \sup_{(y,y^*)\in\mathrm{Gr}(T)} (\langle y^*, x\rangle - \langle y^*, y\rangle + \langle x^*, y\rangle) = \sup_{(y,y^*)\in\mathrm{Gr}(T)} (\langle y^* - x^*, x - y\rangle) + \langle x^*, x\rangle.$$

This function introduced by Simon Fitzpatrick in [6, Definition 3.1] has turned out to be a very convenient and useful tool in monotone operator theory (see, e.g., [2, 26, 27]). Clearly, $\varphi_T$ is proper, convex and lower semicontinuous. Moreover, $\varphi_T$ verifies the following nice properties: for any $\varepsilon \geq 0$,

$$x^* \in T^\varepsilon x \Longleftrightarrow \varphi_T(x, x^*) \leq \varepsilon + \langle x^*, x\rangle,$$

and, if $T$ is maximal monotone (see [6]), then $\varphi_T$ is the minimal convex function such that

$$\langle x^*, x\rangle \leq \varphi_T(x, x^*) \quad \forall\, (x, x^*) \in X \times X^*, \quad \text{and} \quad \langle x^*, x\rangle = \varphi_T(x, x^*) \quad \forall\, (x, x^*) \in \mathrm{Gr}(T).$$

Using these properties one obtains:

**Proposition 2.1.** *Let $T : X \rightrightarrows X^*$ be a monotone operator.*

(*1*) *If $\mathrm{pr}_X$ denotes the usual projection of $X \times X^*$ on $X$, then*

$$\mathrm{pr}_X \operatorname{dom} \varphi_T = \bigcup_{\varepsilon>0} \operatorname{Dom} T^\varepsilon;$$

(*2*) ([5, 27]) *For $T$ maximal monotone and $\varepsilon \geq 0$, the enlargement $T^\varepsilon$ satisfies:*

$$\langle x^* - y^*, x - y\rangle \geq -4\varepsilon \quad \forall\, (x, x^*), (y, y^*) \in \mathrm{Gr}(T^\varepsilon).$$

**Remark 2.2.** An operator $T : X \rightrightarrows X^*$ is called *$\varepsilon$-monotone*, $\varepsilon \geq 0$, if

$$\langle x^* - y^*, x - y\rangle \geq -\varepsilon \quad \forall\, (x, x^*), (y, y^*) \in \mathrm{Gr}(T).$$

This class of operators was introduced and studied by Veselý [31], who showed that any such operator is locally bounded on the interior of its domain $\operatorname{Dom} T$ (loc. cit., Theorem 1 and Corollary 1). In particular, by Proposition 2.1(*2*) any enlargement $T^\varepsilon$, $\varepsilon \geq 0$, of a maximal monotone operator $T$ is locally bounded on the interior of its domain $\operatorname{Dom} T^\varepsilon$.

The next two propositions give examples of monotone operators $T$ which cannot be "$\varepsilon$-enlarged", or, said differently, which are "strongly" maximal monotone in the sense that $T = T^\varepsilon$ for any $\varepsilon \geq 0$. These examples will be helpful in the sequel.

If $L \subset X$ is a linear subspace, we denote by $L^\perp$ the *annihilator* of $L$,

$$L^\perp := \{x^* \in X^* : \text{for all } x \in L, \langle x^*, x\rangle = 0\},$$

and by $N_L : X \rightrightarrows X^*$ the *normal cone mapping to* $L$,

$$N_L(x) := L^\perp, \quad \text{for } x \in L, \quad N_L(x) := \emptyset, \quad \text{for } x \notin L,$$

equivalently, $\mathrm{Gr}(N_L) = L \times L^\perp$.

**Proposition 2.3.** *Let $L \subset X$ be a closed linear subspace. Then $N_L^\varepsilon = N_L$ for any $\varepsilon \geq 0$.*

**Proof.** Let $(v, v^*) \in \mathrm{Gr}(N_L^\varepsilon)$, i.e.,

$$\langle u^* - v^*, u - v\rangle \geq -\varepsilon, \quad \forall\, (u, u^*) \in \mathrm{Gr}(N_L) = L \times L^\perp.$$

Letting $u = 0$ in this inequality gives $\langle u^*, v\rangle \leq \langle v^*, v\rangle + \varepsilon$ for all $u^* \in L^\perp$, hence $\langle u^*, v\rangle = 0$ for all $u^* \in L^\perp$, that is $v \in L$. Letting $u^* = 0$ gives $\langle v^*, v - u\rangle \geq -\varepsilon$ for all $u \in L$, hence $\langle v^*, u\rangle = 0$ for all $u \in L$, showing that $v^* \in L^\perp$. □

Recall that a linear operator $T : \mathrm{Dom}\, T \to X^*$ defined on a linear subspace $\mathrm{Dom}\, T \subset X$ is said to be *anti-symmetric* (or *skew*) if for every $x \in \mathrm{Dom}\, T$ one has $\langle Tx, x\rangle = 0$. In particular, every such $T$ is monotone. When $\mathrm{Dom}\, T$ is dense in $X$, we define (as in [19]) $\mathrm{Dom}\, T^* \subset X^{**}$ to be the set

$$\mathrm{Dom}\, T^* := \{x^{**} \in X^{**} : \mathrm{Dom}\, T \ni y \mapsto \langle x^{**}, Ty\rangle \text{ is continuous}\}.$$

Since $\mathrm{Dom}\, T$ is dense, for every $x^{**} \in \mathrm{Dom}\, T^*$ there is a unique continuous linear functional $T^* x^{**} \in X^*$ characterized by the identity

$$\langle T^* x^{**}, y\rangle = \langle x^{**}, Ty\rangle, \quad \text{for all } y \in \mathrm{Dom}\, T.$$

The linear operator $T^* : \mathrm{Dom}\, T^* \to X^*$ thus defined is called the *adjoint* of $T$. In the following proposition (which extends [19, Lemma 4.4]), we view $X$ as a subspace in $X^{**}$.

**Proposition 2.4.** *Let $T : \mathrm{Dom}\, T \to X^*$ be linear and anti-symmetric with dense domain $\mathrm{Dom}\, T$. Then, $\mathrm{Dom}\, T^* \cap X = \mathrm{pr}_X \operatorname{dom} \varphi_T$ and $T^\varepsilon = -T^*|_{\mathrm{Dom}\, T^\varepsilon}$ for every $\varepsilon \geq 0$. Further, $T = T^\varepsilon$ for every $\varepsilon \geq 0$ if and only if $\mathrm{Dom}\, T = \mathrm{Dom}\, T^* \cap X$. In particular, if $\mathrm{Dom}\, T = X$, then $T = T^\varepsilon$ for every $\varepsilon \geq 0$.*

**Proof.** Let $\hat{x}$ denote the canonical image of $x \in X$ in $X^{**}$. We claim that for $\varepsilon \geq 0$,

$$(x, x^*) \in \mathrm{Gr}(T^\varepsilon) \Longleftrightarrow \langle x^*, x\rangle \geq -\varepsilon \text{ and } T^*\hat{x} = -x^*. \tag{2}$$

Indeed, let $(x, x^*) \in \mathrm{Gr}(T^\varepsilon)$. Then, for all $y \in \mathrm{Dom}\, T$, $\langle x^* - Ty, x - y\rangle \geq -\varepsilon$. In particular, $\langle x^*, x\rangle \geq -\varepsilon$. Moreover, since $\langle Ty, y\rangle = 0$, the former inequality is equivalent to

$$\langle x^*, x\rangle - \langle Ty, x\rangle - \langle x^*, y\rangle \geq -\varepsilon, \quad \forall\, y \in \mathrm{Dom}\, T,$$

and since $\mathrm{Dom}\, T$ is a linear subspace, we derive that

$$\langle Ty, x\rangle + \langle x^*, y\rangle = 0, \quad \forall\, y \in \mathrm{Dom}\, T.$$

Now this equality shows that the linear operator $\mathrm{Dom}\, T \ni y \mapsto \langle Ty, x\rangle$ is continuous, hence $\hat{x} \in \mathrm{Dom}\, T^*$ and $T^*\hat{x} = -x^*$. Conversely, let $(x, x^*)$ be such that $T^*\hat{x} = -x^*$ and $\langle x^*, x\rangle \geq -\varepsilon$. Then, for all $y \in \mathrm{Dom}\, T$, $\langle Ty, x\rangle = \langle T^*\hat{x}, y\rangle = \langle -x^*, y\rangle$, hence

$$\langle x^* - Ty, x - y\rangle = \langle x^*, x\rangle - \langle Ty, x\rangle - \langle x^*, y\rangle + \langle Ty, y\rangle = \langle x^*, x\rangle \geq -\varepsilon,$$

showing that $(x, x^*) \in T^\varepsilon$. This completes the proof of (2).

It follows from (2) that if $x \in \operatorname{Dom} T^\varepsilon$, then $x \in \operatorname{Dom} T^* \cap X$, and, conversely, if $x \in \operatorname{Dom} T^* \cap X$, then $x \in \operatorname{Dom} T^\varepsilon$ for some $\varepsilon > 0$, so $\operatorname{Dom} T^* \cap X = \bigcup_{\varepsilon>0} \operatorname{Dom} T^\varepsilon = \operatorname{pr}_X \operatorname{dom} \varphi_T$ by Proposition 2.1. It is also clear from (2) that if $x \in \operatorname{Dom} T^\varepsilon$, then $T^\varepsilon x = -T^*\hat{x}$.

Further, suppose that $\operatorname{Dom} T^* \cap X = \operatorname{Dom} T$ and let $x^* \in T^\varepsilon x$. Then, $x^* = -T^*\hat{x}$, so $x \in \operatorname{Dom} T^* \cap X = \operatorname{Dom} T$, hence, for all $y \in \operatorname{Dom} T$ we have $\langle -T^*\hat{x}, y\rangle = -\langle Ty, x\rangle = \langle Tx, y\rangle$, because $T$ is anti-symmetric. This implies that $x^* = -T^*\hat{x} = Tx$, for $\operatorname{Dom} T$ is dense. Thus, $T^\varepsilon x = Tx$. Conversely, if $T = T^\varepsilon$ for every $\varepsilon \geq 0$, then $\operatorname{Dom} T = \bigcup_{\varepsilon\geq 0} \operatorname{Dom} T^\varepsilon = \operatorname{Dom} T^* \cap X$. The proof is complete. □

## 3. Extended sum of monotone operators

It is evident that given two proper convex lower semicontinuous functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$, the Minkowski sum of the subdifferentials of $f$ and $g$ is contained in the subdifferential of the sum $f + g$. This inclusion is, in general, strict without additional assumptions. However, the concept of $\varepsilon$-subdifferential allows to have an exact formula for the subdifferential of the sum as it was shown by Hiriart-Urruty and Phelps (other representations of the subdifferential of the sum of convex functions can be found in [1, 8, 10, 16, 21, 28, 29]):

**Theorem 3.1.** ([9]) *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper convex lower semicontinuous functions. Then for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$ one has:*

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon f(x) + \partial_\varepsilon g(x)}^{w^*},$$

*where the notation $\overline{A}^{w^*}$ means the closure of a set $A \subset X^*$ with respect to the weak star topology in $X^*$.*

Let now $T_1, T_2 : X \rightrightarrows X^*$ be monotone operators. Their usual (Minkowski) sum $(T_1 + T_2)(x) = T_1 x + T_2 x$, $x \in X$, is always a monotone operator with domain $\operatorname{Dom} T_1 \cap \operatorname{Dom} T_2$. But this sum is not necessarily maximal monotone, when $T_1$ and $T_2$ are maximal monotone – a classical situation where this lack of maximality occurs is the case of subdifferentials of convex functions. Thus, attempts for finding a notion of generalized sum of monotone operators have been done as, e.g., in [1, 11, 12]. Motivated by the above theorem of Hiriart-Urruty and Phelps, a concept of extended sum, based on the notion of enlargement of a monotone operator, was proposed in [20, 22]:

**Definition 3.2.** ([20, 22]) Let $T_1, T_2 : X \rightrightarrows X^*$ be monotone operators. The *extended sum* of $T_1$ et $T_2$ at the point $x \in X$ is defined by the formula:

$$(T_1 \underset{\text{ext}}{+} T_2)(x) = \bigcap_{\varepsilon>0} \overline{T_1^\varepsilon x + T_2^\varepsilon x}^{w^*}.$$

Obviously, the extended sum of two monotone operators is commutative and contains their usual sum. In general, the extended sum need not be monotone as the following simple example shows:

**Example 3.3.** Let $X = \mathbb{R}$ and consider the operators $T_1, T_2$ defined by: $T_1 x = 0$, if $x = 0$ and $T_1 x = \emptyset$ otherwise, and $T_2 x = x = \partial(\frac{1}{2}x^2)$, $x \in \mathbb{R}$. Then

$$T_1^\varepsilon x = \begin{cases} [-\frac{\varepsilon}{x}, +\infty), & \text{if } x > 0 \\ (-\infty, +\infty), & \text{if } x = 0 \\ (-\infty, -\frac{\varepsilon}{x}], & \text{if } x < 0, \end{cases}$$

while it is known that (see, e.g., [13, 3]) $T_2^\varepsilon x = [\, x - 2\sqrt{\varepsilon}, x + 2\sqrt{\varepsilon}\,]$ for any $x \in \mathbb{R}$. Thus, we obtain that

$$(T_1 \underset{\text{ext}}{+} T_2)(x) = \begin{cases} [\, x, +\infty\,), & \text{if } x > 0 \\ (-\infty, +\infty), & \text{if } x = 0 \\ (-\infty, x], & \text{if } x < 0. \end{cases}$$

The latter operator is easily seen to be non monotone.

However, if both operators are maximal monotone, then the extended sum is always monotone:

**Proposition 3.4.** *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone. Then $T_1 \underset{\text{ext}}{+} T_2$ is monotone.*

**Proof.** Let $(x, x^*), (y, y^*) \in \mathrm{Gr}(T_1 \underset{\text{ext}}{+} T_2)$ and fix some $\varepsilon > 0$. Since $x^* \in \overline{T_1^\varepsilon x + T_2^\varepsilon x}^{w^*}$ and $y^* \in \overline{T_1^\varepsilon y + T_2^\varepsilon y}^{w^*}$ there are nets $\{u_{x,\alpha}^*\} \subset T_1^\varepsilon x$ and $\{v_{x,\alpha}^*\} \subset T_2^\varepsilon x$ such that

$$u_{x,\alpha}^* + v_{x,\alpha}^* \xrightarrow{w^*} x^* \tag{3}$$

and $\{u_{y,\alpha}^*\} \subset T_1^\varepsilon y$ and $\{v_{y,\alpha}^*\} \subset T_2^\varepsilon y$ so that

$$u_{y,\alpha}^* + v_{y,\alpha}^* \xrightarrow{w^*} y^* \tag{4}$$

Applying Proposition 2.1(*2*) we have that for any $\alpha$

$$\langle u_{y,\alpha}^* - u_{x,\alpha}^*, y - x\rangle \geq -4\varepsilon \quad \text{and} \quad \langle v_{y,\alpha}^* - v_{x,\alpha}^*, y - x\rangle \geq -4\varepsilon.$$

Thus, by summing up we obtain that for every $\alpha$

$$\langle (u_{y,\alpha}^* + v_{y,\alpha}^*) - (u_{x,\alpha}^* + v_{x,\alpha}^*), y - x\rangle \geq -8\varepsilon.$$

Passing to the limit and using (3) and (4) yield

$$\langle y^* - x^*, y - x\rangle \geq -8\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we may conclude that

$$\langle y^* - x^*, y - x\rangle \geq 0,$$

which shows that $T_1 \underset{\text{ext}}{+} T_2$ is monotone. The proof is completed. □

The above property allows us to obtain readily several results from [20, 22] which have been proved in a direct way. Namely, we have the following corollaries.

**Corollary 3.5 ([20, 22]).** *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone. If $T_1 + T_2$ is maximal monotone, then $T_1 + T_2 = T_1 \underset{\text{ext}}{+} T_2$.*

The proof of this corollary is obvious since the extended sum contains the usual one. Example 3.11 below shows that the converse in Corollary 3.5 is not true.

Much more important is the next result which shows that without any qualification condition the subdifferential of the sum of two proper convex lower semicontinuous functions is equal to the extended sum of their subdifferentials. This provides a class of operators for which the extended sum is always maximal monotone while the usual one may not be maximal.

**Corollary 3.6 ([20, 22]).** *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then, for any $x \in X$*

$$\partial(f+g)(x) = (\partial f \underset{\text{ext}}{+} \partial g)(x)$$

**Proof.** It follows from Theorem 3.1 that $\partial f \underset{\text{ext}}{+} \partial g$ contains $\partial(f+g)$. Now, because the operators $\partial f$ and $\partial g$ are maximal monotone, $\partial f \underset{\text{ext}}{+} \partial g$ is monotone by Proposition 3.4, hence it must be equal to $\partial(f+g)$ since the latter operator is maximal monotone. □

The next result shows that the usual and extended sum coincide under a qualification condition of Robinson-Rockafellar type. Recall that a subset $A \subset X$ is said to be *absorbing in* $X$ (denoted by $0 \in \operatorname{core}_X A$, or simply $0 \in \operatorname{core} A$) if for any $h \in X$ there is $t > 0$ with $th \in A$.

**Theorem 3.7.** *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators. Suppose that*

$$0 \in \operatorname{core}\,(\operatorname{pr}_X \operatorname{dom} \varphi_{T_1} - \operatorname{pr}_X \operatorname{dom} \varphi_{T_2})\,. \tag{5}$$

*Then:*

(*1*) *For any $\varepsilon \geq 0$ and $x \in X$, $T_1^\varepsilon x + T_2^\varepsilon x$ is $w^*$-closed,*
(*2*) $T_1 + T_2 = T_1 \underset{\text{ext}}{+} T_2$.

**Proof.** (*1*) Let $\varepsilon \geq 0$ and $x \in \operatorname{Dom}(T_1^\varepsilon + T_2^\varepsilon) = \operatorname{Dom} T_1^\varepsilon \cap \operatorname{Dom} T_2^\varepsilon$. Since the set $T_1^\varepsilon x + T_2^\varepsilon x$ is convex, by the Krein-Šmulian theorem it is sufficient to show that if $\{x_\alpha^*\} \subset T_1^\varepsilon x + T_2^\varepsilon x$ is a net such that $\{\|x_\alpha^*\|\}$ is bounded by some $K > 0$ and $x_\alpha^* \xrightarrow{w^*} x^*$, then $x^* \in T_1^\varepsilon x + T_2^\varepsilon x$. Take such a net and for each $\alpha$ let $u_\alpha^* \in T_1^\varepsilon x$ and $v_\alpha^* \in T_2^\varepsilon x$ be so that $x_\alpha^* = u_\alpha^* + v_\alpha^*$.

We claim that $\{u_\alpha^*\}$ (and hence, $\{v_\alpha^*\}$ as well) is bounded. In fact, according to the Banach-Steinhaus theorem, this amounts to showing that for every $h \in X$ the net $\{\langle u_\alpha^*, h\rangle\}$ is bounded from above. So, let $h \in X$. Our qualification assumption produces $t > 0$, $u \in \operatorname{pr}_X \operatorname{dom} \varphi_{T_1}$, and $v \in \operatorname{pr}_X \operatorname{dom} \varphi_{T_2}$ such that $th = u - v$. Then, using Proposition 2.1(*1*), take $\lambda > \varepsilon$, $u^* \in T_1^\lambda u$, and $v^* \in T_2^\lambda v$. Observe that $u_\alpha^* \in T_1^\lambda x$ and $v_\alpha^* \in T_2^\lambda x$, so from Proposition 2.1(*2*) we derive that

$$\begin{aligned}\langle u^*_\alpha, th\rangle &= \langle u^*_\alpha, u-v\rangle = \langle u^*_\alpha, u-x\rangle + \langle u^*_\alpha, x-v\rangle \\ &= \langle u^*_\alpha, u-x\rangle + \langle v^*_\alpha, v-x\rangle + \langle x^*_\alpha, x-v\rangle \\ &\le 4\lambda + \langle u^*, u-x\rangle + 4\lambda + \langle v^*, v-x\rangle + K\|x-v\|,\end{aligned}$$

proving that the net $\{\langle u^*_\alpha, h\rangle\}$ is bounded from above as required.

The net $\{u^*_\alpha\}$ being bounded, by the Banach-Alaoglu theorem there exists a subnet $\{u^*_{\alpha_\beta}\}$ which is $w^*$-convergent to some $\bar{u}^* \in X^*$. Since $T_1^\varepsilon x$ is $w^*$-closed we have $\bar{u}^* \in T_1^\varepsilon x$. The weak star convergence of $x^*_\alpha$ to $x^*$ implies that $v^*_{\alpha_\beta}$ is also $w^*$-convergent to some $\bar{v}^* \in T_2^\varepsilon x$. Thus, passing to the limit, we obtain $x^* = \bar{u}^* + \bar{v}^* \in T_1^\varepsilon x + T_2^\varepsilon x$, which completes the proof of the first assertion.

(*2*) Let $x^* \in (T_1 \underset{\text{ext}}{+} T_2)(x)$. It follows from the first assertion that, for every $k \in \mathbb{N}$,

$$x^* = u^*_k + v^*_k \quad \text{where } u^*_k \in T_1^{1/k}(x) \text{ and } v^*_k \in T_2^{1/k}(x).$$

Proceeding as above, for any $h \in X$ we can find $t > 0$, $\lambda > 1$, $(u, u^*) \in \mathrm{Gr}(T_1^\lambda)$, and $(v, v^*) \in \mathrm{Gr}(T_2^\lambda)$ such that, for every $k \in \mathbb{N}$,

$$\langle u^*_k, th\rangle \quad \le 4\lambda + \langle u^*, u-x\rangle + 4\lambda + \langle v^*, v-x\rangle + \langle x^*, x-v\rangle,$$

showing that the sequence $\{\langle u^*_k, h\rangle\}$ is bounded from above. Hence, there exist subnets $\{u^*_{k_\alpha}\}$ and $\{v^*_{k_\alpha}\}$ such that $u_{k_\alpha} \xrightarrow{w^*} x^*_1$ and $v_{k_\alpha} \xrightarrow{w^*} x^*_2$ for some $x^*_1, x^*_2 \in X^*$. Now, since $u^*_k$ verifies

$$\langle u^*_k - y^*, x-y\rangle \ge -\frac{1}{k} \quad \text{for all } (y, y^*) \in \mathrm{Gr}(T_1),$$

passing to the limit yields

$$\langle x^*_1 - y^*, x-y\rangle \ge 0 \quad \text{for all } (y, y^*) \in \mathrm{Gr}(T_1),$$

from which we derive that $(x, x^*_1) \in \mathrm{Gr}(T_1)$, because $T_1$ is maximal monotone. Similarly, we have $(x, x^*_2) \in \mathrm{Gr}(T_2)$. Thus, $x \in \mathrm{Dom}\, T_1 \cap \mathrm{Dom}\, T_2$ and $x^* = x^*_1 + x^*_2 \in (T_1 + T_2)(x)$, which completes the proof of the second assertion. □

**Remark 3.8.** (a) Assertion (*1*) in Theorem 3.7 is analogue to a result of Thibault [28] dealing with $\varepsilon$-subdifferentials of convex functions. The case $\varepsilon = 0$ in Assertion (*1*) is considered in [30] with a different (a priori stronger) qualification condition.

(b) In case $X$ is reflexive, Assertion (*2*) follows from a recent result of Simons-Zălinescu [26, Theorem 5.5] stating that, under a qualification condition weaker than (5), the (usual) sum $T_1 + T_2$ is maximal monotone; so, the usual and extended sums must coincide by Corollary 3.5.

As a consequence of the preceding results, we obtain the exact sum rule for subdifferentials under the Robinson-Rockafellar condition:

**Corollary 3.9.** *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous functions. If $0 \in \mathrm{core}\,(\mathrm{dom}\, f - \mathrm{dom}\, g)$, then $\partial f + \partial g = \partial(f+g)$.*

**Proof.** For any proper convex lower semicontinuous function $h$ and any $\varepsilon > 0$, we have $\operatorname{dom} h = \operatorname{Dom} \partial_\varepsilon h \subset \operatorname{Dom} \partial^\varepsilon h \subset \operatorname{pr}_X \operatorname{dom} \varphi_{\partial h}$, hence the Robinson-Rockafellar condition implies the qualification condition (5). The result then follows by combining Theorem 3.7 with Corollary 3.6. □

The final result of this section, together with the example which follows, shows that we may have coincidence of the usual and extended sums, without having neither the maximality of the sum nor the qualification condition (5) satisfied.

**Proposition 3.10.** *For $i = 1, 2$, let $T_i : \operatorname{Dom} T_i \subset X \to X^*$ be a linear anti-symmetric maximal monotone mapping with dense domain $\operatorname{Dom} T_i$. Then $T_1 + T_2 = T_1 \underset{\text{ext}}{+} T_2$.*

**Proof.** Let $x^* \in (T_1 \underset{\text{ext}}{+} T_2)(x)$. By Proposition 2.4, for every $\varepsilon > 0$, the set $T_1^\varepsilon x + T_2^\varepsilon x$ is the singleton $\{-T_1^*\hat{x} - T_2^*\hat{x}\}$ with $\langle -T_i^*\hat{x}, x\rangle \geq -\varepsilon$, for $i = 1, 2$. Therefore, we have $x^* = -T_1^*\hat{x} - T_2^*\hat{x}$ with $\langle -T_i^*\hat{x}, x\rangle \geq 0$, for $i = 1, 2$. It follows that, for $i = 1, 2$, it holds (using also the definition of $T^*$ and the anti-symmetry of $T_i$)

$$\langle -T_i^*\hat{x} - Ty, x - y\rangle \geq 0, \quad \forall\, y \in \operatorname{Dom} T_i.$$

The maximality of $T_i$ now implies that $x \in \operatorname{Dom} T_i$ and $-T_i^*\hat{x} = T_i x$, hence $x^* = T_1 x + T_2 x$. □

The following example (built by using an example from Phelps, [18, p. 29]) provides two operators $T_1$ and $T_2$ verifying the conditions of Proposition 3.10 with $T_1 + T_2$ not maximal monotone and (5) not verified.

**Example 3.11.** Let $X$ be the Hilbert space $l_2 \times l_2$ with the usual scalar product and norm and identify $X^*$ with $X$. Let $\operatorname{Dom} T := D \times D$ with

$$D := \{\{x_n\}_{n\in\mathbb{N}} \subset l_2 : \{2^n x_n\}_{n\in\mathbb{N}} \in l_2\},$$

and let $T : \operatorname{Dom} T \to X$ be defined by

$$T(\{x_n\}, \{y_n\}) := (\{2^n y_n\}, -\{2^n x_n\}).$$

Then $T_1 := T$ and $T_2 := -T$ are linear anti-symmetric with common dense domain $\operatorname{Dom} T$. We show that they are maximal monotone. By Proposition 2.4, it suffices to check that $\operatorname{Dom} T = \operatorname{Dom} T^*$. Let $v = (v_1, v_2) \in \operatorname{Dom} T^*$. Then,

$$\langle Tz, v\rangle = \langle z, T^*v\rangle, \quad \forall\, z \in \operatorname{Dom} T.$$

In particular, for $z := (e_n, 0) \in \operatorname{Dom} T$ and $z := (0, e_n) \in \operatorname{Dom} T$, where $\{e_n\}$ is the usual basis in $l_2$, we obtain that, for every $n \in \mathbb{N}$,

$$\langle T(e_n, 0), v\rangle = \langle (e_n, 0), T^*v\rangle \quad \text{and} \quad \langle T(0, e_n), v\rangle = \langle (0, e_n), T^*v\rangle.$$

Hence, we have the following relations for the coordinates of $v_i$ and $(T^*v)_i$, $i = 1, 2$:

$$-2^n v_{2,n} = (T^*v)_{1,n}, \quad 2^n v_{1,n} = (T^*v)_{2,n}, \quad n = 1, 2, \ldots$$

These equalities show that $v \in D \times D = \operatorname{Dom} T$ (and $T^*v = -Tv$). This proves that $\operatorname{Dom} T^* \subset \operatorname{Dom} T$, whence $\operatorname{Dom} T^* = \operatorname{Dom} T$. Thus, $T_1$ and $T_2$ are maximal monotone, so, by Proposition 3.10, $T_1 \underset{\text{ext}}{+} T_2 = T_1 + T_2$. Observe, however, that $T_1 + T_2$ is not maximal monotone, since $T_1 + T_2 \equiv 0$ and $\operatorname{Dom}(T_1 + T_2) = \operatorname{Dom} T \neq X$. The qualification condition (5) is not verified either, by the result of Simons-Zălinescu mentioned in Remark 3.8 (this can also be checked directly: since by Proposition 2.4, $\operatorname{pr}_X \operatorname{dom} \varphi_{T_i} = \operatorname{Dom} T_i^* = \operatorname{Dom} T$, for $i = 1, 2$, the condition (5) amounts to $0 \in \operatorname{core}(\operatorname{Dom} T - \operatorname{Dom} T) = \operatorname{core} \operatorname{Dom} T$, or, equivalently, to $X = \mathbb{R}_+ \operatorname{Dom} T = \operatorname{Dom} T$, which is not true).

## 4. Precompositions of monotone operators

Let $T : X \rightrightarrows X^*$ be a monotone operator in the Banach space $X$ and let $A : Y \to X$ be a continuous linear operator between the Banach spaces $Y$ and $X$. Denoting, as usual, by $A^*$ the adjoint of $A$ which acts between $X^*$ and $Y^*$, it is easily seen that the composition $A^*TA : Y \rightrightarrows Y^*$ is a monotone operator with domain $\operatorname{Dom}(A^*TA) = A^{-1}(\operatorname{Dom} T)$ (the latter is supposed always nonempty in the sequel). Such kind of operators appear, for example, in partial differential equations in divergence form or in some variational inequalities arising in mathematical economics (see, e.g., [14, 23, 15]). Without additional assumptions on $T$ and $A$, the composition $A^*TA$ need not be maximal monotone.

It is known that such compositions could be used to express sums of operators and vice versa, sums can be used to obtain compositions. Indeed, let $T_1, T_2 : X \rightrightarrows X^*$ be two monotone operators. Define $A : X \to X \times X$ by $Ax = (x, x)$ and $T : X \times X \rightrightarrows X^* \times X^*$ by $T(x, y) = T_1x \times T_2y$. One easily sees that $T$ is monotone (and maximal if $T_1, T_2$ are maximal) and, moreover, $T_1 + T_2 = A^*TA$.

Conversely, suppose we are given a monotone operator $T : X \rightrightarrows X^*$ and a continuous linear operator $A : Y \to X$, where $X$ and $Y$ are Banach spaces. Take $Y \times X$ with some usual product norm and consider the operators $N_A, \widetilde{T} : Y \times X \rightrightarrows Y^* \times X^*$ defined by

$$\begin{cases} N_A := N_{\operatorname{Gr}(A)}, \text{ the normal cone mapping to the closed linear subspace } \operatorname{Gr}(A), \\ \widetilde{T}(y, x) := \{0\} \times Tx, \text{ for } (y, x) \in Y \times X. \end{cases}$$

Observe that $N_A$ is maximal monotone with domain $\operatorname{Gr}(A)$, and that $\widetilde{T}$ is (maximal) monotone provided $T$ is so, with domain $Y \times \operatorname{Dom} T$. It is easily verified that

$$N_A(y, Ay) = \{(A^*x^*, -x^*) : x^* \in X^*\}, \quad \forall\, y \in Y, \tag{6}$$

and that

$$y^* \in A^*TAy \iff (y^*, 0) \in (N_A + \widetilde{T})(y, Ay), \tag{7}$$

or, more generally (in fact, equivalently),

$$y^* + A^*x^* \in A^*TAy \iff (y^*, x^*) \in (N_A + \widetilde{T})(y, Ay). \tag{8}$$

The above scheme shows that in a certain sense the operations of sums of monotone operators and precompositions of monotone operators with continuous linear mappings are equivalent. Therefore, it is not surprising that the results concerning either sums or compositions are similar, and often, mutually deducible. The main object of this section is to show that this kind of equivalence holds also for the extended notions related to these operations.

First, we have to introduce the notion of extended composition based on the same idea as for the extended sum. Such a notion was studied for subdifferentials by Fitzpatrick and Simons [7] and in a slightly different form by Penot [17] (for another type of extended composition, in the setting of reflexive Banach spaces and based on the Yosida regularization of the operator $T$, see [15]).

**Definition 4.1.** The *extended composition* of a continuous linear operator $A : Y \to X$ and a monotone operator $T : X \rightrightarrows X^*$ is the operator $(A^*TA)_{\text{ext}}(y) = \bigcap_{\varepsilon>0} \overline{A^*T^\varepsilon Ay}^{w^*}$.

Note that $(A^*TA)_{\text{ext}}$ is indeed an extension of $A^*TA$. The following theorem shows that the equivalence in (8) relating the usual operations of composition and sum is also true for $\varepsilon$-enlargements ($\varepsilon \geq 0$) and for the extended operations.

**Theorem 4.2.** *Let $T : X \rightrightarrows X^*$ be monotone and let $A : Y \to X$ be linear and continuous. Then:*

(*1*) *For any $\varepsilon \geq 0$, we have: $y^* + A^*x^* \in A^*T^\varepsilon Ay \Longleftrightarrow (y^*, x^*) \in (N_A + \widetilde{T}^\varepsilon)(y, Ay)$.*

(*2*) *Consequently: $y^* + A^*x^* \in (A^*TA)_{\text{ext}}(y) \Longleftrightarrow (y^*, x^*) \in (N_A \underset{\text{ext}}{+} \widetilde{T})(y, Ay)$.*

**Proof.** (*1*) By Proposition 2.3, $N_A^\varepsilon = N_A$ and, as easy computations show, for every $(y, x) \in Y \times X$, $\widetilde{T}^\varepsilon(y, x) = \{0\} \times T^\varepsilon x$. Therefore, if $y^* + A^*x^* \in A^*T^\varepsilon Ay$, then $Ay \in \operatorname{Dom} T^\varepsilon$ and $y^* + A^*x^* = A^*v^*$ for some $v^* \in T^\varepsilon Ay$. Hence, using (6),

$$(y^*, x^*) = (A^*v^* - A^*x^*, x^* - v^* + v^*) = (A^*(v^* - x^*), x^* - v^*) + (0, v^*) \in (N_A + \widetilde{T}^\varepsilon)(y, Ay).$$

Conversely, if $(y^*, x^*) \in (N_A + \widetilde{T}^\varepsilon)(y, Ay)$, by (6) there exist $v^* \in T^\varepsilon Ay$ and $u^* \in X^*$ such that $(y^*, x^*) = (A^*u^*, -u^* + v^*)$. This entails $y^* + A^*x^* = A^*u^* + A^*v^* - A^*u^* = A^*v^* \in A^*T^\varepsilon Ay$, completing the proof of Assertion (*1*).

(*2*) Let $y^* + A^*x^* \in (A^*TA)_{\text{ext}}(y)$ and fix $\varepsilon > 0$. By Definition 4.1, there exists a net $\{v_\alpha^*\} \subset T^\varepsilon Ay$ such that $A^*v_\alpha^* \xrightarrow{w^*} y^* + A^*x^*$. By Assertion (*1*), we have

$$(A^*v_\alpha^* - A^*x^*, x^*) \in (N_A^\varepsilon + \widetilde{T}^\varepsilon)(y, Ay), \quad \text{for each } \alpha,$$

so, passing to the limit, we get

$$(y^*, x^*) \in \overline{(N_A^\varepsilon + \widetilde{T}^\varepsilon)(y, Ay)}^{w^*},$$

and since $\varepsilon > 0$ was arbitrary we conclude that $(y^*, x^*) \in (N_A \underset{\text{ext}}{+} \widetilde{T})(y, Ay)$.

Conversely, let $(y^*, x^*) \in (N_A \underset{\text{ext}}{+} \widetilde{T})(y, Ay)$ and fix $\varepsilon > 0$. By Definition 3.2, there exists a net $\{(y_\alpha^*, x_\alpha^*)\} \subset (N_A^\varepsilon + \widetilde{T}^\varepsilon)(y, Ay)$ such that $(y_\alpha^*, x_\alpha^*) \xrightarrow{w^*} (y^*, x^*)$. Since $N_A^\varepsilon = N_A$, by Assertion (*1*), we have

$$y_\alpha^* + A^*x_\alpha^* \in A^*T^\varepsilon Ay, \quad \text{for each } \alpha,$$

so, passing to the limit and using the fact that $A^*$ is continuous from the weak star topology in $X^*$ to the weak star topology in $Y^*$, we get

$$y^* + A^*x^* \in \overline{A^*T^\varepsilon Ay}^{w^*},$$

and since $\varepsilon > 0$ was arbitrary, we conclude that $y^* + A^*x^* \in (A^*TA)_{\text{ext}}(y)$. □

With Theorem 4.2 in hand, it is straightforward to pass from a result on extended sum to its companion result on extended composition. Here are some examples.

**Proposition 4.3.** *Let $T : X \rightrightarrows X^*$ be maximal monotone and $A : Y \to X$ be linear and continuous. Then $(A^*TA)_{\text{ext}}$ is monotone.*

**Proof.** Let $y_i^* \in (A^*TA)_{\text{ext}}(y_i)$, $i = 1, 2$. By Theorem 4.2, $(y_i^*, 0) \in (N_A \underset{\text{ext}}{+} \widetilde{T})(y_i, Ay_i)$, $i = 1, 2$. Since $T$ is maximal, the same is true for $\widetilde{T}$, hence, by Proposition 3.4, $N_A \underset{\text{ext}}{+} \widetilde{T}$ is monotone. Therefore,

$$\langle y_1^* - y_2^*, y_1 - y_2\rangle = \langle (y_1^*, 0) - (y_2^*, 0), (y_1, Ay_1) - (y_2, Ay_2)\rangle \geq 0,$$

proving that $(A^*TA)_{\text{ext}}$ is monotone. □

As we did for sums, we derive from Proposition 4.3 that the usual composition and the extended one coincide if the former is maximal monotone (an example analogue to Example 3.11 shows that the converse is not true):

**Corollary 4.4.** *Let $T : X \rightrightarrows X^*$ be maximal monotone and $A : Y \to X$ be linear and continuous. If $A^*TA$ is maximal monotone, then $A^*TA = (A^*TA)_{\text{ext}}$.*

The “extended composition version" of Corollary 3.6 reads as follows:

**Corollary 4.5 ([7]).** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $A : Y \to X$ be a continuous linear operator with* $\mathrm{R}(A) \cap \operatorname{dom} f \neq \emptyset$. *Then*

$$\partial(f \circ A) = (A^*\partial f A)_{\text{ext}}.$$

**Proof.** Let $T := \partial f$. Then, for $(y, x) \in Y \times X$, $\widetilde{T}(y, x) = \{0\} \times \partial f(x) = \partial F(y, x)$ where $F : Y \times X \to \mathbb{R} \cup \{+\infty\}$ is given by $F(y, x) := f(x)$. It is easily seen (and well known, see, e.g., [9]) that

$$y^* \in \partial(f \circ A)(y) \Longleftrightarrow (y^*, 0) \in \partial(\delta_{\mathrm{Gr}(A)} + F)(y, Ay), \tag{9}$$

where $\delta_{\mathrm{Gr}(A)}$ is the indicator function of $\mathrm{Gr}(A)$, so that $\partial\delta_{\mathrm{Gr}(A)} = N_A$. By Corollary 3.6, $\partial(\delta_{\mathrm{Gr}(A)} + F) = N_A \underset{\text{ext}}{+} \widetilde{T}$. Putting this in (9) and using Theorem 4.2 we get

$$y^* \in \partial(f \circ A)(y) \Longleftrightarrow (y^*, 0) \in (N_A \underset{\text{ext}}{+} \widetilde{T})(y, Ay) \Longleftrightarrow y^* \in (A^*\partial f A)_{\text{ext}}(y),$$

which completes the proof. □

The above result shows a typical situation when the usual (pointwise) composition of a maximal monotone operator with a continuous linear operator is not necessarily maximal monotone, while their extended composition is always maximal, without any qualification condition.

As a further example, we write down the companion of Theorem 3.7:

**Theorem 4.6.** *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator and let $A : Y \to X$ be linear and continuous. Suppose that*

$$0 \in \operatorname{core}_X(\mathrm{R}(A) - \mathrm{pr}_X \operatorname{dom} \varphi_T). \tag{10}$$

*Then:*

(*1*) *For any $\varepsilon \geq 0$ and $y \in Y$, $A^* T^\varepsilon A y$ is $w^*$-closed,*
(*2*) *$A^* T A = (A^* T A)_{\text{ext}}$.*

**Proof.** Easy computations show that (10) is equivalent to

$$\mathrm{Gr}(A) - Y \times \mathrm{pr}_X \operatorname{dom} \varphi_T \;\text{ is absorbing in } Y \times X. \tag{11}$$

Moreover, it follows from Proposition 2.1 and Proposition 2.3 that

$$\mathrm{Gr}(A) = \operatorname{Dom} N_A = \mathrm{pr}_{Y \times X} \operatorname{dom} \varphi_{N_A}$$

and that

$$Y \times \mathrm{pr}_X \operatorname{dom} \varphi_T = \bigcup_{\varepsilon > 0} (Y \times \operatorname{Dom} T^\varepsilon) = \bigcup_{\varepsilon > 0} \operatorname{Dom} \widetilde{T}^\varepsilon = \mathrm{pr}_{Y \times X} \operatorname{Dom} \varphi_{\widetilde{T}}.$$

Hence, (11) is equivalent to

$$\mathrm{pr}_{Y \times X} \operatorname{dom} \varphi_{N_A} - \mathrm{pr}_{Y \times X} \operatorname{dom} \varphi_{\widetilde{T}} \;\text{ is absorbing in } Y \times X. \tag{12}$$

Applying Theorem 3.7, we get that

$$\text{for any } \varepsilon \geq 0 \text{ and } (y, x) \in Y \times X,\ N_A(y, x) + \widetilde{T}^\varepsilon(y, x) \text{ is } w^*\text{-closed}, \tag{13}$$

and

$$N_A + \widetilde{T} = N_A \underset{\text{ext}}{+} \widetilde{T}. \tag{14}$$

Now, Theorem 4.2 and (7) say that (13) and (14) are indeed equivalent to Assertion (*1*) and Assertion (*2*), respectively. □

Combining Theorem 4.6 with Corollary 4.5, we obtain the following well known result:

**Corollary 4.7.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous and let $A : Y \to X$ be linear and continuous. If $0 \in \operatorname{core}(\mathrm{R}(A) - \operatorname{dom} f)$, then $A^* \partial f A = \partial(f \circ A)$.*

Coming back to our program in this section, it remains to show that, symmetrically, extended sums can be viewed as extended compositions, as it is the case for the usual operations. This is the object of the next result.

**Theorem 4.8.** *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone. Define $A : X \to X \times X$ as $Ax := (x, x)$ and $T : X \times X \rightrightarrows X^* \times X^*$ as $T(x, y) := T_1 x \times T_2 y$. Then:*

(*1*) *For any $\varepsilon \geq 0$ and $x \in X$, $A^* T^\varepsilon A x \subset T_1^\varepsilon x + T_2^\varepsilon x \subset A^* T^{2\varepsilon} A x$,*
(*2*) *For any $x \in X$, $(T_1 \underset{\text{ext}}{+} T_2)(x) = (A^* T A)_{\text{ext}}(x)$.*

**Proof.** *(1)* We show first that for any $\varepsilon \geq 0$ and $x, y \in X$, the following chain of inclusions is true:

$$T^{\varepsilon}(x,y) \subset T_1^{\varepsilon}x \times T_2^{\varepsilon}y \subset T^{2\varepsilon}(x,y). \tag{15}$$

Indeed, let $(x^*, y^*) \in T^{\varepsilon}(x,y)$. By the definition of $T^{\varepsilon}$ we have:

$$\langle (x^*, y^*) - (v_1^*, v_2^*), (x,y) - (v_1, v_2)\rangle \geq -\varepsilon, \quad \forall\, ((v_1,v_2),(v_1^*,v_2^*)) \in \mathrm{Gr}(T),$$

which is equivalent to:

$$\langle x^* - v_1^*, x - v_1\rangle + \langle y^* - v_2^*, y - v_2\rangle \geq -\varepsilon, \quad \forall\, (v_i, v_i^*) \in \mathrm{Gr}(T_i), i = 1,2. \tag{16}$$

If $(x, x^*) \in \mathrm{Gr}(T_1)$ by putting $(v_1, v_1^*) = (x, x^*)$ we see that

$$\langle y^* - v_2^*, y - v_2\rangle \geq -\varepsilon, \quad \forall\, (v_2, v_2^*) \in \mathrm{Gr}(T_2).$$

If $(x, x^*) \notin \mathrm{Gr}(T_1)$ by the maximal monotonicity of $T$ there exists $(v_1, v_1^*) \in \mathrm{Gr}(T_1)$ with

$$\langle x^* - v_1^*, x - v_1\rangle < 0,$$

hence from the equation (16) we have again

$$\langle y^* - v_2^*, y - v_2\rangle \geq -\langle x^* - v_1^*, x - v_1\rangle - \varepsilon \geq -\varepsilon, \quad \forall\, (v_2, v_2^*) \in \mathrm{Gr}(T_2).$$

Therefore, $(y, y^*) \in \mathrm{Gr}(T_2^{\varepsilon})$. Following the same argument we obtain also $(x, x^*) \in \mathrm{Gr}(T_1^{\varepsilon})$. Consequently, we have the first inclusion from (15).

Further, let $x^* \in T_1^{\varepsilon}x$ and $y^* \in T_2^{\varepsilon}y$. By the definition of enlargement we have

$$\langle x^* - v_1^*, x - v_1\rangle \geq -\varepsilon, \quad \forall\, (v_1, v_1^*) \in \mathrm{Gr}(T_1)$$

and

$$\langle y^* - v_2^*, y - v_2\rangle \geq -\varepsilon, \quad \forall\, (v_2, v_2^*) \in \mathrm{Gr}(T_2).$$

By adding these two inequalities we obtain

$$\langle (x^*, y^*) - (v_1^*, v_2^*), (x,y) - (v_1, v_2)\rangle \geq -2\varepsilon, \quad \text{for all } ((v_1,v_2),(v_1^*,v_2^*)) \in \mathrm{Gr}(T),$$

i.e., $(x^*, y^*) \in T^{2\varepsilon}(x,y)$. Therefore, (15) is true.

Now using (15), for any $x \in X$ we have

$$A^*T^{\varepsilon}Ax \subset A^*(T_1^{\varepsilon}x \times T_2^{\varepsilon}x) \subset A^*T^{2\varepsilon}Ax,$$

from which Assertion *(1)* follows since $A^*(T_1^{\varepsilon}x \times T_2^{\varepsilon}x) = T_1^{\varepsilon}x + T_2^{\varepsilon}x$.

*(2)* Assertion *(1)* yields that for any $x \in X$,

$$\bigcap_{\varepsilon>0} \overline{A^*T^{\varepsilon}Ax}^{w^*} \subset \bigcap_{\varepsilon>0} \overline{T_1^{\varepsilon}x + T_2^{\varepsilon}x}^{w^*} \subset \bigcap_{\varepsilon>0} \overline{A^*T^{2\varepsilon}Ax}^{w^*},$$

from which we conclude that $(T_1 \underset{\mathrm{ext}}{+} T_2)(x) = (A^*TA)_{\mathrm{ext}}(x)$. □

It is immediate from Theorem 4.8 that the maximality of the extended sum is equivalent to the maximality of its representation as extended composition. As expected, the situation here again is symmetric: the maximality of the extended composition is equivalent to the maximality of its representation as extended sum. Details are given in the following proposition.

**Proposition 4.9.** *Let $T : X \rightrightarrows X^*$ be maximal monotone and let $A : Y \to X$ be linear and continuous. Then, $(A^*TA)_{\text{ext}}$ is maximal monotone if and only if $N_A \underset{\text{ext}}{+} \widetilde{T}$ is maximal monotone.*

**Proof.** Assume that $(A^*TA)_{\text{ext}}$ is maximal. Let $\xi^* := (v^*, u^*) \in Y^* \times X^*$ and $\xi := (v, u) \in Y \times X$ be such that

$$\langle \xi^* - \zeta^*, \xi - \zeta \rangle \geq 0, \quad \forall\, (\zeta, \zeta^*) \in \operatorname{Gr}(N_A \underset{\text{ext}}{+} \widetilde{T}). \tag{17}$$

We have to show that $\xi^* \in (N_A \underset{\text{ext}}{+} \widetilde{T})(\xi)$.

First, fix $\zeta \in \operatorname{Dom}(N_A + \widetilde{T}) \subset \operatorname{Gr}(A)$ and $\beta^* \in \widetilde{T}(\zeta)$. Then, for any $\alpha^* \in \operatorname{Gr}(A)^\perp = N_A(\zeta)$, we can use $\zeta^* := \alpha^* + \beta^* \in (N_A + \widetilde{T})(\zeta)$ in (17) to obtain

$$\langle \xi^*, \xi - \zeta \rangle - \langle \beta^*, \xi - \zeta \rangle \geq \langle \alpha^*, \xi - \zeta \rangle = \langle \alpha^*, \xi \rangle, \quad \forall\, \alpha^* \in \operatorname{Gr}(A)^\perp.$$

We derive that $\langle \alpha^*, \xi \rangle = 0$ for all $\alpha^* \in \operatorname{Gr}(A)^\perp$, hence $\xi \in \operatorname{Gr}(A)$, that is $u = Av$.

Next, take $\zeta := (y, Ay)$ and $\zeta^* := (y^*, 0)$ for $y^* \in (A^*TA)_{\text{ext}}(y)$. According to Theorem 4.2, we have $\zeta^* \in (N_A \underset{\text{ext}}{+} \widetilde{T})(\zeta)$. So, we can use $\zeta$, $\zeta^*$ in (17) to obtain

$$\langle (v^*, u^*) - (y^*, 0), (v, Av) - (y, Ay) \rangle \geq 0, \quad \forall\, (y, y^*) \in \operatorname{Gr}(A^*TA)_{\text{ext}}.$$

Expanding the pairing, we get

$$\langle v^* - y^*, v - y \rangle + \langle u^*, Av - Ay \rangle \geq 0, \quad \forall\, (y, y^*) \in \operatorname{Gr}(A^*TA)_{\text{ext}},$$

and finally, using the definition of the adjoint operator,

$$\langle v^* + A^*u^* - y^*, v - y \rangle \geq 0, \quad \forall\, (y, y^*) \in \operatorname{Gr}(A^*TA)_{\text{ext}}.$$

From the maximality of $(A^*TA)_{\text{ext}}$ we then derive that $v^* + A^*u^* \in (A^*TA)_{\text{ext}}(v)$, which means that $(v^*, u^*) \in (N_A \underset{\text{ext}}{+} \widetilde{T})(v, u)$ by Theorem 4.2 again.

Conversely, assume that $N_A \underset{\text{ext}}{+} \widetilde{T}$ is maximal and let $(z, z^*) \in Y \times Y^*$ be such that

$$\langle z^* - y^*, z - y \rangle \geq 0, \quad \forall\, (y, y^*) \in \operatorname{Gr}(A^*TA)_{\text{ext}}.$$

According to Theorem 4.2, this is equivalent to

$$\langle z^* - y^* - A^*x^*, z - y \rangle \geq 0, \quad \forall\, ((y, Ay), (y^*, x^*)) \in \operatorname{Gr}(N_A \underset{\text{ext}}{+} \widetilde{T}),$$

or,

$$\langle (z^*,0)-(y^*,x^*),(z,Az)-(y,Ay)\rangle \quad \forall\,((y,Ay),(y^*,x^*)) \in \mathrm{Gr}(N_A \underset{\mathrm{ext}}{+} \widetilde{T}).$$

From the maximality of $N_A \underset{\mathrm{ext}}{+} \widetilde{T}$ we conclude that $(z^*,0) \in N_A \underset{\mathrm{ext}}{+} \widetilde{T}(z,Az)$, hence $z^* \in (A^*TA)_{\mathrm{ext}}(z)$, by Theorem 4.2. This completes the proof. □

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